

Lecture Note of Bus 41202, Spring 2006: Value at Risk & Risk Management

What is Value at Risk (VaR)?

- a measure of market risk
- amount a position could decline in a given period
- associated with a given probability

A formal definition:

- time period given: $\Delta t = \ell$
- change in value: $\Delta V(\ell)$
- CDF of the change $F_\ell(x)$
- given probability: p
- a long position:

$$p = Pr[\Delta V(\ell) \leq \text{VaR}] = F_\ell(\text{VaR}).$$

- a short position:

$$\begin{aligned} p &= Pr[\Delta V(\ell) \geq \text{VaR}] \\ &= 1 - Pr[\Delta V(\ell) \leq \text{VaR}] \\ &= 1 - F_\ell(\text{VaR}). \end{aligned}$$

Quantile: x_p is the p th quantile of $F_\ell(x)$ if

$$p = F_\ell(x_p)$$

and $F_\ell(\cdot)$ is continuous.

Factors affect VaR:

1. the probability p .
2. the time horizon ℓ .
3. data frequency.
4. the CDF $F_\ell(x)$.
5. the mark-to-market value of the position.

Why use log returns?

log returns \approx percentage changes.

VaR = Value \times (VaR of log return).

Methods available

1. RiskMetrics
2. Econometric modeling
3. Empirical quantile
4. Traditional extreme value theory (EVT)
5. EVT based on exceedance over a high threshold

Data used in illustrations:

Daily log returns of IBM stock

- span: July 3, 62 to Dec. 31, 98.
- size: 9190 points

Position: long on \$10 millions.

RiskMetrics

- Developed by J.P. Morgan
- r_t given F_{t-1} : $N(0, \sigma_t^2)$
- σ_t^2 follows the special IGARCH(1,1) model

$$\sigma_t^2 = \alpha \sigma_{t-1}^2 + (1 - \alpha) r_{t-1}^2, \quad 1 > \alpha > 0.$$

- VaR = $1.65\sigma_t$ if $p = 0.05$.
- k -horizon: $\text{VaR}[k] = \sqrt{k}\text{VaR}$

The square root of time rule

- Pros: simplicity and transparency
- Cons: model is not adequate

Example: IBM data

Model:

$$r_t = a_t, \quad a_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = 0.9396 \sigma_{t-1}^2 + (1 - 0.9396) a_{t-1}^2$$

Because $r_{9190} = -0.0128$ and $\hat{\sigma}_{9190}^2 = 0.0003472$,
 $\hat{\sigma}_{9190}^2(1) = 0.000336$.

For $p = 0.05$, VaR of $r_t = -1.65 \times \sqrt{0.000336} = -0.03025$

$$\text{VaR} = \$10,000,000 \times 0.03025 = \$302,500.$$

For $p = 0.01$, VaR of $r_t = -2.3262 \times \sqrt{0.000336} = -0.04265$, and
VaR = \$426,500.

Econometric models

- $r_t = \mu_t + a_t$ given F_{t-1}
- μ_t : a mean equation (Ch.2)
- σ_t^2 : a volatility model (Ch. 3 or 4)
- Pros: sound theory
- Cons: a bit complicated.

IBM data:

Case 1: Gaussian

$$r_t = 0.00066 - 0.0247r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = 0.00000389 + 0.0799a_{t-1}^2 + 0.9073\sigma_t^2.$$

From $r_{9189} = -0.00201$, $r_{9190} = -0.0128$ and $\sigma_{9190}^2 = 0.00033455$,
we have

$$\hat{r}_{9190}(1) = 0.00071 \quad \text{and} \quad \hat{\sigma}_{9190}^2(1) = 0.0003211.$$

If $p = 0.05$, then

$$0.00071 - 1.6449 \times \sqrt{0.0003211} = -0.02877.$$

$$\text{VaR} = \$10,000,000 \times 0.02877 = \$287,700.$$

If $p = 0.01$, then the quantile is

$$0.00071 - 2.3262 \times \sqrt{0.0003211} = -0.0409738.$$

$$\text{VaR} = \$409,738.$$

Case 2: Student- t_5

$$r_t = 0.0003 - 0.0335r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = 0.000003 + 0.0559a_{t-1}^2 + 0.9350\sigma_{t-1}^2.$$

