What is the volatility of an asset?
Answer: Conditional standard deviation of the asset return (price)

Why is volatility important?
Has many important applications:

- Option (derivative) pricing, e.g., Black-Scholes formula
- Risk management, e.g. value at risk (VaR)
- Asset allocation, e.g., minimum-variance portfolio; see pages 184-185 of Campbell, Lo and MacKinlay (1997).
- Interval forecasts

A key characteristic: Not directly observable!!

How to calculate volatility?
There are several versions of sample volatility, but the conditional standard deviation is commonly used.

1. Use high-frequency data: French, Schwert & Stambaugh (1987); see Section 3.15.
   - Realized volatility of daily log returns: use intraday high-frequency log returns.
   - Use daily high, low, and closing (log) prices, e.g. range = daily high - daily low.

2. Implied volatility of options data, e.g, VIX of CBOE. Figure 1.
We focus on the econometric modeling first. Use of high frequency data to compute realized volatility will be discussed later.

Note: In most applications, volatility is annualized. This can easily be done by taking care of the data frequency. For instance, if we use daily returns in econometric modeling, then the annualized volatility (in the U.S.) is

\[ \sigma_t^* = \sqrt{252} \sigma_t, \]

where \( \sigma_t \) is the estimated volatility derived from an employed model. If we use monthly returns, then the annualized volatility is

\[ \sigma_t^* = \sqrt{12} \sigma_t, \]

where \( \sigma_t \) is the estimated volatility derived from the employed model for the monthly returns. Our discussion, however, continues to use \( \sigma_t \) for simplicity.

Basic idea of econometric modeling: Shocks of asset returns are NOT serially correlated, but dependent, implying that the serial dependence in asset returns is nonlinear. As shown by the ACF of returns and absolute returns of some assets we discussed so far.

Basic structure

\[ r_t = \mu_t + a_t, \quad \mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i}, \]

Volatility models are concerned with time-evolution of

\[ \sigma_t^2 = \text{Var}(r_t|F_{t-1}) = \text{Var}(a_t|F_{t-1}), \]

the conditional variance of the return \( r_t \).
Consider the daily closing index of the S&P500 index from January 03, 2007 to April 06, 2017. The log returns follow approximately an MA(2) model

\[ r_t = 0.0002 + a_t - 0.109a_{t-1} - 0.053a_{t-2}, \quad \sigma^2 = 0.00017. \]

The residuals show no strong serial correlations. [plot not shown.]

**R Demonstration**

```r
> require(quantmod)
> getSymbols("^GSPC", from="2007-01-03", to="2017-04-06")
[1] "GSPC"
> dim(GSPC)
[1] 2584 6
> head(GSPC)
   GSPC.Open GSPC.High GSPC.Low GSPC.Close GSPC.Volume GSPC.Adjusted
2007-01-03 1418.03 1429.42 1407.86 1416.60 3429160000 1416.60
.. 2007-01-10 1408.70 1415.99 1405.32 1414.85 2764660000 1414.85
> spc <- log(as.numeric(GSPC[,6]))
> rtn <- diff(spc)
> acf(rtn)
```
Figure 2: Time plot of residuals of an MA(2) model fitted to daily log returns of the S&P 500 index from January 3, 2007 to April 06, 2017.

Figure 3: Sample ACF of the squared residuals of an MA(2) model fitted to daily log returns of the S&P 500 index from January 3, 2007 to April 06, 2017.
The residuals are shown in Figure 2.
Is volatility constant over time?
NO! Figure 2 shows a special feature, which is referred to as the volatility clustering.
How to model the evolving volatility? See the ACF of the squared residuals in Figure 3.

Two general categories

- “Fixed function” and
- Stochastic function

of the available information.

