Lecture Note of Bus 41202, Spring 2007:
Value at Risk & Risk Management

Classification of Financial Risk

1. Credit risk
2. Market risk
3. Operational risk

Some techniques for credit risk measurement

1. Long-term credit rating (High to Low)

S&P	Moody	Fitch
AAA	Aaa	AAA
AA	Aa	AA
A	A	A
BBB	Baa	BBB
BB	Ba	BB
B	B	B
CCC	Caa	CCC
CC	Ca	CC
C	C	C
D	D	D

2. Credit quality over time (transition)

S&P One-year transition matrix

(Source: Standard & Poor's, Feb. 1997)

Ini.	Rating at year-end(%)							
Rat.	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	88.5	8.05	0.72	0.06	0.11	0.00	0.00	0.00
AA	0.76	88.3	7.47	0.56	0.05	0.13	0.02	0.00
A	0.08	2.32	87.6	5.02	0.65	0.22	0.01	0.05
BBB	0.03	0.29	5.54	82.5	4.68	1.02	0.11	0.17
BB	0.02	0.11	0.58	7.01	73.8	7.4	0.89	0.98
B	0.00	0.09	0.21	0.39	5.98	72.8	3.42	4.92
CCC	0.17	0.00	0.34	1.02	2.20	9.64	53.13	19.21

3. CreditMetrics (JP Morgan)

4. Altman Z score (Mainly is U.S.)

$$
\begin{aligned}
Z &= 3.3(\text{Earnings before Interest and Taxes [EBIT]/Totl Assets}) \\
&+ 1.0(\text{Sales/Total Assets}) \\
&+ 0.6(\text{Market Value of Equity/Book Value of Debt}) \\
&+ 1.4(\text{Retained Earnings/Total Assets}) \\
&+ 1.2(\text{Working Capital/Total Assets})
\end{aligned}
$$

5. KMV Corporation's credit risk model

We focus on the Market Risk What is Value at Risk (VaR)?

- a measure of market risk
- amount a position could decline in a given period
- associated with a given probability

A formal definition:

- time period given: $\Delta t = \ell$
- change in value: $\Delta V(\ell)$
- CDF of the change $F_\ell(x)$
- given probability: p
- a long position:

$$p = Pr[\Delta V(\ell) \leq \text{VaR}] = F_\ell(\text{VaR}).$$

- a short position:

$$\begin{aligned} p &= Pr[\Delta V(\ell) \geq \text{VaR}] \\ &= 1 - Pr[\Delta V(\ell) \leq \text{VaR}] \\ &= 1 - F_\ell(\text{VaR}). \end{aligned}$$

Quantile: x_p is the pth quantile of $F_\ell(x)$ if

$$p = F_\ell(x_p)$$

and $F_\ell(.)$ is continuous.

Factors affect VaR:

1. the probability p.
2. the time horizon ℓ.
3. data frequency.
4. the CDF $F_\ell(x)$.
5. the mark-to-market value of the position.

Why use log returns?
log returns $\approx$ percentage changes.
VaR = Value $\times$ (VaR of log return).

Methods available

1. RiskMetrics
2. Econometric modeling
3. Empirical quantile
4. Traditional extreme value theory (EVT)
5. EVT based on exceedance over a high threshold

Data used in illustrations:
Daily log returns of IBM stock

- span: July 3, 62 to Dec. 31, 98.
- size: 9190 points

Position: long on \$10 millions.

