

# Teaching Notes #5 (Addendum)

## Robust Control, Time Varying Opportunity Set and Learning

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- Maenhout (2004) also consider an application to portfolio selection with time varying opportunity set.
- His setting is similar to the one we investigated in TN 1 and TN 3 (addendum).
- Maenhout (2004) only investigates the case where the agent maximizes utility from final wealth, but here, using the results in TN 3, we can consider the more interesting case with intermediate consumption
- If one considers learning, however, an immediate question is about the model misspecification: where does the preferences for robustness come in?

- To focus the discussion, consider the case where we believe in the following model

$$\frac{dS}{S} = \mu dt + \sigma dB_t \quad (1)$$

- Assuming  $\mu \sim N(\mu_t, v_t)$  yields

$$\frac{dS}{S} = \mu_t dt + \sigma d\bar{B}_t \quad (2)$$

- where

$$d\mu_t = \sigma_{\mu,t} d\bar{B}_t \quad (3)$$

- and

$$d\bar{B}_t = \frac{1}{\sigma_S} \left\{ \frac{dS}{S} - E_t \left[ \frac{dS}{S} \right] \right\}$$

- The question is: Is the investor worried about the misspecification in the filtering (i.e. equations (2) and (3)) or in the original model (1)?
- Depending on how we answer this question, we would have a different specifications of the intertemporal model
  - If we think that the agent is worried about misspecification of the filtered model, then we are exactly in the setting of section 5: There is a state variable  $\mu_t$  drive by the same BM as the stock return. All the results apply immediately.
  - If we think that the agent is worried about (1) being misspecified, then the question is how we introduce a perturbation in (1) and we apply filtering to the perturbed model. This is not simple.

- In this TN we are going to consider the simple case, where agents are worried about misspecification in the filtered model (2) and (3).
- We can then use a trick to obtain an immediate solution to our problem.
- The following Proposition is in Anderson, Hansen and Sargent (1999) and Maenhout (2004):
- **Proposition:** *An investor with homothetic preferences for robustness  $\psi(J) = \frac{\theta}{(1-\gamma)J}$  and CRRA utility function  $u(c) = c^{1-\gamma}1 - \gamma$  is observationally equivalent to an Epstein-Zin-Weil investor with elasticity of intertemporal substitution (EIS)  $1/\gamma$  and coefficient of relative risk aversion  $\gamma + 1/\theta$ .*
- Given this proposition, we can use back our results in TN 4 (addendum) to obtain an implication for portfolio selection and robust control
- The results are the same, but the interpretation is very different, however.
  - In TN 4 we assumed the elasticity of intertemporal substitution = 1, to have simple equations. Thus, this is equivalent here to the log-utility case  $\gamma = 1$ .
  - In TN 4 we had several values of relative risk aversion  $\gamma_{TN4} = 2, 3, 4$  which therefore correspond to

$$\theta = 1/(\gamma_{TN4} - 1) = 1, 1/2, 1/3.$$

- By using the formulas in TN 4 (addendum) we then have

$$\vartheta_t = \frac{\mu_t - r}{\sigma_S^2 \left(1 + \frac{1}{\theta}\right)} + \frac{\psi_\lambda / \psi \sigma_{\mu,t}}{\left(1 + \frac{1}{\theta}\right) \sigma_S} \quad (4)$$

- where

$$\psi_\lambda / \psi = a_1 + 2a_2 (\mu_t - r) \quad (5)$$

- A lower  $\theta$  implies a lower weight to the “entropy” component in the utility function, and thus a bigger concern about model misspecification
- Thus, a lower  $\theta$  results in a lower weight in stocks, everything else equal.
- Recall that the case of perfect information (studied in Maenhout (2004) has  $\sigma_{\mu,t} = 0$ . This is the same result with obtained in TN 5.
- Also, we can re-interpret the results as due to simply time varying opportunity set, fix  $\sigma_\mu$  to a number, which is positive if we think that positive shocks to returns imply an increase in future expected returns, or negative for the opposite case.

- As in any calibration, the question is then “how large” is  $\theta = 1, 1/2, 1/3$  economically
  - We can gauge this by checking what is the implied level of pessimism that such  $\theta$  generate.
  - The parameter  $\theta$  quantify the amount of perturbation in the return process, and thus by investigating the properties of the perturbed process as a function of  $\theta$  we can gauge whether it is plausible or not.
  - From TN 5, page 284, the optimal perturbation  $h_t$  given  $\theta$  is

$$h = -\frac{1}{\theta(1-\gamma)J(W,t)}J_W\sigma_W$$

- where  $\sigma_W = W\vartheta\sigma_S$  is the diffusion of the wealth process.
- Using the fact that  $J = k(t)W^{1-\gamma}/1-\gamma$  and  $J_W = k(t)W^{-\gamma}$ , the optimal  $\vartheta_t$  in (4) and the fact that  $a_1 = 0$  in (5), we obtain

$$h = -\frac{1}{\theta}(\mu_t - r) \frac{\left(\frac{1}{\sigma_S} + 2a_2\sigma_{\mu,t}\right)}{\left(1 + \frac{1}{\theta}\right)}$$

