

**Lecture Note of Bus 41202, Spring 2007:
GARCH Models. Mr. Ruey Tsay
Conditional Heteroscedastic Models**

What is stock volatility?

Answer: conditional standard deviation of stock returns

Why is volatility important?

Has many important applications

- Option (derivative) pricing, e.g., Black-Scholes formula
- Risk management, e.g. value at risk (VaR)
- Asset allocation, e.g., minimum-variance portfolio; see pages 184-185 of Campbell, Lo and MacKinlay (1997).
- Interval forecasts

A key characteristic: Not directly observable!!

How to calculate volatility?

1. Use high-frequency data: French, Schwert & Stambaugh (1987); see Section 3.15.
 - Realized volatility of daily returns in recent literature.
 - Use daily high, low, and closing prices.
2. Implied volatility of options data, e.g, VIX of CBOE. See Figure 1.
3. Econometric modeling

We focus on the econometric modeling first. Use of high frequency data will be discussed later.

Basic idea of econometric modeling

Shocks of asset returns are NOT serially correlated, but dependent.

See ACF of squared and absolute returns of some stocks

Basic structure

$$r_t = \mu_t + a_t, \quad \mu_t = \phi_0 + \sum_{i=1}^p \phi_i r_{t-i} - \sum_{i=1}^q \theta_i a_{t-i},$$

Volatility models are concerned with time-evolution of

$$\sigma_t^2 = \text{Var}(r_t | F_{t-1}) = \text{Var}(a_t | F_{t-1}).$$

the conditional variance of a return.

Two general categories

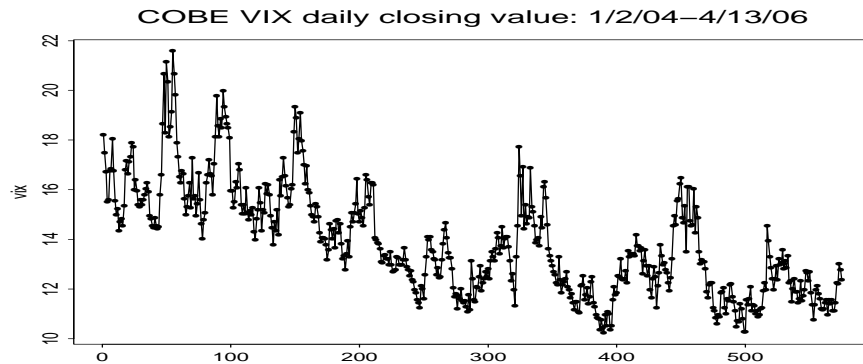


Figure 1: Time plot of the daily closing value of VIX of the COBE: January 2, 2004 to April 13, 2006.

- “Fixed function” and
- Stochastic function

of the available information.

Univariate volatility models

1. Autoregressive conditional heteroscedastic (ARCH) model of Engle (1982),
2. Generalized ARCH (GARCH) model of Bollerslev (1986),
3. GARCH-M models
4. IGARCH models
5. Exponential GARCH (EGARCH) model of Nelson (1991),
6. Threshold GARCH model of Zakoian (1994) or GJR model of Glosten, Jagannathan, and Runkle (1993).
7. Conditional heteroscedastic ARMA (CHARMA) model of Tsay (1987),
8. Random coefficient autoregressive (RCA) model of Nicholls and Quinn (1982)

9. Stochastic volatility (SV) models of Melino and Turnbull (1990), Harvey, Ruiz and Shephard (1994), and Jacquier, Polson and Rossi (1994).

ARCH model

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \dots + \alpha_m a_{t-m}^2,$$

where $\{\epsilon_t\}$ is a sequence of iid r.v. with mean 0 and variance 1, $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i > 0$. Distribution of ϵ_t : Standard normal, standardized Student-t, generalized error dist (GED), or skewed Student-t.

Properties of ARCH models

Consider an ARCH(1) model

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2,$$

where $\alpha_0 > 0$ and $\alpha_1 \geq 0$.

1. $E(a_t) = 0$
2. $\text{Var}(a_t) = \alpha_0 / (1 - \alpha_1)$ if $0 < \alpha_1 < 1$
3. Under normality,

$$m_4 = \frac{3\alpha_0^2(1 + \alpha_1)}{(1 - \alpha_1)(1 - 3\alpha_1^2)},$$

provided $0 < \alpha_1^2 < 1/3$.

The 3rd property implies heavy tails.

Advantages

- Simplicity
- Generates volatility clustering
- Heavy tails (high kurtosis)

Weaknesses

- Symmetric btw positive & negative prior returns
- Restrictive
- Provides no explanation
- Not sufficiently adaptive in prediction

Building an ARCH Model

1. Modeling the mean effect and testing

Use Q-statistics of squared residuals; McLeod and Li (1983) & Engle (1982)

2. Order determination
Use PACF of the squared residuals
3. Estimation: Conditional MLE
4. Model checking: Q-stat of standardized residuals and squared standardized residuals.
Skewness & Kurtosis of standardized residuals.
5. Software: Many available. We use R and S-Plus in class.

