Lecture Note of Bus 41202, Spring 2006: More Volatility Models. Mr. Ruey Tsay

The EGARCH model

Asymmetry in responses to + & − returns:

$$g(\epsilon_t) = \theta\epsilon_t + \gamma[|\epsilon_t| - E(|\epsilon_t|)],$$

with $E[g(\epsilon_t)] = 0$.

To see asymmetry of $g(\epsilon_t)$, rewrite it as

$$g(\epsilon_t) = \begin{cases} (\theta+\gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t \geq 0, \\ (\theta-\gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t < 0. \end{cases}$$

An EGARCH(m, s) model:

$$a_t = \sigma_t\epsilon_t, \ \ln(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \cdots + \beta_{s-1}B^{s-1}}{1 - \alpha_1 B - \cdots - \alpha_m B^m} g(\epsilon_{t-1}).$$

Some features of EGARCH models:

- uses log trans. to relax the positiveness constraint
- asymmetric responses

Consider an EGARCH(1,1) model

$$a_t = \sigma_t\epsilon_t, \qquad (1-\alpha B)\ln(\sigma_t^2) = (1-\alpha)\alpha_0 + g(\epsilon_{t-1}),$$

Under normality, $E(|\epsilon_t|) = \sqrt{2/\pi}$ and the model becomes

$$(1-\alpha B)\ln(\sigma_t^2) = \begin{cases} \alpha_* + (\theta+\gamma)\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0, \\ \alpha_* + (\theta-\gamma)\epsilon_{t-1} & \text{if } \epsilon_{t-1} < 0 \end{cases}$$

where $\alpha_* = (1-\alpha)\alpha_0 - \sqrt{\frac{2}{\pi}}\gamma$.

This is a nonlinear fun. similar to that of the threshold AR model of Tong (1978, 1990).

Specifically, we have

$$\sigma_t^2 = \sigma_{t-1}^{2\alpha} \exp(\alpha_*) \begin{cases} \exp[(\theta+\gamma)\frac{a_{t-1}}{\sqrt{\sigma_{t-1}^2}}] & \text{if } a_{t-1} \geq 0, \\ \exp[(\theta-\gamma)\frac{a_{t-1}}{\sqrt{\sigma_{t-1}^2}}] & \text{if } a_{t-1} < 0. \end{cases}$$

The coefs $(\theta+\gamma)$ & $(\theta-\gamma)$ show the asymmetry in response to positive and negative a_{t-1}. The model is, therefore, nonlinear if $\theta \neq 0$. Thus, θ is referred to as the *leverage* parameter.

Focus on the function $g(\epsilon_{t-1})$. The leverage parameter θ shows the effect of the sign of a_{t-1} whereas γ denotes the magnitude effect.

See Nelson (1991) for an exmaple of EGARCH model.

Another example: Monthly log returns of IBM stock from January 1926 to December 1997 for 864 observations.

An AR(1)-EGARCH(1,1):

$$\begin{aligned} r_t &= 0.0105 + 0.092 r_{t-1} + a_t, \quad a_t = \sigma_t \epsilon_t \\ \ln(\sigma_t^2) &= -5.496 + \frac{g(\epsilon_{t-1})}{1 - .856B}, \\ g(\epsilon_{t-1}) &= -.0795\epsilon_{t-1} + .2647[|\epsilon_{t-1}| - \sqrt{2/\pi}], \end{aligned}$$

Model checking:

For $\tilde{a}_t$: $Q(10) = 6.31(0.71)$ and $Q(20) = 21.4(0.32)$

For $\tilde{a}_t^2$: $Q(10) = 4.13(0.90)$ and $Q(20) = 15.93(0.66)$

Discussion:

Using $\sqrt{2/\pi} \approx 0.7979 \approx 0.8$, we obtain

$$\ln(\sigma_t^2) = -1.0 + 0.856 \ln(\sigma_{t-1}^2) + \begin{cases} 0.1852\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0 \\ -0.3442\epsilon_{t-1} & \text{if } \epsilon_{t-1} < 0. \end{cases}$$

Taking anti-log transformation, we have

$$\sigma_t^2 = \sigma_{t-1}^{2\times 0.856} e^{-1.001} \times \begin{cases} e^{0.1852\epsilon_{t-1}} & \text{if } \epsilon_{t-1} \geq 0 \\ e^{-0.3442\epsilon_{t-1}} & \text{if } \epsilon_{t-1} < 0. \end{cases}$$

For a standardized shock with magnitude 2, (i.e. two standard deviations), we have

$$\frac{\sigma_t^2(\epsilon_{t-1} = -2)}{\sigma_t^2(\epsilon_{t-1} = 2)} = \frac{\exp[-0.3442 \times (-2)]}{\exp(0.1852 \times 2)} = e^{0.318} = 1.374.$$

Therefore, the impact of a negative shock of size two-standard deviations is about 37.4% higher than that of a positive shock of the same size.

