

Solutions to Midterm

All tests are based on the 5% significance level.

**Problem A:** (30 pts) Answer briefly the following questions.

1. For problems 1 to 6, consider the daily log return, in percentages, of the S&P 500 composite index from January 1996 to December 31, 2004 for 2267 data points. Summary statistics of the percentage log returns from SCA and Splus are attached. See Page 1 of the attached output. Is the mean of percentage log returns significantly different from zero? Why?

A: t-ratio = 1.187 < 1.96. Thus, the mean return is not significantly different from zero.

2. Suppose one invested \$1 dollar on the S&P 500 index at the very beginning of 1996 and held the investment until the end of 2004. What was the value of the investment at the market closing on December 31, 2004?

A: total log return =  $0.0299 \times 2267 / 100 = 0.67783$  so that the value =  $\exp(0.67783) \approx 1.97$ .

3. Test the null hypothesis that the skewness of the log returns is zero. Draw your conclusion.

A: t-ratio =  $-0.0939 / 0.0514 = -1.83 < 1.96$  so that the skewness is not significantly different from zero.

4. Given that the last percentage log return was  $-0.134$  (i.e.  $T =$  December 31, 2004), which is the corresponding simple return?

A:  $R_t = \exp(-0.134/100) - 1 = -0.001339 = .1339\%$ .

5. Are the log return serially correlated? You may use Q(10) of the series to answer the question.

A:  $Q(10) = 13.9$  with p-value 0.178 so that there is no serial correlation.

6. Is there any ARCH effect in the log return series? You may use Q(12) of the squared series to answer the question.

A: Yes, Q(12) of squared return is large at 490 (in SCA) and has a p-value of 0.0 from Splus.

7. Give two empirical features of daily log returns of a financial asset.

A: Any two of (a) high kurtosis, (b) volatility clustering, (3) skew to left.

8. What is the purpose of studying kurtosis of an asset return series?

A: Understanding the tail behavior (or risk) of the return.

9. Describe two applications of studying sample autocorrelation function (ACF) of an asset return series.

A: (a) To test serial correlations in the return series, (b) to identify MA order.

10. Describe two methods that can be used to identify the order of an AR model.

A: (a) PACF, (b) Criterion functions such as AIC or BIC.

11. Consider the AR(1) model  $(1 - 0.9B)r_t = 0.2 + a_t$ , where  $\{a_t\}$  is an independent and identically distributed random variables with mean zero and variance 1.0. What is the half-life of the series?

A: Half-life =  $\ln(0.5)/\ln(0.9) = 6.58$  time units.

12. Suppose that the log return  $r_t$  of an asset follows the model below:

$$r_t = 0.02 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = 0.116 + 0.42a_{t-1}^2.$$

Let  $p_t$  be the log price of the asset at time  $t$  and  $p_T(\ell)$  be the  $\ell$ -step ahead forecast of the log price at the forecast origin  $T$ . Then, what is the value of  $p_T(\ell)$  as  $\ell$  increases?

A: 0.02 is the slope of time trend so that  $p_T(\ell) \rightarrow \infty$  as  $\ell$  increases.

13. **For problems 13-14**, consider quarterly series of U.S. unit labor cost from 1947 to 2004. The data were seasonally adjusted and obtained from the Federal Reserve Bank of St Louis. Let  $x_t = (1 - B)ULC_t$  be the first-differenced series of the data at time  $t$ . The model

$$(1 - 0.371B^2)x_t = 0.265 + (1 + 0.171B^4)a_t, \quad \sigma_a = 0.5998,$$

fits the data reasonably well. Under the fitted model, what is the mean of  $x_t$ , i.e.  $E(x_t) = ?$

A:  $E(x_t) = 0.265/(1 - 0.371) = 0.421$ .

14. Does the model imply that there exist business cycles in the unit labor cost? Why?

A: No, because the two roots are real. From  $1 - 0.371x^2$ , we have  $x = \pm 1/\sqrt{(0.371)}$ .