Estimation: Conditional MLE or Quasi MLE

Example: Monthly log returns of Intel stock

R demonstration:

Special Note: R uses “OX” package with “garchOxFit” command to estimate GARCH models. The GARCH order in OX is different from that of the textbook and S-Plus. A GARCH(r, m) model in the textbook and S-Plus is called a GARCH(m, r) model in R. In other words, ARCH order is the 2nd argument in R.

```
> library(fSeries)

> source("garchoxfit_R.txt") %Modification needed to make the program works.

> da=read.table("m-intc7303.txt")
> intc=ts(log(da[,2]+1),frequency=12,start=c(1973,1)) % log returns
> acf(intc)
> acf(intc^2)
> pacf(intc)
> pacf(intc^2)
> Box.test((intc)^2,lag=10,type='Ljung') % Test for ARCH effect.
      Box-Ljung test

data:  (intc)^2
X-squared = 59.7216, df = 10, p-value = 4.091e-09

> m1=garchOxFit(formula.mean=~arma(0,0),formula.var=~garch(0,3),series=intc)
(* Output edited to simplify the handout *)
*****
** SPECIFICATIONS **
*****
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (0, 3) model.
No regressor in the variance
The distribution is a Gauss distribution.
```

Strong convergence using numerical derivatives

Log-likelihood = 233.614

Please wait : Computing the Std Errors ...

Maximum Likelihood Estimation (Std.Errors based on Second derivatives)

	Coefficient	Std.Error	t-value	t-prob
Cst(M)	0.016390	0.0064138	2.555	0.0110
Cst(V)	0.011999	0.0015727	7.630	0.0000
ARCH(Alpha1)	0.215677	0.13180	1.636	0.1026
ARCH(Alpha2)	0.071882	0.048948	1.469	0.1428
ARCH(Alpha3)	0.049396	0.049304	1.002	0.3171

No. Observations :	372	No. Parameters :	5
Mean (Y) :	0.01799	Variance (Y) :	0.01784
Skewness (Y) :	-0.60142	Kurtosis (Y) :	5.92148
Log Likelihood :	233.614	Alpha[1]+Beta[1]:	0.33676

Estimated Parameters Vector :

0.016390; 0.011999; 0.215677; 0.071882; 0.049396

** FORECASTS **

Number of Forecasts: 15

Horizon	Mean	Variance
1	0.01639	0.01414
2	0.01639	0.01532
...	(edited)	
15	0.01639	0.01809

** TESTS **

	Statistic	t-Test	P-Value
Skewness	-0.71214	5.6299	1.8027e-008
Excess Kurtosis	2.9629	11.743	7.6940e-032
Jarque-Bera	167.52	.NaN	4.2100e-037

Information Criterium (to be minimized)

Akaike	-1.229107	Shibata	-1.229462
Schwarz	-1.176434	Hannan-Quinn	-1.208189

Q-Statistics on Standardized Residuals

Q(10) = 11.0752 [0.3516885]

Q(15) = 19.5637 [0.1893145]
 Q(20) = 20.8711 [0.4047564]
 H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

 Q-Statistics on Squared Standardized Residuals

--> P-values adjusted by 3 degree(s) of freedom

Q(10) = 5.55174 [0.5929504]
 Q(15) = 22.8241 [0.0292570]
 Q(20) = 23.8158 [0.1245260]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

 ARCH 1-2 test: F(2,364) = 0.32557 [0.7223]
 ARCH 1-5 test: F(5,358) = 0.32365 [0.8987]
 ARCH 1-10 test: F(10,348) = 0.57556 [0.8339]

 Residual-Based Diagnostic for Conditional Heteroskedasticity of Tse (2001)

RBD(10) = 3.62592 [0.9626488]
 RBD(15) = 16.5646 [0.3455497]
 RBD(20) = 8.58065 [0.9872751]

 P-values in brackets

 Adjusted Pearson Chi-square Goodness-of-fit test

# Cells(g)	Statistic	P-Value(g-1)	P-Value(g-k-1)
40	48.8602	0.133868	0.047509
50	61.8710	0.102550	0.038865
60	74.7742	0.080766	0.032113

Rem.: k = 5 = # estimated parameters

> m1=garchOxFit(formula.mean=~arma(0,0),formula.var=~garch(0,1),series=intc)

(* Output edited *)

** SPECIFICATIONS **

Dependent variable : X

Mean Equation : ARMA (0, 0) model.

No regressor in the mean

Variance Equation : GARCH (0, 1) model.

No regressor in the variance

The distribution is a Gauss distribution.

Strong convergence using numerical derivatives

Log-likelihood = 230.454

Please wait : Computing the Std Errors ...

Maximum Likelihood Estimation (Std.Errors based on Second derivatives)

	Coefficient	Std.Error	t-value	t-prob
Cst(M)	0.016548	0.0061388	2.696	0.0073
Cst(V)	0.012391	0.0015450	8.020	0.0000
ARCH(Alpha1)	0.373638	0.13372	2.794	0.0055

No. Observations :	372	No. Parameters :	3
Mean (Y) :	0.01799	Variance (Y) :	0.01784
Skewness (Y) :	-0.60142	Kurtosis (Y) :	5.92148
Log Likelihood :	230.454	Alpha[1]+Beta[1]:	0.37344

** FORECASTS **

Number of Forecasts: 15

Horizon	Mean	Variance
1	0.01655	0.01383
2	0.01655	0.01755
.... (edited)		
15	0.01655	0.01978

** TESTS **

	Statistic	t-Test	P-Value
Skewness	-0.67997	5.3756	7.6339e-008
Excess Kurtosis	2.4472	9.6989	3.0492e-022
Jarque-Bera	121.49	.NaN	4.1511e-027

Information Criterium (to be minimized)

Akaike	-1.222870	Shibata	-1.222999
Schwarz	-1.191266	Hannan-Quinn	-1.210319

Q-Statistics on Standardized Residuals

Q(10) =	13.6710	[0.1885356]
Q(15) =	22.2135	[0.1023272]
Q(20) =	23.7844	[0.2519364]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