Univariate volatility models discussed:

1. Autoregressive conditional heteroscedastic (ARCH) model of Engle (1982),
2. Generalized ARCH (GARCH) model of Bollerslev (1986),
3. GARCH-M models,
4. IGARCH models (used by RiskMetrics),
5. Exponential GARCH (EGARCH) model of Nelson (1991),

7. Asymmetric power ARCH (APARCH) models of Ding, Granger and Engle (1994), [TGARCH and GJR models are special cases of APARCH models.]

8. Stochastic volatility (SV) models of Melino and Turnbull (1990), Harvey, Ruiz and Shephard (1994), and Jacquier, Polson and Rossi (1994).

ARCH model

\[ a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2, \]

where \( \{\epsilon_t\} \) is a sequence of iid r.v. with mean 0 and variance 1, \( \alpha_0 > 0 \) and \( \alpha_i \geq 0 \) for \( i > 0 \).

Distribution of \( \epsilon_t \): Standard normal, standardized Student-t, generalized error dist (ged), or their skewed counterparts.

Properties of ARCH models

Consider an ARCH(1) model

\[ a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2, \]

where \( \alpha_0 > 0 \) and \( \alpha_1 \geq 0 \).

1. \( \mathrm{E}(a_t) = 0 \)

2. \( \mathrm{Var}(a_t) = \alpha_0/(1 - \alpha_1) \) if \( 0 < \alpha_1 < 1 \)

3. Under normality,

\[ m_4 = \frac{3\alpha_0^2(1 + \alpha_1)}{(1 - \alpha_1)(1 - 3\alpha_1^2)}, \]

provided \( 0 < \alpha_1^2 < 1/3 \).
The 3rd property implies heavy tails.

**Advantages**

- Simplicity
- Generates volatility clustering
- Heavy tails (high kurtosis)

**Weaknesses**

- Symmetric between positive & negative prior returns
- Restrictive
- Provides no explanation
- Not sufficiently adaptive in prediction

**Building an ARCH Model**

1. Modeling the mean effect and testing for ARCH effects
   
   $H_0$: no ARCH effects versus $H_a$ : ARCH effects
   
   Use Q-statistics of squared residuals; McLeod and Li (1983) & Engle (1982)

2. Order determination
   
   Use PACF of the squared residuals. (In practice, simply try some reasonable order).

3. Estimation: Conditional MLE

R provides many plots for model checking and for presenting the results.

5. Software: We use R with the package fGarch. (Other software available).

**Estimation**: Conditional MLE or Quasi MLE

**Special Note**: In this course, we estimate volatility models using the R package **fGarch** with **garchFit** command. The program is easy to use and allows for several types of innovational distributions: The default is Gaussian (**norm**), standardized Student-\(t\) distribution (**std**), generalized error distribution (**ged**), skew normal distribution (**snorm**), skew Student-\(t\) (**sstd**), skew generalized error distribution (**sged**), and standardized inverse normal distribution (**snig**). Except for the inverse normal distribution, other distribution functions are discussed in the textbook. Readers should check the book for details about the density functions and their parameters.

**Example**: Monthly log returns of Intel stock

**R demonstration**: The **fGarch** package. Output edited.

```r
> library(fGarch)
> da=read.table("m-intc7303.txt",header=T)
> head(da)
   date   rtn
1 19730131 0.01005
.....
6 19730629 0.13333
> intc=log(da$rtn+1)  # log returns
> acf(intc)
> acf(intc^2)
> pacf(intc^2)
> Box.test(intc^2,lag=10,type='Ljung')
   Box-Ljung test

data:  intc^2
X-squared = 59.7216, df = 10, p-value = 4.091e-09
```
> m1=garchFit(~garch(3,0),data=intc,trace=F) <= trace=F reduces the amount of output.
> summary(m1)

Title: GARCH Modelling
Call: garchFit(formula = ~garch(3, 0), data = intc, trace = F)

Mean and Variance Equation:
  data ~ garch(3, 0)
  [data = intc]