15. Give two weaknesses of the GARCH-type of models for modeling asset volatility.

A: Any two of (a) symmetric response to past positive and negative returns, (b) restrictive, (c) providing no explanation of volatility clustering, (d) no adaptive in forecasting.

**Problem B.** (20 pts) This problem is concerned with the analysis of quarterly earnings per share of the Procter & Gamble (PG) Company from 1992 to the first quarter of 2003 for 45 data points. The data were obtained from First Call. Log transformation was taken to stabilize the variability of the earnings. Computer output is attached; p.2-6 of output. Due to strong seasonal pattern, which results in models that are close to being non-invertible, we analyze the seasonally differenced series in Splus. Let  $x_t$  be the logarithm of quarterly earnings per share and  $w_t = (1 - B^4)x_t$ . Thus, SCA uses  $x_t$  and Splus uses  $w_t$ .

1. (5 points) Because ACF of the log earnings shows strong seasonal pattern, the seasonal difference  $(1 - B^4)$  is taken. The ACF of the seasonally differenced data indicates no further differencing is necessary. Write down the fitted model for the  $x_t$  series (not the differenced  $w_t$ ).

A:  $(1 - 0.47B)(1 - B^4)x_t = 0.0508 + (1 - 0.307B^4)a_t, \sigma_a = 0.0502.$

2. (4 points) Is the AR coefficient of the fitted model statistically significant? Why?

A: Yes, the t-ratio is 3.26, which is greater than the critical value 1.96.

3. (4 points) Is there any serial correlation in the residuals of the fitted model? Use Q(12) of the ACF of residuals to answer the question. [Hint: for a chi-square distribution with  $m$  degrees of freedom, the expected value is  $m$ .]

A:  $Q(12) = 8.8$  which is less than  $E(\chi_{10}^2) = 10$  so that p-value  $> 0.05$ .

4. (4 points) Let  $T = 45$  be the forecast origin. What are the 1-step and 2-step ahead forecasts of the fitted model (after taking anti-log transformation)?

A:  $x_{45}(1) = 0.912$  and  $x_{45}(2) = 3.95$  (from SCA). For Splus,  $x_{45}(1) = \exp(0.0087 - .1743) = 0.847$ ,  $x_{45}(2) = \exp(-.012479 + 1.278152) = 3.55$ .

5. (3 points) Give an interpretation of the estimated constant 0.0508 of the fitted model for  $x_t$ .

A; Slope of time trend.

**Problem C.** (20 pts) Consider the daily log returns, in percentages, of the Wal-Mart stock and the S&P 500 index from January 1999 to December 2004 with sample size  $T = 1508$ . We employ the *market model*:

$$r_t = \alpha + \beta r_{m,t} + e_t,$$

where  $r_t$  and  $r_{m,t}$  are Wal-Mart stock return and S&P 500 index return, respectively. Splu's output is attached; page 6 of output. Answer the following questions.

1. (4 points) Write down the fitted market model.

A:  $r_t = 0.0205 + 0.9606r_{m,t} + e_t$ .

2. (4 points) The Ljung-Box statistics of the ACF of residuals show some minor serial correlations, but the ACFs are relatively small so we ignore the serial correlations and perform the ARCH effect test. Is there an ARCH effect in the residuals of the fitted market model?

A: Yes, archTest gives a p-value about 0.0.

3. (4 points) We employ a GARCH(1,1) model (called "m2" in the output). Write down the fitted model. Comment on the fitted model.

A: Let  $r_t$  and  $r_{m,t}$  be the Wal-Mart stock return and S&P 500 index return, respectively. The model is

$$\begin{aligned} r_t &= 0.9386r_{m,t} + a_t = 0.9386 + \sigma_t \epsilon_t, & \epsilon_t &\sim t_{6.41} \\ \sigma_t^2 &= -4.29 \times 10^{-5} + 0.0377a_{t-1}^2 + 0.963\sigma_{t-1}^2. \end{aligned}$$

The negative estimate of  $\alpha_0$  does not make sense, but it is statistically not different from 0. Also, the  $\hat{\alpha}_1 + \hat{\beta}_1 \approx 1$  so that the fitted model indicates an IGARCH(1,1) model without the constant.