Q-Statistics on Squared Standardized Residuals

--> P-values adjusted by 1 degree(s) of freedom
Q(10) = 12.2592 [0.1990864]

```

Q( 15) = 29.6965 [0.0084008]
Q( 20) = 31.0595 [0.0397697]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
-----
ARCH 1-2 test:    F(2,366) = 2.5957 [0.0760]
ARCH 1-5 test:    F(5,360) = 1.4194 [0.2164]
ARCH 1-10 test:   F(10,350)= 1.1103 [0.3533]
-----

> names(m1)
[1] "x"          "csts"          "cond.dist"     "arma.orders"  "arfima"
[6] "garch.orders" "arch.in.mean" "model"         "ox"           "call"
[11] "residuals"   "condvars"     "coef"         "title"       "description"

> par(mfcol=c(2,1)) % To plot the returns and volatilities on the sample page
> plot(intc,type='l')
> plot(sqrt(m1$condvars),type='l')
> par(mfcol=c(1,1))
> sresi=m1$residuals/sqrt(m1$condvars) % Compute standardized residuals
> qqnorm(sresi) % Obtain normal probability plot (ideal case: a straight line)
> qqline(sresi) % Impose a straight line on the QQ-plot.

```

S-Plus demnstration:

```

> da=read.table("m-intc7303.txt")
> dim(da)
> intc=log(da[,2]+1) %Compute log returns.
> autocorTest(intc,lag.n=12) % Test serial correlation using Q(12).
Test for Autocorrelation: Ljung-Box

```

Null Hypothesis: no autocorrelation

```

Test Statistics:
Test Stat 18.5664
p.value 0.0995

```

Dist. under Null: chi-square with 12 degrees of freedom

Total Observ.: 372

```

> archTest(intc,lag.n=12) % Test ARCH effect using 12 lags.
Test for ARCH Effects: LM Test

```

Null Hypothesis: no ARCH effects

```

Test Statistics:
Test Stat 43.5041
p.value 0.0000

```

Dist. under Null: chi-square with 12 degrees of freedom

```
> acf(intc^2,lag.max=12,type='partial') % Compute PACF of squared series.  
Call: acf(x = intc^2, lag.max = 12, type = "partial")
```

Partial Correlation matrix:

```
   lag   intc  
1    1  0.1402  
2    2  0.1703  
3    3  0.1557 % First 3 lags are relatively large.  
4    4  0.0633  
5    5  0.0407  
.....  
10   10 -0.0143  
11   11 -0.0399  
12   12  0.1614 % Lag-12 is also relatively large, but its order is high.
```

```
> arch3=garch(intc~1,~garch(3,0)) % Fit a Gaussian ARCH(3) model  
> summary(arch3)
```

Call:

```
garch(formula.mean = intc ~ 1, formula.var = ~ garch(3, 0))
```

Mean Equation: $\text{intc} \sim 1$

Conditional Variance Equation: $\sim \text{garch}(3, 0)$

Conditional Distribution: gaussian

Estimated Coefficients:

```
-----  
          Value Std.Error t value Pr(>|t|)  
C 0.01713  0.006626  2.5860  0.01009  
A 0.01199  0.001107 10.8325  0.00000  
ARCH(1) 0.17874  0.080294  2.2260  0.02662  Whis is the fitted model?  
ARCH(2) 0.07720  0.050552  1.5271  0.12760  
ARCH(3) 0.05722  0.076928  0.7438  0.45749  
-----
```

AIC(5) = -456.5791

BIC(5) = -436.9846

Normality Test:

```
-----  
Jarque-Bera P-value Shapiro-Wilk  P-value  
173.1      0      0.9696 0.0002337
```

Ljung-Box test for standardized residuals:

Statistic	P-value	Chi ² -d.f.
12.79	0.3848	12

Ljung-Box test for squared standardized residuals:

Statistic	P-value	Chi ² -d.f.
21.42	0.04453	12

Lagrange multiplier test:

Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8
0.1646	-0.05844	0.1577	0.2978	0.4671	0.8066	1.037	1.449

Lag 9	Lag 10	Lag 11	Lag 12	C
0.02206	-0.8262	3.857	-0.3651	-0.5014

TR ²	P-value	F-stat	P-value
20.67	0.05549	1.993	0.09808

```
> arch1=garch(intc~1,~garch(1,0)) % SImplify to an ARCH(1) model.  
> summary(arch1)
```

Call:

```
garch(formula.mean = intc ~ 1, formula.var = ~ garch(1, 0))
```

Mean Equation: intc ~ 1

Conditional Variance Equation: ~ garch(1, 0)

Conditional Distribution: gaussian

Estimated Coefficients:

	Value	Std.Error	t value	Pr(> t)
C	0.01741	0.006231	2.794	5.475e-03
A	0.01258	0.001246	10.091	0.000e+00
ARCH(1)	0.35258	0.088515	3.983	8.189e-05

AIC(3) = -454.4589

BIC(3) = -442.7022

Normality Test:

```
Jarque-Bera P-value Shapiro-Wilk P-value
120.9      0      0.9713 0.0008877
```

Ljung-Box test for standardized residuals:

```
-----
Statistic P-value Chi^2-d.f.
15.37  0.2221      12
```

Ljung-Box test for squared standardized residuals:

```
-----
Statistic P-value Chi^2-d.f.
26.28 0.009796      12
```

