Lecture Note of Bus 41202, Spring 2008: Univariate Volatility Models. Mr. Ruey Tsay

Conditional Heteroscedastic Models

What is stock volatility?
Answer: conditional standard deviation of stock returns

Why is volatility important?
Has many important applications

- Option (derivative) pricing, e.g., Black-Scholes formula
- Risk management, e.g. value at risk (VaR)
- Asset allocation, e.g., minimum-variance portfolio; see pages 184-185 of Campbell, Lo and MacKinlay (1997).
- Interval forecasts

A key characteristic: Not directly observable!!

How to calculate volatility?

1. Use high-frequency data: French, Schwert & Stambaugh (1987); see Section 3.15.
   - Realized volatility of daily returns in recent literature.
   - Use daily high, low, and closing prices.
2. Implied volatility of options data, e.g, VIX of CBOE. See Figure 1.
3. Econometric modeling

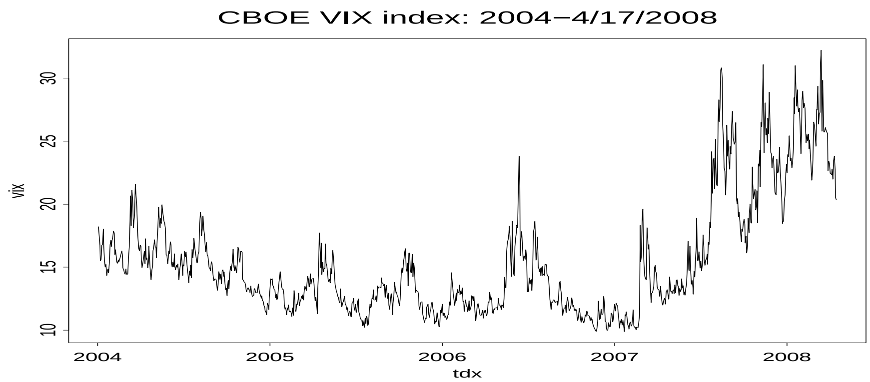

Figure 1: Time plot of the daily closing value of VIX of the CBOE: January 2, 2004 to April 17, 2008.

We focus on the econometric modeling first. Use of high frequency data will be discussed later.

Basic idea of econometric modeling

Shocks of asset returns are NOT serially correlated, but dependent. As shown by the ACF of returns and absolute returns of some assets we discussed so far.

Basic structure

$$r_t = \mu_t + a_t, \quad \mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i},$$

Volatility models are concerned with time-evolution of

$$\sigma_t^2 = \text{Var}(r_t|F_{t-1}) = \text{Var}(a_t|F_{t-1}).$$

the conditional variance of a return.

Revisit the daily closing index of the S&P500 index from 1950 to 2008. The log returns follow an MA(2) model

$$r_t = 0.0003 + a_t + 0.072a_{t-1} - 0.032a_{t-2}.$$

How about the volatility?
Is volatility constant over time?
NO! See the ACF of squared residuals!
How to model the evolving volatility?
Two general categories

- "Fixed function" and
- Stochastic function

of the available information.

Univariate volatility models

1. Autoregressive conditional heteroscedastic (ARCH) model of Engle (1982),
2. Generalized ARCH (GARCH) model of Bollerslev (1986),
3. GARCH-M models
4. IGARCH models
5. Exponential GARCH (EGARCH) model of Nelson (1991),
6. Threshold GARCH model of Zakoian (1994) or GJR model of Glosten, Jagannathan, and Runkle (1993).
7. Conditional heteroscedastic ARMA (CHARMA) model of Tsay (1987),

8. Random coefficient autoregressive (RCA) model of Nicholls and Quinn (1982)

9. Stochastic volatility (SV) models of Melino and Turnbull (1990), Harvey, Ruiz and Shephard (1994), and Jacquier, Polson and Rossi (1994).

ARCH model

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2,$$

where $\{\epsilon_t\}$ is a sequence of iid r.v. with mean 0 and variance 1, $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i > 0$.
Distribution of ϵ_t: Standard normal, standardized Student-t, generalized error dist (GED), or skewed Student-t.

Properties of ARCH models

Consider an ARCH(1) model

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2,$$

where $\alpha_0 > 0$ and $\alpha_1 \geq 0$.

1. $\mathrm{E}(a_t) = 0$

2. $\mathrm{Var}(a_t) = \alpha_0/(1 - \alpha_1)$ if $0 < \alpha_1 < 1$

3. Under normality,

$$m_4 = \frac{3\alpha_0^2(1 + \alpha_1)}{(1 - \alpha_1)(1 - 3\alpha_1^2)},$$

   provided $0 < \alpha_1^2 < 1/3$.

The 3rd property implies heavy tails.

Advantages

- Simplicity
- Generates volatility clustering
- Heavy tails (high kurtosis)

Weaknesses

- Symmetric btw positive & negative prior returns
- Restrictive
- Provides no explanation
- Not sufficiently adaptive in prediction

Building an ARCH Model

1. Modeling the mean effect and testing

   Use Q-statistics of squared residuals; McLeod and Li (1983) & Engle (1982)

2. Order determination

   Use PACF of the squared residuals

3. Estimation: Conditional MLE
4. Model checking: Q-stat of standardized residuals and squared standardized residuals. Skewness & Kurtosis of standardized residuals.
5. Software: Many available. We use R and S-Plus in class.