- Remember that the original model under learning

$$\begin{aligned} \frac{dS_t}{S} &= \mu_t dt + \sigma_S d\bar{B}_t \\ d\mu_t &= \sigma_{\mu,t} d\bar{B}_t \end{aligned}$$

- The perturbed model (misspecification) is

$$\begin{aligned} \frac{dS_t}{S} &= \mu_t dt + \sigma_S (d\widehat{B}_t + h_t) \\ d\mu_t &= \sigma_{\mu,t} (d\widehat{B}_t + h_t) \end{aligned}$$

– This translates into

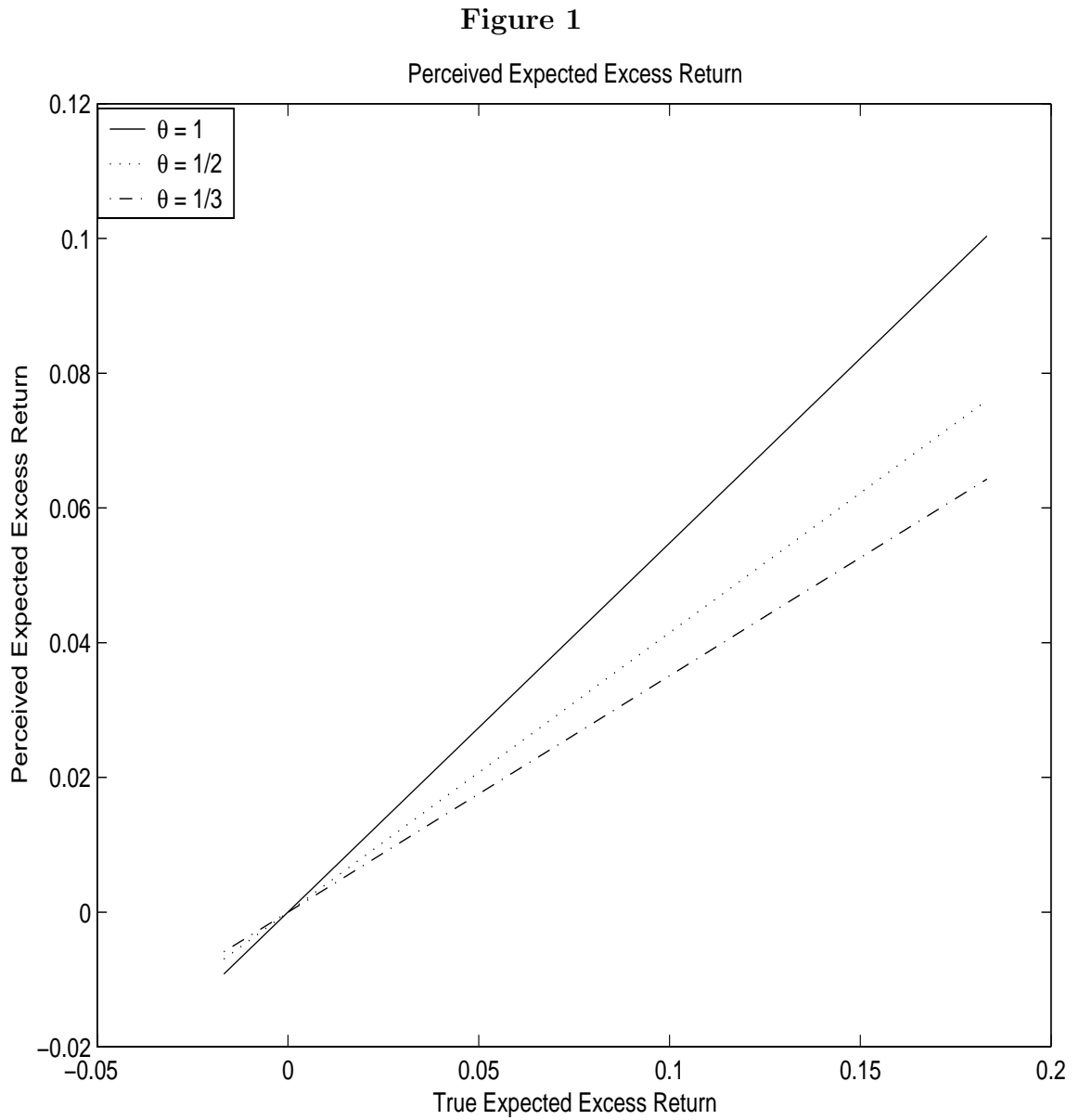
$$\begin{aligned}\frac{dS_t}{S} &= (\mu_t + \sigma_S h_t) dt + \sigma_S d\widehat{B}_t \\ d\mu_t &= \sigma_{\mu,t} h_t dt + \sigma_{\mu,t} d\widehat{B}_t\end{aligned}$$

