

## Lecture Note of Bus 41202, Spring 2008: More Volatility Models. Mr. Ruey Tsay

### The EGARCH model

Asymmetry in responses to + & - returns:

$$g(\epsilon_t) = \theta\epsilon_t + \gamma[|\epsilon_t| - E(|\epsilon_t|)],$$

with  $E[g(\epsilon_t)] = 0$ .

To see asymmetry of  $g(\epsilon_t)$ , rewrite it as

$$g(\epsilon_t) = \begin{cases} (\theta + \gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t \geq 0, \\ (\theta - \gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t < 0. \end{cases}$$

An EGARCH( $m, s$ ) model:

$$a_t = \sigma_t \epsilon_t, \quad \ln(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \dots + \beta_{s-1} B^{s-1}}{1 - \alpha_1 B - \dots - \alpha_m B^m} g(\epsilon_{t-1}).$$

Some features of EGARCH models:

- uses log trans. to relax the positiveness constraint
- asymmetric responses

Consider an EGARCH(1,1) model

$$a_t = \sigma_t \epsilon_t, \quad (1 - \alpha B) \ln(\sigma_t^2) = (1 - \alpha)\alpha_0 + g(\epsilon_{t-1}),$$

Under normality,  $E(|\epsilon_t|) = \sqrt{2/\pi}$  and the model becomes

$$(1 - \alpha B) \ln(\sigma_t^2) = \begin{cases} \alpha_* + (\theta + \gamma)\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0, \\ \alpha_* + (\theta - \gamma)\epsilon_{t-1} & \text{if } \epsilon_{t-1} < 0 \end{cases}$$

where  $\alpha_* = (1 - \alpha)\alpha_0 - \sqrt{2/\pi}\gamma$ .

This is a nonlinear fun. similar to that of the threshold AR model of Tong (1978, 1990).

Specifically, we have

$$\sigma_t^2 = \sigma_{t-1}^{2\alpha} \exp(\alpha_*) \begin{cases} \exp[(\theta + \gamma) \frac{a_{t-1}}{\sqrt{\sigma_{t-1}^2}}] & \text{if } a_{t-1} \geq 0, \\ \exp[(\theta - \gamma) \frac{a_{t-1}}{\sqrt{\sigma_{t-1}^2}}] & \text{if } a_{t-1} < 0. \end{cases}$$

The coefs  $(\theta + \gamma)$  &  $(\theta - \gamma)$  show the asymmetry in response to positive and negative  $a_{t-1}$ . The model is, therefore, nonlinear if  $\theta \neq 0$ . Thus,  $\theta$  is referred to as the *leverage* parameter.

Focus on the function  $g(\epsilon_{t-1})$ . The leverage parameter  $\theta$  shows the effect of the sign of  $a_{t-1}$  whereas  $\gamma$  denotes the magnitude effect.

See Nelson (1991) for an exmample of EGARCH model.

**Another example:** Monthly log returns of IBM stock from January 1926 to December 1997 for 864 observations.

For textbook, an AR(1)-EGARCH(1,1) is obtained (RATS program):

$$\begin{aligned} r_t &= 0.0105 + 0.092r_{t-1} + a_t, & a_t &= \sigma_t \epsilon_t \\ \ln(\sigma_t^2) &= -5.496 + \frac{g(\epsilon_{t-1})}{1 - .856B}, \\ g(\epsilon_{t-1}) &= -.0795\epsilon_{t-1} + .2647[|\epsilon_{t-1}| - \sqrt{2/\pi}], \end{aligned}$$

Model checking:

For  $\tilde{a}_t$ :  $Q(10) = 6.31(0.71)$  and  $Q(20) = 21.4(0.32)$

For  $\tilde{a}_t^2$ :  $Q(10) = 4.13(0.90)$  and  $Q(20) = 15.93(0.66)$

Discussion:

Using  $\sqrt{2/\pi} \approx 0.7979 \approx 0.8$ , we obtain

$$\ln(\sigma_t^2) = -1.0 + 0.856 \ln(\sigma_{t-1}^2) + \begin{cases} 0.1852\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0 \\ -0.3442\epsilon_{t-1} & \text{if } \epsilon_{t-1} < 0. \end{cases}$$

Taking anti-log transformation, we have

$$\sigma_t^2 = \sigma_{t-1}^{2 \times 0.856} e^{-1.001} \times \begin{cases} e^{0.1852\epsilon_{t-1}} & \text{if } \epsilon_{t-1} \geq 0 \\ e^{-0.3442\epsilon_{t-1}} & \text{if } \epsilon_{t-1} < 0. \end{cases}$$

For a standardized shock with magnitude 2, (i.e. two standard deviations), we have

$$\frac{\sigma_t^2(\epsilon_{t-1} = -2)}{\sigma_t^2(\epsilon_{t-1} = 2)} = \frac{\exp[-0.3442 \times (-2)]}{\exp(0.1852 \times 2)} = e^{0.318} = 1.374.$$

Therefore, the impact of a negative shock of size two-standard deviations is about 37.4% higher than that of a positive shock of the same size.