Forecasting: some recursive formula available

Another parameterization of EGARCH models

$$\ln(\sigma_t^2) = \alpha_0 + \alpha_1 \frac{|a_{t-1}| + \gamma_1 a_{t-1}}{\sigma_{t-1}} + \beta_1 \ln(\sigma_{t-1}^2),$$

where γ_1 denotes the leverage effect.

S-Plus demonstration

```
> spfit=garch(x~1,~egarch(1,1),leverage=T) % Fit an EGARCH(1,1) model
> summary(spfit)
Call: garch(formula.mean = x ~ 1, formula.var =  ~ egarch(1, 1), leverage = T)

Mean Equation: x ~ 1
Conditional Variance Equation:  ~ egarch(1, 1)
Conditional Distribution:  gaussian
--------------------------------------------------------------
```

```
Estimated Coefficients:
--------------------------------------------------------------
            Value Std.Error  t value  Pr(>|t|)  % Model explanation:
       C  0.006901  0.001608    4.293 1.986e-05  Mean equ: r(t) = 0.0069 + sigma(t)*e(t)
       A -0.379589  0.052316   -7.256 9.592e-13  Let h(t) = ln(sigma(t)**2)
 ARCH(1)  0.228222  0.029438    7.753 2.798e-14  Volatility equ:
GARCH(1)  0.966338  0.007759  124.542 0.000e+00  h(t) = -0.38 + +.97h(t-1) +
  LEV(1) -0.292277  0.090273   -3.238 1.255e-03  .23*[|e(t-1)|-.29*e(t-1)]/sigma(t-1).
--------------------------------------------------------------
AIC(5) = -2533.294, BIC(5) = -2509.922

Normality Test:
--------------------------------------------------------------
 Jarque-Bera P-value Shapiro-Wilk P-value
       102.5       0       0.9844  0.2536

Ljung-Box test for standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
     11.69  0.4706         12

Ljung-Box test for squared standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
     15.51  0.2148         12

Lagrange multiplier test:
--------------------------------------------------------------
   Lag 1  Lag 2   Lag 3   Lag 4   Lag 5   Lag 6 Lag 7 Lag 8
 -0.7505 0.2304 -0.2817 -0.6344 -0.2022 -0.8465 1.758 2.201

  Lag 9 Lag 10 Lag 11 Lag 12       C
 0.5231  2.278 0.4728  0.144 -0.9919

  TR^2 P-value F-stat P-value
 15.82  0.1996  1.468  0.2428
```

R demonstration: The EGARCH(m, s) model is an EGARCH($m, s+1$) model in R. In addition, R did not provide estimation of EGARCH(1,0) model for the IBM returns so that an EGARCH(1,1) model is used. The estimated ARCH coefficient is essentially zero so that the result is close to an EGARCH(1,0) model.

```
> library("fSeries")
> source("garchOxFit.R")
> ibm=scan(file="m-ibmln.dat")
> m1=garchOxFit(formula.mean=~arma(1,0),formula.var=~egarch(1,1),series=ibm)
 ********************
 ** SPECIFICATIONS **
 ********************
Dependent variable : X
Mean Equation : ARMA (1, 0) model.
No regressor in the mean
Variance Equation : EGARCH (1, 1) model.
 No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1122.3

 Maximum Likelihood Estimation (Std.Errors based on Numerical OPG matrix)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)               0.011524  0.0021491    5.362  0.0000
AR(1)                0.089016   0.039132    2.275  0.0232
Cst(V)               0.0000001.4653e+005     0.00  1.0000
ARCH(Alpha1)        -0.024870    0.22773  -0.1092  0.9131
GARCH(Beta1)         0.997120  0.0068464    145.6  0.0000
EGARCH(Theta1)      -0.031399   0.017869   -1.757  0.0793
EGARCH(Theta2)       0.336494   0.083801    4.015  0.0001

No. Observations :       864  No. Parameters  :         7
Mean (Y)         :   0.01189  Variance (Y)    :   0.00439
Skewness (Y)     :  -0.22061  Kurtosis (Y)    :   5.05331
Log Likelihood   :  1122.295

Warning : To avoid numerical problems, the estimated parameter
Cst(V), and its std.Error have been multiplied by 10^4.
```