– The expected excess return under the perceived model (i.e. the worse case scenario) is

$$E^h \left[ \frac{dS}{S} - r \right] = (\mu_t - r) \left( 1 - \frac{1}{\theta} \frac{1}{\left(1 + \frac{1}{\theta}\right)} (1 + \sigma_{\mu,t} \sigma_S 2a_2) \right)$$

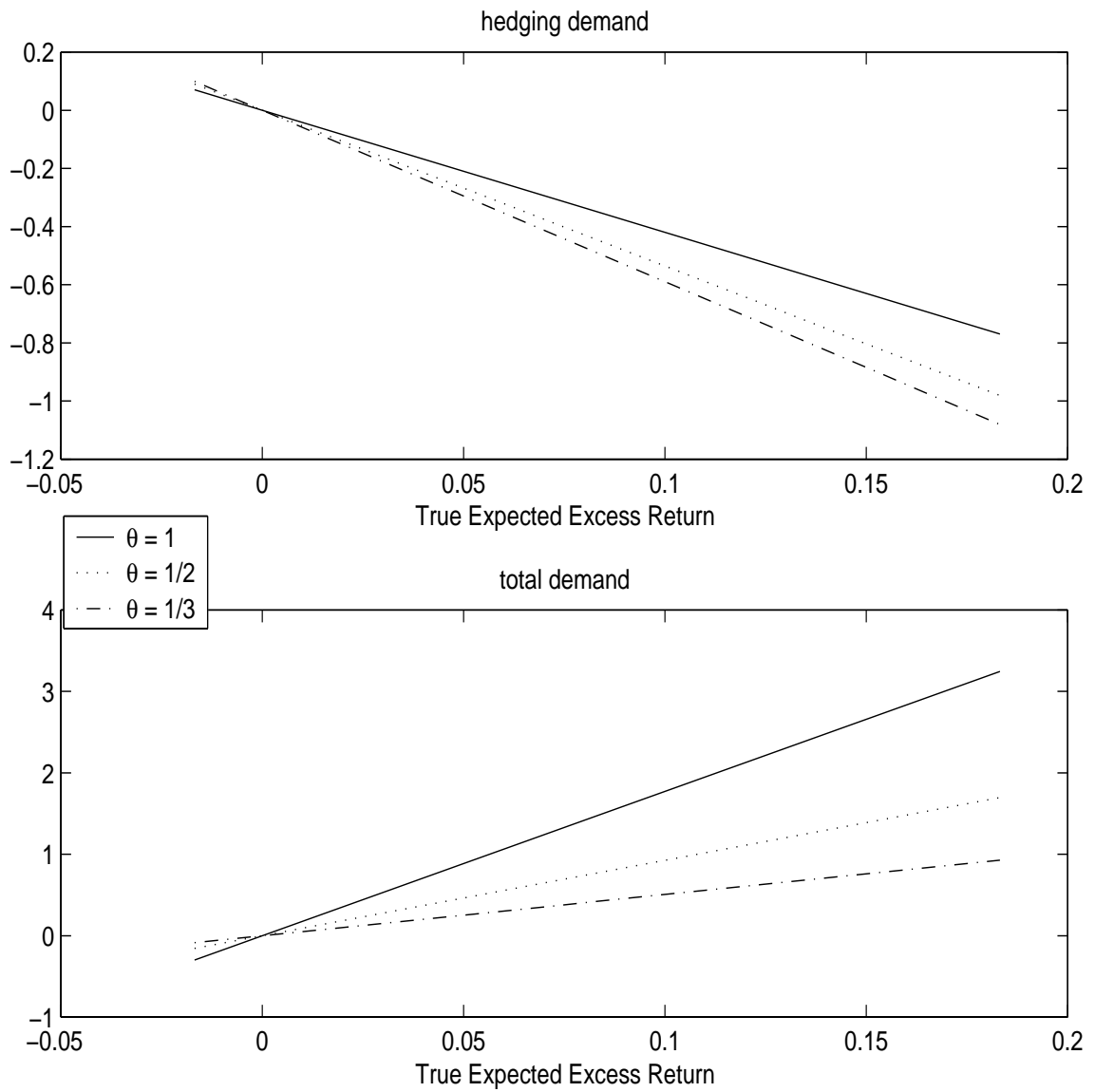
- The case of perfect information (studied in Maenhout (2004)) has  $\sigma_{\mu,t} = 0$  and we obtain his formula as a special case.
- The case where  $\theta = \infty$  corresponds to the case of no preference for robustness, and in fact we have  $E^h \left[ \frac{dS}{S} - r \right] \rightarrow \mu_t - r$ .
- We can plot the distorted (or perceived) expected returns for various levels of  $\theta$  and obtain an economically meaningful measure of whether a particular  $\theta$  is plausible or not.
- The following figures use the same calibration of TN4 (addendum)

- Figure 1 plots the relation between perceived expected return and true expected return.



- Figure 2 plots the hedging demand and the total demand of the asset against the true expected return  $\mu_t - r$ .

**Figure 2**



- Figure 3 plots the hedging demand and the total demand of the asset against the perceived expected return  $\mu_t - r$ .

**Figure 3**

