Some alternative methods:

- Moving window estimates
- Use of high-frequency financial data
- Use of daily open, high, low and closing prices (or log prices)

**Moving window**
A simple approach to capture time-varying feature of the volatility.
Hard to determine the size of the window.

**Demonstration:** Use the `quantmod` package to download the daily trading information of SPDR S&P 500 from January 3, 2003 to April 30, 2016. The tick symbol is `SPY`. Use the adjusted index value to compute daily log returns of SPY. A R script, `mvwindow.R`, is available on the course web.

**Instructions:**

1. Download the data and save it in your R working directory.
2. Compile the program using the command: `source("mvwindow.R")`
3. To run the program: `mvol=mvwindow(rt,size)`, where “rt” denotes the return series and “size” is the size of the moving window.
4. The output is the volatility, i.e., $\sigma_t$, stored in `sigma.t`.

Demonstration shown in class.

**Use of High-Frequency Data**
Suppose we like to estimate the monthly volatility of a stock return. Data: Daily returns
Let \( r_t^m \) be the \( t \)-th month log return.
Let \( \{r_{t,i}\}_{i=1}^n \) be the daily log returns within the \( t \)-th month.
Using properties of log returns, we have
\[
r_t^m = \sum_{i=1}^n r_{t,i}.
\]
Assuming that the conditional variance and covariance exist, we have
\[
\text{Var}(r_t^m | F_{t-1}) = \sum_{i=1}^n \text{Var}(r_{t,i} | F_{t-1}) + 2 \sum_{i<j} \text{Cov}(r_{t,i}, r_{t,j})|F_{t-1}),
\]
where \( F_{t-1} = \) the information available at month \( t - 1 \) (inclusive). Further simplification is possible under additional assumptions.
If \( \{r_{t,i}\} \) is a white noise series, then
\[
\text{Var}(r_t^m | F_{t-1}) = n \text{Var}(r_{t,1}),
\]
where \( \text{Var}(r_{t,1}) \) can be estimated from the daily returns \( \{r_{t,i}\}_{i=1}^n \) by
\[
\hat{\sigma}^2 = \frac{\sum_{i=1}^n (r_{t,i} - \bar{r}_t)^n}{n - 1},
\]
where \( \bar{r}_t \) is the sample mean of the daily log returns in month \( t \) (i.e., \( \bar{r}_t = \sum_{i=1}^n r_{t,i}/n \)).
The estimated monthly volatility is then
\[
\hat{\sigma}^2_m = \frac{n}{n - 1} \sum_{i=1}^n (r_{t,i} - \bar{r}_t)^2 \approx \sum_{i=1}^n (r_{t,i} - \bar{r}_t)^2.
\]
If \( \{r_{t,i}\} \) follows an MA(1) model, then
\[
\text{Var}(r_t^m | F_{t-1}) = n \text{Var}(r_{t,1}) + 2(n - 1)\text{Cov}(r_{t,1}, r_{t,2}),
\]
which can be estimated by
\[
\hat{\sigma}^2_m = \frac{n}{n - 1} \sum_{i=1}^n (r_{t,i} - \bar{r}_t)^2 + 2 \sum_{i=1}^{n-1} (r_{t,i} - \bar{r}_t)(r_{t,i+1} - \bar{r}_t).
\]
Figure 1: Time plots of estimated monthly volatility for the log returns of S&P 500 index from January 1980 to December 1999: (a) assumes that the daily log returns form a white noise series, (b) assumes that the daily log returns follow an MA(1) model, and (c) uses monthly returns from January 1962 to December 1999 and a GARCH(1,1) model.

Advantage: Simple

Weaknesses:

- Models for daily returns \( \{r_{t,i}\} \) are unknown.
- Typically, 21 or 22 trading days in a month, resulting in a small sample size.

See Figure 1 for an illustration; Ex 3.6 of the text.

**Realized integrated volatility**

If the sample mean \( \bar{r}_t \) is zero, then \( \hat{\sigma}_m^2 \approx \sum_{i=1}^{n} r_{t,i}^2 \).

