

Lecture Note of Bus 41202, Spring 2007:

Multivariate Time Series Analysis

Forcus on two series (Bivariate)

Time series:

$$\mathbf{X}_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}.$$

Data: $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_T$.

Weak stationarity:

$$E(\mathbf{X}_t) = \boldsymbol{\mu} \quad \text{Cov}(\mathbf{X}_t, \mathbf{X}_{t-j}) = \boldsymbol{\Gamma}_j$$

are time-invariant

Autocovariance matrix: Lag- ℓ

$$\begin{aligned} \boldsymbol{\Gamma}_\ell &= E[(\mathbf{X}_t - \boldsymbol{\mu})(\mathbf{X}_{t-\ell} - \boldsymbol{\mu})'] \\ &= \begin{bmatrix} E(x_{1t} - \mu_1)(x_{1,t-\ell} - \mu_1) & E(x_{1t} - \mu_1)(x_{2,t-\ell} - \mu_2) \\ E(x_{2t} - \mu_2)(x_{1,t-\ell} - \mu_1) & E(x_{2t} - \mu_2)(x_{2,t-\ell} - \mu_2) \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_{11}(\ell) & \Gamma_{12}(\ell) \\ \Gamma_{21}(\ell) & \Gamma_{22}(\ell) \end{bmatrix}. \end{aligned}$$

Not symmetric if $\ell \neq 0$. Consider $\boldsymbol{\Gamma}_1$:

- $\Gamma_{12}(1)$: $\text{Cov}(x_{1t}, x_{2,t-1})$
- $\Gamma_{21}(1)$: $\text{Cov}(x_{2t}, x_{1,t-1})$

Let the diagonal matrix \mathbf{D} be

$$\mathbf{D} = \begin{bmatrix} \text{std}(x_{1t}) & 0 \\ 0 & \text{std}(x_{2t}) \end{bmatrix} = \begin{bmatrix} \sqrt{\Gamma_{11}(0)} & 0 \\ 0 & \sqrt{\Gamma_{22}(0)} \end{bmatrix}.$$

Cross-Correlation matrix:

$$\boldsymbol{\rho}_\ell = \mathbf{D}^{-1} \boldsymbol{\Gamma}_\ell \mathbf{D}^{-1}$$

Thus, $\rho_{ij}(\ell)$ is the cross-correlation between x_{it} and $x_{j,t-\ell}$.

From stationarity:

$$\boldsymbol{\Gamma}_\ell = \boldsymbol{\Gamma}'_{-\ell}, \quad \boldsymbol{\rho}_\ell = \boldsymbol{\rho}'_{-\ell}.$$

For instance, $\text{cor}(x_{1t}, x_{2,t-1}) = \text{cor}(x_{2t}, x_{1,t+1})$.

Testing for serial dependence

Multivariate version of Ljung-Box $Q(m)$ statistics available.

$H_o : \boldsymbol{\rho}_1 = \cdots = \boldsymbol{\rho}_m = \mathbf{0}$ vs $H_a : \boldsymbol{\rho}_i \neq \mathbf{0}$ for some i

$$Q_2(m) = T^2 \sum_{\ell=1}^m \frac{1}{T-\ell} \text{tr}(\hat{\boldsymbol{\Gamma}}'_\ell \hat{\boldsymbol{\Gamma}}_0^{-1} \hat{\boldsymbol{\Gamma}}_\ell \hat{\boldsymbol{\Gamma}}_0^{-1})$$

which is $\chi^2_{k^2 m}$. Note tr denotes sum of the diagonal elements.

Vector Autoregressive Models(VAR)

VAR(1) model for two return series:

$$\begin{bmatrix} r_{1t} \\ r_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{10} \\ \phi_{20} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} r_{1,t-1} \\ r_{2,t-1} \end{bmatrix} + \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix},$$

where $\mathbf{a}_t = (a_{1t}, a_{2t})'$ is a sequence of iid bivariate normal random vectors with mean zero and covariance matrix

$$\text{Cov}(\mathbf{a}_t) = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

where $\sigma_{12} = \sigma_{21}$.

