Key concept: Ito’s lemma

Stock Options:

• A contract giving its holder the right, but not obligation, to trade shares of a common stock by a certain date for a specified price. (In US, a contract involves 100 shares.)

• Call option: to buy
• Put option: to sell

• Specified price: strike price \( K \)
• date: expiration \( T \) (measured in years)

Note: You can also write a call or put option (underwrite).

Factors affecting the price of an option

• Current stock price: \( P_t \)
• time to expiration: \( T - t \)
• Risk-free interest rate: \( r \) per annum
• Stock volatility: \( \sigma \) annualized

Payoff for European options (exercised at \( T \) only)
Call option:

\[ V(P_T) = (P_T - K)_+ = \begin{cases} 
    P_T - K & \text{if } P_T > K \\
    0 & \text{if } P_T \leq K 
\end{cases} \]

The holder only exercises her option if \( P_T > K \) (buys the stock via exercising the option and sells the stock on the market).

Put option:

\[ V(P_T) = (K - P_T)_+ = \begin{cases} 
    K - P_T & \text{if } P_T < K \\
    0 & \text{if } P_T \geq K 
\end{cases} \]

The holder only exercises her option if \( P_T < K \) (buys the stock from the market and sells it via option).

**Mathematical framework**

- Stock (log) price follows a diffusion equation, i.e. a continuous-time continuous stochastic process such as

\[ dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dw_t, \]

where \( \mu(x_t, t) \) and \( \sigma(x_t, t) \) are the drift and diffusion coefficient, respectively, and \( w_t \) is a standard Brownian motion (or Wiener process).

- In a complete market, use hedging to derive the price of an option (no arbitrage argument).

- In an incomplete market (e.g. existence of jumps), specify risk and a hedging strategy to minimize the risk.
Stochastic processes

- Wiener process (or Standard Brownian motion)
  
  - notation: $w_t$
  
  - initial value: $w_0 = 0$
  
  - small increments are independent and normal
    
    time points: $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$
    
    \[
    \{\Delta w_i = w_{t_i} - w_{t_{i-1}}\} \text{ are independent}
    \]
    
    \[
    \Delta w_t = w_{t+\Delta t} - w_t \sim N(0, \Delta t).
    \]
  
  - property: $w_t \sim N(0, t)$
  
  - zero drift and rate of variance change is 1. That is,
    
    \[
    dw_t = 0dt + 1dw_t.
    \]
  
  - A simple way to understand Wiener processes is to do simulation. In R or S-Plus, this can be achieved by using:
    
    \[
    n=5000
    \]
    
    \[
    at = \text{rnorm}(n)
    \]
    
    \[
    wt = \text{cumsum}(at)/\text{sqrt}(n)
    \]
    
    \[
    \text{plot}(wt, \text{type}='l')
    \]
    
    Repeat the above commands to generate lots of “wt” series.

- Generalized Wiener process
  
  \[
  dx_t = \mu dt + \sigma dw_t,
  \]
  
  where the drift $\mu$ & rate of volatility change $\sigma$ are constant.
• Ito’s process

\[ dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dw_t, \]

where both drift and volatility are time-varying.

• Geometric Brownian motion

\[ dP_t = \mu P_t dt + \sigma P_t dw_t, \]

so that \( \mu(P_t, t) = \mu P_t \) and \( \sigma(P_t, t) = \sigma P_t \) with \( \mu \) and \( \sigma \) being constant.

Illustration: Four simulated standard Brownian motions. key feature: variability increases with time.

Assume that the price of a stock follows a geometric Brownian motion. What is the distribution of the log return?

To answer this question, we need Ito’s calculus.

Review of differentiation

\( G(x) \): a differentiable function of \( x \).

What is \( dG(x) \)?

Taylor expansion:

\[ \Delta G \equiv G(x + \Delta x) - G(x) = \frac{\partial G}{\partial x} \Delta x \]

\[ + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{1}{6} \frac{\partial^3 G}{\partial x^3} (\Delta x)^3 + \cdots. \]

Letting \( \Delta x \to 0 \), we have

\[ dG = \frac{\partial G}{\partial x} dx. \]
Figure 1: Time plots of four simulated Wiener processes
How about $G(x, y)$?

