Lecture Note of Bus 41202, Spring 2006: Multivariate Volatility Models

Multivariate Volatility Models

How do the correlations between asset returns change over time?

Forcus on two series (Bivariate)

Two asset return series:

$$\boldsymbol{r}_t = \begin{bmatrix} r_{1t} \\ r_{2t} \end{bmatrix}.$$

Data: $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_T$.

Basic concept

Let F_{t-1} denote the information available at time $t-1$.

Partition the return as

$$\boldsymbol{r}_t = \boldsymbol{\mu}_t + \boldsymbol{a}_t, \quad \boldsymbol{a}_t = \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\epsilon}_t$$

where $\boldsymbol{\mu}_t = E(\boldsymbol{r}_t|F_{t-1})$ is the predictable component, and

$$\text{Cov}(\boldsymbol{a}_t|F_{t-1}) = \boldsymbol{\Sigma}_t = \begin{bmatrix} \sigma_{11,t} & \sigma_{12,t} \\ \sigma_{21,t} & \sigma_{22,t} \end{bmatrix},$$

$\{\boldsymbol{\epsilon}_t\}$ are iid 2-dimensional random vectors with mean zero and identity covariance matrix.

Multivariate volatility modeling

Study time evolution of $\{\boldsymbol{\Sigma}_t\}$.

$\boldsymbol{\Sigma}_t$ is symmetric, i.e. $\sigma_{12,t} = \sigma_{21,t}$
There are 3 variables in $\boldsymbol{\Sigma}_t$.
If k asset returns, $\boldsymbol{\Sigma}_t$ has $k(k+1)/2$ variables.

Requirement

$\boldsymbol{\Sigma}_t$ must be positive definite for all t,

$$\sigma_{11,t} > 0, \quad \sigma_{22,t} > 0, \quad \sigma_{11,t}\sigma_{22,t} - \sigma_{12,t}^2 > 0.$$

The time-varying correlation between r_{1t} and r_{2t} is

$$\rho_{12,t} = \frac{\sigma_{12,t}}{\sqrt{\sigma_{11,t}\sigma_{22,t}}}.$$

Some complications

- Positiveness requirement is not easy to meet
- Too many series to model

Some simple models available

- Exponentially weighted covariance
- Diagonal model
- BEKK model
- Others, e.g. Dynamic correlation models

Exponentially weighted model

$$\boldsymbol{\Sigma}_t = (1-\lambda)\boldsymbol{a}_{t-1}\boldsymbol{a}'_{t-1} + \lambda\boldsymbol{\Sigma}_{t-1},$$

where $0 < \lambda < 1$. That is,

$$\boldsymbol{\Sigma}_t = (1-\lambda)\sum_{i=1}^{\infty}\lambda^{i-1}\boldsymbol{a}_{t-i}\boldsymbol{a}'_{t-i}.$$

```
cov1 = EWMA.cov(rtn, lambda=0.96) % lambda given.
cov2 = mgarch(rtn~1,~ewma1,trace=F) % Estimate lambda
```

Diagonal VEC model

May not be positive definite.

Model elements of $\boldsymbol{\Sigma}_t$ separately

For instance, DVEC(1,1) model

$$\begin{aligned}
\sigma_{11,t} &= c_{11} + \alpha_{11}a^2_{1,t-1} + \beta_{11}\sigma_{11,t-1} \\
\sigma_{12,t} &= c_{12} + \alpha_{12}a_{1,t-1}a_{2,t-1} + \beta_{11}\sigma_{12,t-1} \\
\sigma_{22,t} &= c_{22} + \alpha_{22}a^2_{2,t-1} + \beta_{22}\sigma_{22,t-1}
\end{aligned}$$

In S-Plus, c is called “A”, α is ARCH(1), β is GARCH(1).

```
fit = mgarch(rtn~1,~dvec(1,1))
summary(fit)
names(fit)  % see what are stored.
  %In particular, fit$R.t stores correlations
```

BEKK model Engle and Kroner (1995)

Simple BEKK(1,1) model

$$\boldsymbol{\Sigma}_t = \boldsymbol{A}_0\boldsymbol{A}_0' + \boldsymbol{A}_1(\boldsymbol{a}_{t-1}\boldsymbol{a}_{t-1}')\boldsymbol{A}_1' + \boldsymbol{B}_1\boldsymbol{\Sigma}_{t-1}\boldsymbol{B}_1'$$

where $\boldsymbol{A}_0$ is a lower triangular matrix, $\boldsymbol{A}_1$ and $\boldsymbol{B}_1$ are square matrices without restrictions.