Forecasting: some recursive formula available

**Another parameterization** of EGARCH models

$$\ln(\sigma_t^2) = \alpha_0 + \alpha_1 \frac{|a_{t-1}| + \gamma_1 a_{t-1}}{\sigma_{t-1}} + \beta_1 \ln(\sigma_{t-1}^2),$$

where  $\gamma_1$  denotes the leverage effect.

### S-Plus demonstration

```
> spfit=garch(x~1,~egarch(1,1),leverage=T) % Fit an EGARCH(1,1) model
> summary(spfit)
Call: garch(formula.mean = x ~ 1, formula.var = ~ egarch(1, 1), leverage = T)

Mean Equation: x ~ 1
Conditional Variance Equation: ~ egarch(1, 1)
Conditional Distribution: gaussian
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Estimated Coefficients:
-----
              Value Std.Error   t value Pr(>|t|) % Model explanation:
C    0.006901  0.001608    4.293 1.986e-05 Mean equ: r(t) = 0.0069 + sigma(t)*e(t)
A   -0.379589  0.052316   -7.256 9.592e-13 Let h(t) = ln(sigma(t)**2)
ARCH(1) 0.228222  0.029438    7.753 2.798e-14 Volatility equ:
GARCH(1) 0.966338  0.007759  124.542 0.000e+00 h(t) = -0.38 + +.97h(t-1) +
LEV(1) -0.292277  0.090273   -3.238 1.255e-03 .23*[|e(t-1)|-.29*e(t-1)]/sigma(t-1).
-----
AIC(5) = -2533.294, BIC(5) = -2509.922

Normality Test:
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Jarque-Bera P-value Shapiro-Wilk P-value
102.5      0      0.9844  0.2536
```

Ljung-Box test for standardized residuals:

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Statistic	P-value	Chi <sup>2</sup> -d.f.
11.69	0.4706	12

Ljung-Box test for squared standardized residuals:

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Statistic	P-value	Chi <sup>2</sup> -d.f.
15.51	0.2148	12

Lagrange multiplier test:

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Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8
-0.7505	0.2304	-0.2817	-0.6344	-0.2022	-0.8465	1.758	2.201

Lag 9	Lag 10	Lag 11	Lag 12	C
0.5231	2.278	0.4728	0.144	-0.9919

TR <sup>2</sup>	P-value	F-stat	P-value
15.82	0.1996	1.468	0.2428

**R demonstration:** It is a bit more complicated to fit an EGARCH model in R. Several factors are involved. First, an EGARCH( $m, s$ ) model in the textbook is an EGARCH( $m, s+1$ ) model in R. Thus, an EGARCH(1,1) model in R also contains the ARCH parameter. Second, the names alpha and beta are interchanged between textbook and R. Third, the default option of parameter constraints in R requires the constant term in the volatility equation to be positive. This is incorrect because EGARCH uses logarithm of volatility. Thus, one needs to use “unconstrained” estimation to fit an EGARCH model. This involves editing the “GarchOxModelling.ox” file in the OX/lib directory. Finally, THETA1 in R output is the leverage parameter. [Alternatively, you may create a separate GarchOxModelling.ox file and garchoxfit-R.txt file to perform the EGARCH estimation in R.] I demonstrate the R estimation below for the monthly log returns of IBM stock.

```
> library("fSeries")
> source("garchoxfit_R.txt")
```

```

> ibm=scan(file='m-ibmln.dat')
> m1=garchOxFit(formula.mean=~arma(1,0),formula.var=~egarch(1,1),series=ibm)
*****
** SPECIFICATIONS **
*****
Dependent variable : X
Mean Equation : ARMA (1, 0) model.
No regressor in the mean
Variance Equation : EGARCH (1, 1) model.
No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1157

```

```

Maximum Likelihood Estimation (Std.Errors based on Numerical OPG matrix)

```

	Coefficient	Std.Error	t-value	t-prob
Cst(M)	0.011489	0.0024738	4.644	0.0000
AR(1)	0.091821	0.039852	2.304	0.0215
Cst(V)	-54956.513427	663.64	-82.81	0.0000
ARCH(Alpha1)	0.312637	0.29194	1.071	0.2845
GARCH(Beta1)	0.766558	0.053626	14.29	0.0000
EGARCH(Theta1)	-0.075240	0.027518	-2.734	0.0064
EGARCH(Theta2)	0.251740	0.067992	3.703	0.0002

```

No. Observations :      864  No. Parameters   :          7
Mean (Y)          :  0.01189  Variance (Y)    :  0.00439
Skewness (Y)     : -0.22061  Kurtosis (Y)   :  5.05331
Log Likelihood   : 1157.004

```

Warning : To avoid numerical problems, the estimated parameter Cst(V), and its std.Error have been multiplied by 10<sup>4</sup>.

