Teaching Notes #1 (Addendum)

An Example of Portfolio Selection with time-varying opportunity set in Complete Markets

Pietro Veronesi
Graduate School of Business
University of Chicago
Business 35909 Spring 2005

This Version: November 17, 2005

1 The Portfolio Problem

- Consider the following portfolio problem with time-varying expected returns.
- Let $(\beta, \mathbf{S})$ be processes of the form

$$d\beta_t = r_t \beta_t dt \text{ with } \beta_0 > 0 \tag{1}$$

$$dS_t^i = \mu_t^i S_t^i dt + S_t^i \boldsymbol{\sigma}_t^i d\mathbf{B}_t \text{ with } S_0^i > 0 \tag{2}$$

- where r_t, $\boldsymbol{\mu}_t = \left(\mu_t^1, ..., \mu_t^d\right)$ and $\boldsymbol{\sigma}^i$ are bounded, adapted processes.

- For concreteness, assume that $\mu_t = \left(\mu_t^1, ..., \mu_t^d\right)$ follow a continuous time, VAR process

$$d\boldsymbol{\mu}_t = \left(\mathbf{A}_0 + \mathbf{A}_1 \boldsymbol{\mu}_t\right) dt + \boldsymbol{\Sigma} d\mathbf{B}_t$$

- Notice that the set of Brownian motions moving $\boldsymbol{\mu}_t$ is the same as the ones moving S_t.
- Instead, assume $r_t = r$ is constant and that $\boldsymbol{\sigma}_t^i = \boldsymbol{\sigma}$ is also constant, for every i.
- Assume again that $\{\boldsymbol{\sigma}\}$ is invertible.

1.1 The dynamics of the market price of risk

- Define by $\boldsymbol{\lambda}_t = \boldsymbol{\mu}_t - r\mathbf{1}_d$ the excess return process.
- Let the $d \times 1$ market price of risk process

$$\boldsymbol{\nu}_t = \boldsymbol{\sigma}^{-1} \boldsymbol{\lambda}_t$$

- Clearly, also $\boldsymbol{\nu}_t$ follows a VAR process

$$\begin{aligned} d\boldsymbol{\nu}_t &= \boldsymbol{\sigma}^{-1} d\boldsymbol{\mu}_t \\ &= \left(\boldsymbol{\sigma}^{-1}\mathbf{A}_0 + \boldsymbol{\sigma}^{-1}\mathbf{A}_1\boldsymbol{\mu}_t\right) dt + \boldsymbol{\sigma}^{-1}\boldsymbol{\Sigma} d\mathbf{B}_t \\ &= \left(\widetilde{\mathbf{A}}_0 + \widetilde{\mathbf{A}}_1 \boldsymbol{\nu}_t\right) dt + \widetilde{\boldsymbol{\Sigma}} d\mathbf{B}_t \end{aligned}$$

- with $\widetilde{\mathbf{A}}_0 = \boldsymbol{\sigma}^{-1}\mathbf{A}$, $\widetilde{\mathbf{A}}_1 = \boldsymbol{\sigma}^{-1}\mathbf{A}_1\boldsymbol{\sigma}$ and $\widetilde{\Sigma} = \boldsymbol{\sigma}^{-1}\boldsymbol{\Sigma}$.

- **Result**: Given an initial condition $\boldsymbol{\nu}_0 = \widehat{\boldsymbol{\nu}}$, then for $\tau > 0$

$$\boldsymbol{\nu}_\tau \sim N\left(\boldsymbol{\alpha}\left(\boldsymbol{\nu}_0, \tau\right), \mathbf{S}\left(\tau\right)\right)$$

- where

$$\begin{aligned} \boldsymbol{\alpha}\left(\boldsymbol{\nu}_0, \tau\right) &= \boldsymbol{\Psi}\left(\tau\right)\boldsymbol{\nu}_0 + \boldsymbol{\zeta}\left(\tau\right) \\ \mathbf{S}\left(\tau\right) &= \int_0^\tau \boldsymbol{\Psi}\left(\tau - s\right)\widetilde{\boldsymbol{\Sigma}}\widetilde{\boldsymbol{\Sigma}}'\boldsymbol{\Psi}\left(\tau - s\right)' ds \\ \boldsymbol{\zeta}\left(\tau\right) &= \int_0^\tau \boldsymbol{\Psi}\left(\tau - s\right)\widetilde{\mathbf{A}}_0 ds \end{aligned}$$

- and $\boldsymbol{\Psi}\left(\tau\right)$ solves the system of differential equation

$$\frac{d\boldsymbol{\Psi}\left(t\right)}{dt} = \widetilde{\mathbf{A}}_1\boldsymbol{\Psi}\left(t\right)$$

  with initial condition $\boldsymbol{\Psi}\left(0\right) = \mathbf{I}$.

- If $\widetilde{\mathbf{B}}$ has distinct and real eigenvalues, then the solution is

$$\boldsymbol{\Psi}\left(\tau\right) = \mathbf{U}\exp\left(\boldsymbol{\Lambda}\cdot\tau\right)\mathbf{U}^{-1}$$

  where, $\boldsymbol{\Lambda}$ is the diagonal matrix with $\widetilde{\mathbf{A}}_1$ eigenvalues on the principal diagonal, $\mathbf{U}$ is the matrix of the associated eigenvectors, and $\exp\left(\boldsymbol{\Lambda}\cdot T\right)$ is the diagonal matrix with $e^{\lambda_i T}$ in its $ii-$th position.