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial y} \Delta y$$

$$+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial y} \Delta x \Delta y + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} (\Delta y)^2 + \cdots.$$ 

Taking limit as $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy.$$ 

Stochastic differentiation

Now, consider $G(x_t, t)$ with $x_t$ being an Ito’s process.

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t$$

$$+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial t} \Delta x \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} (\Delta t)^2 + \cdots.$$ 

A discretized version of the Ito’s process is

$$\Delta x = \mu_* \Delta t + \sigma_* \epsilon \sqrt{\Delta t},$$

where $\mu_* = \mu(x_t, t)$ and $\sigma_* = \sigma(x_t, t)$. Therefore,

$$(\Delta x)^2 = \mu_*^2 (\Delta t)^2 + \sigma_*^2 \epsilon^2 \Delta t + 2 \mu_* \sigma_* \epsilon (\Delta t)^{3/2}$$

$$= \sigma_*^2 \epsilon^2 \Delta t + H(\Delta t).$$

Thus, $(\Delta x)^2$ contains a term of order $\Delta t$.

$$E(\sigma_*^2 \epsilon^2 \Delta t) = \sigma_*^2 \Delta t,$$

$$\Var(\sigma_*^2 \epsilon^2 \Delta t) = E[\sigma_*^4 \epsilon^4 (\Delta t)^2] - [E(\sigma_*^2 \epsilon^2 \Delta t)]^2 = 2 \sigma_*^4 (\Delta t)^2,$$
where we use $E(e^4) = 3$. These two properties show that
\[
\sigma^2 \epsilon^2 \Delta t \to \sigma^2 \Delta t \quad \text{as} \quad \Delta t \to 0.
\]
Consequently,
\[
(\Delta x)^2 \to \sigma^2 dt \quad \text{as} \quad \Delta t \to 0.
\]
Using this result, we have
\[
dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma^2 \Delta t
\]
\[
= \left( \frac{\partial G}{\partial x} \mu_* + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma_*^2 \right) dt + \frac{\partial G}{\partial x} \sigma_* dw_t.
\]
This is the well-known Ito’s lemma.

**Example.** Let $G(w_t, t) = w_t^2$. What is $dG(w_t, t)$?

**Answer:** Here $\mu_* = 0$ and $\sigma_* = 1$.

\[
\frac{\partial G}{\partial w_t} = 2w_t, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{\partial^2 G}{\partial w_t^2} = 2.
\]

Therefore,
\[
dw_t^2 = (2w_t \times 0 + 0 + \frac{1}{2} \times 2 \times 1) dt + 2w_t dw_t = dt + 2w_t dw_t.
\]

If $P_t$ follows a geometric Brownian motion, what is the model for $\ln(P_t)$?

**Answer:** Let $G(P_t, t) = \ln(P_t)$. we have

\[
\frac{\partial G}{\partial P_t} = \frac{1}{P_t}, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} = \frac{1-1}{2 P_t^2}.
\]
Consequently, via Ito’s lemma, we obtain
\[ d \ln(P_t) = \left( \frac{1}{P_t} \mu P_t + \frac{1}{2} \frac{1}{P_t^2} \sigma^2 P_t^2 \right) dt + \frac{1}{P_t} \sigma P_t dw_t \]
\[ = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dw_t. \]
Thus, \( \ln(P_t) \) follows a generalized Wiener Process with drift rate \( \mu - \sigma^2/2 \) and variance rate \( \sigma^2 \).
The log return from \( t \) to \( T \) is normal with mean \( (\mu - \sigma^2/2)(T - t) \) and variance \( \sigma^2(T - t) \).

**Estimation of \( \mu \) and \( \sigma \)**
Assume that \( n \) log returns are available, say \( \{r_t | t = 1, \cdots, n\} \).