....(edited)

```
TR^2 P-value F-stat P-value
22.18 0.03552 2.149 0.0761
```

```
> names(arch1)
```

```
[1] "residuals" "sigma.t" "df.residual" "coef" "model"
[6] "cond.dist" "likelihood" "opt.index" "cov" "prediction"
[11] "call" "asymp.sd" "series"
```

```
> sresi=arch1$residuals/arch1$sigma.t %Compute standardized residuals
```

```
> autocorTest(sresi,lag.n=12) % Repeat the default output.
```

```
Test for Autocorrelation: Ljung-Box
```

```
Null Hypothesis: no autocorrelation
```

```
Test Statistics:
```

```
Test Stat 15.3651
p.value 0.2221
```

```
> autocorTest(sresi^2,lag.n=12)
```

```
Test for Autocorrelation: Ljung-Box
```

```
Null Hypothesis: no autocorrelation
```

```
Test Statistics:
```

```
Test Stat 26.2798
p.value 0.0098 % Same as the default output.
```

```
> autocorTest(sresi^2,lag.n=10)
```

```
Test for Autocorrelation: Ljung-Box
```

```
Null Hypothesis: no autocorrelation
```

Test Statistics:

Test Stat 12.7024
p.value 0.2408

```
> predict(arch1,5) % Prediction 1-step to 5-step ahead forecasts.  
$series.pred:  
[1] 0.01741175 0.01741175 0.01741175 0.01741175 0.01741175 %return
```

```
$sigma.pred:  
[1] 0.1181940 0.1322976 0.1369244 0.1385189 0.1390767 % volatility
```

```
$asypm.sd: % Unconditional variance of a(t).  
[1] 0.1393796
```

```
> qqnorm(sresi) % Normal probability plot to check normal assumption  
> qqline(sresi) % add line to help read the plot.
```

From the R output, we obtain that, under normality,

$$r_t = 0.0165 + a_t, \quad \sigma_t^2 = 0.012 + 0.374a_{t-1}^2.$$

Model checking:

Standardized shocks $\{\tilde{a}_t\}$

$Q(10) = 13.67(0.19)$

For $\{\tilde{a}_t^2\}$

$Q(10) = 12.26(0.20)$, but $Q(15) = 29.70(.01)$

Implications

- Expected monthly log return is about 1.7%.
- $\hat{\alpha}_1^2 = 0.374^2 < 1/3$ so that 4th moment exists.

From the S-Plus output, we obtain that, under normality,

$$r_t = 0.0174 + a_t, \quad \sigma_t^2 = 0.013 + 0.353a_{t-1}^2.$$

Model checking:

Standardized shocks $\{\tilde{a}_t\}$

$Q(12) = 15.37(0.22)$

For $\{\tilde{a}_t^2\}$

$Q(12) = 21.28(.01)$.

The two programs give similar, but not identical, results.

Next, consider Student-t innovations.

R demonstration

```

*****
** SPECIFICATIONS **
*****
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (0, 1) model.
No regressor in the variance
The distribution is a Student distribution, with 5.99816 degrees of freedom.

```

```

Strong convergence using numerical derivatives
Log-likelihood = 243.116
Please wait : Computing the Std Errors ...

```

Maximum Likelihood Estimation (Std.Errors based on Second derivatives)

	Coefficient	Std.Error	t-value	t-prob
Cst(M)	0.021513	0.0060427	3.560	0.0004
Cst(V)	0.013332	0.0019620	6.795	0.0000
ARCH(Alpha1)	0.268092	0.12224	2.193	0.0289
Student(DF)	5.998160	1.6661	3.600	0.0004

```

No. Observations :      372  No. Parameters   :          4
Mean (Y)          :   0.01799  Variance (Y)    :   0.01784
Skewness (Y)     :  -0.60142  Kurtosis (Y)   :   5.92148
Log Likelihood   :   243.116  Alpha[1]+Beta[1]:   0.26809

```

```

*****
** FORECASTS **
*****
Number of Forecasts: 15

```

Horizon	Mean	Variance
1	0.02151	0.01453
2	0.02151	0.01723
.... (edited)		
15	0.02151	0.01821

```

*****
** TESTS **
*****

```

	Statistic	t-Test	P-Value
Skewness	-0.68834	5.4417	5.2763e-008
Excess Kurtosis	2.5502	10.107	5.1483e-024
Jarque-Bera	130.18	.NaN	5.3972e-029

Information Criterium (to be minimized)

Akaike	-1.285572	Shibata	-1.285800
Schwarz	-1.243433	Hannan-Quinn	-1.268837

Q-Statistics on Standardized Residuals

Q(10) = 14.2606 [0.1614340]
Q(15) = 23.2423 [0.0791288]
Q(20) = 24.7769 [0.2101018]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

Q-Statistics on Squared Standardized Residuals

--> P-values adjusted by 1 degree(s) of freedom
Q(10) = 15.0259 [0.0902279]
Q(15) = 33.5172 [0.0024250]
Q(20) = 35.0263 [0.0138654]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

ARCH 1-2 test: F(2,366) = 3.1174 [0.0454]*
ARCH 1-5 test: F(5,360) = 1.8061 [0.1108]
ARCH 1-10 test: F(10,350)= 1.2727 [0.2443]

P-values in brackets

```
> sresi=m1$residuals/m1$condvars^.5  
> pacf(sresi^2)
```

```
> m2=garch0xFit(formula.mean=~arma(0,0),formula.var=~garch(0,2),series=intc,cond.dist="t")
```

```
*****  
** SPECIFICATIONS **  
*****
```

Dependent variable : X

Mean Equation : ARMA (0, 0) model.

No regressor in the mean

Variance Equation : GARCH (0, 2) model.

No regressor in the variance

The distribution is a Student distribution, with 6.09662 degrees of freedom.

Strong convergence using numerical derivatives

Log-likelihood = 245.913

Please wait : Computing the Std Errors ...