- Clearly, we need $\lambda_i \leq 0$ to ensure that the solution does not explode.

- In this case, we have that the Novikov's condition is satisfied:

$$E\left[\exp\left(\frac{1}{2}\int_0^T \boldsymbol{\nu}_t'\boldsymbol{\nu}_t dt\right)\right] < \infty$$

- Thus,

$$\xi_t = \exp\left(-\frac{1}{2}\int_0^t \boldsymbol{\nu}_u' \boldsymbol{\nu}_u du - \int_0^t \boldsymbol{\nu}_u' d\mathbf{B}_u\right)$$

defines a P-martingale.

1.2 The Optimal Consumption Plan

- Define the *state-price deflator*

$$\begin{aligned} \pi_t &= e^{-rt}\xi_t \quad (3) \\ &= \exp\left(-\left(\int_0^t r + \frac{1}{2}\boldsymbol{\nu}_u' \boldsymbol{\nu}_u du\right) - \int_0^t \boldsymbol{\nu}_u' d\mathbf{B}_u\right) \end{aligned}$$

- We then have that the optimal consumption is given by

$$C_t^* = \mathcal{I}_u(\lambda \pi_t, t) \quad (4)$$

- where $\mathcal{I}_u$ is the inverse of the utility functions.
- In addition, by defining again

$$\widehat{w}(\lambda) = E\left(\int_0^T \pi_t \mathcal{I}_u(\lambda \pi_t, t)\, dt\right) \quad (5)$$

- the solution to λ^* is given by the equality $\widehat{w}(\lambda) = w$.
- Assume for instance a power utility

$$u(C, t) = e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma}$$

- this implies

$$u_c = e^{-\rho t} C_t^{-\gamma}$$

- and thus

$$\mathcal{I}_u(x,t) = e^{-\frac{\rho}{\gamma}t} x^{-\frac{1}{\gamma}}$$

- Hence

$$\begin{aligned}
C_t^* &= e^{-\frac{\rho}{\gamma}t}(\lambda \pi_t)^{-\frac{1}{\gamma}} \\
&= e^{-\frac{\rho}{\gamma}t}\left(\lambda \exp\left(-\int_0^t r + \frac{1}{2}\boldsymbol{\nu}_u'\boldsymbol{\nu}_u du - \int_0^t \boldsymbol{\nu}_u' d\mathbf{B}_u\right)\right)^{-\frac{1}{\gamma}} \\
&= \lambda^{-\frac{1}{\gamma}} \exp\left(\int_0^t \frac{1}{\gamma}(r-\rho) + \frac{1}{2\gamma}\boldsymbol{\nu}_u'\boldsymbol{\nu}_u du + \frac{1}{\gamma}\int_0^t \boldsymbol{\nu}_u' d\mathbf{B}_u\right)
\end{aligned}$$

- where λ is a constant determined by the budget constraint.
- The process for log consumption $c_t = \log(C_t)$ is then given by

$$c_t = -\frac{1}{\gamma}\log(\lambda) + \left(\int_0^t \frac{1}{\gamma}(r-\rho) + \frac{1}{2\gamma}\boldsymbol{\nu}_u'\boldsymbol{\nu}_u du + \frac{1}{\gamma}\int_0^t \boldsymbol{\nu}_u' d\mathbf{B}_u\right)$$

- implying

$$dc_t = \left(\frac{1}{\gamma}(r-\rho) + \frac{1}{2\gamma}\boldsymbol{\nu}_t'\boldsymbol{\nu}_t\right) dt + \frac{1}{\gamma}\boldsymbol{\nu}_t' d\mathbf{B}_t$$

1.3 The optimal portfolio weights

- Consider now all the processes under Q: Define the new Brownian motion

$$d\widehat{\mathbf{B}}_t = d\mathbf{B}_t + \boldsymbol{\nu}_t dt$$

- so that, under Q, we have

$$\begin{aligned} dc_t &= \left(\frac{1}{\gamma}(r-\rho) - \frac{1}{2\gamma}\boldsymbol{\nu}_t'\boldsymbol{\nu}_t\right)dt + \frac{1}{\gamma}\boldsymbol{\nu}_t' d\widehat{\mathbf{B}}_t \qquad (6) \\ d\boldsymbol{\nu}_t &= \left(\widetilde{\mathbf{A}}_0 + \widehat{\mathbf{A}}_1\boldsymbol{\nu}_t\right)dt + \widetilde{\boldsymbol{\Sigma}} d\widehat{\mathbf{B}}_t \end{aligned}$$

- with $\widehat{\mathbf{A}}_1 = \widetilde{\mathbf{A}}_1 - \widetilde{\boldsymbol{\Sigma}}$.
- We now find the portfolio weights that support C_t^*.
- Recall the steps (caveat: we can work under Q or under P. Last time we did it under P. Now we do it under Q):
- First, define the Q-martingale

$$M_t = E^Q\left[\int_0^T \beta_u^{-1} C_u du\right] \qquad (7)$$

- We know that $M_0 = w =$ wealth at time 0.
- From the martingale representation theorem, we know that there exists $\widehat{\boldsymbol{\eta}}_t$ such that

$$dM_t = \widehat{\boldsymbol{\eta}}_t d\widehat{\mathbf{B}}_t$$

- The (discounted) wealth is given by

$$\widehat{W}_t = \beta_t^{-1} W_t = E_t^Q \left[\int_t^T \beta_u^{-1} C_u du \right] = M_t - J_t \tag{8}$$