Estimation: Conditional MLE or Quasi MLE
Example: Monthly log returns of Intel stock

R demonstration:

Special Note: R uses "OX" package with "garchOxFit" command to estimate GARCH models. The GARCH order in OX is different from that of the textbook and S-Plus. A GARCH(r, m) model in the textbook and S-Plus is called a GARCH(m, r) model in R. In other words, ARCH order is the 2nd argument in R.

```
> library(fSeries)

> source("garchoxfit_R.txt")  %Modification needed to make the program works.

> da=read.table("m-intc7303.txt")
> intc=ts(log(da[,2]+1),frequency=12,start=c(1973,1))  % log returns
> acf(intc)
> acf(intc^2)
> pacf(intc)
> pacf(intc^2)
> Box.test((intc)^2,lag=10,type='Ljung')  % Test for ARCH effect.
        Box-Ljung test

data:  (intc)^2
X-squared = 59.7216, df = 10, p-value = 4.091e-09

> m1=garchOxFit(formula.mean=~arma(0,0),formula.var=~garch(0,3),series=intc)
(* Output edited to simplify the handout *)
 ********************
 ** SPECIFICATIONS **
 ********************
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (0, 3) model.
 No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 233.614
Please wait : Computing the Std Errors ...

 Maximum Likelihood Estimation (Std.Errors based on Second derivatives)
                 Coefficient  Std.Error  t-value  t-prob
Cst(M)              0.016390  0.0064138    2.555  0.0110
Cst(V)              0.011999  0.0015727    7.630  0.0000
ARCH(Alpha1)        0.215677    0.13180    1.636  0.1026
ARCH(Alpha2)        0.071882   0.048948    1.469  0.1428
```

```
ARCH(Alpha3)         0.049396   0.049304    1.002  0.3171

No. Observations :       372  No. Parameters  :         5
Mean (Y)         :   0.01799  Variance (Y)    :   0.01784
Skewness (Y)     :  -0.60142  Kurtosis (Y)    :   5.92148
Log Likelihood   :   233.614  Alpha[1]+Beta[1]:   0.33676

Estimated Parameters Vector :
 0.016390; 0.011999; 0.215677; 0.071882; 0.049396

***************
** FORECASTS **
***************
Number of Forecasts: 15

Horizon       Mean   Variance
      1    0.01639    0.01414
      2    0.01639    0.01532
      ... (edited)
     15    0.01639    0.01809
---------------

***********
** TESTS **
***********
                   Statistic       t-Test      P-Value
Skewness            -0.71214       5.6299  1.8027e-008
Excess Kurtosis       2.9629       11.743  7.6940e-032
Jarque-Bera           167.52         .NaN  4.2100e-037
---------------
Information Criterium (to be minimized)
Akaike         -1.229107  Shibata       -1.229462
Schwarz         -1.176434  Hannan-Quinn    -1.208189
---------------
Q-Statistics on Standardized Residuals
  Q( 10) =  11.0752   [0.3516885]
  Q( 15) =  19.5637   [0.1893145]
  Q( 20) =  20.8711   [0.4047564]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
--------------
Q-Statistics on Squared Standardized Residuals
  --> P-values adjusted by 3 degree(s) of freedom
  Q( 10) =  5.55174   [0.5929504]
  Q( 15) =  22.8241   [0.0292570]
  Q( 20) =  23.8158   [0.1245260]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
```

```
--------------
ARCH 1-2 test:    F(2,364) =  0.32557 [0.7223]
ARCH 1-5 test:    F(5,358) =  0.32365 [0.8987]
ARCH 1-10 test:   F(10,348)=  0.57556 [0.8339]
--------------
Residual-Based Diagnostic for Conditional Heteroskedasticity of Tse (2001)

  RBD(10) =  3.62592   [0.9626488]
  RBD(15) =  16.5646   [0.3455497]
  RBD(20) =  8.58065   [0.9872751]
------------------------------------------------
P-values in brackets

---------------
Adjusted Pearson Chi-square Goodness-of-fit test

# Cells(g)  Statistic      P-Value(g-1)     P-Value(g-k-1)
    40       48.8602         0.133868          0.047509
    50       61.8710         0.102550          0.038865
    60       74.7742         0.080766          0.032113
Rem.: k = 5 = # estimated parameters

> m1=garchOxFit(formula.mean=~arma(0,0),formula.var=~garch(0,1),series=intc)
 (* Output edited *)
 ********************
 ** SPECIFICATIONS **
 ********************
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (0, 1) model.
 No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 230.454
Please wait : Computing the Std Errors ...

 Maximum Likelihood Estimation (Std.Errors based on Second derivatives)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)               0.016548  0.0061388    2.696  0.0073
Cst(V)               0.012391  0.0015450    8.020  0.0000
ARCH(Alpha1)         0.373638    0.13372    2.794  0.0055

No. Observations :       372  No. Parameters  :         3
Mean (Y)         :   0.01799  Variance (Y)    :   0.01784
```

```
Skewness (Y)     :  -0.60142  Kurtosis (Y)    :   5.92148
Log Likelihood   :   230.454  Alpha[1]+Beta[1]:   0.37344

***************
** FORECASTS **
***************
Number of Forecasts: 15

Horizon       Mean   Variance
      1    0.01655    0.01383
      2    0.01655    0.01755
       .... (edited)
     15    0.01655    0.01978
---------------

***********
** TESTS **
***********
                   Statistic       t-Test      P-Value
Skewness            -0.67997       5.3756  7.6339e-008
Excess Kurtosis       2.4472       9.6989  3.0492e-022
Jarque-Bera           121.49         .NaN  4.1511e-027
---------------
Information Criterium (to be minimized)
Akaike         -1.222870  Shibata        -1.222999
Schwarz         -1.191266  Hannan-Quinn    -1.210319
---------------
Q-Statistics on Standardized Residuals
  Q( 10) =  13.6710   [0.1885356]
  Q( 15) =  22.2135   [0.1023272]
  Q( 20) =  23.7844   [0.2519364]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
--------------
Q-Statistics on Squared Standardized Residuals
  --> P-values adjusted by 1 degree(s) of freedom
  Q( 10) =  12.2592   [0.1990864]
  Q( 15) =  29.6965   [0.0084008]
  Q( 20) =  31.0595   [0.0397697]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
--------------
ARCH 1-2 test:    F(2,366) =   2.5957 [0.0760]
ARCH 1-5 test:    F(5,360) =   1.4194 [0.2164]
ARCH 1-10 test:   F(10,350)=   1.1103 [0.3533]
--------------

> names(m1)
```

```
 [1] "x"             "csts"          "cond.dist"    "arma.orders"  "arfima"
 [6] "garch.orders" "arch.in.mean" "model"        "ox"           "call"
[11] "residuals"    "condvars"     "coef"         "title"        "description"

> par(mfcol=c(2,1)) % To plot the returns and volatilities on the sample page
> plot(intc,type='l')
> plot(sqrt(m1$condvars),type='l')
> par(mfcol=c(1,1))
> sresi=m1$residuals/sqrt(m1$condvars) % Compute standardized residuals
> qqnorm(sresi) % Obtain normal probability plot (ideal case: a straight line)
> qqline(sresi) % Impose a straight line on the QQ-plot.
```