RiskMetrics

- Developed by J.P. Morgan
- r_t given F_{t-1}: $N(0, \sigma_t^2)$
- σ_t^2 follows the special IGARCH(1,1) model

$$\sigma_t^2 = \alpha\sigma_{t-1}^2 + (1-\alpha)r_{t-1}^2, \quad 1 > \alpha > 0.$$

- VaR $= 1.65\sigma_t$ if $p = 0.05$.
- k-horizon: VaR$[k] = \sqrt{k}$VaR

  The square root of time rule
- Pros: simplicity and transparence
- Cons: model is not adequate

Example: IBM data

Model:

$$\begin{aligned} r_t &= a_t, \quad a_t = \sigma_t\epsilon_t, \\ \sigma_t^2 &= 0.9396\sigma_{t-1}^2 + (1-0.9396)a_{t-1}^2 \end{aligned}$$

Because $r_{9190} = -0.0128$ and $\hat{\sigma}_{9190}^2 = 0.0003472$,
$\hat{\sigma}_{9190}^2(1) = 0.000336$.
For $p = 0.05$, VaR of $r_t = -1.65 \times \sqrt{0.000336} = -0.03025$

$$\text{VaR} = \$10,000,000 \times 0.03025 = \$302,500.$$

For $p = 0.01$, VaR of $r_t = -2.3262 \times \sqrt{0.000336} = -0.04265$, and VaR = \$426,500.

Econometric models

- $r_t = \mu_t + a_t$ given F_{t-1}
- μ_t: a mean equation (Ch.2)
- σ_t^2: a volatility model (Ch. 3 or 4)
- Pros: sound theory
- Cons: a bit complicated.

IBM data:

Case 1: Gaussian

$$r_t = 0.00066 - 0.0247 r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = 0.00000389 + 0.0799 a_{t-1}^2 + 0.9073 \sigma_t^2.$$

From $r_{9189} = -0.00201$, $r_{9190} = -0.0128$ and $\sigma_{9190}^2 = 0.00033455$, we have

$$\hat{r}_{9190}(1) = 0.00071 \quad \text{and} \quad \hat{\sigma}_{9190}^2(1) = 0.0003211.$$

If $p = 0.05$, then

$$0.00071 - 1.6449 \times \sqrt{0.0003211} = -0.02877.$$

VaR = \$10,000,000×0.02877 = \$287,700.

If $p = 0.01$, then the quantile is

$$0.00071 - 2.3262 \times \sqrt{0.0003211} = -0.0409738.$$

VaR = \$409,738.

Case 2: Student-t_5

$$r_t = 0.0003 - 0.0335 r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = 0.000003 + 0.0559 a_{t-1}^2 + 0.9350 \sigma_{t-1}^2.$$

From the data, $r_{9189} = -0.00201$, $r_{9190} = -0.0128$ and $\sigma_{9190}^2 = 0.000349$, we have

$$\hat{r}_{9190}(1) = 0.000367 \quad \text{and} \quad \hat{\sigma}_{9190}^2(1) = 0.0003386.$$

If $p = 0.05$, then the quantile is

$$0.000367 - 1.5608\sqrt{0.0003386} = -0.028354.$$

VaR = $\$10,000,000 \times 0.028352 = \$283{,}520$.
If $p = 0.01$, the quantile is

$$0.000367 - (3.3649/\sqrt{5/3})\sqrt{0.0003386} = -0.0475943.$$

VaR = \$475,943.
Discussion:

- Effects of heavy-tails seen with $p = 0.01$.
- Multiple step-ahead forecasts are needed.

Example 7.3 (continued). 15-day horizon.

$\hat{r}_{9190}[15] = 0.00998$ and $\sigma_t[15] = 0.0047948$.

If $p = 0.05$, the quantile is $0.00998 - 1.6449\sqrt{0.0047948} = -0.1039191$.

15-day VaR = \$10,000,000×0.1039191 = \$1,039,191.

RiskMetrics: VaR = $\$287,700 \times \sqrt{15}$ = \$1,114,257.

Empirical quantile

Sample of log returns: $\{r_t | t = 1, \cdots, n\}$.

Order statistics:

$$r_{(1)} \le r_{(2)} \le \cdots \le r_{(n)}$$

$r_{(i)}$ as the ith order statistic of the sample.

$r_{(1)}$ is the sample minimum

$r_{(n)}$ the sample maximum.