Conditional Distribution: norm

Coefficient(s):
  mu  omega  alpha1  alpha2  alpha3
  0.016572  0.012043  0.208649  0.071837  0.049045

Std. Errors:
  based on Hessian

Error Analysis:
  Estimate  Std. Error  t value  Pr(>|t|)
  mu      0.016572    0.006423  2.5800  0.00988 **
  omega   0.012043    0.001579  7.6272  2.4e-14 ***
  alpha1  0.208649    0.129177  1.6150  0.10626
  alpha2  0.071837    0.048551  1.4800  0.13897
  alpha3  0.049045    0.048847  1.0040  0.31536

---

Standardised Residuals Tests:
  Statistic  p-Value
  Jarque-Bera Test R Chi^2  169.7731  0
  Shapiro-Wilk Test R W  0.9606957  1.970413e-08
  Ljung-Box Test R Q(10)  10.97025  0.3598405
  Ljung-Box Test R Q(15)  19.59024  0.1882211
  Ljung-Box Test R Q(20)  20.82192  0.40768
  Ljung-Box Test R^2 Q(10)  5.376602  0.864644
  Ljung-Box Test R^2 Q(15)  22.73460  0.08993976
  Ljung-Box Test R^2 Q(20)  23.70577  0.255481
  LM Arch Test R TR^2  20.48506  0.05844884

Information Criterion Statistics:
  AIC     BIC     SIC     HQIC
  -1.228111 -1.175437 -1.228466 -1.207193

> m1=garchFit(~garch(1,0),data=intc,trace=F)
> summary(m1)

Title: GARCH Modelling
Call: garchFit(formula = ~garch(1, 0), data = intc, trace = F)

Mean and Variance Equation:
   data ~ garch(1, 0)
   [data = intc]

Conditional Distribution: norm

Coefficient(s):
   mu     omega     alpha1
  0.016570  0.012490  0.363447

Std. Errors:
   based on Hessian

Error Analysis:
   Estimate   Std. Error    t value  Pr(>|t|)
   mu    0.016570   0.006161    2.689     0.00716 **
   omega 0.012490   0.001549   8.061   6.66e-16 ***
   alpha1 0.363447  0.131598   2.762     0.00575 **

---

Log Likelihood:
   230.2423      normalized:  0.6189309

Standardised Residuals Tests:

<table>
<thead>
<tr>
<th>Test</th>
<th>Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>R Chi^2</td>
<td>122.4040</td>
</tr>
<tr>
<td>Shapiro-Wilk Test</td>
<td>R W</td>
<td>0.9647629</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R Q(10)</td>
<td>13.72604</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R Q(15)</td>
<td>22.31714</td>
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<tr>
<td>Ljung-Box Test</td>
<td>R Q(20)</td>
<td>23.88257</td>
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<td>Ljung-Box Test</td>
<td>R^2 Q(10)</td>
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<tr>
<td>Ljung-Box Test</td>
<td>R^2 Q(15)</td>
<td>30.11276</td>
</tr>
<tr>
<td>Ljung-Box Test</td>
<td>R^2 Q(20)</td>
<td>31.46404</td>
</tr>
<tr>
<td>LM Arch Test</td>
<td>R TR^2</td>
<td>22.036</td>
</tr>
</tbody>
</table>

Information Criterion Statistics:

<table>
<thead>
<tr>
<th>AIC</th>
<th>BIC</th>
<th>SIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.221733</td>
<td>-1.190129</td>
<td>-1.221861</td>
<td>-1.209182</td>
</tr>
</tbody>
</table>

> plot(m1)
Make a plot selection (or 0 to exit):

1: Time Series
2: Conditional SD
3: Series with 2 Conditional SD Superimposed
The fitted ARCH(1) model is

\[ r_t = 0.0176 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \]
\[ \sigma_t^2 = 0.0125 + 0.363 \sigma_{t-1}^2. \]

Model checking statistics indicate that there are some higher order dependence in the volatility, e.g., see Q(15) for the squared standardized residuals. It turns out that a GARCH(1,1) model fares better for the data.