⇒ Use cumulative sum of squares of daily log returns within a month as an estimate of monthly volatility.
Consider tick-by-tick data: Apply the idea to *intraday log returns* and obtain realized integrated volatility.

Assume daily log return \( r_t = \sum_{i=1}^{n} r_{t,i} \). The quantity

\[
RV_t = \sum_{i=1}^{n} r_{t,i}^2,
\]

is called the *realized* volatility of \( r_t \).

**Advantages:** simplicity and using intraday information

**Weaknesses:**

- Effects of market micro-structure noises
- Overlook overnight volatilities.

**Further discussion**

1. In-filled asymptotic argument. Let \( \Delta \) be the sampling interval, as \( \Delta \to 0 \), the sample size goes to infinity.

   Under the assumption that the \( \Delta \)-interval log returns, e.g. 5-minute returns, are independent and identically distributed, then \( \sum_{j=1}^{n} r_{t,j}^2 \) converges to the variance of the daily log return \( r_t \).

   (Quadratic variation)

2. In practice, however, there are micro-structure noises that affect the estimate such as the bid-ask bounce. In fact, it can be shown that as \( \Delta \) goes to zero, the observed sum of squares of \( \Delta \)-interval returns goes to infinity.

**What next?** Two approaches have been proposed:

(a) Optimal sampling interval: Bandi and Russell (2006). Find an optimal \( \Delta \). Or equivalently, the optimal sample size \( n^* \)
= 6.5 hours/Δ can be chosen as

\[ n^* \approx \left[ \frac{Q}{(\hat{\sigma}_{\text{noise}}^2)^2} \right]^{1/3}, \]

where \( Q = \frac{M}{3} \sum_{j=1}^{M} r_{t,j}^4 \) and \( \hat{\sigma}_{\text{noise}}^2 = \frac{1}{M} \sum_{j=1}^{M} r_{t,j}^2 \), where \( M \) is the number of daily quotes available for the underlying stock and the returns \( r_{t,j} \) are computed from the mid-point of the bid and ask quotes.

(b) Sub-sampling: Zhang et al. (2006). Choose \( \Delta \) between 10 to 20 minutes. Compute integrated volatility for each of the possible \( \Delta \)-interval return series. Then, compute the average. In fact, the authors propose a so-called two scales realized volatility (TSRV) estimate. The form is

\[ \text{RV} = a_n \times \text{ARV}_K - b_n \times \text{ARV}_J, \]

where \( \text{ARV}_i \) denotes the average realized volatility of time interval \( i \), \( a_n \) is a real number approaching 1 and \( b_n = a_n \times n_K/n_J \), and \( n_K = (n - K + 1)/K \) with \( n \) is the number of transactions within the day. \( J \) can be 1 or \( J << K \). When \( J = 1 \), the second term can be regarded as estimate of the noise. When \( K \) is much larger than \( J \), the second term is typically small.

**Use of Daily Open, High, Low and Close Prices**

Figure 2 shows a time plot of price versus time for the \( t \)th trading day. Define

- \( C_t \) = the closing price of the \( t \)th trading day;
- \( O_t \) = the opening price of the \( t \)th trading day;
Figure 2: Time plot of price over time: scale for price is arbitrary.

- $f$ = fraction of the day (in interval $[0,1]$) that trading is closed;
- $H_t$ = the highest price of the $t$th trading period;
- $L_t$ = the lowest price of the $t$th trading period;
- $F_{t-1}$ = public information available at time $t - 1$.

The conventional variance (or volatility) is $\sigma_t^2 = E[(C_t - C_{t-1})^2 | F_{t-1}]$.

