

Lecture Note of Bus 41202, Spring 2008: Value at Risk & Risk Management

Classification of Financial Risk

1. Credit risk
2. Market risk
3. Operational risk

We start with the market risk, because

- more data are available
- easier to understand
- the idea applicable to other types of risk.

What is Value at Risk (VaR)?

- a measure of minimum loss of a given asset
- amount a position could decline in a given period, associated with a given probability (or confidence level)

A formal definition:

- time period given: $\Delta t = \ell$
- change in value: $\Delta V(\ell)$
- CDF of the change $F_\ell(x)$

- given (tail) probability: p
- a long position:

$$p = Pr[\Delta V(\ell) \leq \text{VaR}] = F_\ell(\text{VaR}).$$

- a short position:

$$\begin{aligned} p &= Pr[\Delta V(\ell) \geq \text{VaR}] \\ &= 1 - Pr[\Delta V(\ell) \leq \text{VaR}] \\ &= 1 - F_\ell(\text{VaR}). \end{aligned}$$

We can study VaR by simply focusing on the loss function.

Quantile: x_p is the p th quantile of $F_\ell(x)$ if

$$p = F_\ell(x_p)$$

and $F_\ell(\cdot)$ is continuous.

Factors affect VaR:

1. the probability p .
2. the time horizon ℓ .
3. data frequency.
4. the CDF $F_\ell(x)$. (or CDF of loss)
5. the mark-to-market value of the position.

Why use log returns?

log returns \approx percentage changes.

VaR = Value \times (VaR of log return).

Methods available for market risk

1. RiskMetrics
2. Econometric modeling
3. Empirical quantile
4. Traditional extreme value theory (EVT)
5. EVT based on exceedance over a high threshold

Data used in illustrations:

Daily log returns of IBM stock

- span: July 3, 62 to Dec. 31, 98.
- size: 9190 points

Position: long on \$10 millions.

RiskMetrics

- Developed by J.P. Morgan
- r_t given F_{t-1} : $N(0, \sigma_t^2)$
- σ_t^2 follows the special IGARCH(1,1) model

$$\sigma_t^2 = \alpha \sigma_{t-1}^2 + (1 - \alpha) r_{t-1}^2, \quad 1 > \alpha > 0.$$

- VaR = $1.65\sigma_t$ if $p = 0.05$.
- k -horizon: $\text{VaR}[k] = \sqrt{k}\text{VaR}$

The square root of time rule

- Pros: simplicity and transparency
- Cons: model is not adequate

Example: IBM data

Model:

$$r_t = a_t, \quad a_t = \sigma_t \epsilon_t,$$

$$\sigma_t^2 = 0.9396\sigma_{t-1}^2 + (1 - 0.9396)a_{t-1}^2$$

Because $r_{9190} = -0.0128$ and $\hat{\sigma}_{9190}^2 = 0.0003472$,

$$\hat{\sigma}_{9190}^2(1) = 0.000336.$$

For $p = 0.05$, VaR of $r_t = -1.65 \times \sqrt{0.000336} = -0.03025$

$$\text{VaR} = \$10,000,000 \times 0.03025 = \$302,500.$$

For $p = 0.01$, VaR of $r_t = -2.3262 \times \sqrt{0.000336} = -0.04265$, and
 VaR = \$426,500.

Econometric models

- $r_t = \mu_t + a_t$ given F_{t-1}
- μ_t : a mean equation (Ch.2)
- σ_t^2 : a volatility model (Ch. 3 or 4)

- Pros: sound theory
- Cons: a bit complicated.

IBM data:

Case 1: Gaussian

$$r_t = 0.00066 - 0.0247r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = 0.00000389 + 0.0799a_{t-1}^2 + 0.9073\sigma_{t-1}^2.$$

From $r_{9189} = -0.00201$, $r_{9190} = -0.0128$ and $\sigma_{9190}^2 = 0.00033455$, we have

$$\hat{r}_{9190}(1) = 0.00071 \quad \text{and} \quad \hat{\sigma}_{9190}^2(1) = 0.0003211.$$

If $p = 0.05$, then

$$0.00071 - 1.6449 \times \sqrt{0.0003211} = -0.02877.$$

VaR = \$10,000,000 \times 0.02877 = \$287,700.

If $p = 0.01$, then the quantile is

$$0.00071 - 2.3262 \times \sqrt{0.0003211} = -0.0409738.$$

VaR = \$409,738.