Next, consider Student-t innovations.

\textbf{R demonstration}
Figure 4: QQ-plot for standardized residuals of an ARCH(1) model with Gaussian innovations for monthly log returns of INTC stock: 1973 to 2003.

```r
> m2=garchFit(~garch(1,0),data=intc,cond.dist="std",trace=F)
> summary(m2)
Title: GARCH Modelling

Call: garchFit(formula = ~garch(1, 0), data = intc, cond.dist = "std",
              trace = F)

Mean and Variance Equation:
data ~ garch(1, 0)
  [data = intc]

Conditional Distribution:  std <====== Standardized Student-t.

Coefficient(s):
mu  omega  alpha1  shape
  0.021571  0.013424  0.259867  5.988579

Error Analysis:

            Estimate Std. Error t value  Pr(>|t|)
mu     0.021571   0.006054   3.563  0.000366 ***
omega  0.013424   0.001968  6.820 9.09e-12 ***
alpha1 0.259867   0.119901  2.167  0.030209 *
```

>
Log Likelihood:
242.9678 normalized: 0.6531391

Standardised Residuals Tests:

<table>
<thead>
<tr>
<th>Test</th>
<th>Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>R Chi^2</td>
<td>130.8931 0</td>
</tr>
<tr>
<td>Shapiro-Wilk Test</td>
<td>R W</td>
<td>0.9637529 5.744026e-08</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(10)</td>
<td>14.31288 0.1591926</td>
<td></td>
</tr>
<tr>
<td>Ljung-Box Test R Q(15)</td>
<td>23.34043 0.07717449</td>
<td></td>
</tr>
<tr>
<td>Ljung-Box Test R Q(20)</td>
<td>24.87286 0.2063387</td>
<td></td>
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<tr>
<td>Ljung-Box Test R^2 Q(10)</td>
<td>15.35917 0.1195054</td>
<td></td>
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<tr>
<td>Ljung-Box Test R^2 Q(15)</td>
<td>33.96318 0.003446127</td>
<td></td>
</tr>
<tr>
<td>Ljung-Box Test R^2 Q(20)</td>
<td>35.46828 0.01774746</td>
<td></td>
</tr>
<tr>
<td>LM Arch Test R TR^2</td>
<td>24.11517 0.01961957</td>
<td></td>
</tr>
</tbody>
</table>

Information Criterion Statistics:

- AIC: -1.284773
- BIC: -1.242634
- SIC: -1.285001
- HQIC: -1.268039

> plot(m2)
Make a plot selection (or 0 to exit):

1: Time Series
2: Conditional SD
3: Series with 2 Conditional SD Superimposed
4: ACF of Observations
5: ACF of Squared Observations
6: Cross Correlation
7: Residuals
8: Conditional SDs
9: Standardized Residuals
10: ACF of Standardized Residuals
11: ACF of Squared Standardized Residuals
12: Cross Correlation between r^2 and r
13: QQ-Plot of Standardized Residuals

Selection: 13  The plot shows that the model needs further improvements.

> predict(m2,5)  Prediction

<table>
<thead>
<tr>
<th>meanForecast</th>
<th>meanError</th>
<th>standardDeviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0.02157100</td>
<td>0.1207911</td>
<td>0.1207911</td>
</tr>
<tr>
<td>2 0.02157100</td>
<td>0.1312069</td>
<td>0.1312069</td>
</tr>
<tr>
<td>3 0.02157100</td>
<td>0.1337810</td>
<td>0.1337810</td>
</tr>
<tr>
<td>4 0.02157100</td>
<td>0.1344418</td>
<td>0.1344418</td>
</tr>
</tbody>
</table>
The fitted model with Student-\( t \) innovations is
\[
 r_t = 0.0216 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon \sim t_{5.99}
\]
\[
 \sigma_t^2 = 0.0134 + 0.260a_{t-1}^2.
\]

We use \( t_{5.99} \) to denote the standardized Student-\( t \) distribution with 5.99 d.f.