Pros: positive definite

Cons: Many parameters, dynamic relations require further study

```
fit2=mgarch(rtn~1,~bekk(1,1))
summary(fit2)
names(fit2)
```

Consider the monthly log returns of GM stock and the S&P500 index from 1950 to 2002 for 638 data points.

```
*** Demonstration of S-Plus commands for Chapter 9

> gm=gmspln5002$gm
> sp=gmspln5002$sp
> par(mfcol=c(2,1))
> plot(gm,type='l')
> plot(sp,type='l')

> gmsp=cbind(gm,sp) % create a 2-dim return series.
> fit1=mgarch(gmsp~1,~dvec(1,1),trace=F)
> summary(fit1)
mgarch(formula.mean=gmsp~1,formula.var=~dvec(1,1),trace=F)

Mean Equation: gmsp ~ 1
```

```
Conditional Variance Equation:  ~ dvec(1, 1)
Conditional Distribution:  gaussian
-------------------------------------------------------
Estimated Coefficients:
-------------------------------------------------------
                   Value  Std.Error t value   Pr(>|t|)
          C(1) 0.0104749 0.00243210   4.307 1.919e-005
          C(2) 0.0068800 0.00153509   4.482 8.790e-006
       A(1, 1) 0.0001400 0.00004496   3.113 1.933e-003
       A(2, 1) 0.0001073 0.00004952   2.168 3.055e-002
       A(2, 2) 0.0001033 0.00003944   2.619 9.018e-003
 ARCH(1; 1, 1) 0.0831688 0.01826946   4.552 6.367e-006
 ARCH(1; 2, 1) 0.0385129 0.01622698   2.373 1.792e-002
 ARCH(1; 2, 2) 0.0802303 0.02089382   3.840 1.355e-004
GARCH(1; 1, 1) 0.8916125 0.02274390  39.202 0.000e+000
GARCH(1; 2, 1) 0.9029094 0.03801639  23.751 0.000e+000
GARCH(1; 2, 2) 0.8656909 0.03383195  25.588 0.000e+000
-------------------------------------------------------
AIC(11) = -4269.022
BIC(11) = -4220.015

Normality Test:
-------------------------------------------------------
   Jarque-Bera    P-value Shapiro-Wilk P-value
gm       44.42 2.266e-010       0.9869  0.6761
sp       21.52 2.122e-005       0.9892  0.8954

Ljung-Box test for standardized residuals:
-------------------------------------------------------
```

```
   Statistic P-value Chi^2-d.f.
gm     19.67 0.07364         12
sp     19.98 0.06740         12

Ljung-Box test for squared standardized residuals:
-------------------------------------------------------
   Statistic P-value Chi^2-d.f.
gm    15.652  0.2077         12
sp     4.999  0.9580         12

Lagrange multiplier test:
-------------------------------------------------------
      Lag 1   Lag 2   Lag 3   Lag 4  Lag 5   Lag 6
gm -0.06115 -0.2803 -1.4642 -0.4926 1.1335 -1.6898
sp -0.02731 -0.4904 -0.5857 -0.9897 0.5748  0.0973

     Lag 7  Lag 8  Lag 9  Lag 10 Lag 11  Lag 12      C
gm -0.2745 1.2502 0.9645 -0.8264 2.1744 0.08219 0.4325
sp -0.1627 0.4246 0.3809  1.4249 0.2165 0.12434 0.5229

     TR^2 P-value F-stat P-value
gm 15.035  0.2395 1.4006  0.2743
sp  4.739  0.9661 0.4341  0.9909

> fit2=mgarch(gmsp~1,~bekk(1,1))  % BEKK model
> summary(fit2)
mgarch(formula.mean=gmsp~1,formula.var=~bekk(1,1))

Mean Equation: gmsp ~ 1
```

```
Conditional Variance Equation:  ~ bekk(1, 1)
Conditional Distribution:  gaussian
-------------------------------------------------------
Estimated Coefficients:
-------------------------------------------------------
                     Value Std.Error   t value   Pr(>|t|)
          C(1)  0.0101123  0.002697   3.74967 1.934e-004
          C(2)  0.0065510  0.001697   3.86051 1.248e-004
       A(1, 1)  0.0119214  0.007089   1.68176 9.311e-002