S-Plus demnstration:

```
> da=read.table("m-intc7303.txt")
> dim(da)
> intc=log(da[,2]+1)   %Compute log returns.
> autocorTest(intc,lag.n=12) % Test serial correlation using Q(12).
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:
Test Stat 18.5664
  p.value  0.0995

Dist. under Null: chi-square with 12 degrees of freedom
   Total Observ.: 372
> archTest(intc,lag.n=12)  % Test ARCH effect using 12 lags.
Test for ARCH Effects: LM Test

Null Hypothesis: no ARCH effects

Test Statistics:
Test Stat 43.5041
  p.value  0.0000

Dist. under Null: chi-square with 12 degrees of freedom

> acf(intc^2,lag.max=12,type='partial')  % Compute PACF of squared series.
Call: acf(x = intc^2, lag.max = 12, type = "partial")

Partial Correlation matrix:
   lag    intc
 1   1  0.1402
 2   2  0.1703
```

```
 3   3  0.1557   % First 3 lags are relatively large.
 4   4  0.0633
 5   5  0.0407
 .....
10  10 -0.0143
11  11 -0.0399
12  12  0.1614  % Lag-12 is also relatively large, but its order is high.

> arch3=garch(intc~1,~garch(3,0)) % Fit a Gaussian ARCH(3) model
> summary(arch3)

Call:
garch(formula.mean = intc ~ 1, formula.var =  ~ garch(3, 0))

Mean Equation: intc ~ 1

Conditional Variance Equation:  ~ garch(3, 0)

Conditional Distribution:  gaussian
--------------------------------------------------------------
Estimated Coefficients:
--------------------------------------------------------------
          Value Std.Error t value Pr(>|t|)
      C 0.01713  0.006626  2.5860  0.01009
      A 0.01199  0.001107 10.8325  0.00000
ARCH(1) 0.17874  0.080294  2.2260  0.02662  Whis is the fitted model?
ARCH(2) 0.07720  0.050552  1.5271  0.12760
ARCH(3) 0.05722  0.076928  0.7438  0.45749
--------------------------------------------------------------
AIC(5) = -456.5791
BIC(5) = -436.9846

Normality Test:
--------------------------------------------------------------
 Jarque-Bera P-value Shapiro-Wilk   P-value
       173.1       0       0.9696 0.0002337

Ljung-Box test for standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
     12.79  0.3848         12

Ljung-Box test for squared standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
     21.42 0.04453         12
```

```
Lagrange multiplier test:
--------------------------------------------------------------
  Lag 1    Lag 2  Lag 3  Lag 4  Lag 5  Lag 6 Lag 7 Lag 8
 0.1646 -0.05844 0.1577 0.2978 0.4671 0.8066 1.037 1.449

  Lag 9  Lag 10  Lag 11  Lag 12       C
 0.02206 -0.8262  3.857 -0.3651 -0.5014

  TR^2 P-value F-stat P-value
 20.67 0.05549  1.993 0.09808

> arch1=garch(intc~1,~garch(1,0)) % SImplify to an ARCH(1) model.
> summary(arch1)

Call:
garch(formula.mean = intc ~ 1, formula.var =  ~ garch(1, 0))

Mean Equation: intc ~ 1

Conditional Variance Equation:  ~ garch(1, 0)

Conditional Distribution:  gaussian
--------------------------------------------------------------
Estimated Coefficients:
--------------------------------------------------------------
          Value Std.Error t value  Pr(>|t|)
      C 0.01741  0.006231   2.794 5.475e-03
      A 0.01258  0.001246  10.091 0.000e+00
ARCH(1) 0.35258  0.088515   3.983 8.189e-05
--------------------------------------------------------------
AIC(3) = -454.4589
BIC(3) = -442.7022

Normality Test:
--------------------------------------------------------------
 Jarque-Bera P-value Shapiro-Wilk   P-value
       120.9       0       0.9713 0.0008877

Ljung-Box test for standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
     15.37  0.2221         12

Ljung-Box test for squared standardized residuals:
--------------------------------------------------------------
```

```
 Statistic  P-value Chi^2-d.f.
     26.28 0.009796         12

 ....(edited)
  TR^2 P-value F-stat P-value
 22.18 0.03552  2.149  0.0761

> names(arch1)
 [1] "residuals"   "sigma.t"     "df.residual" "coef"    "model"
 [6] "cond.dist"   "likelihood"  "opt.index"   "cov"     "prediction"
[11] "call"        "asymp.sd"    "series"
> sresi=arch1$residuals/arch1$sigma.t %Compute standardized residuals
> autocorTest(sresi,lag.n=12)  % Repeat the default output.
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 15.3651
  p.value  0.2221

> autocorTest(sresi^2,lag.n=12)
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 26.2798
  p.value  0.0098    % Same as the default output.

> autocorTest(sresi^2,lag.n=10)
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 12.7024
  p.value  0.2408

> predict(arch1,5)  % Prediction 1-step to 5-step ahead forecasts.
$series.pred:
[1] 0.01741175 0.01741175 0.01741175 0.01741175 0.01741175  %return
```

```
$sigma.pred:
[1] 0.1181940 0.1322976 0.1369244 0.1385189 0.1390767  % volatility

$asymp.sd:   % Unconditional variance of a(t).
[1] 0.1393796

> qqnorm(sresi) % Normal probability plot to check normal assumption
> qqline(sresi) % add line to help read the plot.
```