Case 2: Student- t_5

$$r_t = 0.0003 - 0.0335r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = 0.000003 + 0.0559a_{t-1}^2 + 0.9350\sigma_{t-1}^2.$$

From the data, $r_{9189} = -0.00201$, $r_{9190} = -0.0128$ and $\sigma_{9190}^2 = 0.000349$, we have

$$\hat{r}_{9190}(1) = 0.000367 \quad \text{and} \quad \hat{\sigma}_{9190}^2(1) = 0.0003386.$$

If $p = 0.05$, then the quantile is

$$.000367 - (2.015/\sqrt{5/3})\sqrt{.0003386} = .000367 - 1.5608\sqrt{.0003386} = -.028354.$$

$$\text{VaR} = \$10,000,000 \times 0.028352 = \$283,520.$$

If $p = 0.01$, the quantile is

$$0.000367 - (3.3649/\sqrt{5/3})\sqrt{0.0003386} = -0.0475943.$$

$$\text{VaR} = \$475,943.$$

Discussion:

- Effects of heavy-tails seen with $p = 0.01$.
- Multiple step-ahead forecasts are needed.

Example 7.3 (continued). 15-day horizon.

$$\hat{r}_{9190}[15] = 0.00998 \quad \text{and} \quad \sigma_t[15] = 0.0047948.$$

$$\text{If } p = 0.05, \text{ the quantile is } 0.00998 - 1.6449\sqrt{0.0047948} = -0.1039191.$$

$$15\text{-day VaR} = \$10,000,000 \times 0.1039191 = \$1,039,191.$$

$$\text{RiskMetrics: VaR} = \$287,700 \times \sqrt{15} = \$1,114,257.$$

Empirical quantile

Sample of log returns: $\{r_t | t = 1, \dots, n\}$.

Order statistics:

$$r_{(1)} \leq r_{(2)} \leq \cdots \leq r_{(n)}$$

$r_{(i)}$ as the i th order statistic of the sample.

$r_{(1)}$ is the sample minimum

$r_{(n)}$ the sample maximum.

Idea: Use the empirical quantile to estimate the theoretical quantile of r_t .

For a given probability p , what is the empirical quantile?

If $np = \ell$ is an integer, then it is $r_{(\ell)}$.

If np is not an integer, find the two neighboring integers $\ell_1 < np < \ell_2$ and use interpolation.

The quantile is

$$\hat{x}_p = \frac{p_2 - p}{p_2 - p_1} r_{(\ell_1)} + \frac{p - p_1}{p_2 - p_1} r_{(\ell_2)}.$$

IBM data:

$n = 9190$. If $p = 0.05$, then $np = 459.5$.

5% quantile is $(r_{(459)} + r_{(460)})/2 = -0.021603$.

VaR = \$216,030.

If $p = 0.01$, then $np = 91.9$ and the 1% quantile is

$$\begin{aligned} \hat{x}_{0.01} &= \frac{p_2 - 0.01}{p_2 - p_1} r_{(91)} + \frac{0.01 - p_1}{p_2 - p_1} r_{(92)} \\ &= \frac{.00001}{.00011} (-3.658) + \frac{0.0001}{0.00011} (-3.657) \\ &\approx -3.65709. \end{aligned}$$

VaR is \$365,709.

Extreme value theory: Focus on the tail behavior of r_t .

Review of extreme value theory

A properly normalized $r_{(1)}$ assumes a special distribution:

$$F_*(x) = \begin{cases} 1 - \exp[-(1 + kx)^{1/k}] & \text{if } k \neq 0 \\ 1 - \exp[-\exp(x)] & \text{if } k = 0 \end{cases}$$

for $x < -1/k$ if $k < 0$ and for $x > -1/k$ if $k > 0$.

k : the *shape parameter*

$\alpha = -1/k$: tail index of the distribution.

Classification of distributions:

- Type I: $k = 0$, the Gumbel family. The CDF is

$$F_*(x) = 1 - \exp[-\exp(x)], \quad -\infty < x < \infty. \quad (1)$$

- Type II: $k < 0$, the Fréchet family. The CDF is

$$F_*(x) = \begin{cases} 1 - \exp[-(1 + kx)^{1/k}] & \text{if } x < -1/k \\ 1 & \text{otherwise.} \end{cases} \quad (2)$$

- Type III: $k > 0$, the Weibull family. The CDF here is

$$F_*(x) = \begin{cases} 1 - \exp[-(1 + kx)^{1/k}] & \text{if } x > -1/k \\ 0 & \text{otherwise.} \end{cases}$$

The probability density function (pdf) of the normalized minimum is

$$f_*(x) = \begin{cases} (1 + kx)^{1/k-1} \exp[-(1 + kx)^{1/k}] & \text{if } k \neq 0 \\ \exp[x - \exp(x)] & \text{if } k = 0 \end{cases}$$

where $-\infty < x < \infty$ for $k = 0$, $x < -1/k$ for $k < 0$ and $x > -1/k$ for $k > 0$.