Idea: Use the empirical quantile to estimate the theoretical quantile of r_t.

For a given probability p, what is the empirical quantile?

If $np = \ell$ is an integer, then it is $r_{(\ell)}$.

If np is not an integer, find the two neighboring integers $\ell_1 < np < \ell_2$ and use interpolation.

The quantile is

$$\hat{x}_p = \frac{p - p_1}{p_2 - p_1} r_{(\ell_1)} + \frac{p_2 - p}{p_2 - p_1} r_{(\ell_2)}.$$

IBM data:

$n = 9190$. If $p = 0.05$, then $np = 459.5$.

5% quantile is $(r_{(459)} + r_{(460)})/2 = -0.021603$.

VaR = \$216,030.

If $p = 0.01$, then $np = 91.9$ and the 1% quantile is

$$\begin{aligned}
\hat{x}_{0.01} &= \frac{p_1 - 0.01}{p_2 - p_1} r_{(91)} + \frac{p_2 - 0.01}{p_2 - p_1} r_{(92)} \\
&= \frac{.0001}{.00011}(-3.658) + \frac{0.00001}{0.00011}(-3.657) \\
&\approx -3.658.
\end{aligned}$$

VaR is \$365,800.

Extreme value theory: Focus on the tail behavior of r_t.

Review of extreme value theory

A properly normalized $r_{(1)}$ assumes a special distribution:

$$F_*(x) = \begin{cases} 1 - \exp[-(1 + kx)^{1/k}] & \text{if } k \neq 0 \\ \\ 1 - \exp[-\exp(x)] & \text{if } k = 0 \end{cases}$$

for $x < -1/k$ if $k < 0$ and for $x > -1/k$ if $k > 0$.

k: the *shape parameter*

$\alpha = -1/k$: tail index of the distribution.

Classification of distributions:

- Type I: $k = 0$, the Gumbel family. The CDF is

$$F_*(x) = 1 - \exp[-\exp(x)], \quad -\infty < x < \infty. \tag{1}$$

- Type II: $k < 0$, the Fréchet family. The CDF is

$$F_*(x) = \begin{cases} 1 - \exp[-(1+kx)^{1/k}] & \text{if } x < -1/k \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

- Type III: $k > 0$, the Weibull family. The CDF here is

$$F_*(x) = \begin{cases} 1 - \exp[-(1+kx)^{1/k}] & \text{if } x > -1/k \\ 0 & \text{otherwise.} \end{cases}$$

The probability density function (pdf) of the normalized minimum is

$$f_*(x) = \begin{cases} (1+kx)^{1/k-1} \exp[-(1+kx)^{1/k}] & \text{if } k \neq 0 \\ \\ \exp[x - \exp(x)] & \text{if } k = 0 \end{cases}$$

where $-\infty < x < \infty$ for $k = 0$, $x < -1/k$ for $k < 0$ and $x > -1/k$ for $k > 0$.

How to use the EVT distribution?

If we know the three parameters, we can compute the quantile!

Empirical estimation

Divide the sample into non-overlapping subsamples.

Suppose there are T data points, we devide the data as

$$\{r_1, \cdots, r_n | r_{n+1}, \cdots, r_{2n} | r_{2n+1}, \cdots, r_{3n} | \cdots | r_{(g-1)n+1}, \cdots, r_{ng}\},$$

n: size of subgroup

Idea: find the minimum of each subgroup. These minima are the data used to estimate the three parameters.

Several estimation methods available. We use maximum likelihood estimates.