From the data, $r_{9189} = -0.00201$, $r_{9190} = -0.0128$ and $\sigma_{9190}^2 = 0.000349$, we have

$$\hat{r}_{9190}(1) = 0.000367 \quad \text{and} \quad \hat{\sigma}_{9190}^2(1) = 0.0003386.$$

If $p = 0.05$, then the quantile is

$$0.000367 - 1.5608\sqrt{0.0003386} = -0.028354.$$

$$\text{VaR} = \$10,000,000 \times 0.028352 = \$283,520.$$

If $p = 0.01$, the quantile is

$$0.000367 - (3.3649/\sqrt{5/3})\sqrt{0.0003386} = -0.0475943.$$

$$\text{VaR} = \$475,943.$$

Discussion:

- Effects of heavy-tails seen with $p = 0.01$.
- Multiple step-ahead forecasts are needed.

Example 7.3 (continued). 15-day horizon.

$\hat{r}_{9190}[15] = 0.00998$ and $\sigma_t[15] = 0.0047948$.

If $p = 0.05$, the quantile is $0.00998 - 1.6449\sqrt{0.0047948} = -0.1039191$.

15-day VaR = $\$10,000,000 \times 0.1039191 = \$1,039,191$.

RiskMetrics: VaR = $\$287,700 \times \sqrt{15} = \$1,114,257$.

Empirical quantile

Sample of log returns: $\{r_t | t = 1, \dots, n\}$.

Order statistics:

$$r_{(1)} \leq r_{(2)} \leq \dots \leq r_{(n)}$$

$r_{(i)}$ as the i th order statistic of the sample.

$r_{(1)}$ is the sample minimum

$r_{(n)}$ the sample maximum.

Idea: Use the empirical quantile to estimate the theoretical quantile of r_t .

For a given probability p , what is the empirical quantile?

If $np = \ell$ is an integer, then it is $r_{(\ell)}$.

If np is not an integer, find the two neighboring integers $\ell_1 < np < \ell_2$ and use interpolation.

The quantile is

$$\hat{x}_p = \frac{p - p_1}{p_2 - p_1} r_{(\ell_1)} + \frac{p_2 - p}{p_2 - p_1} r_{(\ell_2)}.$$

IBM data:

$n = 9190$. If $p = 0.05$, then $np = 459.5$.

5% quantile is $(r_{(459)} + r_{(460)})/2 = -0.021603$.

VaR = \$216,030.

If $p = 0.01$, then $np = 91.9$ and the 1% quantile is

$$\begin{aligned}\hat{x}_{0.01} &= \frac{p_1 - 0.01}{p_2 - p_1} r_{(91)} + \frac{p_2 - 0.01}{p_2 - p_1} r_{(92)} \\ &= \frac{.0001}{.00011} (-3.658) + \frac{0.00001}{0.00011} (-3.657) \\ &\approx -3.658.\end{aligned}$$

VaR is \$365,800.

Extreme value theory: Focus on the tail behavior of r_t .

Review of extreme value theory

A properly normalized $r_{(1)}$ assumes a special distribution:

$$F_*(x) = \begin{cases} 1 - \exp[-(1 + kx)^{1/k}] & \text{if } k \neq 0 \\ 1 - \exp[-\exp(x)] & \text{if } k = 0 \end{cases}$$

for $x < -1/k$ if $k < 0$ and for $x > -1/k$ if $k > 0$.

k : the *shape parameter*

$\alpha = -1/k$: tail index of the distribution.

