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Graduate School of Business
Business 41912-01, Spring Quarter 2008, Mr. Ruey S. Tsay

Solutions to Homework Assignment #1

1. Compute the sample means, sample covariance matrix, and sample correlation matrix of the monthly simple returns of IBM, Hewlett-Packard Co. (HPQ), Merrill Lynch (MER), Bank of America (BAC), and Standard and Poor's 500 index from 1988 to 2007.

   Answer: The sample period is from 1988 to 2007 for 20 years.

```
> x=read.table("m-5c8807.txt",header=T)
> mean(x[,2:6])
        ibm         hpq         mer         bac         sp5
0.010665158 0.015270917 0.018631471 0.016057708 0.008211737
> cov(x[,2:6])
            ibm         hpq         mer         bac         sp5
ibm 0.007218919 0.004078220 0.002817018 0.001374108 0.001813088
hpq 0.004078220 0.010770324 0.004534955 0.001893751 0.002358878
mer 0.002817018 0.004534955 0.009191204 0.003992565 0.002566229
bac 0.001374108 0.001893751 0.003992565 0.007167267 0.001724306
sp5 0.001813088 0.002358878 0.002566229 0.001724306 0.001514263
> cor(x[,2:6])
          ibm       hpq       mer       bac       sp5
ibm 1.0000000 0.4625089 0.3458337 0.1910328 0.5483805
hpq 0.4625089 1.0000000 0.4557980 0.2155419 0.5841044
mer 0.3458337 0.4557980 1.0000000 0.4919138 0.6878736
bac 0.1910328 0.2155419 0.4919138 1.0000000 0.5234039
sp5 0.5483805 0.5841044 0.6878736 0.5234039 1.0000000
```

2. Find the maximum and minimum of $\frac{\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}}$, where $\boldsymbol{x} \neq \boldsymbol{0}$ is a 3-dimensional real-valued vector and

$$\boldsymbol{A} = \begin{bmatrix} 1 & .5 & .3 \\ .5 & 1 & .5 \\ .3 & .5 & 1 \end{bmatrix}.$$

   Also, compute the symmetric square-root matrix of $\boldsymbol{A}$.

   Answer:

```
> A=matrix(c(1,.5,.3,.5,1,.5,.3,.5,1),3,3)
> A
     [,1] [,2] [,3]
[1,]  1.0  0.5  0.3
[2,]  0.5  1.0  0.5
[3,]  0.3  0.5  1.0
> e1=eigen(A)
> names(e1)
[1] "values"  "vectors"
> e1$values
[1] 1.8728416 0.7000000 0.4271584
> P=e1$vectors
> L=diag(sqrt(e1$values))
> Ahf=P%*%L%*%t(P)
> Ahf
          [,1]      [,2]      [,3]
[1,] 0.9609432 0.2472687 0.1242832
[2,] 0.2472687 0.9368652 0.2472687
[3,] 0.1242832 0.2472687 0.9609432
> Ahf%*%Ahf   # verification
     [,1] [,2] [,3]
[1,]  1.0  0.5  0.3
[2,]  0.5  1.0  0.5
[3,]  0.3  0.5  1.0
```

The maximum is 1.873 and the minimum is 0.427. The square root matrix is the "Ahf" shown in the output.

3. Problem 4.3 of the textbook, p. 201.

   Answer:

   (a) X_1 and X_2 are dependent because $\text{Cov}(X_1, X_2) = -2$.

   (b) X_2 and X_3 are independent because they have no correlation.

   (c) (X_1, X_2) and X_3 are independent because their covariance is zero.

   (d) $\frac{X_1+X_2}{2}$ and X_3 are independent because they have no correlation. See the output below.

   (e) X_2 and $X_2 - \frac{5}{2}X_1 - X_3$ are dependent because their covariance is not zero. See the output below.

```
> Sig=matrix(c(1,-2,0,-2,5,0,0,0,2),3,3)
> Sig
```

```
     [,1] [,2] [,3]
[1,]    1   -2    0
[2,]   -2    5    0
[3,]    0    0    2
> D=matrix(c(0.5,0,0.5,0,0,1),2,3)
> D
     [,1] [,2] [,3]
[1,]  0.5  0.5    0
[2,]  0.0  0.0    1
> D%*%Sig%*%t(D)
     [,1] [,2]
[1,]  0.5    0
[2,]  0.0    2
> E=matrix(c(0,-2.5,1,1,0,-1),2,3)
> E
     [,1] [,2] [,3]
[1,]  0.0    1    0
[2,] -2.5    1   -1
> E%*%Sig%*%t(E)
     [,1]  [,2]
[1,]    5 10.00
[2,]   10 23.25
```

4. Problem 4.13 of the textbook, p. 203.

   Answer: For part (a), simply apply the result of Problem 4.11 or use the hint of Problem 4.11.