       A(2, 1)  0.0146959  0.004269   3.44261 6.143e-004
       A(2, 2)  0.0007941  0.056368   0.01409 9.888e-001
 ARCH(1; 1, 1)  0.3348499  0.052971   6.32135 4.910e-010
 ARCH(1; 2, 1)  0.0814715  0.039972   2.03820 4.195e-002
 ARCH(1; 1, 2) -0.3081572  0.078010  -3.95023 8.690e-005
 ARCH(1; 2, 2)  0.1630736  0.054573   2.98816 2.916e-003
GARCH(1; 1, 1)  0.9237848  0.020397  45.29098 0.000e+000
GARCH(1; 2, 1) -0.0280710  0.016658  -1.68515 9.245e-002
GARCH(1; 1, 2)  0.0710387  0.047735   1.48818 1.372e-001
GARCH(1; 2, 2)  0.9304805  0.031737  29.31835 0.000e+000
-------------------------------------------------------
AIC(13) = -4241.851
BIC(13) = -4183.933

Normality Test:
-------------------------------------------------------
   Jarque-Bera    P-value Shapiro-Wilk P-value
gm       41.87 8.086e-010       0.9901  0.9418
sp       36.84 1.001e-008       0.9868  0.6660
```

```
Ljung-Box test for standardized residuals:
--------------------------------------------------------------
   Statistic P-value Chi^2-d.f.
gm     21.68 0.04123         12
sp     20.98 0.05067         12

Ljung-Box test for squared standardized residuals:
--------------------------------------------------------------
   Statistic P-value Chi^2-d.f.
gm    20.540 0.05754         12
sp     7.639 0.81264         12

Lagrange multiplier test:
--------------------------------------------------------------
     Lag 1   Lag 2   Lag 3   Lag 4 Lag 5   Lag 6   Lag 7
gm 0.09856 -0.1232 -1.1413 -0.4392 1.279 -1.6702 -0.2185
sp 0.77406 -0.4373 -0.4633 -0.2767 1.181 -0.2194  0.6637

     TR^2 P-value F-stat P-value
gm 19.903 0.06895 1.8690  0.1201
sp  6.963 0.86002 0.6402  0.8942

>fit3=mgarch(gmsp~1,~ccc(1,1)) %Constant correlations!
>summary(fit3)
mgarch(formula.mean = gmsp ~ 1,formula.var=~ccc(1,1))

Mean Equation: gmsp ~ 1
Conditional Variance Equation:  ~ ccc(1, 1)
Conditional Distribution:  gaussian
```

```
---------------------------------------------------------------
Estimated Coefficients:
---------------------------------------------------------------
                      Value  Std.Error t value   Pr(>|t|)
           C(1) 0.01049566 0.00241094   4.353 1.564e-005
           C(2) 0.00735053 0.00152773   4.811 1.876e-006
        A(1, 1) 0.00007338 0.00004950   1.482 1.387e-001
        A(2, 2) 0.00007252 0.00003277   2.213 2.727e-002
 ARCH(1; 1, 1) 0.11404920 0.02685448   4.247 2.493e-005
 ARCH(1; 2, 2) 0.09837054 0.02328916   4.224 2.755e-005
GARCH(1; 1, 1) 0.88296850 0.02786099  31.692 0.000e+000
GARCH(1; 2, 2) 0.86944221 0.02880646  30.182 0.000e+000

Estimated Conditional Constant Correlation Matrix:
---------------------------------------------------------------
       gm     sp
gm 1.0000 0.6482
sp 0.6482 1.0000

Standard Errors:
        gm      sp
gm      NA 0.01906
sp 0.01906      NA
---------------------------------------------------------------
AIC(8) = -4238.222
BIC(8) = -4202.58

Normality Test:
---------------------------------------------------------------
```

```
   Jarque-Bera    P-value Shapiro-Wilk P-value
gm       35.44 2.015e-008       0.9867  0.6564
sp       36.80 1.022e-008       0.9852  0.4492

Ljung-Box test for standardized residuals:
-------------------------------------------------------
   Statistic P-value Chi^2-d.f.
gm     18.05 0.11415         12