From the R output, we obtain that, under normality,

$$r_t = 0.0165 + a_t, \quad \sigma_t^2 = 0.012 + 0.374 a_{t-1}^2.$$

Model checking:
Standardized shocks $\{\tilde{a}_t\}$
$Q(10) = 13.67(0.19)$
For $\{\tilde{a}_t^2\}$
$Q(10) = 12.26(0.20)$, but $Q(15) = 29.70(.01)$
Implications

- Expected monthly log return is about 1.7%.
- $\hat{\alpha}_1^2 = 0.374^2 < 1/3$ so that 4th moment exists.

From the S-Plus output, we obtain that, under normality,

$$r_t = 0.0174 + a_t, \quad \sigma_t^2 = 0.013 + 0.353 a_{t-1}^2.$$

Model checking:
Standardized shocks $\{\tilde{a}_t\}$
$Q(12) = 15.37(0.22)$
For $\{\tilde{a}_t^2\}$
$Q(12) = 21.28(.01)$.
The two programs give similar, but not identical, results.

Next, consider Student-t innovations.
R demonstration

```
 ********************
 ** SPECIFICATIONS **
 ********************
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (0, 1) model.
 No regressor in the variance
The distribution is a Student distribution, with 5.99816 degrees of freedom.

Strong convergence using numerical derivatives
Log-likelihood = 243.116
Please wait : Computing the Std Errors ...

 Maximum Likelihood Estimation (Std.Errors based on Second derivatives)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)               0.021513  0.0060427    3.560  0.0004
Cst(V)               0.013332  0.0019620    6.795  0.0000
ARCH(Alpha1)         0.268092    0.12224    2.193  0.0289
Student(DF)          5.998160     1.6661    3.600  0.0004

No. Observations :       372  No. Parameters  :         4
Mean (Y)         :   0.01799  Variance (Y)    :   0.01784
Skewness (Y)     :  -0.60142  Kurtosis (Y)    :   5.92148
Log Likelihood   :   243.116  Alpha[1]+Beta[1]:   0.26809

***************
** FORECASTS **
***************
Number of Forecasts: 15

Horizon       Mean   Variance
      1    0.02151    0.01453
      2    0.02151    0.01723
      .... (edited)
     15    0.02151    0.01821
---------------

***********
** TESTS **
***********
                   Statistic       t-Test      P-Value
Skewness            -0.68834       5.4417  5.2763e-008
Excess Kurtosis       2.5502       10.107  5.1483e-024
Jarque-Bera           130.18         .NaN  5.3972e-029
---------------
```

```
Information Criterium (to be minimized)
Akaike         -1.285572  Shibata        -1.285800
Schwarz         -1.243433  Hannan-Quinn    -1.268837
---------------
Q-Statistics on Standardized Residuals
  Q( 10) =  14.2606   [0.1614340]
  Q( 15) =  23.2423   [0.0791288]
  Q( 20) =  24.7769   [0.2101018]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
--------------
Q-Statistics on Squared Standardized Residuals
  --> P-values adjusted by 1 degree(s) of freedom
  Q( 10) =  15.0259   [0.0902279]
  Q( 15) =  33.5172   [0.0024250]
  Q( 20) =  35.0263   [0.0138654]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
--------------
ARCH 1-2 test:    F(2,366) =   3.1174 [0.0454]*
ARCH 1-5 test:    F(5,360) =   1.8061 [0.1108]
ARCH 1-10 test:   F(10,350)=   1.2727 [0.2443]
--------------
P-values in brackets

> sresi=m1$residuals/m1$condvars^.5
> pacf(sresi^2)

> m2=garchOxFit(formula.mean=~arma(0,0),formula.var=~garch(0,2),series=intc,cond.dist="t")

 ********************
 ** SPECIFICATIONS **
 ********************
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (0, 2) model.
 No regressor in the variance
The distribution is a Student distribution, with 6.09662 degrees of freedom.

Strong convergence using numerical derivatives
Log-likelihood = 245.913
Please wait : Computing the Std Errors ...

 Maximum Likelihood Estimation (Std.Errors based on Second derivatives)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)               0.022100  0.0060294    3.665  0.0003
Cst(V)               0.012420  0.0018284    6.793  0.0000
```

```
ARCH(Alpha1)         0.184359    0.10805    1.706  0.0888
ARCH(Alpha2)         0.110735   0.068533    1.616  0.1070   <== insigh. at 5%
Student(DF)          6.096618     1.6919    3.603  0.0004

No. Observations :        372  No. Parameters  :        5
Mean (Y)         :    0.01799  Variance (Y)    :   0.01784
Skewness (Y)     :   -0.60142  Kurtosis (Y)    :   5.92148
Log Likelihood   :    245.913  Alpha[1]+Beta[1]:   0.29509

> qqplot(rt(10000,6.1),sresi)  % qq-plot for student-t dist.
> qqline(sresi)
```