   For part (b), apply the result of Problem 4.12 to get

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})^{-1} & \boldsymbol{0} \\ \boldsymbol{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ \boldsymbol{0}' & \boldsymbol{I} \end{bmatrix}.$$

   In addition, partition $(\boldsymbol{x}-\boldsymbol{\mu})'$ as $[(\boldsymbol{x}_1-\boldsymbol{\mu}_1)', (\boldsymbol{x}_2-\boldsymbol{\mu}_2)']$. Direct multiplication would show the result.

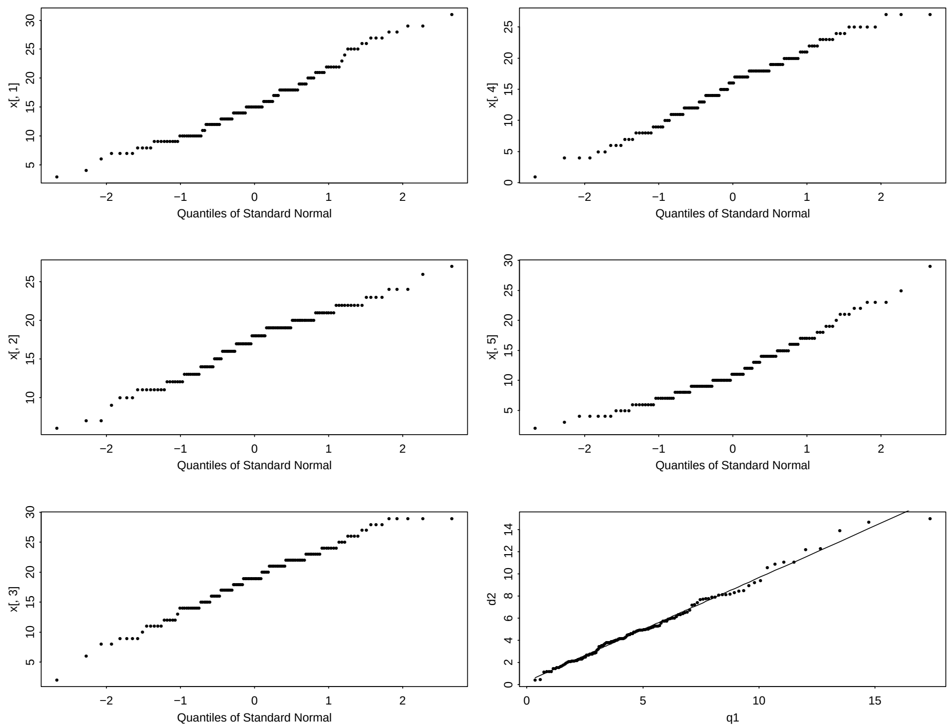
   For part (c), let $k_i = \dim(\boldsymbol{x}_i)$. Using parts (a) and (b), we can write the probability density function of $\boldsymbol{x}$ as

$$f(\boldsymbol{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{(k_1+k_2)/2}|\boldsymbol{\Sigma}_{22}|^{1/2}|\boldsymbol{\Sigma}_{11}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}|^{1/2}} \times$$

$$\exp[-\frac{1}{2}(\boldsymbol{x}_1-\boldsymbol{\mu}_1-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2-\boldsymbol{\mu}_2))'(\boldsymbol{\Sigma}_{11}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})^{-1}(\boldsymbol{x}_1-\boldsymbol{\mu}_1-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2-\boldsymbol{\mu}_2))]\times$$

$$\exp[-\frac{1}{2}(\boldsymbol{x}_2-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2-\boldsymbol{\mu}_2)],$$

Figure 1: QQ-plots for Problem 4.39



which is the product of $f(\boldsymbol{x}_2|\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22}) \times f(\boldsymbol{x}_1|\boldsymbol{x}_2, \boldsymbol{\mu}, \boldsymbol{\Sigma})$. Consequently, $\boldsymbol{x}_2 \sim N_{k_2}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$ and $\boldsymbol{X}_1|\boldsymbol{X}_2 = \boldsymbol{x}_2 \sim N_{k_1}(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})$.

5. Parts (a) and (b) of Problem 4.39 of the textbook, p. 207.

   Answer: The normality assumption seems reasonable. See the attached figure for individual and joint QQ-plots.