IBM data:

n	g	Scale α_n	Location β_n	Shape Par. k_n
(a) Minimal returns				
21	437	0.823(0.035)	−1.902(0.044)	−0.197(0.036)
63	145	0.945(0.077)	−2.583(0.090)	−0.335(0.076)
126	72	1.147(0.131)	−3.141(0.153)	−0.330(0.101)
252	36	1.542(0.242)	−3.761(0.285)	−0.322(0.127)
(b) Maximal returns				
21	437	0.931(0.039)	2.184(0.050)	−0.168(0.036)
63	145	1.157(0.087)	3.012(0.108)	−0.217(0.066)
126	72	1.292(0.158)	3.471(0.181)	−0.349(0.130)
252	36	1.624(0.271)	4.475(0.325)	−0.264(0.186)

EVT to VaR: Use a two-step procedure, because of the division into subgroup.

VaR for r_t:

$$\text{VaR} = \begin{cases} \beta_n - \frac{\alpha_n}{k_n}\left\{1 - [-n\ln(1-p)]^{k_n}\right\} & \text{if } k_n \neq 0 \\ \\ \beta_n + \alpha_n \ln[-n\ln(1-p)] & \text{if } k_n = 0. \end{cases}$$

For IBM data, if $n = 63$ (quarterly minima), then $\hat{\alpha}_n = 0.945$, $\hat{\beta}_n = -2.583$, and $\hat{k}_n = -0.335$. If $p = 0.01$, the VaR is

$$\text{VaR} = -2.583 - \frac{0.945}{-0.335}\left\{1 - [-63\ln(1-0.01)]^{-0.335}\right\}$$

$$= -3.04969$$

VaR is \$304,969.

If $p = 0.05$, then VaR is \$166,641.

For $n = 21$, the results are:

VaR = \$340,013 for $p = 0.01$;

VaR = \$184,127 for $p = 0.05$.

Discussion:

- Results depend on the choice of n
- VaR seems low, but it might be due to the choice of p.

  If $p = 0.001$, then

  VaR = \$546,641 for the Gaussian AR(2)-GARCH(1,1) model

  VaR = \$666,590 for the extreme value theory with $n = 21$.

Summary of IBM data:

Position = \$10 millions.

If $p = 0.05$, then

1. \$302,500 for the RiskMetrics,
2. \$287,200 for an AR(2)-GARCH(1,1) model,
3. \$283,520 for an AR(2)-GARCH(1,1) with t_5
4. \$216,030 using the empirical quantile, and
5. \$184,127 for EVT with $n = 21$.

$p = 0.01$, then

1. \$426,500 for the RiskMetrics,
2. \$409,738 for an AR(2)-GARCH(1,1) model,
3. \$475,943 for an AR(2)-GARCH(1,1) model with t_5
4. \$365,800 for empirical quantile, and
5. \$340,013 for EVT with $n = 21$.

If $p = 0.001$, then

1. \$566,443 for the RiskMetrics,
2. \$546,641 for an AR(2)-GARCH(1,1) model,
3. \$836,341 for an AR(2)-GARCH(1,1) model with t_5
4. \$780,712 for quantile, and
5. \$666,590 for EVT with $n = 21$.

Multi-period VaR with EVT

$$\text{VaR}(\ell) = \ell^{1/\alpha}\text{VaR} = \ell^{-k}\text{VaR}$$

where α is the tail index and k is the shape parameter.
For IBM data with $p = 0.05$,

$$\text{VaR}(30) = (30)^{0.335}\text{VaR} = 3.125 \times \$184,127 = \$575,397.$$

New approach to VaR

Based on Exceedances over a high threshold

Idea: frequency of big returns and their magnitudes are important.

Statistical theory:

Two-dimensional Poisson process

Two possible cases:

Homogeneous: parameters are fixed over time

Non-homogeneous case: parameters are time-varying, according to some explanatory variables.