Comparison with normal innovations:

- Using a heavy-tailed dist for \( \epsilon_t \) reduces the ARCH effect.
- The difference between the models is small for this particular instance.

You may try other distributions for \( \epsilon_t \).

**GARCH Model**

\[
a_t = \sigma_t \epsilon_t,
\]
\[
\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2
\]

where \( \{\epsilon_t\} \) is defined as before, \( \alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0, \) and \( \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1. \)

Re-parameterization:

Let \( \eta_t = a_t^2 - \sigma_t^2 \). \( \{\eta_t\} \) un-correlated series.

The GARCH model becomes
\[
a_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^{s} \beta_j \eta_{t-j}.
\]

This is an ARMA form for the squared series \( a_t^2 \).

Use it to understand properties of GARCH models, e.g. moment equations, forecasting, etc.
Focus on a GARCH(1,1) model

\[ \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \]

- Weak stationarity: \( 0 \leq \alpha_1, \beta_1 \leq 1, (\alpha_1 + \beta_1) < 1. \)
- Volatility clusters
- Heavy tails: if \( 1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0, \) then
  \[ \frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} > 3. \]
- For 1-step ahead forecast,
  \[ \sigma_h^2(1) = \alpha_0 + \alpha_1 a_h^2 + \beta_1 \sigma_h^2. \]

For multi-step ahead forecasts, use \( a_t^2 = \sigma_t^2 \epsilon_t^2 \) and rewrite the model as

\[ \sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1) \sigma_t^2 + \alpha_1 \sigma_t^2 (\epsilon_t^2 - 1). \]

2-step ahead volatility forecast

\[ \sigma_h^2(2) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2(1). \]

In general, we have

\[ \sigma_h^2(\ell) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2(\ell - 1), \quad \ell > 1. \]

This result is exactly the same as that of an ARMA(1,1) model with AR polynomial \( 1 - (\alpha_1 + \beta_1)B. \)

**Example:** Monthly excess returns of S&P 500 index starting from 1926 for 792 observations.
The fitted of a Gaussian AR(3) model
\[
\tilde{r}_t = r_t - 0.0062 \\
\tilde{r}_t = .089\tilde{r}_{t-1} - .024\tilde{r}_{t-2} - .123\tilde{r}_{t-3} + .007 + a_t,
\]
\[
\hat{\sigma}_a^2 = 0.00333.
\]
For the GARCH effects, use a GARCH(1,1) model, we have
A joint estimation:
\[
\begin{align*}
r_t &= 0.032r_{t-1} - 0.030r_{t-2} - 0.011r_{t-3} + 0.0077 + a_t \\
\sigma_t^2 &= 7.98 \times 10^{-5} + 0.853\sigma_{t-1}^2 + 0.124a_{t-1}^2.
\end{align*}
\]
Implied unconditional variance of \(a_t\) is
\[
\frac{0.0000798}{1 - 0.853 - 0.1243} = 0.00352
\]
close to the expected value. All AR coefficients are statistically insignificant.
A simplified model:
\[
r_t = 0.00745 + a_t, \sigma_t^2 = 8.06 \times 10^{-5} + 0.854\sigma_{t-1}^2 + 0.122a_{t-1}^2.
\]
Model checking:
For \(\tilde{a}_t\): \(Q(10) = 11.22(0.34)\) and \(Q(20) = 24.30(0.23)\).
For \(\tilde{a}_t^2\): \(Q(10) = 9.92(0.45)\) and \(Q(20) = 16.75(0.67)\).
Forecast: 1-step ahead forecast:
\[
\sigma_h^2(1) = 0.00008 + 0.854\sigma_h^2 + 0.122a_h^2
\]
<table>
<thead>
<tr>
<th>Horizon</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
<tr>
<td>Return</td>
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<td>.0074</td>
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<td>.0074</td>
</tr>
<tr>
<td>Volatility</td>
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<td>.054</td>
<td>.054</td>
<td>.054</td>
<td>.054</td>
<td>.059</td>
</tr>
</tbody>
</table>