S-Plus demonstration

```
> m1=garch(intc~1,~garch(1,0),cond.dist="t") %Use student-t innovations
> summary(m1)

Call:
garch(formula.mean=intc ~ 1,formula.var= ~ garch(1, 0),cond.dist="t")

Mean Equation: intc ~ 1

Conditional Variance Equation:  ~ garch(1, 0)

Conditional Distribution:  t
 with estimated parameter 6.159751 and standard error 1.647094
--------------------------------------------------------------
Estimated Coefficients:
--------------------------------------------------------------
          Value Std.Error t value  Pr(>|t|)
      C 0.02213  0.006010   3.681 2.666e-04
      A 0.01338  0.001965   6.809 4.002e-11
ARCH(1) 0.24916  0.115574   2.156 3.174e-02
--------------------------------------------------------------
AIC(4) = -477.9073
BIC(4) = -462.2317

Normality Test:
--------------------------------------------------------------
 Jarque-Bera P-value Shapiro-Wilk   P-value
       128.9       0       0.9707 0.0005601

Ljung-Box test for standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
      16.1  0.1868         12
```

```
Ljung-Box test for squared standardized residuals:
--------------------------------------------------------------
 Statistic  P-value Chi^2-d.f.
     29.91 0.002882         12

> tresi=m1$residuals/m1$sigma.t
> autocorTest(tresi,lag.n=12)
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 16.0974
  p.value  0.1868

> autocorTest(tresi^2,lag.n=12)
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 29.9089
  p.value  0.0029

> autocorTest(tresi^2,lag.n=10) % use 10 lags
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 15.6545
  p.value  0.1100   % The result confirms that lag-12 is significant.
                    % See below PACF of squared residuals.
> predict(m1,5)  % Prediction
$series.pred:
[1] 0.02212715 0.02212715 0.02212715 0.02212715 0.02212715

$sigma.pred:
[1] 0.1204767 0.1303599 0.1327079 0.1332865 0.1334302

$asymp.sd:
[1] 0.1334779
```

```
attr(, "class"):
[1] "predict.garch"
> acf(tresi^2,type='partial',lag.max=12)
Call: acf(x = tresi^2, lag.max = 12, type = "partial")

Partial Correlation matrix:
   lag   tresi
 1   1 -0.0352
 2   2  0.1273
 3   3  0.0643
  .....
11  11 -0.0634
12  12  0.1639
> m2=garch(intc~1,~garch(2,0),cond.dist="t") % Increase the order
> summary(m2)

Mean Equation: intc ~ 1

Conditional Variance Equation:  ~ garch(2, 0)

Conditional Distribution:  t
 with estimated parameter 6.02561 and standard error 1.565027
--------------------------------------------------------------
Estimated Coefficients:
--------------------------------------------------------------
          Value Std.Error t value  Pr(>|t|)
      C 0.02264  0.006005   3.770 1.899e-04
      A 0.01247  0.001918   6.500 2.627e-10
ARCH(1) 0.17457  0.105950   1.648 1.003e-01
ARCH(2) 0.12109  0.092888   1.304 1.932e-01   % Insignificant
--------------------------------------------------------------
AIC(5) = -481.6892
BIC(5) = -462.0947
```

Question: What is the fitted ARCH(1) model in R?

$$r_t = 0.022 + a_t, \quad \sigma_t^2 = 0.013 + 0.268 a_{t-1}^2,$$

and the t-distribution has 6.00 d.f.

Question: What is the fitted ARCH(1) model in S-Plus?

$$r_t = 0.022 + a_t, \quad \sigma_t^2 = 0.013 + 0.249 a_{t-1}^2,$$

and the t-distribution has 6.16 d.f.

Comparison with normal models:

- Using a heavy-tailed dist for ϵ_t reduces the ARCH effect
- The difference between the models is small for this particular instance.

You may try the generalized error distribution.

GARCH Model

$$a_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

where $\{\epsilon_t\}$ is defined as before, $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\Sigma_{i=1}^{\max(m,s)}(\alpha_i + \beta_i) < 1$.
Re-parameterization:
Let $\eta_t = a_t^2 - \sigma_t^2$. $\{\eta_t\}$ un-correlated series.
The GARCH model becomes

$$a_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^{s} \beta_j \eta_{t-j}.$$

This is an ARMA form for the squared series a_t^2.
Use it to understand properties of GARCH models, e.g. moment equations, forecasting, etc.
Focus on a GARCH(1,1) model

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

- Weak stationarity: $0 \leq \alpha_1, \beta_1 \leq 1, (\alpha_1 + \beta_1) < 1$.
- Volatility clusters
- Heavy tails: if $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$, then

$$\frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} > 3.$$

- For 1-step ahead forecast,

$$\sigma_h^2(1) = \alpha_0 + \alpha_1 a_h^2 + \beta_1 \sigma_h^2.$$

For multi-step ahead forecasts, use $a_t^2 = \sigma_t^2 \epsilon_t^2$ and rewrite the model as

$$\sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2 + \alpha_1 \sigma_t^2(\epsilon_t^2 - 1).$$

2-step ahead volatility forecast

$$\sigma_h^2(2) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_h^2(1).$$

In general, we have

$$\sigma_h^2(\ell) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_h^2(\ell - 1), \quad \ell > 1.$$

This result is exactly the same as that of an ARMA(1,1) model with AR polynomial $1 - (\alpha_1 + \beta_1)B$.