sp     22.72 0.03016         12

Ljung-Box test for squared standardized residuals:
-------------------------------------------------------
   Statistic P-value Chi^2-d.f.
gm     16.98  0.1504         12
sp     16.66  0.1627         12

Lagrange multiplier test:
----------------------------------------------------
     Lag 1    Lag 2    Lag 3   Lag 4  Lag 5   Lag 6
gm -0.3586 -0.69005 -1.81246 -0.7803 0.6568 -1.8299
sp  1.3149 -0.06239 -0.04997 -0.1384 0.8093  0.3792

    TR^2 P-value F-stat P-value
gm 16.28  0.1788  1.520  0.2214
sp 14.64  0.2618  1.363  0.2937

> fit4=mgarch(gmsp~1,~egarch(1,1),leverage=T)
> summary(fit4)
mgarch(formula.mean = gmsp ~ 1,formula.var=~egarch(1,1),
```

```
  leverage = T)

Mean Equation: gmsp ~ 1
Conditional Variance Equation:  ~ egarch(1, 1)
Conditional Distribution:  gaussian
----------------------------------------------------
Estimated Coefficients:
----------------------------------------------------
                  Value Std.Error t value  Pr(>|t|)
         C(1)  0.009157  0.003196   2.865 0.0043037
         C(2)  0.006447  0.002157   2.988 0.0029151
      A(1, 1) -0.415912  0.158767  -2.620 0.0090141
      A(2, 2) -0.719478  0.364322  -1.975 0.0487219
 ARCH(1; 1, 1)  0.219687  0.064281   3.418 0.0006724
 ARCH(1; 2, 2)  0.169399  0.075000   2.259 0.0242454
GARCH(1; 1, 1)  0.954872  0.023042  41.440 0.0000000
GARCH(1; 2, 2)  0.907813  0.051497  17.629 0.0000000
  LEV(1; 1, 1) -0.136060  0.107624  -1.264 0.2066189
  LEV(1; 2, 2) -0.374486  0.224256  -1.670 0.0954339
----------------------------------------------------
AIC(10) = -3910.694
BIC(10) = -3866.142

Normality Test:
----------------------------------------------------
   Jarque-Bera    P-value Shapiro-Wilk   P-value
gm       28.43 6.717e-007       0.9893 0.9011985
sp      222.96 0.000e+000       0.9762 0.0007066
```

```
Ljung-Box test for standardized residuals:
--------------------------------------------------------------------
   Statistic P-value Chi^2-d.f.
gm    19.568 0.07572         12
sp     9.854 0.62873         12

Ljung-Box test for squared standardized residuals:
--------------------------------------------------------------------
   Statistic P-value Chi^2-d.f.
gm    19.683 0.07332         12
sp     5.395 0.94346         12

Lagrange multiplier test:
--------------------------------------------------------------------
      Lag 1    Lag 2   Lag 3   Lag 4    Lag 5  Lag 6
gm -0.05071 -0.24765 -1.6586 -0.5954 0.906238 -1.718
sp -0.23843  0.08391 -0.4111 -0.4491 0.009386 -1.073

     TR^2 P-value F-stat P-value
gm 18.667  0.0969 1.7493  0.1475
sp  5.123  0.9538 0.4695  0.9832

*** Demonstration of forecasts
> predict(fit2,5)
$series.pred:
             gm          sp
[1,] 0.01011233 0.006550963
[2,] 0.01011233 0.006550963
[3,] 0.01011233 0.006550963
```

```
[4,] 0.01011233 0.006550963
[5,] 0.01011233 0.006550963

$sigma.pred:
            gm         sp
[1,] 0.1161069 0.05758898
[2,] 0.1149695 0.05642254
[3,] 0.1138556 0.05538088
[4,] 0.1127669 0.05445162
[5,] 0.1117043 0.05362331

$R.pred:

, , gm
     gm        sp
[1,]  1 0.5406667
[2,]  1 0.5322805
[3,]  1 0.5243709
[4,]  1 0.5169547
[5,]  1 0.5100440

, , sp
            gm sp
[1,] 0.5406667  1
[2,] 0.5322805  1
[3,] 0.5243709  1
[4,] 0.5169547  1
[5,] 0.5100440  1
```