Rewrite the model as

$$\begin{aligned}r_{1t} &= \phi_{10} + \phi_{11}r_{1,t-1} + \phi_{12}r_{2,t-1} + a_{1t} \\r_{2t} &= \phi_{20} + \phi_{21}r_{1,t-1} + \phi_{22}r_{2,t-1} + a_{2t}\end{aligned}$$

Thus, ϕ_{11} and ϕ_{12} denotes the dependence of r_{1t} on the past returns $r_{1,t-1}$ and $r_{2,t-1}$, respectively.

Unidirectional dependence

For the VAR(1) model, if $\phi_{12} = 0$, but $\phi_{21} \neq 0$, then

- r_{1t} does not depend on $r_{2,t-1}$, but
- r_{2t} depends on $r_{1,t-1}$,

implying that knowing $r_{1,t-1}$ is helpful in predicting r_{2t} , but $r_{2,t-1}$ is not helpful in forecasting r_{1t} .

$\{r_{1t}\}$ is an *input*, $\{r_{2t}\}$ is the *output* variable.

A **Granger** causality relation.

If $\sigma_{12} = 0$, then r_{1t} and r_{2t} are not concurrently correlated.

Stationarity condition

Write the VAR(1) model as

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\Phi}\mathbf{r}_{t-1} + \mathbf{a}_t.$$

$\{\mathbf{r}_t\}$ is stationary if zeros of the polynomial

$$|\mathbf{I} - \boldsymbol{\Phi}x|$$

are greater than 1 in modulus.

(A generalization of univariate case)

Mean of \mathbf{r}_t satisfies

$$(\mathbf{I} - \mathbf{\Phi})\boldsymbol{\mu} = \boldsymbol{\phi}_0, \quad \text{or}$$

$$\boldsymbol{\mu} = (\mathbf{I} - \mathbf{\Phi})^{-1}\boldsymbol{\phi}_0$$

if the inverse exists.

Covariance matrices:

$$\text{Cov}(\mathbf{r}_t) = \sum_{i=0}^{\infty} \mathbf{\Phi}^i \boldsymbol{\Sigma} (\mathbf{\Phi}^i)'$$

$$\boldsymbol{\Gamma}_\ell = \mathbf{\Phi} \boldsymbol{\Gamma}_{\ell-1} \text{ for } \ell > 0.$$

Can be generalized to higher order models.

Building VAR models

- Order selection: use AIC or BIC (page 356) or a stepwise χ^2 test Eq. (8.18)

For instance, test VAR(1) vs VAR(2).

- Estimation: use ordinary least squares method
- Model checking: as univariate case
- Forecasting: similar to univariate case

Simple AR models are sufficient to model asset returns.

Co-integration

Basic ideas

- x_{1t} and x_{2t} are unit-root nonstationary
- a linear combination of x_{1t} and x_{2t} is unit-root stationary

That is, x_{1t} and x_{2t} share a single unit root!

Why is it of interest?

Stationary series is *mean reverting*.

Long term forecasts of the “linear” combination converge to a mean value, implying that the long-term forecasts of x_{1t} and x_{2t} must be linearly related.

Example. Consider the exchange-traded funds (ETF) of U.S. Real Estate. We focus on the Dow Jones i-share (IYR) and Vanguard REIT fund (VNQ) from October 2004 to May 2007. The daily adjusted prices of the two funds are shown in Figure 1. What can be said about the two prices? Is there any arbitrage opportunity between the two funds?

The two series all have a unit root (based on ADF test). Are they co-integrated?

Co-integration test

Several tests available, e.g. Johansen’s test.

Basic idea

Consider a univariate AR(2) model

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + a_t.$$

Let $\Delta x_t = x_t - x_{t-1}$.