Example: Monthly excess returns of S&P 500 index starting from 1926 for 792 observations.
The fitted of a Gaussian AR(3) model

$$r_t = .088r_{t-1} - .023r_{t-2} - .123r_{t-3} + .007 + a_t,$$

$\hat{\sigma}_a^2 = 0.00333$.
For the GARCH effects, use a GARCH(1,1) model (R output).
A joint estimation:

$$\begin{aligned} r_t &= 0.032r_{t-1} - 0.030r_{t-2} - 0.010r_{t-3} + 0.0076 + a_t \\ \sigma_t^2 &= .00008 + .853\sigma_{t-1}^2 + 0.125a_{t-1}^2. \end{aligned}$$

Implied unconditional variance of a_t is

$$\frac{0.0000794}{1 - 0.85298 - 0.1247} = 0.00356$$

close to the expected value.
A simplified model:

$$r_t = 0.0074 + a_t, \sigma_t^2 = .00008 + .854\sigma_{t-1}^2 + .122a_{t-1}^2.$$

Model checking:
For $\tilde{a}_t$: $Q(10) = 11.22(0.34)$ and $Q(20) = 24.29(0.23)$.
For $\tilde{a}_t^2$: $Q(10) = 9.92(0.27)$ and $Q(20) = 16.74(0.54)$.
Forecast: 1-step ahead forecast:

$$\sigma_h^2(1) = 0.00008 + 0.854\sigma_h^2 + 0.122a_h^2$$

Horizon	1	2	3	4	5	∞
Return	.0074	.0074	.0074	.0074	.0074	.0074
Volatility	.053	.052	.052	.051	.051	.050

R demonstration:

```
> library("fSeries")
> source("garchOxFit_R.txt")
> sp5=scan(file="sp500.dat")
> plot(sp5,type='l')
> m1=arima(sp5,order=c(3,0,0))
> m1
Call:
arima(x = sp5, order = c(3, 0, 0))

Coefficients:
         ar1      ar2      ar3  intercept
      0.0890  -0.0238  -0.1229     0.0062
s.e.  0.0353   0.0355   0.0353     0.0019

sigma^2 estimated as 0.00333:  log likelihood = 1135.25,  aic = -2260.5

> x=ts(sp5)
> m2=garchOxFit(formula.mean=~arma(3,0),formula.var=~garch(1,1),series=x)
 ********************
 ** SPECIFICATIONS **
 ********************
Dependent variable : X
Mean Equation : ARMA (3, 0) model.
```

```
No regressor in the mean
Variance Equation : GARCH (1, 1) model.
 No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1272.18

 Maximum Likelihood Estimation (Std.Errors based on Second derivatives)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)               0.007639  0.0015225    5.017  0.0000
AR(1)                0.031987   0.038368   0.8337  0.4047
AR(2)               -0.030276   0.038407  -0.7883  0.4308
AR(3)               -0.010637   0.037558  -0.2832  0.7771
Cst(V)               0.793989    0.28174    2.818  0.0050
ARCH(Alpha1)         0.124710   0.022615    5.514  0.0000
GARCH(Beta1)         0.852981   0.021891    38.97  0.0000

No. Observations :       792  No. Parameters  :         7
Mean (Y)         :   0.00614  Variance (Y)    :   0.00341
Skewness (Y)     :   0.41134  Kurtosis (Y)    :  12.30025
Log Likelihood   :  1272.183  Alpha[1]+Beta[1]:   0.97749

Warning : To avoid numerical problems, the estimated parameter
Cst(V), and its std.Error have been multiplied by 10^4.

***************
** FORECASTS **
***************
Number of Forecasts: 15

Horizon       Mean   Variance
      1    0.01248   0.002889
      2   0.005195   0.002824
       ....
     15   0.007639   0.002101
---------------

***********
** TESTS **
***********
                   Statistic       t-Test      P-Value
Skewness            -0.38960       4.4847  7.3025e-006
Excess Kurtosis       1.2654       7.2920  3.0543e-013
Jarque-Bera           72.877         .NaN  1.4963e-016
---------------
```

```
Information Criterium (to be minimized)
Akaike         -3.194906  Shibata        -3.195060
Schwarz         -3.153590  Hannan-Quinn    -3.179027
---------------
Q-Statistics on Standardized Residuals
  --> P-values adjusted by 3 degree(s) of freedom
  Q( 10) =  11.5651   [0.1157988]
  Q( 15) =  17.7852   [0.1223710]
  Q( 20) =  24.1110   [0.1164494]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
--------------
Q-Statistics on Squared Standardized Residuals
  --> P-values adjusted by 2 degree(s) of freedom
  Q( 10) =  10.3145   [0.2436438]
  Q( 15) =  14.2072   [0.3594178]
  Q( 20) =  16.7703   [0.5389502]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
--------------
ARCH 1-2 test:    F(2,785) =  0.69609 [0.4988]
ARCH 1-5 test:    F(5,779) =  0.62957 [0.6772]
ARCH 1-10 test:   F(10,769)=   1.0519 [0.3976]
--------------
```