- where

$$J_t = \int_0^t \beta_u^{-1} C_u du$$

- Thus, the process for the discounted wealth is

$$d\widehat{W}_t = -\beta_t^{-1} C_t dt + \widehat{\boldsymbol{\eta}}_t d\widehat{\mathbf{B}}_t \tag{9}$$

- We also have that the wealth is always equal to the total amount invested in stocks and bonds, which must satisfy the budget constraint

$$\widehat{W}_t = \theta_t^0 + \boldsymbol{\theta}_t \widehat{\mathbf{S}}_t = \int_0^t \boldsymbol{\theta}_u d\widehat{\mathbf{S}}_t - \int_0^t \beta_u^{-1} C_u du$$

- where $\widehat{\mathbf{S}}_t$ is a martingale under Q

$$d\widehat{\mathbf{S}}_t = \mathbf{I}_{\widehat{S}} \boldsymbol{\sigma} d\widehat{\mathbf{B}}_t$$

- Thus, the process for the discounted wealth under Q is

$$d\widehat{W}_t = -\beta_t^{-1} C_t + \boldsymbol{\theta}_t \mathbf{I}_{\widehat{S}} \boldsymbol{\sigma} d\widehat{\mathbf{B}}_t$$

- which, comparing with (9), yields immediately

$$\boldsymbol{\theta}_t \mathbf{I}_{\widehat{S}} \boldsymbol{\sigma} = \widehat{\boldsymbol{\eta}}_t$$

2 How do we get $\widehat{\boldsymbol{\eta}}_t$ practically?

2.1 The (discounted) wealth process

- Notice that we can rewrite (8) as

$$\begin{aligned} \widehat{W}_t &= E_t^Q\left[\int_t^T \beta_u^{-1} C_u du\right] \\ &= \beta_t^{-1} C_t E_t^Q\left[\int_t^T \frac{\beta_u^{-1}}{\beta_t^{-1}} \frac{C_u}{C_t} du\right] \\ &= \beta_t^{-1} C_t E_t^Q\left[\int_t^T \frac{\beta_u^{-1}}{\beta_t^{-1}} e^{c_u - c_t} du\right] \end{aligned}$$

- From the process for optimal consumption (6), we see that the conditional expectation $E_t^Q\left[\int_t^T \frac{\beta_u^{-1}}{\beta_t^{-1}} e^{c_u - c_t} du\right]$ depends only on $\boldsymbol{\nu}_t$.

- In other words, we can define the function

$$F\left(\boldsymbol{\nu}_t, t; T\right) = E_t^Q\left[\int_t^T \frac{\beta_\tau^{-1}}{\beta_t^{-1}} e^{c_\tau - c_t} du\right] \quad (10)$$

- and therefore the process

$$\widehat{W}_t = \beta_t^{-1} C_t F\left(\boldsymbol{\nu}_t, t; T\right)$$

- Thus, using Ito's lemma, the diffusion part of the discounted wealth process $d\widehat{W}_t$ must be given by

$$\widehat{\boldsymbol{\sigma}}'_W = \widehat{W}_t\left(\frac{1}{\gamma}\boldsymbol{\nu}'_t + \frac{1}{F}\sum_{i=1}^{n} \frac{\partial F}{\partial \nu^i} \widetilde{\boldsymbol{\sigma}}^i\right)$$

- with $\widetilde{\boldsymbol{\sigma}}^i = \left[\widetilde{\boldsymbol{\Sigma}}\right]_i$, the $i-$th row of $\widetilde{\boldsymbol{\Sigma}}$.
- In other words, from (9) we must have $\widehat{\boldsymbol{\eta}}_t = \widehat{\boldsymbol{\sigma}}'_W$.
- This yields

$$\boldsymbol{\theta}_t \mathbf{I}_{\widehat{S}} \boldsymbol{\sigma} = \widehat{W}_t \left(\frac{1}{\gamma} \boldsymbol{\nu}'_t + \frac{1}{F} \sum_{i=1}^{n} \frac{\partial F}{\partial \nu^i} \widetilde{\boldsymbol{\sigma}}^i \right)$$

- or

$$\boldsymbol{\theta}_t \mathbf{I}_{\widehat{S}} = \widehat{W}_t \left(\frac{1}{\gamma} \boldsymbol{\nu}'_t + \frac{1}{F} \sum_{i=1}^{n} \frac{\partial F}{\partial \nu^i} \widetilde{\boldsymbol{\sigma}}^i \right) \boldsymbol{\sigma}^{-1}$$

2.2 Myopic and Hedging Demand

- In terms of fraction of wealth, and recalling $\boldsymbol{\nu}_t = \boldsymbol{\sigma}^{-1} \left(\boldsymbol{\mu}_t - r \mathbf{1}_n \right)$

$$\boldsymbol{\vartheta}'_t = \frac{\boldsymbol{\theta}_t \mathbf{I}_{\widehat{S}}}{\widehat{W}_t} = \frac{\boldsymbol{\theta}_t \mathbf{I}_S}{W_t} = \frac{\left(\boldsymbol{\mu}_t - r \mathbf{1}_n \right)'}{\gamma} \left(\boldsymbol{\sigma} \boldsymbol{\sigma}' \right)^{-1} + \sum_{i=1}^{n} \frac{1}{F} \frac{\partial F}{\partial \nu^i} \widetilde{\boldsymbol{\sigma}}^i \boldsymbol{\sigma}^{-1} \tag{11}$$

- The first term on the RHS is the usual "myopic term": Higher excess return increase the portfolio holding, while higher risk and higher risk aversion decreases it.
- The second term is the hedging demand component. Notice first we can write

$$\widetilde{\boldsymbol{\sigma}}^i \boldsymbol{\sigma}^{-1} = \left(\widetilde{\boldsymbol{\sigma}}^i \boldsymbol{\sigma} \right) \left(\boldsymbol{\sigma} \boldsymbol{\sigma}' \right)^{-1}$$

- In addition, it turns out (see later) that

$$\frac{1}{F}\frac{\partial F}{\partial \nu^i} = -\frac{J_{W\nu^i}}{WJ_{WW}}$$

- where $J(W,t;T) = E_t\left[e^{-\rho(\tau-t)}\frac{C_\tau^{1-\gamma}}{1-\gamma}\right]$ is the indirect utility function.