Classification of distributions:

- Type I: $k = 0$, the Gumbel family. The CDF is

$$F_*(x) = 1 - \exp[-\exp(x)], \quad -\infty < x < \infty. \quad (1)$$

- Type II: $k < 0$, the Fréchet family. The CDF is

$$F_*(x) = \begin{cases} 1 - \exp[-(1 + kx)^{1/k}] & \text{if } x < -1/k \\ 1 & \text{otherwise.} \end{cases} \quad (2)$$

- Type III: $k > 0$, the Weibull family. The CDF here is

$$F_*(x) = \begin{cases} 1 - \exp[-(1 + kx)^{1/k}] & \text{if } x > -1/k \\ 0 & \text{otherwise.} \end{cases}$$

The probability density function (pdf) of the normalized minimum is

$$f_*(x) = \begin{cases} (1 + kx)^{1/k-1} \exp[-(1 + kx)^{1/k}] & \text{if } k \neq 0 \\ \exp[x - \exp(x)] & \text{if } k = 0 \end{cases}$$

where $-\infty < x < \infty$ for $k = 0$, $x < -1/k$ for $k < 0$ and $x > -1/k$ for $k > 0$.

How to use the EVT distribution?

If we know the three parameters, we can compute the quantile!

Empirical estimation

Divide the sample into non-overlapping subsamples.

Suppose there are T data points, we divide the data as

$$\{r_1, \dots, r_n | r_{n+1}, \dots, r_{2n} | r_{2n+1}, \dots, r_{3n} | \dots | r_{(g-1)n+1}, \dots, r_{ng}\},$$

n : size of subgroup

Idea: find the minimum of each subgroup. These minima are the data used to estimate the three parameters.

Several estimation methods available. We use maximum likelihood estimates.

IBM data:

n	g	Scale α_n	Location β_n	Shape Par. k_n
(a) Minimal returns				
21	437	0.823(0.035)	-1.902(0.044)	-0.197(0.036)
63	145	0.945(0.077)	-2.583(0.090)	-0.335(0.076)
126	72	1.147(0.131)	-3.141(0.153)	-0.330(0.101)
252	36	1.542(0.242)	-3.761(0.285)	-0.322(0.127)
(b) Maximal returns				
21	437	0.931(0.039)	2.184(0.050)	-0.168(0.036)
63	145	1.157(0.087)	3.012(0.108)	-0.217(0.066)
126	72	1.292(0.158)	3.471(0.181)	-0.349(0.130)
252	36	1.624(0.271)	4.475(0.325)	-0.264(0.186)

EVT to VaR: Use a two-step procedure, because of the division into subgroup.

VaR for r_t :

$$\text{VaR} = \begin{cases} \beta_n - \frac{\alpha_n}{k_n} \left\{ 1 - [-n \ln(1 - p)]^{k_n} \right\} & \text{if } k_n \neq 0 \\ \beta_n + \alpha_n \ln[-n \ln(1 - p)] & \text{if } k_n = 0. \end{cases}$$

For IBM data, if $n = 63$ (quarterly minima), then $\hat{\alpha}_n = 0.945$, $\hat{\beta}_n = -2.583$, and $\hat{k}_n = -0.335$. If $p = 0.01$, the VaR is

$$\text{VaR} = -2.583 - \frac{0.945}{-0.335} \left\{ 1 - [-63 \ln(1 - 0.01)]^{-0.335} \right\}$$

$$= -3.04969$$

VaR is \$304,969.

If $p = 0.05$, then VaR is \$166,641.

For $n = 21$, the results are:

VaR = \$340,013 for $p = 0.01$;

VaR = \$184,127 for $p = 0.05$.

Discussion:

- Results depend on the choice of n
- VaR seems low, but it might be due to the choice of p .

If $p = 0.001$, then

VaR = \$546,641 for the Gaussian AR(2)-GARCH(1,1) model

VaR = \$666,590 for the extreme value theory with $n = 21$.

Summary of IBM data:

Position = \$10 millions.

If $p = 0.05$, then

1. \$302,500 for the RiskMetrics,
2. \$287,200 for an AR(2)-GARCH(1,1) model,
3. \$283,520 for an AR(2)-GARCH(1,1) with t_5
4. \$216,030 using the empirical quantile, and
5. \$184,127 for EVT with $n = 21$.

$p = 0.01$, then

1. \$426,500 for the RiskMetrics,
2. \$409,738 for an AR(2)-GARCH(1,1) model,
3. \$475,943 for an AR(2)-GARCH(1,1) model with t_5
4. \$365,800 for empirical quantile, and
5. \$340,013 for EVT with $n = 21$.