IBM data: homogeneous model

Thr.	Exc.	Shape Par. k	Log(Scale) $\ln(\alpha)$	Location β
(a) Original log returns				
3.0%	175	−0.30697(0.09015)	0.30699(0.12380)	4.69204(0.19058)
2.5%	310	−0.26418(0.06501)	0.31529(0.11277)	4.74062(0.18041)
2.0%	554	−0.18751(0.04394)	0.27655(0.09867)	4.81003(0.17209)
(b) Removing the sample mean				
3.0%	184	−0.30516(0.08824)	0.30807(0.12395)	4.73804(0.19151)
2.5%	334	−0.28179(0.06737)	0.31968(0.12065)	4.76808(0.18533)
2.0%	590	−0.19260(0.04357)	0.27917(0.09913)	4.84859(0.17255)

VaR calculation:

$$\text{VaR} = \begin{cases} \beta + \frac{\alpha}{k}\left\{1 - [-T\ln(1-p)]^k\right\} & \text{if } k \neq 0 \\ \\ \beta + \alpha \ln[-T\ln(1-p)] & \text{if } k = 0 \end{cases}$$

where $T = 252$, the number trading days in a year.

IBM data: VaR of 5% & 1%

- Case I: original returns

  1. $\eta = 3.0\%$: \$228,239 & \$359.303.
  2. $\eta = 2.5\%$: \$219,106 & \$361,119.
  3. $\eta = 2.0\%$: \$212,981 & \$368.552.

- Case II: remove sample mean

  1. $\eta = 3.0\%$: \$232,094 & \$363,697.
  2. $\eta = 2.5\%$: \$225,782 & \$364,254.
  3. $\eta = 2.0\%$: \$217,740 & \$372,372.

Non-homogeneous case:

$$\begin{aligned} k_t &= \gamma_0 + \gamma_1 x_{1t} + \cdots + \gamma_v x_{vt} \equiv \gamma_0 + \boldsymbol{\gamma}' \boldsymbol{x}_t \\ \ln(\alpha_t) &= \delta_0 + \delta_1 x_{1t} + \cdots + \delta_v x_{vt} \equiv \delta_0 + \boldsymbol{\delta}' \boldsymbol{x}_t \\ \beta_t &= \theta_0 + \theta_1 x_{1t} + \cdots + \theta_v x_{vt} \equiv \theta_0 + \boldsymbol{\theta}' \boldsymbol{x}_t. \end{aligned}$$

For IBM data, explanatory variables include past volatilities, etc.

See Chapter 7 for more details and estimation results.

Illustration:

For December 31, 1998, we have $x_{3,9190} = 0$, $x_{4,9190} = 0.9737$ and $x_{5,9190} = 1.9766$. The parameters become

$$k_{9190} = -0.01195, \quad \ln(\alpha_{9190}) = 0.19331, \quad \beta_{9190} = 6.105.$$

If $p = 0.05$, then quantile = 3.03756% and

$$\text{VaR} = \$10,000,000 \times 0.0303756 = \$303,756.$$

If $p = 0.01$, then Var is $497,425.

For December 30, 1998, we have $x_{3,9189} = 1$, $x_{4,9189} = 0.9737$ and $x_{5,9189} = 1.8757$ and

$$k_{9189} = -0.2500, \quad \ln(\alpha_{9189}) = 0.52385, \quad \beta_{9189} = 5.8834.$$

The 5% VaR becomes

$$\text{VaR} = \$10,000,000 \times 0.0269139 = \$269,139.$$

If p 0.01, then VaR becomes $448,323.

R and S-Plus Demonstration:

Both packages use the library: **evir**. For R, download the library first before using the commands.

```
*** Please note that the shape parameter "k" of chapter 7
    is denoted by "minus xi" in R and S-Plus.
*** Also, the program ‘‘evir’’ uses maxima (right tail) so that
    one should use ‘‘minus returns’’ for the left tail.
***
(* Command line starts with > *)
(* Output is edited to simplify the handout *)

(* Generate CDF of Weibull, Frechet, & Gumbel dists *)
> z=seq(-5,5,length=200)
```

```
 <= just get a sequence of numbers in [-5,5],equally spaced.
> z[1:10]
 [1] -5.0000 -4.94975 -4.89949 -4.84925 -4.79899 -4.74874
 [7] -4.698492 -4.648241 -4.597990 -4.547739

(* Use the command "pgev" to obtain the probability
  (CDF) of generalized extreme value dist. *)
> cdf.f=ifelse((z > -2),pgev(z,xi=0.5),0)
   <== Frechet dist for z > -2 only, because xi=0.5.
> cdf.w=ifelse((z < 2), pgev(z,xi=-0.5),1)
   <== Weibull dist for z < 2 only.
> cdf.g=exp(-exp(-z))

> plot(z,cdf.w,type='l',xlab='z',ylab='H(z)')
> lines(z,cdf.f,type='l',lty=2)
> lines(z,cdf.g,lty=3)
> legend(-5,1,legend=c("Weibull H(-0.5,0,1)",
  "Frechet H(0.5,0.1)","Gumbel H(0,0,1)"),lty=1:3)

(*Use the command "dgev" to obtain the pdf of
    generalized extreme value dist *)
> pdf.f=ifelse((z > -2),dgev(z,xi=0.5),0)
> pdf.w=ifelse((z < 2),dgev(z,xi=-0.5),0)
> pdf.g=exp(-exp(-z))*exp(-z)
> plot(z,pdf.w,type='l',xlab='z',ylab='density')
> lines(z,pdf.f,lty=2)
> lines(z,pdf.g,lty=3)
> legend(-5.25,0.4,legend=c("Weibull H(-0.5,0,1)",
  "Frechet H(0.5,0,1)","Gumbel H(0,0,1)"),lty=1:3)
```

```
>

(* Other related commands include "qgev" and "rgev"
   that gives quantiles and generates random
   variates from generalized extreme value
   distribution. The parameters must be given in
   using these two commands. *)
(* For example, to obtain the 95th quantile, use below *)

> qgev(0.95,xi=0.5,mu=0,sigma=1)
[1] 6.830793
```