- Thus, the hedging demand is given by

$$\text{Hedging Demand} = -\sum_{i=1}^{n}\frac{J_{W\nu^i}}{WJ_{WW}}\left(\widetilde{\boldsymbol{\sigma}}^i\boldsymbol{\sigma}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}$$

- I give an intuition of this term below in the context of a specific example.

2.3 How do we compute $F(\boldsymbol{\nu}_t, t; T)$ in (10) ?

- From contingent claim valuation: Consider a security that pays out C_t over time as dividend.

- Under Q, its value is

$$V(C_t, \boldsymbol{\nu}_t, t; T) = E_t^Q\left[\int_t^T e^{-r(\tau-t)}C_\tau d\tau\right] = C_t F(\boldsymbol{\nu}_t, t; T)$$

- The total expected return (under Q) on this security must equal the risk free rate, so that

$$E_t^Q[dV] + Cdt = Vrdt \qquad (12)$$

- From Ito's Lemma

$$\begin{aligned} E_t^Q[dV] &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial C} C_t \mu_C(\boldsymbol{\nu}_t) + \sum_{i=1}^{d} \frac{\partial V}{\partial \nu^i} \mu_i(\boldsymbol{\nu}_t) \\ &\quad + \sum_{i=1}^{d} \frac{\partial^2 V}{\partial C \partial \nu^i} C_t \frac{\boldsymbol{\nu}_t}{\gamma} \widetilde{\boldsymbol{\sigma}}_i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 V}{\partial \nu^i \partial \nu^j} \widetilde{\boldsymbol{\sigma}}_i \widetilde{\boldsymbol{\sigma}}_j' \end{aligned}$$

- where

$$\mu_C(\boldsymbol{\nu}_t) = \frac{1}{\gamma}(r-\rho) + \frac{\boldsymbol{\nu}_t' \boldsymbol{\nu}_t}{2\gamma}\left(\frac{1}{\gamma} - 1\right) \tag{13}$$

$$\mu_i(\boldsymbol{\nu}_t) = \widetilde{\mathbf{A}}_{0,i} + \widehat{\mathbf{A}}_{1,i} \boldsymbol{\nu}_t \tag{14}$$

- Finally, since

$$\begin{aligned} \frac{\partial V}{\partial t} &= C \frac{\partial F}{\partial t}; \ \frac{\partial V}{\partial C} = F; \ \frac{\partial V}{\partial \nu^i} = C \frac{\partial F}{\partial \nu^i} \\ \frac{\partial^2 V}{\partial C \partial \nu^i} &= \frac{\partial F}{\partial \nu^i}; \ \frac{\partial^2 V}{\partial \nu^i \partial \nu^j} = \frac{\partial^2 F}{\partial \nu^i \partial \nu^j} C \end{aligned}$$

- substituting everything into (12), we find

$$\begin{aligned} Fr &= 1 + \frac{\partial F}{\partial t} + F \mu_C(\boldsymbol{\nu}_t) + \sum_{i=1}^{d} \frac{\partial F}{\partial \nu^i} \mu_i(\boldsymbol{\nu}_t) + \sum_{i=1}^{d} \frac{\partial F}{\partial \nu^i} \widetilde{\boldsymbol{\sigma}}_i \frac{\boldsymbol{\nu}_t}{\gamma} \\ &\quad + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 F}{\partial \nu^i \partial \nu^j} \widetilde{\boldsymbol{\sigma}}_i \widetilde{\boldsymbol{\sigma}}_j' \end{aligned}$$

- Or

$$\begin{aligned} F(r - \mu_C(\boldsymbol{\nu}_t)) &= 1 + \frac{\partial F}{\partial t} + \sum_{i=1}^{d} \frac{\partial F}{\partial \nu^i} \left(\widetilde{\mathbf{A}}_{0,i} + \left(\widehat{\mathbf{A}}_{1,i} + \frac{\widetilde{\boldsymbol{\sigma}}_i}{\gamma} \right) \boldsymbol{\nu}_t \right) \\ &\quad + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 F}{\partial \nu^i \partial \nu^j} \widetilde{\boldsymbol{\sigma}}_i \widetilde{\boldsymbol{\sigma}}_j' \end{aligned} \tag{15}$$

- with final condition

$$F\left(\boldsymbol{\nu}_T, T; T\right) = 0.$$

- If a solution is not available, the PDE can often be computed numerically.
- From F, one can then obtain $\boldsymbol{\eta}_t$ and, in addition, also consumption. In fact, recall that

$$\widehat{W}_t = \beta_t^{-1} C_t F\left(\boldsymbol{\nu}_t, t; T\right)$$

- which gives

$$C_t = W_t F\left(\boldsymbol{\nu}_t, t; T\right)^{-1}$$

2.4 The Hedging Demand Again

- We can finally show that

$$\frac{1}{\partial F} \partial \nu^i = -\frac{J_{W\nu^i}}{W J_{WW}} \tag{16}$$

- From the Bellman equation (see TN 1), we have that the first order conditions with respect to consumption C is

$$J_W = u_c\left(C_t^*\right)$$

- where $C_t^* = C(W_t, \boldsymbol{\nu}_t) = W_t F(\boldsymbol{\nu}_t, t; T)^{-1}$ is the optimal policy function.