\textit{R demonstration:}
> sp5=scan("sp500.txt")
Read 792 items
> pacf(sp5)
> m1=arima(sp5,order=c(3,0,0))
> m1
Call: arima(x = sp5, order = c(3, 0, 0))

Coefficients:
   ar1  ar2  ar3 intercept
       0.0890  -0.0238  -0.1229    0.0062
s.e.  0.0353   0.0355   0.0353 0.0019

sigma^2 estimated as 0.00333: log likelihood = 1135.25, aic=-2260.5
> m2=garchFit(~arma(3,0)+garch(1,1),data=sp5,trace=F)
> summary(m2)

Title:  GARCH Modelling
Call:
garchFit(formula = ~arma(3,0)+garch(1,1), data = sp5, trace = F)

Mean and Variance Equation:
data ~ arma(3, 0) + garch(1, 1)
[ data = sp5 ]

Conditional Distribution: norm

Error Analysis:

|        | Estimate | Std. Error | t value | Pr(>|t|) |
|--------|----------|------------|---------|---------|
| mu     | 7.708e-03| 1.607e-03  | 4.798   | 1.61e-06 *** |
| ar1    | 3.197e-02| 3.837e-02  | 0.833   | 0.40473 |
| ar2    | -3.026e-02| 3.841e-02 | -0.788  | 0.43076 |
| ar3    | -1.065e-02| 3.756e-02 | -0.284  | 0.77677 |
| omega  | 7.975e-05| 2.810e-05  | 2.838   | 0.00454 ** |
| alpha1 | 1.242e-01| 2.247e-02  | 5.529   | 3.22e-08 *** |
| beta1  | 8.530e-01| 2.183e-02  | 39.075  | <2e-16 *** |

Log Likelihood:
1272.179 normalized: 1.606287

Standardised Residuals Tests:

<table>
<thead>
<tr>
<th></th>
<th>Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarque-Bera Test</td>
<td>R Chi^2</td>
<td>73.04842 1.110223e-16</td>
</tr>
<tr>
<td>Shapiro-Wilk Test</td>
<td>R W</td>
<td>0.985797 5.961994e-07</td>
</tr>
<tr>
<td>Ljung-Box Test R Q(10)</td>
<td>11.56744</td>
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<td>Ljung-Box Test R Q(15)</td>
<td>17.78747</td>
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<tr>
<td>Ljung-Box Test R Q(20)</td>
<td>24.11916</td>
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</tr>
<tr>
<td>Ljung-Box Test R^2 Q(10)</td>
<td>10.31614</td>
<td>0.4132089</td>
</tr>
</tbody>
</table>
Ljung-Box Test R^2 Q(15) 14.22819 0.5082978
Ljung-Box Test R^2 Q(20) 16.79404 0.6663038
LM Arch Test R TR^2 13.34305 0.3446075

Information Criterion Statistics:
\begin{align*}
\text{AIC} & \quad \text{BIC} & \quad \text{SIC} & \quad \text{HQIC} \\
-3.194897 & \quad -3.153581 & \quad -3.195051 & \quad -3.179018
\end{align*}

> m2=garchFit(~garch(1,1),data=sp5,trace=F)
> summary(m2)
Title: GARCH Modelling
Call: garchFit(formula = ~garch(1, 1), data = sp5, trace = F)

Mean and Variance Equation:
data ~ garch(1, 1)
\[\text{[data = sp5]}\]