S-Plus demonstration:

```
> x=scan(file='sp500.dat')
> spfit=garch(x~ar(3),~garch(1,1)) % Fit an AR(3) + GARCH(1,1) model.
> summary(spfit)
 Call: garch(formula.mean = x ~ ar(3), formula.var =  ~ garch(1, 1))

Mean Equation: x ~ ar(3)
Conditional Variance Equation:  ~ garch(1, 1)
Conditional Distribution:  gaussian
--------------------------------------------------------------
Estimated Coefficients:
--------------------------------------------------------------
              Value Std.Error  t value  Pr(>|t|)
       C  7.751e-03 1.603e-03   4.8359 1.595e-06
   AR(1)  3.267e-02 3.849e-02   0.8488 3.903e-01 % AR coefs are insign. at 5%
   AR(2) -2.884e-02 3.823e-02  -0.7543 4.509e-01
   AR(3) -8.407e-03 3.550e-02  -0.2368 8.129e-01
       A  8.374e-05 2.436e-05   3.4382 6.164e-04
 ARCH(1)  1.213e-01 2.030e-02   5.9774 3.439e-09
GARCH(1)  8.523e-01 1.969e-02  43.2803 0.000e+00
--------------------------------------------------------------
AIC(7) = -2526.239, BIC(7) = -2493.517
```

```
Normality Test:
--------------------------------------------------------------
 Jarque-Bera  P-value Shapiro-Wilk P-value
       72.25 2.22e-16       0.9817 0.04185

Ljung-Box test for standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
     11.78  0.4636         12

Ljung-Box test for squared standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
     13.44   0.338         12

> spfit=garch(x~1,~garch(1,1)) % A refined model
> summary(spfit)
Call: garch(formula.mean = x ~ 1, formula.var =  ~ garch(1, 1))

Mean Equation: x ~ 1
Conditional Variance Equation:  ~ garch(1, 1)
Conditional Distribution:  gaussian
--------------------------------------------------------------
Estimated Coefficients:
--------------------------------------------------------------
             Value Std.Error t value  Pr(>|t|)
       C 7.647e-03 1.545e-03   4.950 9.096e-07
       A 8.561e-05 2.412e-05   3.549 4.097e-04
 ARCH(1) 1.216e-01 1.974e-02   6.159 1.165e-09
GARCH(1) 8.511e-01 1.899e-02  44.809 0.000e+00
--------------------------------------------------------------
AIC(4) = -2530.821, BIC(4) = -2512.122

Normality Test:
--------------------------------------------------------------
 Jarque-Bera P-value Shapiro-Wilk P-value
       81.58       0       0.9809  0.0201

Ljung-Box test for standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
     11.99  0.4468         12

Ljung-Box test for squared standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
```

```
     13.11  0.3609         12

Lagrange multiplier test:
--------------------------------------------------------------
   Lag 1  Lag 2   Lag 3   Lag 4   Lag 5  Lag 6 Lag 7 Lag 8
 -0.9755 0.5875 -0.4926 -0.8138 -0.1367 -1.018 1.497 1.859

  Lag 9 Lag 10 Lag 11 Lag 12      C
 0.5532  1.758 0.2104 0.1441 -0.947

  TR^2 P-value F-stat P-value
 13.15  0.3583  1.216  0.3824

> predict(spfit,5)
$series.pred:
[1] 0.007647292 0.007647292 0.007647292 0.007647292 0.007647292
$sigma.pred:
[1] 0.05358398 0.05365175 0.05371758 0.05378154 0.05384369

> mean(x)
[1] 0.006143056  % Point forecasts are higher than sample mean!
```

Compare the Splus result with that of R!

Turn to Student-t innovation.

Estimation of degrees of freedom:

$$r_t = 0.0085 + a_t,$$

$$\sigma_t^2 = .00012 + .113a_{t-1}^2 + .842\sigma_{t-1}^2,$$

where the estimated degrees of freedom is 6.99.

Forecasting evaluation

Not easy to do; see Andersen and Bollerslev (1998).

IGARCH model

An IGARCH(1,1) model:

$$a_t = \sigma_t\epsilon_t, \sigma_t^2 = \alpha_0 + \beta_1\sigma_{t-1}^2 + (1-\beta_1)a_{t-1}^2.$$

For the monthly excess returns of the S&P 500 index, we have

$$r_t = .007 + a_t, \sigma_t^2 = .0001 + .806\sigma_{t-1}^2 + .194a_{t-1}^2$$

For an IGARCH(1,1) model,

$$\sigma_h^2(\ell) = \sigma_h^2(1) + (\ell - 1)\alpha_0, \quad \ell \geq 1,$$

where h is the forecast origin.
Effect of $\sigma_h^2(1)$ on future volatilities is persistent, and the volatility forecasts form a straight line with slope α_0. See Nelson (1990) for more info.
Special case: $\alpha_0 = 0$.
used in RiskMetrics to VaR calculation.
Example: An IGARCH(1,1) model for the monthly excess returns of S&P500 index from 1926 to 1991 is given below via R.

$$r_t = 0.0074 + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = .00005 + .143 a_{t-1}^2 + .857 \sigma_{t-1}^2.$$