- Differentiating both sides with respect to W and, say, ν^i yields the equalities

$$
\begin{aligned}
J_{WW} &= u_{cc} \frac{\partial C^*}{\partial W} = F(\boldsymbol{\nu}_t, t; T)^{-1} \\
J_{W\nu^i} &= u_{cc} \frac{\partial C^*}{\partial \nu^i} = -W F(\boldsymbol{\nu}_t, t; T)^{-2} \frac{\partial F}{\partial \nu^i}
\end{aligned}
$$

- It is immediate to verify that the RHS and LHS of (16) coincide.

3 The case $\gamma = 1$

- Notice that if $\gamma = 1$, then $F(\boldsymbol{\nu}_t, t; T) = F(t; T)$ is the solution to the PDE.

- In fact, in this case, we obtain

$$F(t;T)\rho = 1 + F'(t;T)$$

which yields

$$F(t;T) = He^{\rho t} + \frac{1}{\rho}$$

- The final condition

$$F(T;T) = He^{\rho T} + \frac{1}{\rho} = 0$$

- yields

$$H = \frac{e^{-\rho T}}{\rho}$$

- Thus

$$F(t;T) = \frac{1}{\rho}\left(1 - e^{-\rho(T-t)}\right)$$

- This implies that the consumption to wealth ratio is deterministic and given by

$$C_t = \frac{W_t \rho}{\left(1 - e^{-\rho(T-t)}\right)}$$

- In addition, the optimal portfolio choice is

$$\boldsymbol{\vartheta}_t' = \frac{(\boldsymbol{\mu}_t - r\mathbf{1}_n)'}{\gamma} (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}$$

4 The Solution in the Univariate Case for $\gamma > 1$

- Consider the univariate case ($d = 1$).
- In this case, the portfolio holding of the market is given by

$$\vartheta_t = \frac{\mu_t - r}{\gamma\sigma^2} + \left(\frac{1}{F}\frac{\partial F}{\partial \nu}\right)\frac{\widetilde{\sigma}}{\sigma}$$

- where F has to satisfy the PDE (15) becomes

$$F\left(r - \mu_C\left(\nu_t\right)\right) = 1 + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \nu}\left(\widetilde{A}_0 + \left(\widehat{A}_1 + \frac{\widetilde{\sigma}}{\gamma}\right)\nu_t\right) + \frac{1}{2}\frac{\partial^2 F}{\partial \nu^2}\widetilde{\sigma}^2 \tag{17}$$

- Unfortunately, it does not have a closed form solution.
- Wachter (2002) however finds a nice way out: Rather than considering a claim to the whole sequence of consumption plan $\{C_\tau\}_t^T$, that is, the security

$$V(C, \nu, t; T) = E^Q\left[\int_0^T e^{-r(\tau - t)} C_\tau d\tau\right]$$

she first considers a set of claims, each to the exact consumption "coupon" C_τ paid at that particular τ, for $\tau \in [0, T]$.

- By no arbitrage, the value of the security paying the process $\{C_\tau\}$ will be the sum (i.e. the integral) of all these individual claims.

- Let the value of each claim to the "coupon" C_tau be given by

$$v(C,\nu,t;\tau) = E^Q\left[e^{-r(\tau-t)}C_\tau\right].$$

- The homogeneity discussed earlier entails that $v(C,\nu,t;\tau) = Cf(\nu,t;\tau)$ for some $f(.)$.

- Under Q also this claim must earn the risk free rate

$$E^Q[dv] = rvdt$$

- Ito's Lemma then gives

$$rv = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial C}C_t\mu_C(\nu_t) + \frac{\partial v}{\partial \nu}\left(\widehat{A}_0 + \widehat{A}_1\nu_t\right) + \frac{1}{2}\frac{\partial^2 v}{\partial \nu^2}\widehat{\sigma}^2 + \frac{\partial^2 v}{\partial C\partial \nu}C\frac{\nu_t}{\gamma}\widehat{\sigma}$$

- with final condition $v(C_\tau,\nu_\tau,\tau;\tau) = C_\tau$

- Recall also that under Q

$$\mu_C(\nu_t) = \frac{1}{\gamma}(r-\rho) + \frac{\nu_t^2}{2\gamma}\left(\frac{1}{\gamma}-1\right)$$

- As before, we can substitute the following quantities

$$\frac{\partial v}{\partial t} = C\frac{\partial f}{\partial t}; \quad \frac{\partial v}{\partial C} = f; \quad \frac{\partial v}{\partial \nu} = C\frac{\partial f}{\partial \nu}$$
$$\frac{\partial^2 v}{\partial \nu^2} = C\frac{\partial^2 f}{\partial \nu^2}; \quad \frac{\partial^2 v}{\partial C \partial \nu} = \frac{\partial f}{\partial \nu}$$

