Lecture Note of Bus 41202, Spring 2006: Multivariate Time Series Analysis

Forcus on two series (Bivariate)

Time series:

$$\boldsymbol{X}_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}.$$

Data: $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_T$.

Weak stationarity:

$$E(\boldsymbol{X}_t) = \boldsymbol{\mu} \quad \mathrm{Cov}(\boldsymbol{X}_t, \boldsymbol{X}_{t-j}) = \boldsymbol{\Gamma}_j$$

are time-invariant

Autocovariane matrix: Lag-ℓ

$$\boldsymbol{\Gamma}_\ell = E[(\boldsymbol{X}_t - \boldsymbol{\mu})(\boldsymbol{X}_{t-\ell} - \boldsymbol{\mu})']$$

$$= \begin{bmatrix} E(x_{1t} - \mu_1)(x_{1,t-\ell} - \mu_1) & E(x_{1t} - \mu_1)(x_{2,t-\ell} - \mu_2) \\ E(x_{2t} - \mu_2)(x_{1,t-\ell} - \mu_1) & E(x_{2t} - \mu_2)(x_{2,t-\ell} - \mu_2) \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_{11}(\ell) & \Gamma_{12}(\ell) \\ \Gamma_{21}(\ell) & \Gamma_{22}(\ell) \end{bmatrix}.$$

Not symmetric if $\ell \neq 0$. Consider $\boldsymbol{\Gamma}_1$:

- $\Gamma_{12}(1)$: $\mathrm{Cov}(x_{1t}, x_{2,t-1})$
- $\Gamma_{21}(1)$: $\mathrm{Cov}(x_{2t}, x_{1,t-1})$

Let the diagonal matrix $\boldsymbol{D}$ be

$$\boldsymbol{D} = \begin{bmatrix} \text{std}(x_{1t}) & 0 \\ 0 & \text{std}(x_{2t}) \end{bmatrix} = \begin{bmatrix} \sqrt{\Gamma_{11}(0)} & 0 \\ 0 & \sqrt{\Gamma_{22}(0)} \end{bmatrix}.$$

Cross-Correlation matrix:

$$\boldsymbol{\rho}_\ell = \boldsymbol{D}^{-1}\boldsymbol{\Gamma}_\ell \boldsymbol{D}^{-1}$$

Thus, $\rho_{ij}(\ell)$ is the cross-correlation between x_{it} and $x_{j,t-\ell}$.

From stationarity:

$$\boldsymbol{\Gamma}_\ell = \boldsymbol{\Gamma}'_{-\ell}, \quad \boldsymbol{\rho}_\ell = \boldsymbol{\rho}'_{-\ell}.$$

For instance, $\text{cor}(x_{1t}, x_{2,t-1}) = \text{cor}(x_{2t}, x_{1,t+1})$.

Testing for serial dependence

Multivariate version of Ljung-Box $Q(m)$ statistics available.

$H_o : \boldsymbol{\rho}_1 = \cdots = \boldsymbol{\rho}_m = \mathbf{0}$ vs $H_a : \boldsymbol{\rho}_i \neq \mathbf{0}$ for some i

$$Q_2(m) = T^2 \sum_{\ell=1}^{m} \frac{1}{T-\ell} tr(\hat{\boldsymbol{\Gamma}}'_\ell \hat{\boldsymbol{\Gamma}}_0^{-1} \hat{\boldsymbol{\Gamma}}_\ell \hat{\boldsymbol{\Gamma}}_0^{-1})$$

which is $\chi^2_{km^2}$. Note tr denotes sum of diagonal elements.

Vector Autoregressive Models(VAR)

VAR(1) model for two return series:

$$\begin{bmatrix} r_{1t} \\ r_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{10} \\ \phi_{20} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} r_{1,t-1} \\ r_{2,t-1} \end{bmatrix} + \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix},$$

where $\boldsymbol{a}_t = (a_{1t}, a_{2t})'$ is a sequence of iid bivariate normal random vectors with mean zero and covariance matrix

$$\text{Cov}(\boldsymbol{a}_t) = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

where $\sigma_{12} = \sigma_{21}$.

Rewrite the model as

$$\begin{aligned} r_{1t} &= \phi_{10} + \phi_{11} r_{1,t-1} + \phi_{12} r_{2,t-1} + a_{1t} \\ r_{2t} &= \phi_{20} + \phi_{21} r_{1,t-1} + \phi_{22} r_{2,t-1} + a_{1t} \end{aligned}$$

Thus, ϕ_{11} and ϕ_{12} denotes the dependence of r_{1t} on the past returns $r_{1,t-1}$ and $r_{2,t-1}$, respectively.

Unidirectional dependence

For the VAR(1) model, if $\phi_{12} = 0$, but $\phi_{21} \neq 0$, then

- r_{1t} does not depend on $r_{2,t-1}$, but
- r_{2t} depends on $r_{1,t-1}$,

implying that knowning $r_{1,t-1}$ is helpful in predicting r_{2t}, but $r_{2,t-1}$ is not helpful in forecasting r_{1t}.

$\{r_{1t}\}$ is an *input*, $\{r_{2t}\}$ is the *output* variable.