R demonstration

```
> m2=garchOxFit(formula.mean=~arma(0,0),formula.var=~igarch(1,1),series=sp5)
 ********************
 ** SPECIFICATIONS **
 ********************
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : IGARCH (1, 1) model.
 No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1268.24

 Maximum Likelihood Estimation (Std.Errors based on Second derivatives)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)               0.007416  0.0015254    4.861  0.0000
Cst(V)               0.512441    0.17527    2.924  0.0036
ARCH(Alpha1)         0.142948   0.021444    6.666  0.0000
GARCH(Beta1)         0.857252

No. Observations :       792  No. Parameters  :         4
```

```
Mean (Y)          :   0.00614  Variance (Y)    :   0.00341
Skewness (Y)      :   0.41134  Kurtosis (Y)    :  12.30025
Log Likelihood    :  1268.238

Warning : To avoid numerical problems, the estimated parameter
Cst(V), and its std.Error have been multiplied by 10^4.

***************
** FORECASTS **
***************
Number of Forecasts: 15

Horizon       Mean   Variance
      1   0.007416   0.003079
      2   0.007416   0.003079
      ......
     15   0.007416   0.003079
---------------

***********
** TESTS **
***********
                   Statistic       t-Test      P-Value
Skewness            -0.40577       4.6708  3.0001e-006
Excess Kurtosis       1.2112       6.9797  2.9578e-012
Jarque-Bera           70.146         .NaN  5.8618e-016
---------------
Information Criterium (to be minimized)
Akaike         -3.195044  Shibata        -3.195073
Schwarz         -3.177338  Hannan-Quinn    -3.188239
---------------
Q-Statistics on Standardized Residuals
  Q( 10) =  10.8290   [0.3709943]
  Q( 15) =  17.6387   [0.2821344]
  Q( 20) =  23.6418   [0.2583909]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
--------------
Q-Statistics on Squared Standardized Residuals
  --> P-values adjusted by 2 degree(s) of freedom
  Q( 10) =  10.0513   [0.2614457]
  Q( 15) =  13.4740   [0.4119030]
  Q( 20) =  15.9382   [0.5968584]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]
--------------
ARCH 1-2 test:    F(2,785) =   1.1501 [0.3171]
ARCH 1-5 test:    F(5,779) =  0.79599 [0.5527]
```

```
ARCH 1-10 test:   F(10,769)=   1.0004 [0.4413]
```

The GARCH-M model

$$r_t = \mu + c\sigma_t^2 + a_t, \quad a_t = \sigma_t \epsilon_t, \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where c is referred to as risk premium, which is expected to be positive.

Example: A GARCH(1,1)-M model for the monthly excess returns of S&P 500 index from January 1926 to December 1991. The fitted model is

$$r_t = 0.0054 + 1.01\sigma_t^2 + a_t, \sigma_t^2 = .00008 + .123a_{t-1}^2 + .852\sigma_{t-1}^2.$$

Std err of risk premium is 0.818.

R demonstration

```
> m3=garchOxFit(formula.mean=~arma(0,0),formula.var=~garch(1,1),series=sp5,arch.in.mean=T)
 ********************
 ** SPECIFICATIONS **
 ********************
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (1, 1) model.
            in-mean
 No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1270.1

 Maximum Likelihood Estimation (Std.Errors based on Second derivatives)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)               0.005420  0.0023602    2.297  0.0219
Cst(V)               0.829654    0.29321    2.830  0.0048
ARCH(Alpha1)         0.123127   0.022286    5.525  0.0000
GARCH(Beta1)         0.852256   0.022400    38.05  0.0000
ARCH-in-mean(var)    1.008013    0.88853    1.134  0.2569

Warning : To avoid numerical problems, the estimated parameter
Cst(V), and its std.Error have been multiplied by 10^4.
```

S-Plus demonstration

```
> spfit=garch(x~1+var.in.mean,~garch(1,1))
> summary(spfit)
garch(formula.mean = x ~ 1 + var.in.mean, formula.var =  ~ garch(1, 1))

Mean Equation: x ~ 1 + var.in.mean
Conditional Variance Equation:  ~ garch(1, 1)
Conditional Distribution:  gaussian
--------------------------------------------------------------
Estimated Coefficients:
--------------------------------------------------------------
                 Value Std.Error t value  Pr(>|t|)
           C 5.487e-03 2.262e-03   2.426 7.747e-03
ARCH-IN-MEAN 1.088e+00 8.182e-01   1.330 9.203e-02
           A 8.764e-05 2.507e-05   3.496 2.494e-04
     ARCH(1) 1.227e-01 2.047e-02   5.993 1.571e-09
    GARCH(1) 8.494e-01 1.958e-02  43.390 0.000e+00
--------------------------------------------------------------
AIC(5) = -2530.136, BIC(5) = -2506.763

Normality Test:
--------------------------------------------------------------
 Jarque-Bera P-value Shapiro-Wilk  P-value
       79.85       0       0.9801 0.009548

Ljung-Box test for standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
     13.43  0.3385         12

Ljung-Box test for squared standardized residuals:
--------------------------------------------------------------
 Statistic P-value Chi^2-d.f.
     11.83  0.4598         12

Lagrange multiplier test:
--------------------------------------------------------------
   Lag 1  Lag 2  Lag 3   Lag 4   Lag 5   Lag 6 Lag 7 Lag 8
 -0.9655 0.4582 -0.509 -0.7114 0.03989 -0.9966 1.444 1.656

  Lag 9 Lag 10 Lag 11 Lag 12       C
 0.4562  1.713 0.2978 0.1433 -0.9991

  TR^2 P-value F-stat P-value
 11.85  0.4575  1.094  0.4738
```

```
> predict(spfit,5)
$series.pred:
[1] 0.008621675 0.008629477 0.008637062 0.008644436 0.008651603
$sigma.pred:
[1] 0.05368237 0.05374914 0.05381396 0.05387690 0.05393801
$asymp.sd:
[1] 0.05602398
```