Conditional Distribution: norm

Error Analysis:
\begin{align*}
\text{Estimate} & \quad \text{Std. Error} & \quad \text{t value} & \quad \text{Pr(>|t|)} \\
\text{mu} & \quad 7.450e-03 & \quad 1.538e-03 & \quad 4.845 & \quad 1.27e-06 *** \\
\text{omega} & \quad 8.061e-05 & \quad 2.833e-05 & \quad 2.845 & \quad 0.00444 ** \\
\text{alpha1} & \quad 1.220e-01 & \quad 2.202e-02 & \quad 5.540 & \quad 3.02e-08 *** \\
\text{beta1} & \quad 8.544e-01 & \quad 2.175e-02 & \quad 39.276 & \quad < 2e-16 *** \\
\end{align*}
---

Log Likelihood:
1269.455 normalized: 1.602848

Standardised Residuals Tests:
\begin{align*}
\text{Statistic} & \quad \text{p-Value} \\
\text{Jarque-Bera Test R} & \quad \text{Chi}^2 \quad 80.32111 \quad 0 \\
\text{Shapiro-Wilk Test R} & \quad \text{W} \quad 0.9850517 \quad 3.141228e-07 \\
\text{Ljung-Box Test R} & \quad \text{Q(10)} \quad 11.22050 \quad 0.340599 \\
\text{Ljung-Box Test R} & \quad \text{Q(15)} \quad 17.99703 \quad 0.262822 \\
\text{Ljung-Box Test R} & \quad \text{Q(20)} \quad 24.29896 \quad 0.2295768 \\
\text{Ljung-Box Test R^2} & \quad \text{Q(10)} \quad 9.920157 \quad 0.4475259 \\
\text{Ljung-Box Test R^2} & \quad \text{Q(15)} \quad 14.21124 \quad 0.509572 \\
\text{Ljung-Box Test R^2} & \quad \text{Q(20)} \quad 16.75081 \quad 0.6690903 \\
\text{LM Arch Test R} & \quad \text{TR^2} \quad 13.04872 \quad 0.3655092 \\
\end{align*}

Information Criterion Statistics:
\begin{align*}
\text{AIC} & \quad \text{BIC} & \quad \text{SIC} & \quad \text{HQIC} \\
-3.195594 & \quad -3.171985 & \quad -3.195645 & \quad -3.186520
\end{align*}

> plot(m2)
Make a plot selection (or 0 to exit):

1: Time Series
2: Conditional SD
3: Series with 2 Conditional SD Superimposed
4: ACF of Observations
5: ACF of Squared Observations
6: Cross Correlation
7: Residuals
8: Conditional SDs
9: Standardized Residuals
10: ACF of Standardized Residuals
11: ACF of Squared Standardized Residuals
12: Cross Correlation between $r^2$ and $r$
13: QQ-Plot of Standardized Residuals

Selection: 3

> predict(m2,6)

<table>
<thead>
<tr>
<th>meanForecast</th>
<th>meanError</th>
<th>standardDeviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.007449721</td>
<td>0.05377242</td>
<td>0.05377242</td>
</tr>
<tr>
<td>0.007449721</td>
<td>0.05388567</td>
<td>0.05388567</td>
</tr>
<tr>
<td>0.007449721</td>
<td>0.05399601</td>
<td>0.05399601</td>
</tr>
<tr>
<td>0.007449721</td>
<td>0.05410353</td>
<td>0.05410353</td>
</tr>
<tr>
<td>0.007449721</td>
<td>0.05420829</td>
<td>0.05420829</td>
</tr>
<tr>
<td>0.007449721</td>
<td>0.05431038</td>
<td>0.05431038</td>
</tr>
</tbody>
</table>

Turn to Student-$t$ innovation. (R output omitted.)

Estimation of degrees of freedom:

$$ r_t = 0.0085 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t_7 $$

$$ \sigma_t^2 = .000125 + .113a_{t-1}^2 + .842\sigma_{t-1}^2, $$

where the estimated degrees of freedom is 7.00.

