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Graduate School of Business
Business 41912, Spring Quarter 2004, Mr. Ruey S. Tsay

Solutions to Midterm

1. This question can be answered following the lecture note or textbook.

   (a) Using the square-root matrix $\boldsymbol{\Sigma}^{1/2}$ such that $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2}$, we can take the transformation
   $$\boldsymbol{\Sigma}^{-1/2}\boldsymbol{Y} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{Z}\boldsymbol{\beta} + \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}.$$
   Rewrite the above equation as
   $$\boldsymbol{Y}_* = \boldsymbol{Z}_*\boldsymbol{\beta} + \boldsymbol{\epsilon}_*,$$
   where $\boldsymbol{\epsilon}_* = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\epsilon}$ such that $\mathrm{Cov}(\boldsymbol{\epsilon}_*) = \boldsymbol{I}$, the identity matrix. Then, the estimate of $\boldsymbol{\beta}$ is
   $$\hat{\boldsymbol{\beta}} = (\boldsymbol{Z}_*'\boldsymbol{Z}_*)^{-1}\boldsymbol{Z}_*'\boldsymbol{Y}_* = (\boldsymbol{Z}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Z})^{-1}(\boldsymbol{Z}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}).$$
   The residuals in this case is
   $$\hat{\boldsymbol{\epsilon}} = \boldsymbol{Y} - \hat{\boldsymbol{Y}} = \boldsymbol{Y} - \boldsymbol{Z}\hat{\boldsymbol{\beta}} = (\boldsymbol{I} - \boldsymbol{H}_*)\boldsymbol{Y},$$
   where $\boldsymbol{H}_* = \boldsymbol{Z}'(\boldsymbol{Z}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Z})^{-1}\boldsymbol{Z}\boldsymbol{\Sigma}^{-1}$. It is easy to show that $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $\mathrm{Cov}(\boldsymbol{\epsilon})$ $= \boldsymbol{\Sigma} - \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Z})^{-1}\boldsymbol{Z}'$. Hence, ϵ_i is normally distributed with mean zero and variance γ_{ii}, where γ_{ii} is the ith diagonal element of $\mathrm{Cov}(\boldsymbol{\epsilon})$.

   (b) No, they are typically correlated with $\mathrm{Cov}(\hat{\epsilon}_i, \hat{\epsilon}_j) = -h_{ij}\sigma^2$, where h_{ij} is the (i, j)th element of $\boldsymbol{H}$.

   (c) $\hat{\boldsymbol{Y}}'\hat{\boldsymbol{\epsilon}} = \hat{\boldsymbol{\beta}}'\boldsymbol{Z}'\hat{\boldsymbol{\epsilon}} = 0$, because $\boldsymbol{Z}'\hat{\boldsymbol{\epsilon}} = \mathbf{0}$.

2. Note that $\boldsymbol{X} = \boldsymbol{C}\boldsymbol{Z}$, where
   $$\boldsymbol{C} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) $\boldsymbol{X} \sim N_2(\boldsymbol{C}\boldsymbol{\mu}, \boldsymbol{C}\boldsymbol{I}\boldsymbol{C}')$, taht is, $\boldsymbol{X} \sim N_2(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$, where
   $$\boldsymbol{\mu}_x = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \boldsymbol{\Sigma}_x = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

   (b) $\boldsymbol{X} \sim N_2(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$ because Z_3 and $\boldsymbol{X}$ are independent.

   (c) $X_1 \sim N(3, 2)$.

   (d) Using $Y = (1, 1, -1)\boldsymbol{W}$, where $\boldsymbol{W} = (\boldsymbol{X}', Z_3)$ which is multivariate normal with mean $(3, 5, 3)$ and covariance matrix $\mathrm{diag}(\boldsymbol{\Sigma}_x, 1)$, we obtain $Y \sim N(5, 7)$.

3. $\boldsymbol{X}_t$ is an MA(1) process. It is easy to obtain that $\text{Cov}(\boldsymbol{X}_t) = \boldsymbol{\Sigma} + \boldsymbol{\theta}\boldsymbol{\Sigma}\boldsymbol{\theta}'$, $\boldsymbol{\Gamma}_1 = -\boldsymbol{\theta}\boldsymbol{\Sigma}$, $\boldsymbol{\Gamma}_{-1} = -\boldsymbol{\Sigma}\boldsymbol{\theta}'$, and $\boldsymbol{\Gamma}_k = \boldsymbol{0}$ for $|k| > 1$.

   (a) $E(\bar{\boldsymbol{X}}_n) = \frac{1}{n}E(\sum_{i=1}^n \boldsymbol{X}_i) = \frac{1}{n}\sum_{i=1}^n E(\boldsymbol{X}_i) = \boldsymbol{c}$.

   (b) $\text{Cov}(\sqrt{n}\bar{\boldsymbol{X}}_n) = \frac{1}{n}\text{Cov}(\sum_{i=1}^n X_i) = \frac{1}{n}[n\boldsymbol{\Gamma}_0 + (n-1)\boldsymbol{\Gamma}_1 + (n-1)\boldsymbol{\Gamma}_{-1}]$.

   (c) Taking the limit of part (b), we have $\boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_{-1} = \boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\Sigma}\boldsymbol{\theta}' - \boldsymbol{\theta}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\boldsymbol{\theta}'$.

4. Use the attache doutput.

   (a) $(96.46, 42.91, 35.37, 14.51, 25.63, 9.57, 9.71)'$ and $(99.34, 43.74, 39.31, 14.66, 30.00, 9.66, 9.37)'$.

   (b) Based on the S-Plus output, the two means are significantly different at the 5% level.

   (c) Based on the maximization lemma, the linear combination is given by the vector a in Matlab output. That is, $(.006, .151, -.854, .268, -.383, -2.187, 2.971)'$.

   (d) You may use the t-dist as in Section 6.5 to construct the simultaneous C.I. as

$$d_i \pm t_{63}(.05/(7 \times 2 \times 1))\sqrt{\frac{2 \times s_{ii}}{35}},$$

   where d_i is the ith element of the mean difference and s_{ii} is the (i,i)th element of the pooled covariance estimate given in the Matlab output. The

   (e) Normality, independent samples, and equal covariance matrices.