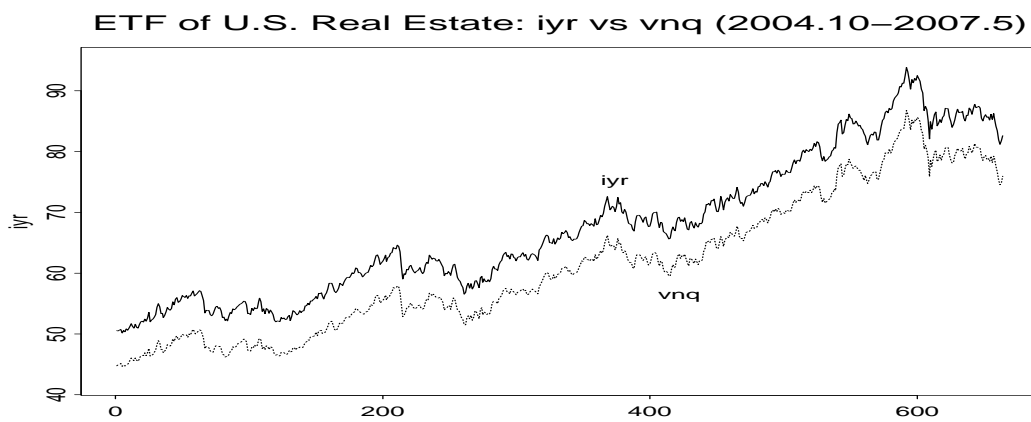


Figure 1: Daily prices of IYR and VNQ from October 2004 to May 2007

Subtract x_{t-1} from both sides and rearrange terms

$$\Delta x_t = \gamma x_{t-1} + \phi_1^* \Delta x_{t-1} + a_t,$$

where $\phi_1^* = -\phi_2$ and $\gamma = \phi_2 + \phi_1 - 1$.

x_t is unit-root nonstationary if and only if $\gamma = 0$.

Testing for x_t has a unit root is equivalent to testing for $\gamma = 0$ in the above model.

VAR(p) case:

$$\mathbf{X}_t = \Phi_1 \mathbf{X}_{t-1} + \dots + \Phi_p \mathbf{X}_{t-p} + \mathbf{a}_t.$$

Let $\mathbf{Y}_t = \mathbf{X}_t - \mathbf{X}_{t-1}$.

Rewrite the model as

$$\mathbf{Y}_t = \Omega \mathbf{X}_{t-1} + \sum_{i=1}^{p-1} \Phi_i^* \mathbf{Y}_{t-i} + \mathbf{a}_t, \quad (1)$$

where

$$\begin{aligned} \Phi_{p-1}^* &= -\Phi_p \\ \Phi_{p-2}^* &= -\Phi_{p-1} - \Phi_p \\ &\vdots \\ \Phi_1^* &= -\Phi_2 - \dots - \Phi_p \\ \Omega &= \Phi_p + \dots + \Phi_1 - I. \end{aligned}$$

This is the *Error-Correction* Model (ECM).

To test for co-integration:

- Fit the model in Eq. (1),

- Test for the rank of Ω .

If \mathbf{X}_t is k dimensional, and rank of Ω is m , then we have $k - m$ unit roots in \mathbf{X}_t .

There are m linear combinations of \mathbf{X}_t that are unit-root stationary.

If Ω has rank m , then

$$\Omega = \alpha\beta$$

where α is a $k \times m$ and β is a $m \times k$ full-rank matrix.

$\mathbf{Z}_t = \beta\mathbf{X}_t$ is unit-root stationary.

β is the co-integrating vector.

Discussion

- ECM formulation is useful
- Co-integration tests have some weaknesses, e.g. robustness
- Co-integration overlooks the effect of scale of the series

Demonstration of VAR models: U.S. quarterly GDP and unemployment rates starting from 1948 with 228 observations.

```
**** S-Plus: Quarterly U.S. gdp and unemployment rate ****
> da=read.table("q-gdpun.txt")
> dim(da)
[1] 228 5
> tdx=c(1:228)/4+1948
> par(mfcol=c(2,1))
> plot(tdx,da[,4],xlab='year',ylab='ln',type='l')
> title(main='Log(gdp)')
> plot(tdx,da[,5],xlab='year',ylab='rate',type='l')
> title(main='unemployment rate')
> gdp=diff(da[,4])
```

```

> un=diff(da[,5])
> plot(tdx[2:228],gdp,xlab='year',ylab='growth',type='l')
> title(main='GDP growth rate')
> plot(tdx[2:228],un,xlab='year',ylab='unemp',type='l')
> title(main='Changes in unemployment rate')