**Statistical theory:**
Estimate the mean and variance by the sample mean and variance.
\[ \bar{r} = \frac{\sum_{t=1}^{n} r_t}{n}, \]
\[ s_r^2 = \frac{1}{n-1} \sum_{t=1}^{n} (r_t - \bar{r})^2. \]

**Remember the length of time intervals!**
Let \( \Delta \) be the length of time intervals measured in years.
Then, the distribution of \( r_t \) is
\[ r_t \sim N[(\mu - \sigma^2/2)\Delta, \sigma^2\Delta]. \]
We obtain the estimates
\[ \hat{\sigma} = \frac{s_r}{\sqrt{\Delta}}. \]
\[
\hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = \frac{\bar{r}}{\Delta} + \frac{s_r^2}{2\Delta}.
\]

**Example.** Daily log returns of IBM stock in 1998.
The data show \( \bar{r} = 0.002276 \) and \( s_r = 0.01915 \).
Since \( \Delta = 1/252 \) year, we obtain that
\[
\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = 0.3040, \quad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.6198.
\]
Thus, the estimated expected return was 61.98% and the standard
deviation was 30.4% per annum for IBM stock in 1998.

**Example.** Daily log returns of Cisco stock in 1999.
Data show \( \bar{r} = 0.00332 \) and \( s_r = 0.026303 \),
Also, \( Q(12) = 10.8 \). Therefore, we have
\[
\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = \frac{0.026303}{\sqrt{1.0/252.0}} = 0.418, \quad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.924.
\]
Expected return was 92.4% per annum
Estimated s.d. was 41.8% per annum.

Data show \( \bar{r} = -0.00301 \) and \( s_r = 0.05192 \).
Therefore, \( \hat{\sigma} = 0.818 \hat{\mu} = -0.412 \).
Time-varying nature of mean and volatility is clearly shown.

**Distributions of stock prices**
If the price follows
\[
dP_t = \mu P_t dt + \sigma P_t dw_t,
\]
then,
\[
\ln(P_T) - \ln(P_t) \sim N \left( \left( \mu - \frac{\sigma^2}{2} \right) (T - t), \sigma^2 (T - t) \right).
\]
Consequently, given \( P_t \),
\[
\ln(P_T) \sim N \left( \ln(P_t) + \left( \mu - \frac{\sigma^2}{2} \right) (T - t), \sigma^2 (T - t) \right),
\]
and we obtain (log-normal dist; ch. 1)
\[
E(P_T) = P_t \exp[\mu(T - t)],
\]
\[
\text{Var}(P_T) = P_t^2 \exp[2\mu(T - t)] \{ \exp[\sigma^2(T - t)] - 1 \}.
\]
The result can be used to make inference about \( P_T \).
Simulation is often used to study the behavior of \( P_T \).

**Black-Scholes equation**

- Price of stock: \( P_t \) is a Geo. B. Motion
- Price of derivative: \( G_t = G(P_t, t) \) contingent the stock
- Risk neutral world: expected returns are given by the risk-free interest rate (no arbitrage)

From Ito’s lemma:
\[
dG_t = \left( \frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) dt + \frac{\partial G_t}{\partial P_t} \sigma P_t dw_t.
\]
A discretized version of the set-up:
\[
\Delta P_t = \mu P_t \Delta t + \sigma P_t \Delta w_t,
\]
Consider the portfolio:

- short on derivative
- long $\frac{\partial G_t}{\partial P_t}$ shares of the stock.

Value of the portfolio is

$$V_t = -G_t + \frac{\partial G_t}{\partial P_t} P_t.$$

The change in value is

$$\Delta V_t = -\Delta G_t + \frac{\partial G_t}{\partial P_t} \Delta P_t.$$

by substitution, we have

$$\Delta V_t = \left( -\frac{\partial G_t}{\partial t} - \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t.$$

**No stochastic** component involved.

The portfolio must be riskless during a small time interval.

$$\Delta V_t = r V_t \Delta t$$

where $r$ is the risk-free interest rate. We then have

$$\left( \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t = r \left( G_t - \frac{\partial G_t}{\partial P_t} P_t \right) \Delta t.$$
the Black-Scholes differential equ. for derivative pricing.

**Example.** A forward contract on a stock (no dividend). Here

\[ G_t = P_t - K \exp[-r(T - t)] \]

where \( K \) is the delivery price. We have

\[
\frac{\partial G_t}{\partial t} = -rK \exp[-r(T - t)], \quad \frac{\partial G_t}{\partial P_t} = 1, \quad \frac{\partial^2 G_t}{\partial P_t^2} = 0. 
\]

Substituting these quantities into LHS yields

\[
-rK \exp[-r(T - t)] + rP_t = r\{P_t - K \exp[-r(T - t)]\},
\]

which equals RHS.