How to use the EVT distribution?

If we know the three parameters, we can compute the quantile!

Empirical estimation

Divide the sample into non-overlapping subsamples.

Suppose there are T data points, we divide the data as

$$\{r_1, \dots, r_n | r_{n+1}, \dots, r_{2n} | r_{2n+1}, \dots, r_{3n} | \dots | r_{(g-1)n+1}, \dots, r_{ng}\},$$

n: size of subgroup

Idea: find the minimum of each subgroup. These minima are the data used to estimate the three parameters.

Several estimation methods available. We use maximum likelihood estimates.

IBM data:

n	g	Scale α_n	Location β_n	Shape Par. k_n
(a) Minimal returns				
21	437	0.823(0.035)	-1.902(0.044)	-0.197(0.036)
63	145	0.945(0.077)	-2.583(0.090)	-0.335(0.076)
126	72	1.147(0.131)	-3.141(0.153)	-0.330(0.101)
252	36	1.542(0.242)	-3.761(0.285)	-0.322(0.127)
(b) Maximal returns				
21	437	0.931(0.039)	2.184(0.050)	-0.168(0.036)
63	145	1.157(0.087)	3.012(0.108)	-0.217(0.066)
126	72	1.292(0.158)	3.471(0.181)	-0.349(0.130)
252	36	1.624(0.271)	4.475(0.325)	-0.264(0.186)

EVT to VaR: Use a two-step procedure, because of the division into subgroup.

VaR for r_t :

$$\text{VaR} = \begin{cases} \beta_n - \frac{\alpha_n}{k_n} \left\{ 1 - [-n \ln(1 - p)]^{k_n} \right\} & \text{if } k_n \neq 0 \\ \beta_n + \alpha_n \ln[-n \ln(1 - p)] & \text{if } k_n = 0. \end{cases}$$

For IBM data, if $n = 63$ (quarterly minima), then $\hat{\alpha}_n = 0.945$, $\hat{\beta}_n = -2.583$, and $\hat{k}_n = -0.335$. If $p = 0.01$, the VaR is

$$\begin{aligned} \text{VaR} &= -2.583 - \frac{0.945}{-0.335} \left\{ 1 - [-63 \ln(1 - 0.01)]^{-0.335} \right\} \\ &= -3.04969 \end{aligned}$$

VaR is \$304,969.

If $p = 0.05$, then VaR is \$166,641.

For $n = 21$, the results are:

VaR = \$340,013 for $p = 0.01$;

VaR = \$184,127 for $p = 0.05$.

Discussion:

- Results depend on the choice of n
- VaR seems low, but it might be due to the choice of p .

If $p = 0.001$, then

VaR = \$546,641 for the Gaussian AR(2)-GARCH(1,1) model

VaR = \$666,590 for the extreme value theory with $n = 21$.

Summary of IBM data:

Position = \$10 millions.

If $p = 0.05$, then

1. \$302,500 for the RiskMetrics,
2. \$287,200 for an AR(2)-GARCH(1,1) model,
3. \$283,520 for an AR(2)-GARCH(1,1) with t_5
4. \$216,030 using the empirical quantile, and
5. \$184,127 for EVT with $n = 21$.

$p = 0.01$, then

1. \$426,500 for the RiskMetrics,

2. \$409,738 for an AR(2)-GARCH(1,1) model,
3. \$475,943 for an AR(2)-GARCH(1,1) model with t_5
4. \$365,800 for empirical quantile, and
5. \$340,013 for EVT with $n = 21$.

If $p = 0.001$, then

1. \$566,443 for the RiskMetrics,
2. \$546,641 for an AR(2)-GARCH(1,1) model,
3. \$836,341 for an AR(2)-GARCH(1,1) model with t_5
4. \$780,712 for quantile, and
5. \$666,590 for EVT with $n = 21$.

Multi-period VaR with EVT

$$\text{VaR}(\ell) = \ell^{1/\alpha}\text{VaR} = \ell^{-k}\text{VaR}$$

where α is the tail index and k is the shape parameter.

For IBM data with $p = 0.05$ and $n = 21$,

$$\text{VaR}(30) = (30)^{0.197}\text{VaR} = 1.954 \times \$184,127 = \$359,841.$$

New approach to VaR

Based on Exceedances over a high threshold

Idea: frequency of big returns and their magnitudes are important.