If $p = 0.001$, then

1. \$566,443 for the RiskMetrics,
2. \$546,641 for an AR(2)-GARCH(1,1) model,
3. \$836,341 for an AR(2)-GARCH(1,1) model with t_5
4. \$780,712 for quantile, and
5. \$666,590 for EVT with $n = 21$.

Multi-period VaR with EVT

$$\text{VaR}(\ell) = \ell^{1/\alpha} \text{VaR} = \ell^{-k} \text{VaR}$$

where α is the tail index and k is the shape parameter.

For IBM data with $p = 0.05$,

$$\text{VaR}(30) = (30)^{0.335} \text{VaR} = 3.125 \times \$184,127 = \$575,397.$$

New approach to VaR

Based on Exceedances over a high threshold

Idea: frequency of big returns and their magnitudes are important.

Statistical theory:

Two-dimensional Poisson process

Two possible cases:

Homogeneous: parameters are fixed over time

Non-homogeneous case: parameters are time-varying, according to some explanatory variables.

IBM data: homogeneous model

Thr.	Exc.	Shape Par. k	Log(Scale) $\ln(\alpha)$	Location β
(a) Original log returns				
3.0%	175	-0.30697(0.09015)	0.30699(0.12380)	4.69204(0.19058)
2.5%	310	-0.26418(0.06501)	0.31529(0.11277)	4.74062(0.18041)
2.0%	554	-0.18751(0.04394)	0.27655(0.09867)	4.81003(0.17209)
(b) Removing the sample mean				
3.0%	184	-0.30516(0.08824)	0.30807(0.12395)	4.73804(0.19151)
2.5%	334	-0.28179(0.06737)	0.31968(0.12065)	4.76808(0.18533)
2.0%	590	-0.19260(0.04357)	0.27917(0.09913)	4.84859(0.17255)

VaR calculation:

$$\text{VaR} = \begin{cases} \beta + \frac{\alpha}{k} \{1 - [-T \ln(1 - p)]^k\} & \text{if } k \neq 0 \\ \beta + \alpha \ln[-T \ln(1 - p)] & \text{if } k = 0 \end{cases}$$

where $T = 252$, the number trading days in a year.

IBM data: VaR of 5% & 1%

- Case I: original returns
 1. $\eta = 3.0\%$: \$228,239 & \$359.303.
 2. $\eta = 2.5\%$: \$219,106 & \$361,119.
 3. $\eta = 2.0\%$: \$212,981 & \$368.552.

- Case II: remove sample mean
 1. $\eta = 3.0\%$: \$232,094 & \$363,697.
 2. $\eta = 2.5\%$: \$225,782 & \$364,254.
 3. $\eta = 2.0\%$: \$217,740 & \$372,372.

Non-homogeneous case:

$$\begin{aligned}k_t &= \gamma_0 + \gamma_1 x_{1t} + \cdots + \gamma_v x_{vt} \equiv \gamma_0 + \boldsymbol{\gamma}' \mathbf{x}_t \\ \ln(\alpha_t) &= \delta_0 + \delta_1 x_{1t} + \cdots + \delta_v x_{vt} \equiv \delta_0 + \boldsymbol{\delta}' \mathbf{x}_t \\ \beta_t &= \theta_0 + \theta_1 x_{1t} + \cdots + \theta_v x_{vt} \equiv \theta_0 + \boldsymbol{\theta}' \mathbf{x}_t.\end{aligned}$$

For IBM data, explanatory variables include past volatilities, etc. See Chapter 7 for more details and estimation results.

Illustration:

For December 31, 1998, we have $x_{3,9190} = 0$, $x_{4,9190} = 0.9737$ and $x_{5,9190} = 1.9766$. The parameters become

$$k_{9190} = -0.01195, \quad \ln(\alpha_{9190}) = 0.19331, \quad \beta_{9190} = 6.105.$$

If $p = 0.05$, then quantile = 3.03756% and

$$\text{VaR} = \$10,000,000 \times 0.0303756 = \$303,756.$$

If $p = 0.01$, then Var is \$497,425.