Maximum Likelihood Estimation (Std.Errors based on Second derivatives)

	Coefficient	Std.Error	t-value	t-prob
Cst(M)	0.022100	0.0060294	3.665	0.0003
Cst(V)	0.012420	0.0018284	6.793	0.0000

ARCH(Alpha1)	0.184359	0.10805	1.706	0.0888	
ARCH(Alpha2)	0.110735	0.068533	1.616	0.1070	<= insigh. at 5%
Student(DF)	6.096618	1.6919	3.603	0.0004	

```

No. Observations :      372  No. Parameters :      5
Mean (Y)          :  0.01799  Variance (Y)      :  0.01784
Skewness (Y)     : -0.60142  Kurtosis (Y)   :  5.92148
Log Likelihood   :  245.913  Alpha[1]+Beta[1]:  0.29509

```

```

> qqplot(rt(10000,6.1),sresi) % qq-plot for student-t dist.
> qqline(sresi)

```

S-Plus demonstration

```

> m1=garch(intc~1,~garch(1,0),cond.dist="t") %Use student-t innovations
> summary(m1)

```

Call:

```
garch(formula.mean=intc ~ 1,formula.var= ~ garch(1, 0),cond.dist="t")
```

Mean Equation: intc ~ 1

Conditional Variance Equation: ~ garch(1, 0)

Conditional Distribution: t

with estimated parameter 6.159751 and standard error 1.647094

Estimated Coefficients:

	Value	Std.Error	t value	Pr(> t)
C	0.02213	0.006010	3.681	2.666e-04
A	0.01338	0.001965	6.809	4.002e-11
ARCH(1)	0.24916	0.115574	2.156	3.174e-02

AIC(4) = -477.9073

BIC(4) = -462.2317

Normality Test:

Jarque-Bera	P-value	Shapiro-Wilk	P-value
128.9	0	0.9707	0.0005601

Ljung-Box test for standardized residuals:

Statistic	P-value	Chi ² -d.f.
16.1	0.1868	12

Ljung-Box test for squared standardized residuals:

```
-----  
Statistic  P-value  Chi^2-d.f.  
      29.91  0.002882      12
```

```
> tresi=m1$residuals/m1$sigma.t  
> autocorTest(tresi,lag.n=12)  
Test for Autocorrelation: Ljung-Box
```

Null Hypothesis: no autocorrelation

Test Statistics:

```
Test Stat 16.0974  
p.value  0.1868
```

```
> autocorTest(tresi^2,lag.n=12)  
Test for Autocorrelation: Ljung-Box
```

Null Hypothesis: no autocorrelation

Test Statistics:

```
Test Stat 29.9089  
p.value  0.0029
```

```
> autocorTest(tresi^2,lag.n=10) % use 10 lags  
Test for Autocorrelation: Ljung-Box
```

Null Hypothesis: no autocorrelation

Test Statistics:

```
Test Stat 15.6545  
p.value  0.1100  % The result confirms that lag-12 is significant.  
              % See below PACF of squared residuals.
```

```
> predict(m1,5) % Prediction  
$series.pred:  
[1] 0.02212715 0.02212715 0.02212715 0.02212715 0.02212715
```

```
$sigma.pred:  
[1] 0.1204767 0.1303599 0.1327079 0.1332865 0.1334302
```

```
$asymp.sd:  
[1] 0.1334779
```

```

attr(, "class"):
[1] "predict.garch"
> acf(tresi^2,type='partial',lag.max=12)
Call: acf(x = tresi^2, lag.max = 12, type = "partial")

Partial Correlation matrix:
  lag  tresi
1   1 -0.0352
2   2  0.1273
3   3  0.0643
  ....
11  11 -0.0634
12  12  0.1639
> m2=garch(intc~1,~garch(2,0),cond.dist="t") % Increase the order
> summary(m2)

```

Mean Equation: $\text{intc} \sim 1$

Conditional Variance Equation: $\sim \text{garch}(2, 0)$

Conditional Distribution: t
with estimated parameter 6.02561 and standard error 1.565027

Estimated Coefficients:

	Value	Std.Error	t value	Pr(> t)	
C	0.02264	0.006005	3.770	1.899e-04	
A	0.01247	0.001918	6.500	2.627e-10	
ARCH(1)	0.17457	0.105950	1.648	1.003e-01	
ARCH(2)	0.12109	0.092888	1.304	1.932e-01	% Insignificant

AIC(5) = -481.6892

BIC(5) = -462.0947

Question: What is the fitted ARCH(1) model in R?

$$r_t = 0.022 + a_t, \quad \sigma_t^2 = 0.013 + 0.268a_{t-1}^2,$$

and the t-distribution has 6.00 d.f.

Question: What is the fitted ARCH(1) model in S-Plus?

$$r_t = 0.022 + a_t, \quad \sigma_t^2 = 0.013 + 0.249a_{t-1}^2,$$

and the t-distribution has 6.16 d.f.

Comparison with normal models:

- Using a heavy-tailed dist for ϵ_t reduces the ARCH effect

- The difference between the models is small for this particular instance.

You may try the generalized error distribution.

GARCH Model

$$a_t = \sigma_t \epsilon_t,$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^m \alpha_i a_{t-i}^2 + \sum_{j=1}^s \beta_j \sigma_{t-j}^2$$

where $\{\epsilon_t\}$ is defined as before, $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1$.

Re-parameterization:

Let $\eta_t = a_t^2 - \sigma_t^2$. $\{\eta_t\}$ un-correlated series.

The GARCH model becomes

$$a_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^s \beta_j \eta_{t-j}.$$

This is an ARMA form for the squared series a_t^2 .

Use it to understand properties of GARCH models, e.g. moment equations, forecasting, etc.