Some alternatives:
- $\hat{\sigma}_{0,t}^2 = (C_t - C_{t-1})^2$,
\[
\hat{\sigma}_{1,t}^2 = \frac{(O_{t-1} - C_{t-1})^2}{2f} + \frac{(C_t - O_t)^2}{2(1-f)}, \quad 0 < f < 1;
\]
\[
\hat{\sigma}_{2,t}^2 = \frac{(H_t - L_t)^2}{4\ln(2)} \approx 0.3607(H_t - L_t)^2;
\]
\[
\hat{\sigma}_{3,t}^2 = 0.17\frac{(O_t - C_{t-1})^2}{f} + 0.83\frac{(H_t - L_t)^2}{(1-f)4\ln(2)}, \quad 0 < f < 1;
\]
\[
\hat{\sigma}_{5,t}^2 = 0.5(H_t - L_t)^2 - [2\ln(2) - 1](C_t - O_t)^2,
\]
which is \(\approx 0.5(H_t - L_t)^2 - 0.386(C_t - O_t)^2;\)
\[
\hat{\sigma}_{6,t}^2 = 0.12\frac{(O_t - C_{t-1})^2}{f} + 0.88\frac{\hat{\sigma}_{5,t}^2}{1-f}, \quad 0 < f < 1.
\]

A more precise, but complicated, estimator \(\hat{\sigma}_{4,t}^2\) was also considered. But it is close to \(\hat{\sigma}_{5,t}^2\).

Defining the efficiency factor of a volatility estimator as
\[
\text{Eff}(\hat{\sigma}_{i,t}^2) = \frac{\text{Var}(\hat{\sigma}_{0,t}^2)}{\text{Var}(\hat{\sigma}_{i,t}^2)},
\]
Garman and Klass (1980) found that \(\text{Eff}(\hat{\sigma}_{i,t}^2)\) is approximately 2, 5.2, 6.2, 7.4 and 8.4 for \(i = 1, 2, 3, 5\) and 6, respectively, for the simple diffusion model entertained.

For log-return volatility, one takes the logarithms of the Open, High, Low and Close prices.

Define
\[
\cdot \ o_t = \ln(O_t) - \ln(C_{t-1}) \text{ be the normalized open;}
\]
\[
\cdot \ u_t = \ln(H_t) - \ln(O_t) \text{ be the normalized high;}
\]
\[
\cdot \ d_t = \ln(L_t) - \ln(O_t) \text{ be the normalized low;}
\]
\[
\cdot \ c_t = \ln(C_t) - \ln(O_t) \text{ be the normalized close.}
\]
Suppose that there are \( n \) days of data available and the volatility is constant over the period. Yang and Zhang (2000) recommend the estimate

\[
\hat{\sigma}_{yz}^2 = \hat{\sigma}_o^2 + k\hat{\sigma}_c^2 + (1 - k)\hat{\sigma}_{rs}^2
\]

as a robust estimator of the volatility, where

\[
\hat{\sigma}_o^2 = \frac{1}{n - 1} \sum_{t=1}^{n} (o_t - \bar{o})^2 \quad \text{with} \quad \bar{o} = \frac{1}{n} \sum_{t=1}^{n} o_t,
\]

\[
\hat{\sigma}_c^2 = \frac{1}{n - 1} \sum_{t=1}^{n} (c_t - \bar{c})^2 \quad \text{with} \quad \bar{c} = \frac{1}{n} \sum_{t=1}^{n} c_t,
\]

\[
\hat{\sigma}_{rs}^2 = \frac{1}{n} \sum_{t=1}^{n} [u_t(u_t - c_t) + d_t(d_t - c_t)],
\]

\[
k = \frac{0.34}{1.34 + (n + 1)/(n - 1)}.
\]

This estimate seems to perform reasonably well.

**Remark:** One must consider the stock split in the above calculation.

Some work using daily range. For log returns, daily range is defined as

\[
r_t = \ln(H_t) - \ln(L_t).
\]

This is related to the **duration models** to be discussed later in high-frequency data.

**Takeaway**

Some alternative approaches to volatility estimation are currently under intensive study. It is rather early to assess the impact of these methods. It is a good idea in general to use more information. However, regulations and institutional effects need to be considered.