A **Granger** causality relation.

If $\sigma_{12} = 0$, then r_{1t} and r_{2t} are not concurrently correlationed.

Stationarity condition

Write the VAR(1) model as

$$\boldsymbol{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\Phi} \boldsymbol{r}_{t-1} + \boldsymbol{a}_t.$$

$\{\boldsymbol{r}_t\}$ is stationary if zeros of the polynomial

$$|\boldsymbol{I} - \boldsymbol{\Phi} x|$$

are greater than 1 in modulus.

(A generalization of univariate case)

Mean of $\boldsymbol{r}_t$ satisfies

$$(\boldsymbol{I} - \boldsymbol{\Phi})\boldsymbol{\mu} = \boldsymbol{\phi}_0, \quad \text{or}$$

$$\boldsymbol{\mu} = (\boldsymbol{I} - \boldsymbol{\Phi})^{-1}\boldsymbol{\phi}_0$$

if the inverse exists.

Covariance matrices:

$\text{Cov}(\boldsymbol{r}_t) = \Sigma_{i=0}^{\infty} \boldsymbol{\Phi}^i \boldsymbol{\Sigma} (\boldsymbol{\Phi}^i)'$

$\boldsymbol{\Gamma}_\ell = \boldsymbol{\Phi}\boldsymbol{\Gamma}_{\ell-1}$ for $\ell > 0$.

Can be generalized to higher order models.

Building VAR models

- Order selection: use AIC or a stepwise χ^2 test Eq. (8.18)

  For instance, test VAR(1) vs VAR(2).

- Estimation: use ordinary least squares method

- Model checking: as univariate case

- Forecasting: similar to univariate case

Simple AR models are sufficient to model asset returns.

Co-integration

Basic ideas

- x_{1t} and x_{2t} are unit-root nonstationary
- a linear combination of x_{1t} and x_{2t} is unit-root stationary

That is, x_{1t} and x_{2t} share a single unit root!

Why is it of interest?

Stationary series is *mean reverting.*

Long term forecasts of the "linear" combination converge to a mean value, implying that the long-term forecasts of x_{1t} and x_{2t} must be linearly related.

Co-integration test

Several tests available, e.g. Johansen's test.

Basic idea

Consider a univariaet AR(2) model

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + a_t.$$

Let $\Delta x_t = x_t - x_{t-1}$.

Substract x_{t-1} from both sides and rearrange terms

$$\Delta x_t = \gamma x_{t-1} + \phi_1^* \Delta x_{t-1} + a_t,$$

where $\phi_1^* = -\phi_2$ and $\gamma = \phi_2 + \phi_1 - 1$.

x_t is unit-root nonstationary if and only if $\gamma = 0$.

Testing for x_t has a unit root is equivalent to testing for $\gamma = 0$ in the above model.

VAR(p) case:

$$\boldsymbol{X}_t = \boldsymbol{\Phi}_1 \boldsymbol{X}_{t-1} + \cdots + \boldsymbol{\Phi}_p \boldsymbol{X}_{t-p} + \boldsymbol{a}_t.$$

Let $\boldsymbol{Y}_t = \boldsymbol{X}_t - \boldsymbol{X}_{t-1}$.
Rewrite the model as

$$\boldsymbol{Y}_t = \boldsymbol{\Omega} \boldsymbol{X}_{t-1} + \sum_{i=1}^{p-1} \boldsymbol{\Phi}_i^* \boldsymbol{Y}_{t-i} + \boldsymbol{a}_t, \quad (1)$$

where

$$\begin{aligned}
\boldsymbol{\Phi}_{p-1}^* &= -\boldsymbol{\Phi}_p \\
\boldsymbol{\Phi}_{p-2}^* &= -\boldsymbol{\Phi}_{p-1} - \boldsymbol{\Phi}_p \\
\vdots &= \vdots \\
\boldsymbol{\Phi}_1^* &= -\boldsymbol{\Phi}_2 - \cdots - \boldsymbol{\Phi}_p \\
\boldsymbol{\Omega} &= \boldsymbol{\Phi}_p + \cdots + \boldsymbol{\Phi}_1 - \boldsymbol{I}.
\end{aligned}$$

This is the *Error-Correction* Model (ECM).

To test for co-integration:

- Fit the model in Eq. (1),
- Test for the rank of $\boldsymbol{\Omega}$.

If $\boldsymbol{X}_t$ is k dimensional, and rank of $\boldsymbol{\Omega}$ is m, then we have $k-m$ unit roots in $\boldsymbol{X}_t$.
There are m linear combinations of $\boldsymbol{X}_t$ that are unit-root stationary.

If $\boldsymbol{\Omega}$ has rank m, then

$$\boldsymbol{\Omega} = \boldsymbol{\alpha}\boldsymbol{\beta}$$

where $\boldsymbol{\alpha}$ is a $k \times m$ and $\boldsymbol{\beta}$ is a $m \times k$ full-rank matrix.

$\boldsymbol{Z}_t = \boldsymbol{\beta}\boldsymbol{X}_t$ is unit-root stationary.

$\boldsymbol{\beta}$ is the co-integrating vector.