- Substituting also $v = Cf$, $\mu_C(\nu)$ and deleting C on both sides yields

$$rf = \frac{\partial f}{\partial t} + f\left(\frac{1}{\gamma}(r-\rho) + \frac{\nu_t^2}{2\gamma}\left(\frac{1}{\gamma} - 1\right)\right) + \frac{\partial f}{\partial \nu}\left(\widehat{A}_0 + \widehat{A}_1 \nu_t\right)$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial \nu^2}\widehat{\sigma}^2 + \frac{\partial f}{\partial \nu}\frac{\nu_t}{\gamma}\widehat{\sigma}$$

- with final condition $f(\nu_\tau, \tau; \tau) = 1$
- Fortunately, a "solution" to this PDE instead exists.
- **How can we find it constructively?**
- Use the **method of undetermined coefficients**
  - This methodology is extensively used to obtain the prices of Fixed Income Securities.
- Conjecture (this comes with experience)

$$f(\nu, t; \tau) = \exp\left(a_0(t;\tau) + a_1(t;\tau)\nu + a_2(t;\tau)\nu^2\right)$$

- Then we obtain

$$\frac{\partial f}{\partial t} = \left(a_0' + a_1'\nu_t + a_2'\nu_t^2\right) f$$

$$\begin{aligned} \frac{\partial f}{\partial \nu} &= (a_1 + 2a_2\nu_t)f \\ \frac{\partial^2 f}{\partial \nu^2} &= (2a_2 + (a_1 + 2a_2\nu_t)^2)f \end{aligned}$$

- Substitute and delete f on both sides, to find

$$\begin{aligned} r &= a_0' + a_1'\nu + a_2'\nu^2 + \left(\frac{1}{\gamma}(r-\rho) + \frac{\nu_t^2}{2\gamma}\left(\frac{1}{\gamma} - 1\right)\right) \\ &+ (a_1 + 2a_2\nu)\left(\widehat{A}_0 + \widehat{A}_1\nu + \frac{\nu_t}{\gamma}\widehat{\sigma}\right) \\ &+ \frac{1}{2}(2a_2 + (a_1 + 2a_2\nu)^2)\widehat{\sigma}^2 \end{aligned}$$

- Finally, bunch up together terms in ν, ν^2 etc.
- One obtains:

$$\begin{aligned} 0 &= a_0' - r + \left(\frac{1}{\gamma}(r-\rho)\right) + a_1\widehat{A}_0 + \frac{1}{2}\left(2a_2 + a_1^2\right)\widehat{\sigma}^2 + \\ &+ \left(a_1' + a_1\left(\widehat{A}_1 + \frac{\widehat{\sigma}}{\gamma}\right) + 2a_2\widehat{A}_0 + 2a_1a_2\widehat{\sigma}^2\right)\nu_t \\ &+ \left(a_2' + \frac{1}{2\gamma}\left(\frac{1}{\gamma} - 1\right) + 2a_2\left(\widehat{A}_1 + \frac{\widehat{\sigma}}{\gamma}\right) + 2a_2^2\widehat{\sigma}^2\right)\nu_t^2 \end{aligned}$$

- This equation is satisfied if the following system of ODEs is satisfied

$$\begin{aligned} a_2' + 2a_2\left(\widehat{A}_1 + \frac{\widehat{\sigma}}{\gamma}\right) + 2a_2^2\widehat{\sigma}^2 + \frac{1}{2\gamma}\left(\frac{1}{\gamma} - 1\right) &= 0 \\ a_1' + a_1\left(\widehat{A}_1 + \frac{\widehat{\sigma}}{\gamma}\right) + 2a_2\widehat{A}_0 + 2a_1a_2\widehat{\sigma}^2 &= 0 \\ a_0' - r + \frac{1}{\gamma}(r-\rho) + a_1\widehat{A}_0 + \frac{1}{2}\left(2a_2 + a_1^2\right)\widehat{\sigma}^2 &= 0 \end{aligned}$$

- Note that the system can be easily solved recursively: Solve the first equation (for a_2), then plug in the solution into the second equation (for a_1) and then finally, obtain the solution of the last equation (for a_0)
- It is possible to find exact closed formulas for a_2 and a_1. However, this is as easy to obtain numerically.
- ODEs are infinitely simpler than PDE, as you can solve then backward: Start with the final condition $a_0(\tau) = a_1(\tau) = a_2(\tau) = 0$ and then simply move backwards. Below are the details.
- Once we have the solution for $a_0(t;\tau)$, $a_1(t;\tau)$ and $a_2(t;\tau)$ for every $\tau \in [0,T]$, the function $F(\nu,t;T)$ can be obtained easily.
- In fact, recall that

$$\begin{aligned} V(C,\nu,t;T) &= CF(\nu,t;T) \\ &= E_t^Q\left[\int_t^T e^{-r(\tau-t)} C_\tau d\tau\right] \\ &= \int_t^T E_t^Q\left[e^{-r(\tau-t)} C_\tau\right] d\tau \\ &= \int_t^T v(C,\nu,t;\tau) d\tau \\ &= C\int_t^T f(\nu,t;\tau) d\tau \end{aligned}$$

- This implies that simply

$$F(\nu, t; T) = \int_t^T f(\nu, t; \tau) d\tau$$
$$= \int_t^T \exp\left(a_0(t; \tau) + a_1(t; \tau)\nu_t + a_2(t; \tau)\nu_t^2\right) d\tau$$

- The portfolio holdings require the computation of

$$\frac{\partial F}{\partial \nu} = \int_t^T (a_1(t; \tau) + a_2(t; \tau)\nu_t) \exp\left(a_0(t; \tau) + a_1(t; \tau)\nu_t + a_2(t; \tau)\nu_t^2\right) d\tau$$