Statistical theory:

Two-dimensional Poisson process

Two possible cases:

Homogeneous: parameters are fixed over time

Non-homogeneous case: parameters are time-varying, according to some explanatory variables.

IBM data: homogeneous model

Thr.	Exc.	Shape Par. k	Log(Scale) $\ln(\alpha)$	Location β
(a) Original log returns				
3.0%	175	-0.30697(0.09015)	0.30699(0.12380)	4.69204(0.19058)
2.5%	310	-0.26418(0.06501)	0.31529(0.11277)	4.74062(0.18041)
2.0%	554	-0.18751(0.04394)	0.27655(0.09867)	4.81003(0.17209)
(b) Removing the sample mean				
3.0%	184	-0.30516(0.08824)	0.30807(0.12395)	4.73804(0.19151)
2.5%	334	-0.28179(0.06737)	0.31968(0.12065)	4.76808(0.18533)
2.0%	590	-0.19260(0.04357)	0.27917(0.09913)	4.84859(0.17255)

VaR calculation:

$$\text{VaR} = \begin{cases} \beta + \frac{\alpha}{k} \{1 - [-T \ln(1 - p)]^k\} & \text{if } k \neq 0 \\ \beta + \alpha \ln[-T \ln(1 - p)] & \text{if } k = 0 \end{cases}$$

where $T = 252$, the number trading days in a year.

IBM data: VaR of 5% & 1%

- Case I: original returns
 1. $\eta = 3.0\%$: \$228,239 & \$359.303.
 2. $\eta = 2.5\%$: \$219,106 & \$361,119.
 3. $\eta = 2.0\%$: \$212,981 & \$368.552.
- Case II: remove sample mean
 1. $\eta = 3.0\%$: \$232,094 & \$363,697.
 2. $\eta = 2.5\%$: \$225,782 & \$364,254.
 3. $\eta = 2.0\%$: \$217,740 & \$372,372.

Non-homogeneous case:

$$\begin{aligned}k_t &= \gamma_0 + \gamma_1 x_{1t} + \cdots + \gamma_v x_{vt} \equiv \gamma_0 + \boldsymbol{\gamma}' \mathbf{x}_t \\ \ln(\alpha_t) &= \delta_0 + \delta_1 x_{1t} + \cdots + \delta_v x_{vt} \equiv \delta_0 + \boldsymbol{\delta}' \mathbf{x}_t \\ \beta_t &= \theta_0 + \theta_1 x_{1t} + \cdots + \theta_v x_{vt} \equiv \theta_0 + \boldsymbol{\theta}' \mathbf{x}_t.\end{aligned}$$

For IBM data, explanatory variables include past volatilities, etc.

See Chapter 7 for more details and estimation results.

Illustration:

For December 31, 1998, we have $x_{3,9190} = 0$, $x_{4,9190} = 0.9737$ and $x_{5,9190} = 1.9766$. The parameters become

$$k_{9190} = -0.01195, \quad \ln(\alpha_{9190}) = 0.19331, \quad \beta_{9190} = 6.105.$$

If $p = 0.05$, then quantile = 3.03756% and

$$\text{VaR} = \$10,000,000 \times 0.0303756 = \$303,756.$$

If $p = 0.01$, then Var is \$497,425.

For December 30, 1998, we have $x_{3,9189} = 1$, $x_{4,9189} = 0.9737$ and $x_{5,9189} = 1.8757$ and

$$k_{9189} = -0.2500, \quad \ln(\alpha_{9189}) = 0.52385, \quad \beta_{9189} = 5.8834.$$

The 5% VaR becomes

$$\text{VaR} = \$10,000,000 \times 0.0269139 = \$269,139.$$

If $p = 0.01$, then VaR becomes \$448,323.

R and S-Plus Demonstration:

Both packages use the library: **evir**. For R, download the library first before using the commands.

```
*** Please note that the shape parameter "k" of chapter 7
    is denoted by "minus xi" in evir.
*** Also, the program ‘‘evir’’ uses maxima (right tail) so that
    one should use ‘‘minus returns’’ for the left tail.
***
(* Command line starts with > *)
(* Output is edited to simplify the handout *)

(* Generate CDF of Weibull, Frechet, & Gumbel dists *)
> z=seq(-5,5,length=200)
```

```

<= just get a sequence of numbers in [-5,5],equally spaced.
> z[1:10]
[1] -5.0000 -4.94975 -4.89949 -4.84925 -4.79899 -4.74874
[7] -4.698492 -4.648241 -4.597990 -4.547739