Discussion

- ECM formulation is useful
- Co-integration tests have some weaknesses, e.g. robustness
- Co-integration overlooks the effect of scale of the series

Demonstration of VAR models: U.S. quarterly GDP and unemployment rates starting from 1948 with 228 observations.

```
**** S-Plus: Quarterly U.S. gdp and unemployment rate ****
> da=read.table("q-gdpun.txt")
> dim(da)
[1] 228   5
> tdx=c(1:228)/4+1948
> par(mfcol=c(2,1))
> plot(tdx,da[,4],xlab='year',ylab='ln',type='l')
> title(main='Log(gdp)')
> plot(tdx,da[,5],xlab='year',ylab='rate',type='l')
> title(main='unemployment rate')
> gdp=diff(da[,4])
> un=diff(da[,5])
> plot(tdx[2:228],gdp,xlab='year',ylab='growth',type='l')
> title(main='GDP growth rate')
> plot(tdx[2:228],un,xlab='year',ylab='unemp',type='l')
> title(main='Changes in unemployment rate')

> acf(gdp) % compute ACFs
> acf(un)

> x=cbind(gdp,un)
> zt=data.frame(x)
> ord=VAR(zt,lag.max=8)   % Find AR order
> ord$ar.order
```

```
[1] 1
> fit=VAR(x~ar(1))

> summary(fit)
Call:
VAR(formula = x ~ ar(1))

Coefficients:
                 gdp       un
(Intercept)   0.0069   0.1074
  (std.err)   0.0010   0.0310
   (t.stat)   7.0636   3.4619

   gdp.lag1   0.1814 -12.1629
  (std.err)   0.0880   2.7891
   (t.stat)   2.0615  -4.3608

    un.lag1  -0.0054   0.4078
  (std.err)   0.0022   0.0708
   (t.stat)  -2.3961   5.7595

Regression Diagnostics:
                  gdp     un
     R-squared 0.1319 0.4397
Adj. R-squared 0.1241 0.4347
  Resid. Scale 0.0094 0.2971

Information Criteria:
       logL        AIC        BIC         HQ
   758.2854 -1504.5707 -1484.0475 -1496.2884

                   total residual
Degree of freedom:   226      223

> names(fit)
 [1] "R"      "coef"    "fitted"    "residuals" "Sigma"   "df.resid"
 [7] "rank"   "call"    "ar.order" "n.na"      "terms"   "Y0"
> dim(fit$residuals)
[1] 226   2
> acf(fit$residuals[,1])   % compute residuals ACFs
> acf(fit$residuals[,2])

*** R: Multivariate Time Series Analysis ***
*** Very limited capability ***
```

```
> da=read.table("q-gdpun.txt")
> dim(da)
[1] 228   5
> gdp=da[,4]
> un=da[,5]
> tdx=c(1:228)/4+1948
> par(mfcol=c(2,1))
> plot(tdx,gdp,type='l',xlab='year',ylab='gdp')
> plot(tdx,un,type='l',xlab='year',ylab='un')
> x1=diff(gdp)
> x2=diff(un)
> x=cbind(x1,x2)
> dim(x)
[1] 227   2
> fit=ar(x,order.max=8)
> fit$order
[1] 2   % a vector AR(2) model is selected.
> fit
Call:
ar(x = x, order.max = 8)

$ar
, , 1
         gdp        un
gdp   0.1435 -0.008765
un  -11.5745  0.452652

, , 2
         gdp        un
gdp   0.1730  0.008553
un  -10.0953 -0.299139

$var.pred
           gdp        un
gdp  8.402e-05 -0.001706
un  -1.706e-03  0.081848

> names(fit)
 [1] "order"      "ar"         "var.pred"   "x.mean"     "aic"
 [6] "n.used"     "order.max"  "partialacf" "resid"      "method"
[11] "series"     "frequency"  "call"
> fit$method
[1] "Yule-Walker"
```

```
> fit=ar(x,order.max=10,method=c("ols"))  %Use ordinary least squares method
> fit$order
[1] 8
> fit
Call:
ar(x = x, order.max = 10, method = c("ols"))

$ar
, , 1
         gdp        un
gdp   0.1732 -0.007466
un  -11.8784  0.371661

, , 2
        gdp        un
gdp  0.1130  0.006198
un  -8.3815 -0.180014

, , 3
          gdp         un
gdp -0.003447  0.0002626
un  -4.510161 -0.0841344

, , 4
         gdp       un
gdp  0.09733  0.00704
un  -0.79610 -0.16820

, , 5
         gdp        un
gdp 0.006099 -0.001274
un  4.191216  0.077492

, , 6
        gdp       un
gdp  0.1155 0.002229
un  -3.7503 0.030763

, , 7
         gdp        un
gdp -0.06509 -0.001843
un   1.64065  0.083381

, , 8
```

```
         gdp         un
gdp 0.04539  0.003241
un  0.81069 -0.170200

$x.intercept
        gdp         un
 1.077e-05 -5.426e-03

$var.pred
            gdp         un
gdp  7.096e-05 -0.001409
un  -1.409e-03  0.068373
```