For December 30, 1998, we have $x_{3,9189} = 1$, $x_{4,9189} = 0.9737$ and $x_{5,9189} = 1.8757$ and

$$k_{9189} = -0.2500, \quad \ln(\alpha_{9189}) = 0.52385, \quad \beta_{9189} = 5.8834.$$

The 5% VaR becomes

$$\text{VaR} = \$10,000,000 \times 0.0269139 = \$269,139.$$

If $p = 0.01$, then VaR becomes \$448,323.

R and S-Plus Demonstration:

Both packages use the library: **evir**. For R, download the library first before using the commands.

```
*** Please note that the shape parameter "k" of chapter 7
    is denoted by "minus xi" in R and S-Plus.
*** Also, the program 'evir' uses maxima (right tail) so that
    one should use 'minus returns' for the left tail.
***
(* Command line starts with > *)
(* Output is edited to simplify the handout *)

(* Generate CDF of Weibull, Frechet, & Gumbel dists *)
> z=seq(-5,5,length=200)
```

```

<= just get a sequence of numbers in [-5,5],equally spaced.
> z[1:10]
[1] -5.0000 -4.94975 -4.89949 -4.84925 -4.79899 -4.74874
[7] -4.698492 -4.648241 -4.597990 -4.547739

(* Use the command "pgev" to obtain the probability
  (CDF) of generalized extreme value dist. *)
> cdf.f=ifelse((z > -2),pgev(z,xi=0.5),0)
  <== Frechet dist for z > -2 only, because xi=0.5.
> cdf.w=ifelse((z < 2), pgev(z,xi=-0.5),1)
  <== Weibull dist for z < 2 only.
> cdf.g=exp(-exp(-z))

> plot(z,cdf.w,type='l',xlab='z',ylab='H(z)')
> lines(z,cdf.f,type='l',lty=2)
> lines(z,cdf.g,lty=3)
> legend(-5,1,legend=c("Weibull H(-0.5,0,1)",
  "Frechet H(0.5,0.1)","Gumbel H(0,0,1)"),lty=1:3)

(*Use the command "dgev" to obtain the pdf of
  generalized extreme value dist *)
> pdf.f=ifelse((z > -2),dgev(z,xi=0.5),0)
> pdf.w=ifelse((z < 2),dgev(z,xi=-0.5),0)
> pdf.g=exp(-exp(-z))*exp(-z)
> plot(z,pdf.w,type='l',xlab='z',ylab='density')
> lines(z,pdf.f,lty=2)
> lines(z,pdf.g,lty=3)
> legend(-5.25,0.4,legend=c("Weibull H(-0.5,0,1)",
  "Frechet H(0.5,0,1)","Gumbel H(0,0,1)"),lty=1:3)

```

```
>
```

```
(* Other related commands include "qgev" and "rgev"
   that gives quantiles and generates random
   variates from generalized extreme value
   distribution. The parameters must be given in
   using these two commands. *)
```

```
(* For example, to obtain the 95th quantile, use below *)
```

```
> qgev(0.95,xi=0.5,mu=0,sigma=1)
[1] 6.830793
```