From the output, the estimate of ARCH coefficient is insignificant with p-value of 0.28. For simplicity, we ignore it and the fitted model reduces approximately to

$$\begin{aligned}
 r_t &= 0.011 + 0.092r_{t-1} + a_t, & a_t &= \sigma_t \epsilon_t, \\
 \ln(\sigma_t^2) &= -5.496 + \frac{g(\epsilon_{t-1})}{(1 - 0.767B)}, \\
 g(\epsilon_{t-1}) &= -0.075\epsilon_{t-1} + 0.252[|\epsilon_{t-1}| - 0.8].
 \end{aligned}$$

This is close to what we have before.

**Remark:** Before you are comfortable with changing commands in R for EGARCH model estimation, you may use the GJR model discussed below to estimate the leverage effect.

**The Threshold GARCH (TGARCH) or GJR Model** A TGARCH( $s, m$ ) or GJR( $s, m$ ) model is defined as

$$r_t = \mu_t + a_t, \quad a_t = \sigma_t \epsilon_t \sigma_t = \alpha_0 + \sum_{i=1}^s (\alpha_i + \gamma_i N_{t-i}) a_{t-i}^2 + \sum_{j=1}^m \beta_j \sigma_{t-j}^2,$$

where  $N_{t-i}$  is an indicator variable such that

$$N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

One expects  $\gamma_i$  to be positive so that prior negative returns have higher impact on the volatility.

## R demonstration

```
> m2=garch0xFit(formula.mean=~arma(1,0),formula.var=~gjr(1,1),series=ibm)
*****
** SPECIFICATIONS **
*****
Dependent variable : X
Mean Equation : ARMA (1, 0) model.
No regressor in the mean
Variance Equation : GJR (1, 1) model.
No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1168.27
```

```
Maximum Likelihood Estimation (Std.Errors based on Numerical OPG matrix)
      Coefficient Std.Error  t-value  t-prob
Cst(M)      0.012261  0.0024782   4.948  0.0000
AR(1)       0.108345  0.038208   2.836  0.0047
Cst(V)      3.976257   1.1618    3.422  0.0006
ARCH(Alpha1) 0.053328  0.024655   2.163  0.0308
GARCH(Beta1) 0.806274  0.044067  18.30  0.0000
GJR(Gamma1) 0.090895  0.033665   2.700  0.0071
```

```
No. Observations :      864  No. Parameters :      6
```

Mean (Y) : 0.01189 Variance (Y) : 0.00439  
 Skewness (Y) : -0.22061 Kurtosis (Y) : 5.05331  
 Log Likelihood : 1168.266

Warning : To avoid numerical problems, the estimated parameter Cst(V), and its std.Error have been multiplied by 10<sup>4</sup>.

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\*\* FORECASTS \*\*

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Number of Forecasts: 15

Horizon	Mean	Variance
1	0.005999	0.005012
2	0.01158	0.004536
3	0.01219	0.004106
4	0.01225	0.003716
5	0.01226	0.003363
...		
15	0.01226	0.00124

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\*\* TESTS \*\*

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	Statistic	t-Test	P-Value
Skewness	0.0051867	0.062348	0.95029
Excess Kurtosis	0.98490	5.9264	3.0957e-009
Jarque-Bera	34.925	.NaN	2.6070e-008

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 Information Criterium (to be minimized)

Akaike	-2.690430	Shibata	-2.690526
Schwarz	-2.657364	Hannan-Quinn	-2.677774

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 Q-Statistics on Standardized Residuals

--> P-values adjusted by 1 degree(s) of freedom

Q( 10) = 6.42925 [0.6963067]

Q( 15) = 12.4119 [0.5732594]

Q( 20) = 20.8502 [0.3451387]

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 Q-Statistics on Squared Standardized Residuals

--> P-values adjusted by 2 degree(s) of freedom

Q( 10) = 2.87912 [0.9417129]

Q( 15) = 8.19737 [0.8305081]

Q( 20) = 10.4124 [0.9176103]

For the series of monthly IBM log returns, the fitted GJR model is

$$\begin{aligned} r_t &= 0.012 + 0.108r_{t-1} + a_t, & a_t &= \sigma_t \epsilon_t \\ \sigma_t^2 &= 3.98 \times 10^{-4} + (.053 + .091N_{t-1})a_{t-1}^2 + .806\sigma_{t-1}^2, \end{aligned}$$

where all estimates are significant, and model checking indicates that the fitted model is adequate.