Focus on a GARCH(1,1) model

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

- Weak stationarity: $0 \leq \alpha_1, \beta_1 \leq 1, (\alpha_1 + \beta_1) < 1$.
- Volatility clusters
- Heavy tails: if $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$, then

$$\frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} > 3.$$

- For 1-step ahead forecast,

$$\sigma_h^2(1) = \alpha_0 + \alpha_1 a_h^2 + \beta_1 \sigma_h^2.$$

For multi-step ahead forecasts, use $a_t^2 = \sigma_t^2 \epsilon_t^2$ and rewrite the model as

$$\sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1) \sigma_t^2 + \alpha_1 \sigma_t^2 (\epsilon_t^2 - 1).$$

2-step ahead volatility forecast

$$\sigma_h^2(2) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2(1).$$

In general, we have

$$\sigma_h^2(\ell) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2(\ell - 1), \quad \ell > 1.$$

This result is exactly the same as that of an ARMA(1,1) model with AR polynomial $1 - (\alpha_1 + \beta_1)B$.

Example: Monthly excess returns of S&P 500 index starting from 1926 for 792 observations. The fitted of a Gaussian AR(3) model

$$r_t = .088r_{t-1} - .023r_{t-2} - .123r_{t-3} + .007 + a_t,$$

$$\hat{\sigma}_a^2 = 0.00333.$$

For the GARCH effects, use a GARCH(1,1) model (R output).

A joint estimation:

$$\begin{aligned} r_t &= 0.032r_{t-1} - 0.030r_{t-2} - 0.010r_{t-3} + 0.0076 + a_t \\ \sigma_t^2 &= .00008 + .853\sigma_{t-1}^2 + 0.125a_{t-1}^2. \end{aligned}$$

Implied unconditional variance of a_t is

$$\frac{0.0000794}{1 - 0.85298 - 0.1247} = 0.00356$$

close to the expected value.

A simplified model:

$$r_t = 0.0074 + a_t, \sigma_t^2 = .00008 + .854\sigma_{t-1}^2 + .122a_{t-1}^2.$$

Model checking:

For \tilde{a}_t : $Q(10) = 11.22(0.34)$ and $Q(20) = 24.29(0.23)$.

For \tilde{a}_t^2 : $Q(10) = 9.92(0.27)$ and $Q(20) = 16.74(0.54)$.

Forecast: 1-step ahead forecast:

$$\sigma_h^2(1) = 0.00008 + 0.854\sigma_h^2 + 0.122a_h^2$$

Horizon	1	2	3	4	5	∞
Return	.0074	.0074	.0074	.0074	.0074	.0074
Volatility	.053	.052	.052	.051	.051	.050

R demonstration:

```
> library("fSeries")
> source("garch0xFit_R.txt")
> sp5=scan(file="sp500.dat")
> plot(sp5,type='l')
> m1=arima(sp5,order=c(3,0,0))
> m1
Call:
arima(x = sp5, order = c(3, 0, 0))
```

Coefficients:

```
      ar1      ar2      ar3  intercept
0.0890 -0.0238 -0.1229    0.0062
s.e.  0.0353  0.0355  0.0353    0.0019
```

sigma^2 estimated as 0.00333: log likelihood = 1135.25, aic = -2260.5

```
> x=ts(sp5)
```

```
> m2=garchOxFit(formula.mean=~arma(3,0),formula.var=~garch(1,1),series=x)
```

```
*****
```

```
** SPECIFICATIONS **
```

```
*****
```

Dependent variable : X

Mean Equation : ARMA (3, 0) model.

No regressor in the mean

Variance Equation : GARCH (1, 1) model.

No regressor in the variance

The distribution is a Gauss distribution.

Strong convergence using numerical derivatives

Log-likelihood = 1272.18

Maximum Likelihood Estimation (Std.Errors based on Second derivatives)

	Coefficient	Std.Error	t-value	t-prob
Cst(M)	0.007639	0.0015225	5.017	0.0000
AR(1)	0.031987	0.038368	0.8337	0.4047
AR(2)	-0.030276	0.038407	-0.7883	0.4308
AR(3)	-0.010637	0.037558	-0.2832	0.7771
Cst(V)	0.793989	0.28174	2.818	0.0050
ARCH(Alpha1)	0.124710	0.022615	5.514	0.0000
GARCH(Beta1)	0.852981	0.021891	38.97	0.0000

No. Observations :	792	No. Parameters :	7
Mean (Y) :	0.00614	Variance (Y) :	0.00341
Skewness (Y) :	0.41134	Kurtosis (Y) :	12.30025
Log Likelihood :	1272.183	Alpha[1]+Beta[1]:	0.97749

Warning : To avoid numerical problems, the estimated parameter Cst(V), and its std.Error have been multiplied by 10^4.

```
*****
```

```
** FORECASTS **
```

```
*****
```

Number of Forecasts: 15

Horizon	Mean	Variance
1	0.01248	0.002889
2	0.005195	0.002824
...		
15	0.007639	0.002101

** TESTS **

	Statistic	t-Test	P-Value
Skewness	-0.38960	4.4847	7.3025e-006
Excess Kurtosis	1.2654	7.2920	3.0543e-013
Jarque-Bera	72.877	.NaN	1.4963e-016

Information Criterium (to be minimized)

Akaike	-3.194906	Shibata	-3.195060
Schwarz	-3.153590	Hannan-Quinn	-3.179027

Q-Statistics on Standardized Residuals

--> P-values adjusted by 3 degree(s) of freedom

Q(10) = 11.5651 [0.1157988]

Q(15) = 17.7852 [0.1223710]

Q(20) = 24.1110 [0.1164494]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

Q-Statistics on Squared Standardized Residuals

--> P-values adjusted by 2 degree(s) of freedom

Q(10) = 10.3145 [0.2436438]

Q(15) = 14.2072 [0.3594178]