Example: daily log returns, in percentages, of IBM stock: 1962 to 1998.

```
> library(evir)
> da=read.table("d-ibmln98.dat")
> ibm=da[,1]
> plot(ibm,type='l')
> qqnorm(ibm)  % normal probability plot

> nibm=-ibm   % Focus on the left tail.

> m1=gev(nibm,block=21) % fit gen. extreme value dist.
> m1
$n.all
[1] 9190
$n
[1] 438
$data
[1]3.288 3.619 3.994 3.864 1.824 2.161 1.578 1.535 0.911 1.019
   .....
```

```
[431] 3.502 2.501 4.576 8.456 3.723 1.082 2.910 3.099
$block
[1] 21

$par.ests
       xi     sigma        mu
0.1956199 0.8239793 1.9031998
$par.ses
        xi      sigma          mu
0.03554473 0.03476737 0.04413629

$varcov
              [,1]          [,2]          [,3]
[1,]  1.263428e-03 -2.782725e-05 -0.0004338483
[2,] -2.782725e-05  1.208770e-03  0.0008475859
[3,] -4.338483e-04  8.475859e-04  0.0019480124

$converged
[1] 0
$nllh.final
[1] 654.3337
attr(,"class")
[1] "gev"

> names(m1)
[1] "n.all"   "n"       "data"    "block"    "par.ests"
[6] "par.ses"  "varcov"  "converged" "nllh.final"

> m1$n   % numbers of monthly maximum
```

```
[1] 438
> ymax=m1$data
> hist(ymax)
> ysort=sort(ymax)
> plot(ysort,-log(-log(ppoints(ysort))),xlab='Monthly maximum')
        % Gumbel qq-plot

> plot(m1)  % Model chekcing plots
Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 1

Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 2

Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 0

> 1-pgev(max(ymax),xi=.196,mu=1.90,sigma=.824)
[1] 5.857486e-05 % Prob. that the drop will exceed the maximum.

> rlevel.gev(m1,k.blocks=36) %return level & its 95% conf. interval.
[1] 5.568969 6.158516 6.980167
```

```
** Peak Over the Threshold approach (homogeneous case).

> meplot(nibm)  % mean excess plot

> m2=gpd(nibm,threshold=2.5)
> names(m2)
 [1] "n"          "data"         "threshold"    "p.less.thresh"
 [5] "n.exceed"   "method"       "par.ests"     "par.ses"
 [9] "varcov"     "information" "converged"    "nllh.final"
> m2$threshold
[1] 2.5
> m2$n.exceed
[1] 310

> m2   % Obtain all output
$n
[1] 9190
$data
[1] 3.288 2.649 2.817 3.619 3.994 3.795 3.784 2.623 3.864 3.149
   .....
[301]2.501 2.835 4.576 4.393 8.456 2.916 3.723 2.910 2.654 3.099

$threshold
[1] 2.5
$p.less.thresh
[1] 0.9662677
$n.exceed
[1] 310
$method
```

```
[1] "ml"

$par.ests
       xi      beta
0.2641593 0.7786761

$par.ses
        xi       beta
0.06659234 0.06714131

$varcov
             [,1]         [,2]
[1,]  0.004434540 -0.002614442
[2,] -0.002614442  0.004507955
$information
[1] "observed"
$converged
[1] 0
$nllh.final
[1] 314.375
attr(,"class")
[1] "gpd"

> plot(m2)  % Model checking. Should see all plots.

Make a plot selection (or 0 to exit):
1: plot: Excess Distribution
2: plot: Tail of Underlying Distribution
3: plot: Scatterplot of Residuals
```

```
4: plot: QQplot of Residuals
Selection: 1
[1] "threshold = 2.5 xi = 0.264 scale = 0.779 location= 2.5"

Make a plot selection (or 0 to exit):
1: plot: Excess Distribution
2: plot: Tail of Underlying Distribution
3: plot: Scatterplot of Residuals
4: plot: QQplot of Residuals
Selection: 0

> shape(nibm)  % A plot showing the stability of the estimates.

> riskmeasures(m2,c(0.95,0.99)) % Compute VaR and expected shortfall.
        p quantile    sfall
[1,] 0.95 2.208932 3.162654
[2,] 0.99 3.616487 5.075507
```

Additional information on applying extreme value theory to value at risk calculation.

To traditional approach of EVT

Return Level: It is a risk measure based on the idea of subperiods. The g n-subperiod return level, $L_{n,g}$, is the level that is *exceeded* in one out of every g subperiods of length n.

$$P(r_{n,i} < L_{n,g}) = \frac{1}{g},$$

where n is the length of subperiod used in estimating the GEV model and $r_{n,i}$ denotes subperiod minimum. For sufficiently large n,

$$L_{n,g} = \beta_n + \frac{\alpha_n}{k_n}\{[-\ln(1-1/g)]^{k_n} - 1\},$$

where the shape parameter $k_n \neq 0$.
For a short position, the return level is

$$L_{n,g} = \beta_n + \frac{\alpha_n}{k_n}\{1 - [-\ln(1-1/g)]^{1/k_n}\}.$$

Peaks over Threshold

Generalized Pareto Distribution: For simplicity, assume that the shape parameter $k \neq 0$. Consider the extreme value distribution of *maximum* (Eq. (7.29) of the textbook)

$$F_*(r) = \exp\left[-\left(1 - \frac{k(r-\beta)}{\alpha}\right)^{1/k}\right].$$

The distribution of $r \leq x + \eta$ given η, where $x \geq 0$, is

$$\Pr(r \leq x + \eta | r > \eta) \approx 1 - \left(1 - \frac{kx}{\psi(\eta)}\right)^{1/k},$$

where $\psi(\eta) = \alpha - k(\eta - \beta)$, which depends on η.
The distribution with cumulative distribution function

$$G(x) = 1 - \left[1 - \frac{kx}{\psi(\eta)}\right]^{1/k},$$

is called a generalized Pareto distribution (GPD).