- To conclude, we have the following portfolio rule

$$\vartheta_t = \text{Miopic Demand} + \text{Hedging Demand}$$

with

$$\text{Myopic Demand} = \frac{\mu_t - r}{\gamma \sigma^2}$$
$$\text{Hedging Demand} = \left(\frac{1}{F}\frac{\partial F}{\partial \nu}\right)\frac{\widetilde{\sigma}}{\sigma}$$
$$= \frac{\widetilde{\sigma}}{\sigma}\frac{\int_t^T (a_1(t; \tau) + a_2(t; \tau)\nu_t) f(\nu_t, t; \tau) d\tau}{\int_t^T f(\nu_t, t; \tau) d\tau}$$

- Also note that the optimal consumption was given by $C_t = W_t F(\nu, t; T)^{-1}$ implying that the C/W ratio is given by

$$\frac{C}{W} = \frac{1}{\int_t^T f(\nu_t, t; \tau) d\tau}$$

4.1 A Calibration

- An obvious application of the setting above is the one where returns μ_t are predictable from the dividend price ratio.

- We can think then of μ_t to be just

$$\mu_t = \alpha + \beta \log (D_t/P_t)$$

  where α and β are the regression coefficients of some sort of predictive regression

- Note that if $\log(D_t/P_t)$ follows a mean reverting process, so does μ_t and so we are back to the case discussed in this section.

- Also, there is a natural negative correlation between returns and D/P ratio: a negative return implies that P_t decreased. Since dividends do not move much, this implies that D/P went up.

- How do we impose a negative correlation in the model? Just assume that $\widehat{\sigma} < 0$

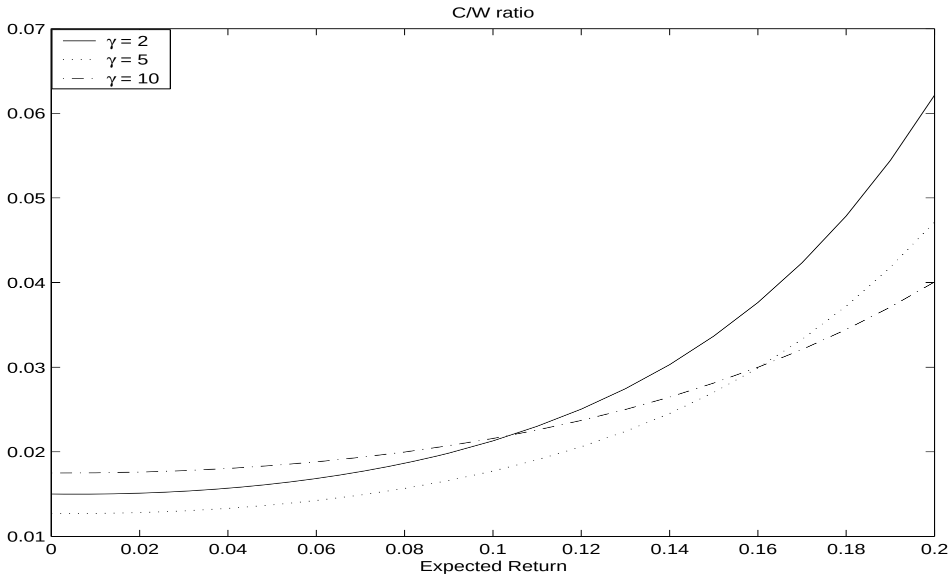
- The following parameters have been used by many, including Barberis (JF 2000), and Wachter (JFQA 2002)

Parameter Choice

Rate of time preference ρ	0.0624
Risk free rate r	0.0168
Volatility of stock prices σ	0.1510
Volatility of ν_t	-0.0655
Mean Reversion A_1	-0.0226
Drift A_0	0.0062

- The following figures show the C/W ratio, Hedging Demand and Total Demand as a function of expected return μ_t

Figure 1: C/W ratio



- Note that the C/W ratio initially declines with γ, as intuition would have it: Higher risk aversion implies a higher desire to save and thus consume less.

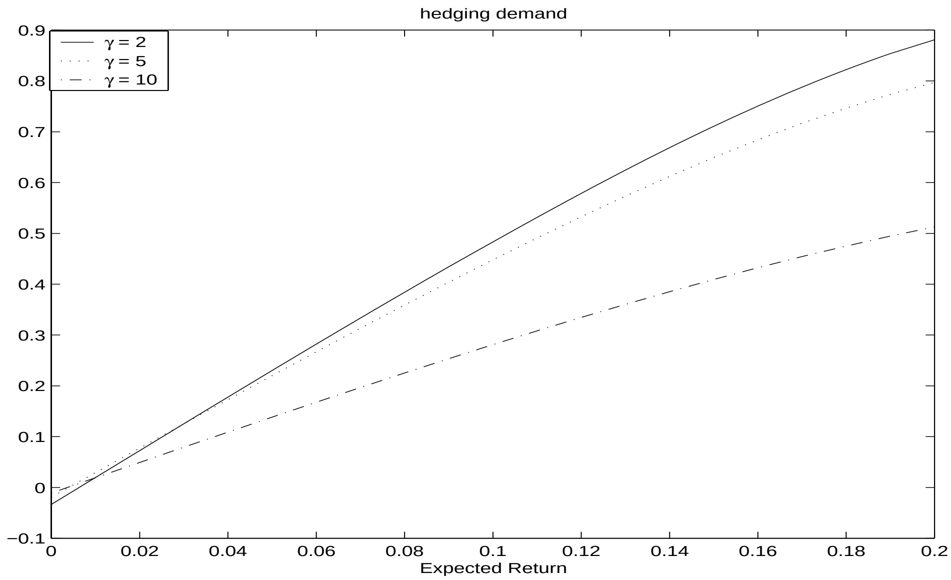
- However, as γ increase, the C/W increases again as the investor elasticity of intertemporal substitution (EIS $= 1/\gamma$) decreases.