4. (6 points) Further analysis indicates that an EGARCH(2,1) model fits the data better. There are two leverage effect parameters. Are these two effects statistically significant? Why? You may test the effect individually.

A: Examine the t-ratio of the two leverage parameter estimates. For Lev(1), the t-ratio is  $-2.129$ . For Lev(2), the t-ratio is  $-1.847$ . In both cases, the p-values are less than 0.05 so that they are both significant. [If you use two-sided tests, then Lev(2) is not significant.]

5. (2 points) What are the 1-step ahead forecasts for the return and its volatility of Wal-Mart stock at the forecast origin  $T = 1508$  using the EGARCH model.

A: zero for return and 0.7082 for volatility.

**Problem D.** (30 pts) Consider the daily log returns, in percentages, of Home Depot stock from January 1995 to December 2004 with 2519 observations. Splus output is attached; page 8 of output. Answer the following questions. The ACFs of the returns are small so that the mean equation consists of a constant term only.

1. (5 points) Consider the fitted GARCH(1,1) model. The volatility equation is

$$\sigma_t^2 = 0.042 + 0.049a_{t-1}^2 + 0.944\sigma_{t-1}^2.$$

Let  $\eta_t = a_t^2 - \sigma_t^2$ . Rewrite the prior volatility equation in an ARMA form for the  $\{a_t^2\}$  series.

A:  $a_t^2 = 0.042 + 0.993a_{t-1}^2 + \eta_t - 0.944\eta_{t-1}$ .

2. (5 points) Write down the fitted EGARCH(1,1) model with leverage effect (both mean and volatility equations and the parameter of the conditional distribution used).

A: The model is

$$\begin{aligned} r_t &= 0.04918 + a_t = 0.04918 + \sigma_t \epsilon_t, \quad \epsilon_t \sim t_{6.68} \\ \ln(\sigma_t^2) &= -0.06316 + 0.1033 \left( \frac{|a_{t-1}| - 0.5464a_{t-1}}{\sigma_{t-1}} \right) + 0.989 \ln(\sigma_{t-1}^2). \end{aligned}$$

3. (4 points) Test the hypotheses  $H_o : v = 5$  vs  $H_a : v \neq 5$ , where  $v$  is the degrees of freedom of the conditional Student  $t$  distribution. Draw your conclusion. [Hint: you may use the usual t-ratio test.]

A; t-ratio =  $\frac{6.684-5}{0.643} = 2.62 > 1.96$ . Thus, reject  $H_o : v = 5$ .

4. (5 points) To better understand the leverage effect, use the fitted EGARCH(1,1) model to calculate the ratio  $\frac{\sigma_t^2(\epsilon=-3)}{\sigma_t^2(\epsilon=3)}$ , where  $\epsilon_t$  denotes the *iid* sequence of the innovations defined in class.

A: From the fitted volatility equation, we have

$$\sigma_t^2 = \exp(-0.06316)(\sigma_{t-1}^2)^{.989} \exp(0.1033|\epsilon_{t-1}| - 0.0564\epsilon_{t-1}).$$

Therefore,

$$\frac{\sigma_t^2(\epsilon = -3)}{\sigma_t^2(\epsilon = 3)} = \frac{\exp(-0.0564(-3))}{\exp(-0.0564(3))} = \exp(0.0564 \times 6) = 1.403.$$

5. (4 points) Used the fitted EGARCH model and  $T = 2519$  as the forecast origin. What are the 1-step ahead forecasts of log return and volatility?

A: Forecast of log return is 0.0492 and forecast of volatility is 1.184.

6. (4 points) Write down the mean equation of the fitted GARCH-M model for the data.

A:  $r_t = 0.058 + 0.00629\sigma_t^2 + a_t$ .

7. (3 points) Based on the GARCH-M model, is the risk premium statistically significant? Why?

A: No, the t-ratio is 0.453 with p-value = 0.65 (two-sided).