Q(20) = 16.7703 [0.5389502]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

ARCH 1-2 test: F(2,785) = 0.69609 [0.4988]

ARCH 1-5 test: F(5,779) = 0.62957 [0.6772]

ARCH 1-10 test: F(10,769) = 1.0519 [0.3976]

S-Plus demonstration:

```
> x=scan(file='sp500.dat')
```

```
> spfit=garch(x~ar(3),~garch(1,1)) % Fit an AR(3) + GARCH(1,1) model.
```

```
> summary(spfit)
```

```
Call: garch(formula.mean = x ~ ar(3), formula.var = ~ garch(1, 1))
```

Mean Equation: $x \sim ar(3)$

Conditional Variance Equation: $\sim garch(1, 1)$

Conditional Distribution: gaussian

Estimated Coefficients:

Value	Std.Error	t value	Pr(> t)
-------	-----------	---------	----------

C	7.751e-03	1.603e-03	4.8359	1.595e-06	
AR(1)	3.267e-02	3.849e-02	0.8488	3.903e-01	% AR coefs are insign. at 5%
AR(2)	-2.884e-02	3.823e-02	-0.7543	4.509e-01	
AR(3)	-8.407e-03	3.550e-02	-0.2368	8.129e-01	
A	8.374e-05	2.436e-05	3.4382	6.164e-04	
ARCH(1)	1.213e-01	2.030e-02	5.9774	3.439e-09	
GARCH(1)	8.523e-01	1.969e-02	43.2803	0.000e+00	

AIC(7) = -2526.239, BIC(7) = -2493.517

Normality Test:

Jarque-Bera P-value Shapiro-Wilk P-value
72.25 2.22e-16 0.9817 0.04185

Ljung-Box test for standardized residuals:

Statistic P-value Chi^2-d.f.
11.78 0.4636 12

Ljung-Box test for squared standardized residuals:

Statistic P-value Chi^2-d.f.
13.44 0.338 12

```
> spfit=garch(x~1,~garch(1,1)) % A refined model
> summary(spfit)
Call: garch(formula.mean = x ~ 1, formula.var = ~ garch(1, 1))
```

```
Mean Equation: x ~ 1
Conditional Variance Equation: ~ garch(1, 1)
Conditional Distribution: gaussian
```

Estimated Coefficients:

	Value	Std.Error	t value	Pr(> t)
C	7.647e-03	1.545e-03	4.950	9.096e-07
A	8.561e-05	2.412e-05	3.549	4.097e-04
ARCH(1)	1.216e-01	1.974e-02	6.159	1.165e-09
GARCH(1)	8.511e-01	1.899e-02	44.809	0.000e+00

AIC(4) = -2530.821, BIC(4) = -2512.122

Normality Test:

Jarque-Bera P-value Shapiro-Wilk P-value

81.58 0 0.9809 0.0201

Ljung-Box test for standardized residuals:

Statistic P-value Chi^2-d.f.
11.99 0.4468 12

Ljung-Box test for squared standardized residuals:

Statistic P-value Chi^2-d.f.
13.11 0.3609 12

Lagrange multiplier test:

Lag 1 Lag 2 Lag 3 Lag 4 Lag 5 Lag 6 Lag 7 Lag 8
-0.9755 0.5875 -0.4926 -0.8138 -0.1367 -1.018 1.497 1.859

Lag 9 Lag 10 Lag 11 Lag 12 C
0.5532 1.758 0.2104 0.1441 -0.947

TR^2 P-value F-stat P-value
13.15 0.3583 1.216 0.3824

```
> predict(sffit,5)
$series.pred:
[1] 0.007647292 0.007647292 0.007647292 0.007647292 0.007647292
$sigma.pred:
[1] 0.05358398 0.05365175 0.05371758 0.05378154 0.05384369

> mean(x)
[1] 0.006143056 % Point forecasts are higher than sample mean!
```

Compare the Splus result with that of R!

Turn to Student-t innovation.

Estimation of degrees of freedom:

$$r_t = 0.0085 + a_t,$$

$$\sigma_t^2 = .00012 + .113a_{t-1}^2 + .842\sigma_{t-1}^2,$$

where the estimated degrees of freedom is 6.99.

Forecasting evaluation

Not easy to do; see Andersen and Bollerslev (1998).

IGARCH model

An IGARCH(1,1) model:

$$a_t = \sigma_t \epsilon_t, \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1) a_{t-1}^2.$$

For the monthly excess returns of the S&P 500 index, we have

$$r_t = .007 + a_t, \sigma_t^2 = .0001 + .806\sigma_{t-1}^2 + .194a_{t-1}^2$$

For an IGARCH(1,1) model,

$$\sigma_h^2(\ell) = \sigma_h^2(1) + (\ell - 1)\alpha_0, \quad \ell \geq 1,$$

where h is the forecast origin.

Effect of $\sigma_h^2(1)$ on future volatilities is persistent, and the volatility forecasts form a straight line with slope α_0 . See Nelson (1990) for more info.

Special case: $\alpha_0 = 0$.

used in RiskMetrics to VaR calculation.

