Lecture Note of Bus 41202, Spring 2006: Supplement to VaR

This note provides some additional information on applying extreme value theory to value at risk calculation.

To traditional approach of EVT

Return Level: It is a risk measure based on the idea of subperiods. The g n-subperiod return level, $L_{n,g}$, is the level that is *exceeded* in one out of every g subperiods of length n.

$$P(r_{n,i} < L_{n,g}) = \frac{1}{g},$$

where n is the length of subperiod used in estimating the GEV model and $r_{n,i}$ denotes subperiod minimum. For sufficiently large n,

$$L_{n,g} = \beta_n + \frac{\alpha_n}{k_n}\{[-\ln(1-1/g)]^{k_n} - 1\},$$

where the shape parameter $k_n \neq 0$.

For a short position, the return level is

$$L_{n,g} = \beta_n + \frac{\alpha_n}{k_n}\{1 - [-\ln(1-1/g)]^{1/k_n}\}.$$

Peaks over Threshold

Generalized Pareto Distribution: For simplicity, assume that the shape parameter $k \neq 0$. Consider the extreme value distribution of *maximum* (Eq. (7.29) of the textbook)

$$F_*(r) = \exp\left[-\left(1 - \frac{k(r-\beta)}{\alpha}\right)^{1/k}\right].$$

The distribution of $r \leq x + \eta$ given η, where $x \geq 0$, is

$$\Pr(r \leq x + \eta | r > \eta) \approx 1 - \left(1 - \frac{kx}{\psi(\eta)}\right)^{1/k},$$

where $\psi(\eta) = \alpha - k(\eta - \beta)$, which depends on η.

The distribution with cumulative distribution function

$$G(x) = 1 - \left[1 - \frac{kx}{\psi(\eta)}\right]^{1/k},$$

is called a generalized Pareto distribution (GPD).

Selection of the high threshold

Mean Excess: Given a high threshold η_o, suppose the excess $r - \eta_o$ follows a GPD with parameter k and $\psi(\eta_o)$, where $0 > k > -1$. Then the mean excess over the threshold is

$$E(r - \eta_o | r > \eta_o) = \frac{\psi(\eta_o)}{1 + k}.$$

For any $\eta > \eta_o$, the mean excess function is defined as

$$e(\eta) = E(r - \eta | r > \eta) = \frac{\psi(\eta_o) - k(\eta - \eta_o)}{1 + k}.$$

The fact that, for a given k, $e(\eta)$ is a linear function of η, where $\eta > \eta_o$, provides a simple method to infer the threshold η_o for GPD. Define the empirical mean excess as

$$e_T(\eta) = \frac{1}{N_\eta} \sum_{i=1}^{N_\eta} (r_{t_i} - \eta),$$

where N_η is the number of returns that exceed η and r_{t_i} are the values of the corresponding returns.

The scatterplot $e_T(\eta)$ versus η is called the mean excess plot, which should be linear for $\eta > \eta_o$.
In R or S-Plus, the command is **meplot**.

Use of GPD in VaR

For a given threshold, estimate GPD to obtain parameters k and $\psi(\eta)$. Check the adequacy of the fit; see demonstration. Provided that the model is adequate, the VaR can be computed by

$$\mathrm{VaR}_q = \eta + \frac{\psi(\eta)}{k}\left\{1 - \left[\frac{T}{N_\eta}(1-q)\right]^k\right\},$$

where $q = 1 - p$ with $0 < p < 0.05$, T is the sample size and N_η is the number of exceedances.
Alternatively, one can use the formula in Eq. (7.38) of the textbook when one treats the exceedances and exceeding times as a two-dimensional Poisson process. The VaR results obtained are close.

Expected Shortfall (ES): the expected loss given that the VaR is exceeded. Specifically,

$$E\mathrm{ES}_q = E(r|r > \mathrm{VaR}_q) = \mathrm{VaR}_q + E(r - \mathrm{VaR}_q|r > \mathrm{VaR}_q).$$

For GPD, it turns out that

$$\mathrm{ES}_q = \frac{\mathrm{VaR}_q}{1+k} + \frac{\psi(\eta) + k\eta}{1+k}.$$

In R or S-Plus, the command is **riskmeasures**.