- In the limit, as γ increases, the investor's desire to smooth out consumption is so large that changes in μ_t have no impact on C/W.

- In this case, the average C/W across possible μ_t must be the same as the one under different γ's, implying a higher C/W for low μ_t and lower for high μ_t.

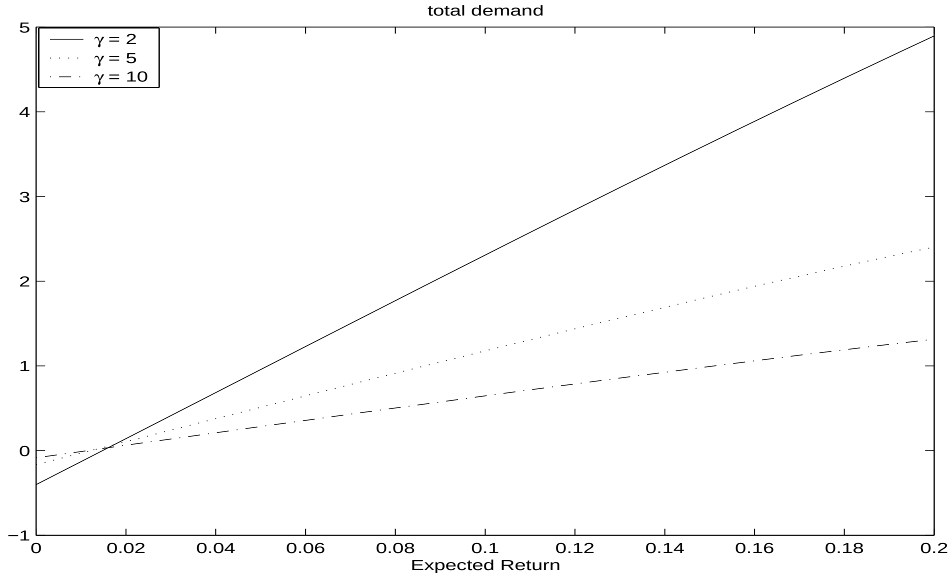
- Later on we will do recursive utility, and see the implications of EIS and risk aversion independently

Figure 2: Hedging Demand



- The hedging demand is positive. The intuition is simple:
  - If we have a bad shock to returns, we have that μ_t increases (intuitively, the D/P increases, implying higher expected return).
  - But a higher μ_t implies that investor now want to buy more of the stock.
  - Anticipating this correlation, the investor buys more of the stock today, compared to the case where the hedging demand is zero.
- This finding is bad news for the portfolio holding puzzle: We already showed that the agent would hold too much of the stock even with simple myopic demand (no time varying investment opportunity set).
- The total demand now of the stock is even higher, deepening the puzzle.

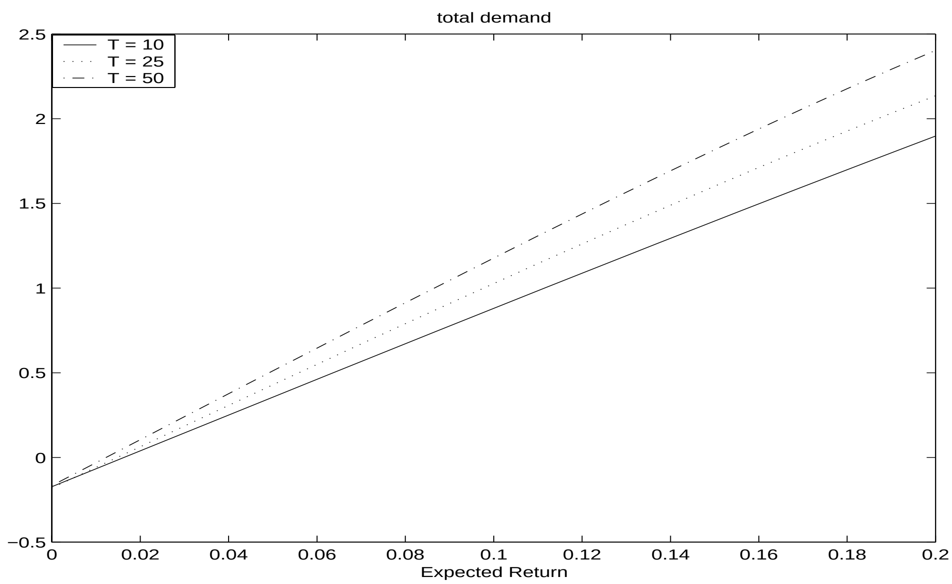
Figure 3: Total Demand



- We will see other channels that would decrease the holding of stocks later on.

- The predictability, however, helps a little to generate asset holding that depend on life cycle
- Using the same parameters, with $\gamma = 5$ but for three different maturities T we obtain the following.

Figure 4: Total Demand



- As it can be see, the shorter the time to "death" the lower the share in stocks, especially if current expected return is high.
- In this case, mean reversion kicks in and the investor is wary about the negative consequences of a decrease in expected returns.