For the output and ignoring the ARCH parameter, the fitted model of R is

$$
\begin{aligned}
r_t &= 0.012 + 0.089 r_{t-1} + a_t, \quad a_t = \sigma_t \epsilon_t, \\
\ln(\sigma_t^2) &= \frac{g(\epsilon_{t-1})}{(1 - 0.997B)}, \\
g(\epsilon_{t-1}) &= -0.031\epsilon_{t-1} + 0.336[|\epsilon_{t-1}| - 0.8].
\end{aligned}
$$

The volatility model, therefore, is

$$\ln(\sigma_t^2) = -.027 + 0.997\ln(\sigma_{t-1}^2) + \begin{cases} 0.305\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0, \\ -0.367\epsilon_{t-1} & \text{if } \epsilon_{t-1} < 0. \end{cases}$$

Compared with prior results, this gives a smaller leverage effect.

The Threshold GARCH (TGARCH) or GJR Model A TGARCH(s, m) or GJR(s, m) model is defined as

$$r_t = \mu_t + a_t, \quad a_t = \sigma_t\epsilon_t\sigma_t^2 = \alpha_0 + \sum_{i=1}^{s}(\alpha_i + \gamma_i N_{t-i})a_{t-i}^2 + \sum_{j=1}^{m}\beta_j\sigma_{t-j}^2,$$

where N_{t-i} is an indicator variable such that

$$N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

One expects γ_i to be positive so that prior negative returns have higher impact on the volatility.

R demonstration

```
> m2=garchOxFit(formula.mean=~arma(1,0),formula.var=~gjr(1,1),series=ibm)
 ********************
 ** SPECIFICATIONS **
 ********************
Dependent variable : X
Mean Equation : ARMA (1, 0) model.
No regressor in the mean
Variance Equation : GJR (1, 1) model.
 No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1168.27

 Maximum Likelihood Estimation (Std.Errors based on Numerical OPG matrix)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)               0.012261  0.0024782    4.948  0.0000
AR(1)                0.108345   0.038208    2.836  0.0047
Cst(V)               3.976257     1.1618    3.422  0.0006
```

```
ARCH(Alpha1)          0.053328   0.024655    2.163  0.0308
GARCH(Beta1)          0.806274   0.044067    18.30  0.0000
GJR(Gamma1)           0.090895   0.033665    2.700  0.0071

No. Observations :       864  No. Parameters  :         6
Mean (Y)         :   0.01189  Variance (Y)    :   0.00439
Skewness (Y)     :  -0.22061  Kurtosis (Y)    :   5.05331
Log Likelihood   :  1168.266

Warning : To avoid numerical problems, the estimated parameter
Cst(V), and its std.Error have been multiplied by 10^4.

Estimated Parameters Vector :
 0.012261; 0.108345; 0.000398; 0.053328; 0.806274; 0.090895

***************
** FORECASTS **
***************
Number of Forecasts: 15

Horizon       Mean   Variance
      1   0.005999   0.005012
      2    0.01158   0.004536
      3    0.01219   0.004106
      4    0.01225   0.003716
      5    0.01226   0.003363
     ...
     15    0.01226    0.00124
***********
** TESTS **
***********
                   Statistic       t-Test      P-Value
Skewness           0.0051867     0.062348      0.95029
Excess Kurtosis      0.98490       5.9264  3.0957e-009
Jarque-Bera           34.925         .NaN  2.6070e-008
---------------
Information Criterium (to be minimized)
Akaike         -2.690430  Shibata        -2.690526
Schwarz         -2.657364  Hannan-Quinn    -2.677774
---------------
Q-Statistics on Standardized Residuals
  --> P-values adjusted by 1 degree(s) of freedom
  Q( 10) =  6.42925   [0.6963067]
  Q( 15) =  12.4119   [0.5732594]
  Q( 20) =  20.8502   [0.3451387]
--------------
```

```
Q-Statistics on Squared Standardized Residuals
  --> P-values adjusted by 2 degree(s) of freedom
  Q( 10) =  2.87912   [0.9417129]
  Q( 15) =  8.19737   [0.8305081]
  Q( 20) =  10.4124   [0.9176103]
```

For the series of monthly IBM log returns, the fitted GJR model is

$$\begin{aligned} r_t &= 0.012 + 0.108 r_{t-1} + a_t, \quad a_t = \sigma_t \epsilon_t \\ \sigma_t^2 &= 3.98 \times 10^{-4} + (.053 + .091 N_{t-1}) a_{t-1}^2 + .806 \sigma_{t-1}^2, \end{aligned}$$

where all estimates are significant, and model checking indicates that the fitted model is adequate.