```

```

(* Use the command "pgev" to obtain the probability
  (CDF) of generalized extreme value dist. *)
> cdf.f=ifelse((z > -2),pgev(z,xi=0.5),0)
  <== Frechet dist for z > -2 only, because xi=0.5.
> cdf.w=ifelse((z < 2), pgev(z,xi=-0.5),1)
  <== Weibull dist for z < 2 only.
> cdf.g=exp(-exp(-z))

> plot(z,cdf.w,type='l',xlab='z',ylab='H(z)')
> lines(z,cdf.f,type='l',lty=2)
> lines(z,cdf.g,lty=3)
> legend(-5,1,legend=c("Weibull H(-0.5,0,1)",
  "Frechet H(0.5,0.1)","Gumbel H(0,0,1)"),lty=1:3)

```

```

(*Use the command "dgev" to obtain the pdf of
  generalized extreme value dist *)
> pdf.f=ifelse((z > -2),dgev(z,xi=0.5),0)
> pdf.w=ifelse((z < 2),dgev(z,xi=-0.5),0)
> pdf.g=exp(-exp(-z))*exp(-z)
> plot(z,pdf.w,type='l',xlab='z',ylab='density')
> lines(z,pdf.f,lty=2)
> lines(z,pdf.g,lty=3)
> legend(-5.25,0.4,legend=c("Weibull H(-0.5,0,1)",
  "Frechet H(0.5,0,1)","Gumbel H(0,0,1)"),lty=1:3)

```

```
>
```

```
(* Other related commands include "qgev" and "rgev"
   that gives quantiles and generates random
   variates from generalized extreme value
   distribution. The parameters must be given in
   using these two commands. *)
```

```
(* For example, to obtain the 95th quantile, use below *)
```

```
> qgev(0.95,xi=0.5,mu=0,sigma=1) % obtain quantile
[1] 6.830793
```

Example: daily log returns, in percentages, of IBM stock: 1962 to 1998.

```
> library(evir)
> da=read.table("d-ibmln98.dat")
> ibm=da[,1]
> plot(ibm,type='l')
> qqnorm(ibm) % normal probability plot

> nibm=-ibm % Focus on the left tail.

> m1=gev(nibm,block=21) % fit gen. extreme value dist.
> m1
$n.all
[1] 9190
$n
[1] 438
$data
[1]3.288 3.619 3.994 3.864 1.824 2.161 1.578 1.535 0.911 1.019
.....
```

```
[431] 3.502 2.501 4.576 8.456 3.723 1.082 2.910 3.099
```

```
$block
```

```
[1] 21
```

```
$par.ests
```

```
      xi      sigma      mu
0.1956199 0.8239793 1.9031998
```

```
$par.ses
```

```
      xi      sigma      mu
0.03554473 0.03476737 0.04413629
```

```
$varcov
```

```
      [,1]      [,2]      [,3]
[1,] 1.263428e-03 -2.782725e-05 -0.0004338483
[2,] -2.782725e-05 1.208770e-03 0.0008475859
[3,] -4.338483e-04 8.475859e-04 0.0019480124
```

```
$converged
```

```
[1] 0
```

```
$nllh.final
```

```
[1] 654.3337
```

```
attr(,"class")
```

```
[1] "gev"
```

```
> names(m1)
```

```
[1] "n.all" "n" "data" "block" "par.ests"
```

```
[6] "par.ses" "varcov" "converged" "nllh.final"
```

```
> m1$n % numbers of monthly maximum
```

```

[1] 438
> ymax=m1$data
> hist(ymax)
> ysort=sort(ymax)
> plot(ysort,-log(-log(ppoints(ysort))),xlab='Monthly maximum')
      % Gumbel qq-plot

> plot(m1) % Model chekcing plots
Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 1

Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 2

Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 0

> 1-pgev(max(ymax),xi=.196,mu=1.90,sigma=.824)
[1] 5.857486e-05 % Prob. that the drop will exceed the maximum.

> rlevel.gev(m1,k.blocks=36) %return level & its 95% conf. interval.
[1] 5.568969 6.158516 6.980167