Example: An IGARCH(1,1) model for the monthly excess returns of S&P500 index from 1926 to 1991 is given below via R.

$$r_t = 0.0074 + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = .00005 + .143a_{t-1}^2 + .857\sigma_{t-1}^2.$$

R demonstration

```
> m2=garchOxFit(formula.mean=~arma(0,0),formula.var=~igarch(1,1),series=sp5)
```

```
*****
```

```
** SPECIFICATIONS **
```

```
*****
```

```
Dependent variable : X
```

```
Mean Equation : ARMA (0, 0) model.
```

```
No regressor in the mean
```

```
Variance Equation : IGARCH (1, 1) model.
```

```
No regressor in the variance
```

```
The distribution is a Gauss distribution.
```

```
Strong convergence using numerical derivatives
```

```
Log-likelihood = 1268.24
```

```
Maximum Likelihood Estimation (Std.Errors based on Second derivatives)
```

	Coefficient	Std.Error	t-value	t-prob
Cst(M)	0.007416	0.0015254	4.861	0.0000
Cst(V)	0.512441	0.17527	2.924	0.0036
ARCH(Alpha1)	0.142948	0.021444	6.666	0.0000
GARCH(Beta1)	0.857252			

No. Observations :	792	No. Parameters :	4
Mean (Y) :	0.00614	Variance (Y) :	0.00341
Skewness (Y) :	0.41134	Kurtosis (Y) :	12.30025
Log Likelihood :	1268.238		

Warning : To avoid numerical problems, the estimated parameter Cst(V), and its std.Error have been multiplied by 10^4.

** FORECASTS **

Number of Forecasts: 15

Horizon	Mean	Variance
1	0.007416	0.003079
2	0.007416	0.003079
.....		
15	0.007416	0.003079

** TESTS **

	Statistic	t-Test	P-Value
Skewness	-0.40577	4.6708	3.0001e-006
Excess Kurtosis	1.2112	6.9797	2.9578e-012
Jarque-Bera	70.146	.NaN	5.8618e-016

Information Criterium (to be minimized)

Akaike	-3.195044	Shibata	-3.195073
Schwarz	-3.177338	Hannan-Quinn	-3.188239

Q-Statistics on Standardized Residuals

Q(10) = 10.8290 [0.3709943]
 Q(15) = 17.6387 [0.2821344]
 Q(20) = 23.6418 [0.2583909]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

Q-Statistics on Squared Standardized Residuals

--> P-values adjusted by 2 degree(s) of freedom
 Q(10) = 10.0513 [0.2614457]
 Q(15) = 13.4740 [0.4119030]
 Q(20) = 15.9382 [0.5968584]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

ARCH 1-2 test: F(2,785) = 1.1501 [0.3171]
 ARCH 1-5 test: F(5,779) = 0.79599 [0.5527]
 ARCH 1-10 test: F(10,769)= 1.0004 [0.4413]

The GARCH-M model

$$r_t = \mu + c\sigma_t^2 + a_t, \quad a_t = \sigma_t\epsilon_t, \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where c is referred to as risk premium, which is expected to be positive.

Example: A GARCH(1,1)-M model for the monthly excess returns of S&P 500 index from January 1926 to December 1991. The fitted model is

$$r_t = 0.0054 + 1.01\sigma_t^2 + a_t, \sigma_t^2 = .00008 + .123a_{t-1}^2 + .852\sigma_{t-1}^2.$$

Std err of risk premium is 0.818.

R demonstration

```
> m3=garchOxFit(formula.mean=~arma(0,0),formula.var=~garch(1,1),series=sp5,arch.in.mean=T)
*****
** SPECIFICATIONS **
*****
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (1, 1) model.
                in-mean
No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1270.1
```

	Coefficient	Std.Error	t-value	t-prob
Cst(M)	0.005420	0.0023602	2.297	0.0219
Cst(V)	0.829654	0.29321	2.830	0.0048
ARCH(Alpha1)	0.123127	0.022286	5.525	0.0000
GARCH(Beta1)	0.852256	0.022400	38.05	0.0000
ARCH-in-mean(var)	1.008013	0.88853	1.134	0.2569

Warning : To avoid numerical problems, the estimated parameter Cst(V), and its std.Error have been multiplied by 10^4 .

S-Plus demonstration

```
> spfit=garch(x~1+var.in.mean,~garch(1,1))
> summary(spfit)
garch(formula.mean = x ~ 1 + var.in.mean, formula.var = ~ garch(1, 1))

Mean Equation: x ~ 1 + var.in.mean
Conditional Variance Equation: ~ garch(1, 1)
Conditional Distribution: gaussian
-----
Estimated Coefficients:
-----
                Value Std.Error t value  Pr(>|t|)
```

	C	5.487e-03	2.262e-03	2.426	7.747e-03
ARCH-IN-MEAN		1.088e+00	8.182e-01	1.330	9.203e-02
	A	8.764e-05	2.507e-05	3.496	2.494e-04
ARCH(1)		1.227e-01	2.047e-02	5.993	1.571e-09
GARCH(1)		8.494e-01	1.958e-02	43.390	0.000e+00

AIC(5) = -2530.136, BIC(5) = -2506.763

Normality Test:

Jarque-Bera P-value Shapiro-Wilk P-value
79.85 0 0.9801 0.009548

Ljung-Box test for standardized residuals:

Statistic P-value Chi^2-d.f.
13.43 0.3385 12

Ljung-Box test for squared standardized residuals:

Statistic P-value Chi^2-d.f.
11.83 0.4598 12

Lagrange multiplier test:

Lag 1 Lag 2 Lag 3 Lag 4 Lag 5 Lag 6 Lag 7 Lag 8
-0.9655 0.4582 -0.509 -0.7114 0.03989 -0.9966 1.444 1.656

Lag 9 Lag 10 Lag 11 Lag 12 C
0.4562 1.713 0.2978 0.1433 -0.9991

TR^2 P-value F-stat P-value
11.85 0.4575 1.094 0.4738

> predict(spfit,5)

\$series.pred:

[1] 0.008621675 0.008629477 0.008637062 0.008644436 0.008651603

\$sigma.pred:

[1] 0.05368237 0.05374914 0.05381396 0.05387690 0.05393801

\$asympt.sd:

[1] 0.05602398