The CHARMA model

Make use of "interaction" btw past shocks

A CHARMA model is defined as

$$r_t = \mu_t + a_t, a_t = \delta_{1t} a_{t-1} + \delta_{2t} a_{t-2} + \cdots + \delta_{mt} a_{t-m} + \eta_t,$$

where $\{\eta_t\}$ is iid $N(0, \sigma_\eta^2)$, $\{\boldsymbol{\delta}_t\} = \{(\delta_{1t}, \cdots, \delta_{mt})'\}$ is a sequence of iid random vectors $D(\mathbf{0}, \mathbf{\Omega})$, $\{\boldsymbol{\delta}_t\} \perp \{\eta_t\}$.

The model can be written as

$$a_t = \boldsymbol{a}_{t-1}' \boldsymbol{\delta}_t + \eta_t,$$

with conditional variance

$$\sigma_t^2 = \sigma_\eta^2 + \boldsymbol{a}_{t-1}' \mathrm{Cov}(\boldsymbol{\delta}_t) \boldsymbol{a}_{t-1}$$

$$= \sigma_\eta^2 + (a_{t-1}, \cdots, a_{t-m}) \mathbf{\Omega} (a_{t-1}, \cdots, a_{t-m})'.$$

Example: Monthly excess returns of S&P 500 index (26-91).

A fitted model is

$$r_t = 0.0068 + a_t,$$

$$\sigma_t^2 = .00136 + (a_{t-1}, a_{t-2}, a_{t-3})\widehat{\mathbf{\Omega}}(a_{t-1}, a_{t-2}, a_{t-3})'$$

where, std errors in parentheses,

$$\hat{\mathbf{\Sigma}} = \begin{bmatrix} 0.121(.036) & -0.062(.028) & 0 \\ -0.062(.028) & 0.191(.025) & 0 \\ 0 & 0 & 0.299(0.042) \end{bmatrix}.$$

Effects of explanatory variables

Can be used in the same manner, i.e. with random coefs.

RCA model

A time series r_t is a RCA(p) model if

$$r_t = \phi_0 + \sum_{i=1}^{p}(\phi_i + \delta_{it})r_{t-i} + a_t.$$

For the model, we have

$$\begin{aligned} \mu_t &= E(a_t|F_{t-1}) = \sum_{i=1}^{p}\phi_i a_{t-i}, \\ \sigma_t^2 &= \sigma_a^2 + (r_{t-1}, \cdots, r_{t-p})\mathbf{\Omega}_\delta(r_{t-1}, \cdots, r_{t-p})'. \end{aligned}$$

Stochastic volatility model

A (simple) SV model is

$$a_t = \sigma_t\epsilon_t, (1 - \alpha_1 B - \cdots - \alpha_m B^m)\ln(\sigma_t^2) = \alpha_0 + v_t$$

where ϵ_t's are iid $N(0,1)$, v_t's are iid $N(0, \sigma_v^2)$, $\{\epsilon_t\}$ and $\{v_t\}$ are independent.

Long-memory SV model

A simple LMSV is

$$a_t = \sigma_t \epsilon_t, \qquad \sigma_t = \sigma \exp(u_t/2), \qquad (1-B)^d u_t = \eta_t$$

where $\sigma > 0$, ϵ_t's are iid $N(0,1)$, η_t's are iid $N(0, \sigma_\eta^2)$ and independent of ϵ_t, and $0 < d < 0.5$.

The model says

$$\begin{aligned} \ln(a_t^2) &= \ln(\sigma^2) + u_t + \ln(\epsilon_t^2) \\ &= [\ln(\sigma^2) + E(\ln \epsilon_t^2)] + u_t + [\ln(\epsilon_t^2) - E(\ln \epsilon_t^2)] \\ &\equiv \mu + u_t + e_t. \end{aligned}$$

Thus, the $\ln(a_t^2)$ series is a Gaussian long-memory signal plus a non-Gaussian white noise; see Breidt, Crato and de Lima (1998).

Application

see Examples 3.4 & 3.5