```

** Peak Over the Threshold approach (homogeneous case).

```
> meplot(nibm) % mean excess plot
```

```
> m2=gpd(nibm,threshold=2.5)
```

```
> names(m2)
```

```
[1] "n"          "data"       "threshold"  "p.less.thresh"  
[5] "n.exceed"  "method"     "par.ests"   "par.ses"  
[9] "varcov"    "information" "converged"  "nllh.final"
```

```
> m2$threshold
```

```
[1] 2.5
```

```
> m2$n.exceed
```

```
[1] 310
```

```
> m2 % Obtain all output
```

```
$n
```

```
[1] 9190
```

```
$data
```

```
[1] 3.288 2.649 2.817 3.619 3.994 3.795 3.784 2.623 3.864 3.149
```

```
.....
```

```
[301]2.501 2.835 4.576 4.393 8.456 2.916 3.723 2.910 2.654 3.099
```

```
$threshold
```

```
[1] 2.5
```

```
$p.less.thresh
```

```
[1] 0.9662677
```

```
$n.exceed
```

```
[1] 310
```

```
$method
```

```
[1] "m1"
```

```
$par.ests
```

```
      xi      beta  
0.2641593 0.7786761
```

```
$par.ses
```

```
      xi      beta  
0.06659234 0.06714131
```

```
$varcov
```

```
          [,1]      [,2]  
[1,] 0.004434540 -0.002614442  
[2,] -0.002614442 0.004507955
```

```
$information
```

```
[1] "observed"
```

```
$converged
```

```
[1] 0
```

```
$nllh.final
```

```
[1] 314.375
```

```
attr("class")
```

```
[1] "gpd"
```

```
> plot(m2) % Model checking. Should see all plots.
```

Make a plot selection (or 0 to exit):

1: plot: Excess Distribution

2: plot: Tail of Underlying Distribution

3: plot: Scatterplot of Residuals

```
4: plot: QQplot of Residuals
Selection: 1
[1] "threshold = 2.5 xi = 0.264 scale = 0.779 location= 2.5"
```

Make a plot selection (or 0 to exit):

```
1: plot: Excess Distribution
2: plot: Tail of Underlying Distribution
3: plot: Scatterplot of Residuals
4: plot: QQplot of Residuals
Selection: 0
```

```
> shape(nibm) % A plot showing the stability of the estimates.
```

```
> riskmeasures(m2,c(0.95,0.99)) % Compute VaR and expected shortfall.
      p quantile      sfall
[1,] 0.95 2.208932 3.162654
[2,] 0.99 3.616487 5.075507
```

Additional information on applying extreme value theory to value at risk calculation.

To traditional approach of EVT

Return Level: It is a risk measure based on the idea of subperiods. The g n -subperiod return level, $L_{n,g}$, is the level that is *exceeded* in one out of every g subperiods of length n .

$$P(r_{n,i} < L_{n,g}) = \frac{1}{g},$$

where n is the length of subperiod used in estimating the GEV model

and $r_{n,i}$ denotes subperiod minimum. For sufficiently large n ,

$$L_{n,g} = \beta_n + \frac{\alpha_n}{k_n} \{ [-\ln(1 - 1/g)]^{k_n} - 1 \},$$

where the shape parameter $k_n \neq 0$.

For a short position, the return level is

$$L_{n,g} = \beta_n + \frac{\alpha_n}{k_n} \{ 1 - [-\ln(1 - 1/g)]^{1/k_n} \}.$$

Peaks over the Threshold (POT)

Generalized Pareto Distribution: For simplicity, assume that the shape parameter $k \neq 0$. Consider the extreme value distribution of *maximum* (Eq. (7.29) of the textbook)

$$F_*(r) = \exp \left[- \left(1 - \frac{k(r - \beta)}{\alpha} \right)^{1/k} \right].$$

The distribution of $r \leq x + \eta$ given $r > \eta$, where $x \geq 0$, is

$$\Pr(r \leq x + \eta | r > \eta) \approx 1 - \left(1 - \frac{kx}{\psi(\eta)} \right)^{1/k},$$

where $\psi(\eta) = \alpha - k(\eta - \beta)$, which depends on η .

The distribution with cumulative distribution function

$$G(x) = 1 - \left[1 - \frac{kx}{\psi(\eta)} \right]^{1/k},$$

is called a generalized Pareto distribution (GPD).

Selection of the high threshold

Mean Excess: Given a high threshold η_o , suppose the excess $r - \eta_o$ follows a GPD with parameter k and $\psi(\eta_o)$, where $0 > k > -1$. Then the mean excess over the threshold is

$$E(r - \eta_o | r > \eta_o) = \frac{\psi(\eta_o)}{1 + k}.$$

For any $\eta > \eta_o$, the mean excess function is defined as

$$e(\eta) = E(r - \eta | r > \eta) = \frac{\psi(\eta_o) - k(\eta - \eta_o)}{1 + k}.$$

The fact that, for a given k , $e(\eta)$ is a linear function of η , where $\eta > \eta_o$, provides a simple method to infer the threshold η_o for GPD. Define the empirical mean excess as

$$e_T(\eta) = \frac{1}{N_\eta} \sum_{i=1}^{N_\eta} (r_{t_i} - \eta),$$

where N_η is the number of returns that exceed η and r_{t_i} are the values of the corresponding returns.