Example: daily log returns, in percentages, of IBM stock: 1962 to 1998.

```
> library(evir)
> da=read.table("d-ibmln98.dat")
> ibm=da[,1]
> plot(ibm,type='l')
> qqnorm(ibm) % normal probability plot

> nibm=-ibm % Focus on the left tail.

> m1=gev(nibm,block=21) % fit gen. extreme value dist.
> m1
$n.all
[1] 9190
$n
[1] 438
$data
[1]3.288 3.619 3.994 3.864 1.824 2.161 1.578 1.535 0.911 1.019
.....
```

```
[431] 3.502 2.501 4.576 8.456 3.723 1.082 2.910 3.099
```

```
$block
```

```
[1] 21
```

```
$par.ests
```

```
      xi      sigma      mu
0.1956199 0.8239793 1.9031998
```

```
$par.ses
```

```
      xi      sigma      mu
0.03554473 0.03476737 0.04413629
```

```
$varcov
```

```
          [,1]          [,2]          [,3]
[1,] 1.263428e-03 -2.782725e-05 -0.0004338483
[2,] -2.782725e-05 1.208770e-03 0.0008475859
[3,] -4.338483e-04 8.475859e-04 0.0019480124
```

```
$converged
```

```
[1] 0
```

```
$nllh.final
```

```
[1] 654.3337
```

```
attr(,"class")
```

```
[1] "gev"
```

```
> names(m1)
```

```
[1] "n.all" "n" "data" "block" "par.ests"
```

```
[6] "par.ses" "varcov" "converged" "nllh.final"
```

```
> m1$n % numbers of monthly maximum
```

```

[1] 438
> ymax=m1$data
> hist(ymax)
> ysort=sort(ymax)
> plot(ysort,-log(-log(ppoints(ysort))),xlab='Monthly maximum')
    % Gumbel qq-plot

> plot(m1) % Model chekcing plots
Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 1

Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 2

Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 0

> 1-pgev(max(ymax),xi=.196,mu=1.90,sigma=.824)
[1] 5.857486e-05 % Prob. that the drop will exceed the maximum.

> rlevel.gev(m1,k.blocks=36) %return level & its 95% conf. interval.
[1] 5.568969 6.158516 6.980167

```

```
** Peak Over the Threshold approach (homogeneous case).
```

```
> meplot(nibm) % mean excess plot
```

```
> m2=gpd(nibm,threshold=2.5)
```

```
> names(m2)
```

```
[1] "n"          "data"        "threshold"   "p.less.thresh"
```

```
[5] "n.exceed"   "method"      "par.ests"    "par.ses"
```

```
[9] "varcov"     "information" "converged"   "nllh.final"
```

```
> m2$threshold
```

```
[1] 2.5
```

```
> m2$n.exceed
```

```
[1] 310
```

```
> m2 % Obtain all output
```

```
$n
```

```
[1] 9190
```

```
$data
```

```
[1] 3.288 2.649 2.817 3.619 3.994 3.795 3.784 2.623 3.864 3.149
```

```
.....
```

```
[301]2.501 2.835 4.576 4.393 8.456 2.916 3.723 2.910 2.654 3.099
```

```
$threshold
```

```
[1] 2.5
```

```
$p.less.thresh
```

```
[1] 0.9662677
```

```
$n.exceed
```

```
[1] 310
```

```
$method
```

```
[1] "ml"
```

```
$par.ests
```

```
      xi      beta  
0.2641593 0.7786761
```

```
$par.ses
```

```
      xi      beta  
0.06659234 0.06714131
```

```
$varcov
```

```
          [,1]      [,2]  
[1,] 0.004434540 -0.002614442  
[2,] -0.002614442 0.004507955
```

```
$information
```

```
[1] "observed"
```

```
$converged
```

```
[1] 0
```

```
$nllh.final
```

```
[1] 314.375
```

```
attr(,"class")
```

```
[1] "gpd"
```

```
> plot(m2) % Model checking. Should see all plots.
```

```
Make a plot selection (or 0 to exit):
```

```
1: plot: Excess Distribution
```

```
2: plot: Tail of Underlying Distribution
```

```
3: plot: Scatterplot of Residuals
```

```
4: plot: QQplot of Residuals
```

```
Selection: 1
```

```
[1] "threshold = 2.5 xi = 0.264 scale = 0.779 location= 2.5"
```

```
Make a plot selection (or 0 to exit):
```

```
1: plot: Excess Distribution
```

```
2: plot: Tail of Underlying Distribution
```

```
3: plot: Scatterplot of Residuals
```

```
4: plot: QQplot of Residuals
```

```
Selection: 0
```

```
> shape(nibm) % A plot showing the stability of the estimates.
```

```
> riskmeasures(m2,c(0.95,0.99)) % Compute VaR and expected shortfall.
```

```
      p quantile      sfall
```

```
[1,] 0.95 2.208932 3.162654
```

```
[2,] 0.99 3.616487 5.075507
```