```

```

> acf(gdp) % compute ACFs
> acf(un)

```

```

> x=cbind(gdp,un)
> zt=data.frame(x)
> ord=VAR(zt,lag.max=8) % Find AR order
> ord$ar.order
[1] 1
> fit=VAR(x~ar(1))

```

```

> summary(fit)
Call:
VAR(formula = x ~ ar(1))

```

Coefficients:

	gdp	un
(Intercept)	0.0069	0.1074
(std.err)	0.0010	0.0310
(t.stat)	7.0636	3.4619
gdp.lag1	0.1814	-12.1629
(std.err)	0.0880	2.7891
(t.stat)	2.0615	-4.3608
un.lag1	-0.0054	0.4078
(std.err)	0.0022	0.0708
(t.stat)	-2.3961	5.7595

Regression Diagnostics:

	gdp	un
R-squared	0.1319	0.4397
Adj. R-squared	0.1241	0.4347
Resid. Scale	0.0094	0.2971

Information Criteria:

logL	AIC	BIC	HQ
758.2854	-1504.5707	-1484.0475	-1496.2884

```
total residual
Degree of freedom: 226 223
```

```
> names(fit)
[1] "R"      "coef"   "fitted" "residuals" "Sigma" "df.resid"
[7] "rank"   "call"   "ar.order" "n.na"      "terms"  "Y0"
> dim(fit$residuals)
[1] 226 2
> acf(fit$residuals[,1]) % compute residuals ACFs
> acf(fit$residuals[,2])
```

```
*** R: Multivariate Time Series Analysis ***
```

```
*** Very limited capability ***
```

```
> da=read.table("q-gdpun.txt")
> dim(da)
[1] 228 5
> gdp=da[,4]
> un=da[,5]
> tdx=c(1:228)/4+1948
> par(mfcol=c(2,1))
> plot(tdx,gdp,type='l',xlab='year',ylab='gdp')
> plot(tdx,un,type='l',xlab='year',ylab='un')
> x1=diff(gdp)
> x2=diff(un)
> x=cbind(x1,x2)
> dim(x)
[1] 227 2
> fit=ar(x,order.max=8)
> fit$order
[1] 2 % a vector AR(2) model is selected.
> fit
Call:
ar(x = x, order.max = 8)
```

```
$ar
```

```
, , 1
```

```
      gdp      un
gdp  0.1435 -0.008765
un  -11.5745  0.452652
```

```
, , 2
```

```
      gdp      un
gdp  0.1730  0.008553
un  -10.0953 -0.299139
```

```

$var.pred
      gdp      un
gdp 8.402e-05 -0.001706
un -1.706e-03 0.081848

> names(fit)
 [1] "order"      "ar"          "var.pred"    "x.mean"     "aic"
 [6] "n.used"     "order.max"   "partialacf"  "resid"      "method"
[11] "series"     "frequency"   "call"
> fit$method
 [1] "Yule-Walker"

> fit=ar(x,order.max=10,method=c("ols")) %Use ordinary least squares method
> fit$order
 [1] 8
> fit
Call:
ar(x = x, order.max = 10, method = c("ols"))

$ar
, , 1
      gdp      un
gdp 0.1732 -0.007466
un -11.8784 0.371661

, , 2
      gdp      un
gdp 0.1130 0.006198
un -8.3815 -0.180014

, , 3
      gdp      un
gdp -0.003447 0.0002626
un -4.510161 -0.0841344

, , 4
      gdp      un
gdp 0.09733 0.00704
un -0.79610 -0.16820

, , 5
      gdp      un
gdp 0.006099 -0.001274

```

un 4.191216 0.077492

, , 6

	gdp	un
gdp	0.1155	0.002229
un	-3.7503	0.030763

, , 7

	gdp	un
gdp	-0.06509	-0.001843
un	1.64065	0.083381

, , 8

	gdp	un
gdp	0.04539	0.003241
un	0.81069	-0.170200

\$x.intercept

	gdp	un
	1.077e-05	-5.426e-03

\$var.pred

	gdp	un
gdp	7.096e-05	-0.001409
un	-1.409e-03	0.068373