The scatterplot $e_T(\eta)$ versus η is called the mean excess plot, which should be linear for $\eta > \eta_o$.

In R or S-Plus, the command is **meplot**.

Use of GPD in VaR

For a given threshold, estimate GPD to obtain parameters k and $\psi(\eta)$. Check the adequacy of the fit; see demonstration. Provided that the model is adequate, the VaR can be computed by

$$\text{VaR}_q = \eta + \frac{\psi(\eta)}{k} \left\{ 1 - \left[\frac{T}{N_\eta} (1 - q) \right]^k \right\},$$

where $q = 1 - p$ with $0 < p < 0.05$, T is the sample size and N_η is the number of exceedances.

Alternatively, one can use the formula in Eq. (7.38) of the textbook when one treats the exceedances and exceeding times as a two-dimensional Poisson process. The VaR results obtained are close.

Expected Shortfall (ES): the expected loss given that the VaR is exceeded. Specifically,

$$EES_q = E(r|r > \text{VaR}_q) = \text{VaR}_q + E(r - \text{VaR}_q|r > \text{VaR}_q).$$

For GPD, it turns out that

$$ES_q = \frac{\text{VaR}_q}{1+k} + \frac{\psi(\eta) + k\eta}{1+k}.$$

In **evir**, the command is **riskmeasures**.

Let $r_{n,i}$ be the maximum of a subperiod of length n . Under the traditional EVT, $r_{n,i}$ follows a generalized extreme value distribution with parameter (xi,sigma,mu).

What is the relationship between quantile of $r_{n,i}$ and the return r_t ?

Let Q be a real number.

$$\begin{aligned} P(r_{n,i} > Q) &= 1 - P(r_{n,i} \leq Q) \\ &= 1 - P(\text{all } r_t \text{ in the subperiod} \leq Q) \\ &= 1 - \prod_{t=1}^n P(r_t \leq Q) \quad (\text{use independence}) \\ &= 1 - [P(r_t \leq Q)]^n \quad (\text{because of same distribution}) \end{aligned}$$

Consequently, let p be a small upper tail probability of r_t and Q be the corresponding quantile. That is,

$$P(r_t \leq Q) = 1 - p$$

From the above equation, we have

$$P(r_{n,i} > Q) = 1 - (1 - p)^n.$$

Therefore,

$$P(r_{n,i} \leq Q) = 1 - P(r_{n,i} > Q) = (1 - p)^n.$$

This means that Q is the $(1 - p)^n$ -th quantile of the generalized extreme value distribution.

Takeaway: For a small probability p , compute $(1 - p)^n$, where n is the length of subperiod, then VaR can be obtained by finding the $(1 - p)^n$ -th quantile of the extreme value distribution.

Credit Risk:

Reference: *Credit Risk Measurement: New Approaches to Value at Risk and Other Paradigms*, 2nd Edition, by Anthony Saunders and Linda Allen, Wiley, 2002.

Some techniques for credit risk measurement

1. Long-term credit rating (High to Low)

S&P	Moody	Fitch
AAA	Aaa	AAA
AA	Aa	AA
A	A	A
BBB	Baa	BBB
BB	Ba	BB
B	B	B
CCC	Caa	CCC
CC	Ca	CC
C	C	C
D	D	D

2. Credit quality over time (transition)

S&P One-year transition matrix

(Source: Standard & Poor's, Feb. 1997)

Ini. Rat.	Rating at year-end(%)							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	88.5	8.05	0.72	0.06	0.11	0.00	0.00	0.00
AA	0.76	88.3	7.47	0.56	0.05	0.13	0.02	0.00
A	0.08	2.32	87.6	5.02	0.65	0.22	0.01	0.05
BBB	0.03	0.29	5.54	82.5	4.68	1.02	0.11	0.17
BB	0.02	0.11	0.58	7.01	73.8	7.4	0.89	0.98
B	0.00	0.09	0.21	0.39	5.98	72.8	3.42	4.92
CCC	0.17	0.00	0.34	1.02	2.20	9.64	53.13	19.21

3. CreditMetrics (JP Morgan & other sponsors)

4. Altman Z score (Mainly is U.S.)

$$\begin{aligned} Z = & 3.3(\text{Earnings before Interest and Taxes [EBIT]}/\text{Totl Assets}) \\ & + 1.0(\text{Sales}/\text{Total Assets}) \\ & + 0.6(\text{Market Value of Equity}/\text{Book Value of Debt}) \\ & + 1.4(\text{Retained Earnings}/\text{Total Assets}) \\ & + 1.2(\text{Working Capital}/\text{Total Assets}) \end{aligned}$$

5. KMV Corporation's credit risk model (now merged with Moody)

CreditMetrics: developed by J.P. Morgan and other sponsors in 1997.