Selection of the high threshold

Mean Excess: Given a high threshold η_o, suppose the excess $r-\eta_o$ follows a GPD with parameter k and $\psi(\eta_o)$, where $0 > k > -1$. Then the mean excess over the threshold is

$$E(r-\eta_o|r>\eta_o) = \frac{\psi(\eta_o)}{1+k}.$$

For any $\eta > \eta_o$, the mean excess function is defined as

$$e(\eta) = E(r-\eta|r>\eta) = \frac{\psi(\eta_o)-k(\eta-\eta_o)}{1+k}.$$

The fact that, for a given k, $e(\eta)$ is a linear function of η, where $\eta > \eta_o$, provides a simple method to infer the threshold η_o for GPD. Define the empirical mean excess as

$$e_T(\eta) = \frac{1}{N_\eta}\sum_{i=1}^{N_\eta}(r_{t_i}-\eta),$$

where N_η is the number of returns that exceed η and r_{t_i} are the values of the corresponding returns.

The scatterplot $e_T(\eta)$ versus η is called the mean excess plot, which should be linear for $\eta > \eta_o$.

In R or S-Plus, the command is **meplot**.

Use of GPD in VaR

For a given threshold, estimate GPD to obtain parameters k and $\psi(\eta)$. Check the adequacy of the fit; see demonstration. Provided that the model is adequate, the VaR can be computed by

$$\text{VaR}_q = \eta + \frac{\psi(\eta)}{k}\left\{1-\left[\frac{T}{N_\eta}(1-q)\right]^k\right\},$$

where $q = 1 - p$ with $0 < p < 0.05$, T is the sample size and N_η is the number of exceedances.
Alternatively, one can use the formula in Eq. (7.38) of the textbook when one treats the exceedances and exceeding times as a two-dimensional Poisson process. The VaR results obtained are close.

Expected Shortfall (ES): the expected loss given that the VaR is exceeded. Specifically,

$$E\text{ES}_q = E(r|r > \text{VaR}_q) = \text{VaR}_q + E(r - \text{VaR}_q|r > \text{VaR}_q).$$

For GPD, it turns out that

$$\text{ES}_q = \frac{\text{VaR}_q}{1+k} + \frac{\psi(\eta) + k\eta}{1+k}.$$

In R or S-Plus, the command is **riskmeasures**.

Let $r_{n,i}$ be the maximum of a subperiod of length n. Under the traditional EVT, $r_{n,i}$ follows a generalized extreme value distribution with parameter (xi,sigma,mu).
What is the relationship between quantile of $r_{n,i}$ and the return r_t?
Let Q be a real number.

$$\begin{aligned}
P(r_{n,i} > Q) &= 1 - P(r_{n,i} \le Q) \\
&= 1 - P(\text{all } r_t \text{ in the subperiod} \le Q) \\
&= 1 - \prod_{t=1}^{n} P(r_t \le Q) \quad \text{(use independence)} \\
&= 1 - [P(r_t \le Q)]^n \quad \text{(because of same distribution)}
\end{aligned}$$

Consequently, let p be a small upper tail probability of r_t and Q be the corresponding quantile. That is,

$$P(r_t \le Q) = 1 - p$$

From the above equation, we have

$$P(r_{n,i} > Q) = 1 - (1 - p)^n.$$

Therefore,

$$P(r_{n,i} \le Q) = 1 - P(r_{n,i} > Q) = (1 - p)^n.$$

This means that Q is the $(1 - p)^n$-th quantile of the generalized extreme value distribution.
Takeaway: For a small probability p, compute $(1 - p)^n$, where n is the length of subperiod, then VaR can be obtained by finding the $(1 - p)^n$-th quantile of the extreme value distribution.