**Black-Scholes formulas**

A European call option: expected payoff

\[ E_\ast[\max(P_T - K, 0)] \]

Price of the call: (current value)

\[ c_t = \exp[-r(T - t)]E_\ast[\max(P_T - K, 0)]. \]

In a risk-neutral world, \( \mu = r \) so that

\[
\ln(P_T) \sim N\left[\ln(P_t) + \left(r - \frac{\sigma^2}{2}\right)(T - t), \sigma^2(T - t)\right].
\]

Let \( g(P_T) \) be the pdf of \( P_T \). Then,

\[ c_t = \exp[-r(T - t)] \int_K^\infty (P_T - K)g(P_T)dP_T. \]
After some algebra (appendix)

\[ c_t = P_t \Phi(h_+) - K \exp[-r(T - t)] \Phi(h_-) \]

where \( \Phi(x) \) is the CDF of \( N(0, 1) \),

\[
\begin{align*}
    h_+ &= \frac{\ln(P_t/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \\
    h_- &= \frac{\ln(P_t/K) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} = h_+ - \sigma \sqrt{T - t}.
\end{align*}
\]

See Chapter 6 for some interpretations of the formula.

For put option:

\[ p_t = K \exp[-r(T - t)] \Phi(-h_-) - P_t \Phi(-h_+). \]

Alternatively, use the put-call parity:

\[ p_t - c_t = K \exp[-r(T - t)] - P_t. \]

**Put-call parity:** Same underlying stock, same strike price, same time to maturity.

<table>
<thead>
<tr>
<th>current date ( t )</th>
<th>Expiration date ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P^* \leq K )</td>
<td>( K &lt; P^* )</td>
</tr>
<tr>
<td>Write call ( c )</td>
<td>( - )</td>
</tr>
<tr>
<td>Buy put ( -p )</td>
<td>( K - P^* )</td>
</tr>
<tr>
<td>Buy stock ( -P )</td>
<td>( P^* )</td>
</tr>
<tr>
<td>Borrow ( Ke^{-r(T-t)} )</td>
<td>( -K )</td>
</tr>
<tr>
<td>Total</td>
<td>( - )</td>
</tr>
</tbody>
</table>
Current amount: $c - p - P + Ke^{-r(T-t)}$.

On the expiration date:

1. If $P^* \leq K$: call is worthless, put is $K - P^*$, stock $P^*$ and owe bank $K$. Net worth = 0.

2. If $K < P^*$: call is $K - P^*$, put is worthless, stock $P^*$ and owe $K$. Net worth = 0.

Under no arbitrage assumption: The initial position should be zero. That is, $c = p + P - Ke^{-r(T-t)}$.

**Example.** $P_t = $80. $\sigma = 20\%$ per annum. $r = 8\%$ per annum.

What is the price of a European call option with a strike price of $90 that will expire in 3 months?

From the assumptions, we have $P_t = 80$, $K = 90$, $T - t = 0.25$, $\sigma = 0.2$ and $r = 0.08$. Therefore,

$$h_+ = \frac{\ln(80/90) + (0.08 + 0.04/2) \times 0.25}{.2\sqrt{0.25}} = -0.9278$$

$$h_- = h_+ - .2\sqrt{.25} = -1.0278.$$ 

It can be found

$$\Phi(-.9278) = 0.1767, \quad \Phi(-1.0278) = 0.1520.$$

Therefore,

$$c_t = 80\Phi(-0.9278) - 90\Phi(-1.0278) \exp(-0.02) = $0.73.$$

The stock price has to rise by $10.73 for the purchaser of the call option to break even.
If $K = \$81$, then

$$c_t = \$80\Phi(0.125775) - \$81\exp(-0.02)\Phi(0.025775) = \$3.49.$$ 

**A note on computer program:** Check the web site of Prof. Joseph Goguen of UCSD.
http://www-cse.ucsd.edu/~goguen/courses/130/SayBlackScholes.html

**Lower bounds of European options:** No dividends.

$$c_t \geq P_t - K\exp[-r(T-t)].$$

Why?