Simply put, CreditMetrics addresses the question:

“How much will one lose on his loans and loan portfolios next year for a given confidence level?”

From the assessment of market risk, the current market value and its volatility of a financial position play an imporant role in VaR calculation. Application of VaR methodology to *nontrabable loans* encounters some immediate problems:

1. The current market value of the loan is not directly observable, because most loans are not traded.
2. No time-series data available to estimate the volatility.

To overcome the difficulties, we make use of

1. Available data on a borrower's credit rating
2. The probability that the rating will change over the next year (the rating transition matrix)
3. Recovery rates on defaulted loans
4. Credit spreads and yields in the bond (or loan) market.

Example: Consider a five-year fixed-rate loan of \$100 million made at 6% annual interest, and the borrower is rated BBB.

Note: The numerical numbers used in this example are from Chapter 6 of the reference book cited above.

Rating migration: One-year transition probabilities for BBB-rated borrower

AAA	AA	A	BBB	BB	B	CCC	Defaulty
0.02	0.33	5.95	86.93	5.30	1.17	0.12	0.18

Valuation Rating change (upgrades and downgrades) will affect the required credit risk spreads or premiums on the loan's remaining cash flows and, hence, the implied market value of the loan.

Downgrade → credit spread premium rises → present value of the loan should fall.

Upgrade has the opposite effect.

return to the example. (after one-year and a credit rating change)

$$P = 6 + \frac{6}{1 + r_{1,1} + s_1} + \frac{6}{(1 + r_{1,2} + s_2)^2} + \frac{6}{(1 + r_{1,3} + s_3)^3} + \frac{106}{(1 + r_{1,4} + s_4)^4},$$

where $r_{1,i}$ are the risk-free rates on zero-coupon U.S. Treasury bonds expected to exist one year into the future and s_i is the annual credit spread on loans of a particular rating class of 1-year, 2-year, 3-year and 4-year maturities (derived from observed spreads in the corporate bond market over Treasuries).

One-year forward zero curves plus credit spreads by credit rating category:

Category	Year 1	Year 2	Year 3	Year 4
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.72	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
BB	5.55	6.02	6.78	7.27
B	6.05	7.02	8.03	8.52
CCC	15.05	15.02	14.03	13.52

Suppose that, during the first year, the borrower gets upgraded from BBB to A. The present value of the loan is

$$P = 6 + \frac{6}{1.0372} + \frac{6}{(1.0432)^2} + \frac{6}{(1.0493)^3} + \frac{106}{(1.0532)^4} = \$108.66.$$

Value of the loan at the end of Year 1, under different rating changes (including first-year coupon):

Rating	AAA	AA	A	BBB	BB	B	CCC	Defaulty
value	109.37	109.19	108.66	107.55	102.02	98.10	83.64	51.13

Calculation of VaR

Year-end Rating	Probability of State(%)	New Loan		Difference	
		Value Plus Coupon (millions)	Probability Weighted Value(\$)	of Value from Mean (\$)	Probability Weighted Diff. Squared
AAA	0.02	109.37	0.02	2.28	0.0010
AA	0.33	109.19	0.36	2.10	0.0146
A	5.95	108.66	6.47	1.57	0.1474
BBB	86.93	107.55	93.49	0.46	0.1853
BB	5.30	102.02	5.41	(5.06)	1.3592
B	1.17	98.10	1.15	(8.99)	0.9446
CCC	0.12	83.64	1.10	(23.45)	0.6598
Default	0.18	51.13	0.09	(55.96)	5.6358

Form the table, the mean value of the loan is \$107.09 (sum of the 4-th column). The variance of the value is 8.9477 (sum of the last column).

Consequently, the standard deviation is $\sigma = \sqrt{8.9477} = 2.99$.

If normal distribution of the loan value is used,

- 5% VaR: $1.65 \times \sigma = \$4.93$
- 1% VaR: $2.33 \times \sigma = \$6.97$

If actual distribution is used,

- 6.77% VaR: $\$107.09 - 102.02 = \5.07
- 1.47% VaR: $\$107.09 - 98.10 = \8.99
- 1% VaR: $\$107.09 - 92.29 = \14.80 .

The 1% number 92.29 is obtained by interpolation as (1.47%, 98.10) and (0.3%, 83.64).