The sample variance of the IBM log returns is about 0.005 and the empirical 2.5% percentile of the data is about  $-0.119$ . If we use these two quantities for  $\sigma_{t-1}^2$  and  $a_{t-1}$ , respectively, then we have

$$\frac{\sigma_t^2(-)}{\sigma_t^2(+)} = \frac{0.0004 + 0.144 \times 0.119^2 + 0.806 \times 0.005}{0.0004 + 0.051 \times 0.119^2 + 0.806 \times 0.005} = 1.26.$$

In this particular case, the negative prior return has about 26% higher impact on the conditional variance.

### The CHARMA model

Make use of “interaction” btw past shocks

A CHARMA model is defined as

$$r_t = \mu_t + a_t, a_t = \delta_{1t}a_{t-1} + \delta_{2t}a_{t-2} + \cdots + \delta_{mt}a_{t-m} + \eta_t,$$

where  $\{\eta_t\}$  is iid  $N(0, \sigma_\eta^2)$ ,  $\{\boldsymbol{\delta}_t\} = \{(\delta_{1t}, \dots, \delta_{mt})'\}$  is a sequence of iid random vectors  $D(\mathbf{0}, \boldsymbol{\Omega})$ ,  $\{\boldsymbol{\delta}_t\} \perp \{\eta_t\}$ .

The model can be written as

$$a_t = \mathbf{a}'_{t-1} \boldsymbol{\delta}_t + \eta_t,$$

with conditional variance

$$\begin{aligned} \sigma_t^2 &= \sigma_\eta^2 + \mathbf{a}'_{t-1} \text{Cov}(\boldsymbol{\delta}_t) \mathbf{a}_{t-1} \\ &= \sigma_\eta^2 + (a_{t-1}, \dots, a_{t-m}) \boldsymbol{\Omega} (a_{t-1}, \dots, a_{t-m})'. \end{aligned}$$

**Example:** Monthly excess returns of S&P 500 index (26-91).

A fitted model is

$$r_t = 0.0068 + a_t,$$

$$\sigma_t^2 = .00136 + (a_{t-1}, a_{t-2}, a_{t-3})\widehat{\mathbf{\Omega}}(a_{t-1}, a_{t-2}, a_{t-3})'$$

where, std errors in parentheses,

$$\widehat{\mathbf{\Sigma}} = \begin{bmatrix} 0.121(.036) & -0.062(.028) & 0 \\ -0.062(.028) & 0.191(.025) & 0 \\ 0 & 0 & 0.299(0.042) \end{bmatrix}.$$

### Effects of explanatory variables

Can be used in the same manner, i.e. with random coefs.

### RCA model

A time series  $r_t$  is a RCA( $p$ ) model if

$$r_t = \phi_0 + \sum_{i=1}^p (\phi_i + \delta_{it})r_{t-i} + a_t.$$

For the model, we have

$$\mu_t = E(a_t | F_{t-1}) = \sum_{i=1}^p \phi_i a_{t-i},$$

$$\sigma_t^2 = \sigma_a^2 + (r_{t-1}, \dots, r_{t-p})\mathbf{\Omega}_\delta(r_{t-1}, \dots, r_{t-p})'.$$

### Stochastic volatility model

A (simple) SV model is

$$a_t = \sigma_t \epsilon_t, (1 - \alpha_1 B - \dots - \alpha_m B^m) \ln(\sigma_t^2) = \alpha_0 + v_t$$

where  $\epsilon_t$ 's are iid  $N(0, 1)$ ,  $v_t$ 's are iid  $N(0, \sigma_v^2)$ ,  $\{\epsilon_t\}$  and  $\{v_t\}$  are independent.

### Long-memory SV model

A simple LMSV is

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t = \sigma \exp(u_t/2), \quad (1 - B)^d u_t = \eta_t$$

where  $\sigma > 0$ ,  $\epsilon_t$ 's are iid  $N(0, 1)$ ,  $\eta_t$ 's are iid  $N(0, \sigma_\eta^2)$  and independent of  $\epsilon_t$ , and  $0 < d < 0.5$ .

The model says

$$\begin{aligned}\ln(a_t^2) &= \ln(\sigma^2) + u_t + \ln(\epsilon_t^2) \\ &= [\ln(\sigma^2) + E(\ln \epsilon_t^2)] + u_t + [\ln(\epsilon_t^2) - E(\ln \epsilon_t^2)] \\ &\equiv \mu + u_t + e_t.\end{aligned}$$

Thus, the  $\ln(a_t^2)$  series is a Gaussian long-memory signal plus a non-Gaussian white noise; see Breidt, Crato and de Lima (1998).

### **Application**

see Examples 3.4 & 3.5