Consider two portfolios:

- A: One European call option plus cash $K\exp[-r(T-t)].$

- B: One share of the stock.

For A: Invest the cash at risk-free interest rate. At time $T$, the value is $K$. If $P_T > K$, the call option is exercised so that the portfolio is worth $P_T$. If $P_T < K$, the call option expires at $T$ and the portfolio is worth $K$. Therefore, the value of the portfolio is $\max(P_T, K)$.

For B: The value at time $T$ is $P_T$.

Thus, portfolio A must be worth more than portfolio B today; that is,

$$c_t + K\exp[-r(T-t)] \geq P_t.$$ 

See Example 6.7 for an application.
Stochastic integral

The formula
\[ \int_0^t dx_s = x_t - x_0 \]
continues hold. In particular,
\[ \int_0^t dw_s = w_t - w_0 = w_t. \]

From
\[ dw_t^2 = dt + 2w_t dw_t \]
we have
\[ w_t^2 = t + 2 \int_0^t w_s dw_s. \]

Therefore,
\[ \int_0^t w_s dw_s = \frac{1}{2}(w_t^2 - t). \]

Different from \( \int_0^t y dy = (y_t^2 - y_0^2)/2. \)

Assume \( x_t \) is a Geo. Brownian motion,
\[ dx_t = \mu x_t dt + \sigma x_t dw_t. \]

Apply Ito’s lemma to \( G(x_t, t) = \ln(x_t) \), we obtain
\[ d\ln(x_t) = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dw_t. \]

Taking integration, we have
\[ \int_0^t d\ln(x_s) = \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t ds + \sigma \int_0^t dw_s. \]

Consequently,
\[ \ln(x_t) = \ln(x_0) + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma w_t, \]
and

\[ x_t = x_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t]. \]

Change \( x_t \) to \( P_t \). The price is

\[ P_t = P_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t]. \]

**Jump diffusion**

Weaknesses of diffusion models:

- no volatility smile (convex function of implied volatility vs strike price)
- fail to capture effects of rare events (tails)

Modification: jump diffusion and stochastic volatility

Jumps are governed by a probability law:

Poisson process: \( X_t \) is a Poisson process if

\[ Pr(X_t = m) = \frac{\lambda^m t^m}{m!} \exp(-\lambda t), \ \lambda > 0. \]

Use a special jump diffusion model by Kou (2002).

\[ \frac{dP_t}{P_t} = \mu dt + \sigma dw_t + d\left( \sum_{i=1}^{n_t} (J_i - 1) \right), \]

- \( w_t \): a Wiener process,
- \( n_t \): a Poisson process with rate \( \lambda \),
- \( \{J_i\} \): iid such that \( X = \ln(J) \) has a double exp. dist. with pdf

\[ f_X(x) = \frac{1}{2\eta} e^{-|x-\kappa|/\eta}, \ \ 0 < \eta < 1. \]
the above three processes are independent.

\[ n_t = \text{the number of jumps in } [0, t] \text{ and Poisson}(\lambda t). \] At the \( i \)th jump, the proportion of price jump is \( J_i - 1 \).

For pdf of double exp. dist., see Figure 6.8 of the text.

Stock price under the jump diffusion model:

\[
P_t = P_0 \exp\left[(\mu - \sigma^2/2)t + \sigma w_t\right] n_t \prod_{i=1}^{n_t} J_i.
\]

This result can be used to obtain the distribution for the return series.

Price of an option: Analytical results available, but complicated.

**Example** \( P_t = \$80 \). \( K = \$81 \). \( r = 0.08 \) and \( T - t = 0.25 \).

Jump: \( \lambda = 10 \), \( \kappa = -0.02 \) and \( \eta = 0.02 \).

We obtain \( c_t = \$3.92 \), which is higher than \$3.49 of Example 6.6.

\( p_t = \$3.31 \), which is also higher.

**Some greeks**: option value \( V \), stock price \( P \)

1. Delta: \( \Delta = \frac{\partial V}{\partial P} \)

2. Gamma: \( \Gamma = \frac{\partial \Delta}{\partial P} \)

3. Theta: \( \Theta = -\frac{\partial V}{\partial t} \).