**Forecasting evaluation**

Not easy to do; see Andersen and Bollerslev (1998).

**IGARCH model**

An IGARCH(1,1) model:

$$ a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1)a_{t-1}^2. $$
For the monthly excess returns of the S&P 500 index, we have

\[ r_t = 0.007 + a_t, \sigma^2_t = 0.0001 + 0.806\sigma^2_{t-1} + 0.194a^2_{t-1} \]

For an IGARCH(1,1) model,

\[ \sigma^2_h(\ell) = \sigma^2_h(1) + (\ell - 1)\alpha_0, \quad \ell \geq 1, \]

where \( h \) is the forecast origin.

Effect of \( \sigma^2_h(1) \) on future volatilities is persistent, and the volatility forecasts form a straight line with slope \( \alpha_0 \). See Nelson (1990) for more info.

Special case: \( \alpha_0 = 0 \). Volatility forecasts become a constant.

This property is used in RiskMetrics to VaR calculation.

**Example**: An IGARCH(1,1) model for the monthly excess returns of S&P500 index from 1926 to 1991 is given below via R.

\[ r_t = 0.0069 + a_t, \quad a_t = \sigma_t\epsilon_t \]
\[ \sigma_t^2 = 0.099a_{t-1}^2 + .901\sigma_{t-1}^2. \]

**R demonstration:** Using R script `Igarch.R`.

```r
> source("Igarch.R")
> sp5=scan(file="sp500.txt")
> Igarch(sp5,include.mean=T)
Estimates: 0.006874402 0.9007153
Maximized log-likelihood: -1258.219
```

Coefficient(s):

| Estimate  | Std. Error | t value | Pr(>|t|) |
|-----------|------------|---------|----------|
| mu        | 0.0068744  | 0.0015402 | 4.46332  | 8.07e-06 *** |
| beta      | 0.9007153  | 0.0158018 | 57.00082 | < 2e-16 *** |

**Another R package:** `rugarch` can be used to fit volatility models too.

```r
> sp5=scan("sp500.txt")
> require(rugarch)
> spec1=ugarchspec(variance.model=list(model="iGARCH",garchOrder=c(1,1)),
mean.model=list(armaOrder=c(0,0)))
> mm=ugarchfit(data=sp5,spec=spec1)
> mm
```

```
*---------------------------------*
<table>
<thead>
<tr>
<th>Conditional Variance Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>GARCH Model : iGARCH(1,1)</td>
</tr>
<tr>
<td>Mean Model : ARFIMA(0,0,0)</td>
</tr>
<tr>
<td>Distribution : norm</td>
</tr>
</tbody>
</table>

Optimal Parameters

| Estimate  | Std. Error | t value | Pr(>|t|) |
|-----------|------------|---------|----------|
| mu        | 0.007417   | 0.001525 | 4.8621   | 0.000001  |
| omega     | 0.000051   | 0.000018 | 2.9238   | 0.003458  |
| alpha1    | 0.142951   | 0.021443 | 6.6667   | 0.000000  |
| beta1     | 0.857049   | NA       | NA       | NA        |

Robust Standard Errors:

| Estimate  | Std. Error | t value | Pr(>|t|) |
|-----------|------------|---------|----------|
| mu        | 0.007417   | 0.001587 | 4.6726   | 0.000003  |
| omega     | 0.000051   | 0.000019 | 2.6913   | 0.007118  |
| alpha1    | 0.142951   | 0.024978 | 5.7230   | 0.000000  |
| beta1     | 0.857049   | NA       | NA       | NA        |
```
LogLikelihood : 1268.238

Information Criteria
------------------------------------
Akaike       -3.1950  
Bayes        -3.1773  
Shibata      -3.1951  
Hannan-Quinn -3.1882  

Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
<table>
<thead>
<tr>
<th></th>
<th>statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lag[1]</td>
<td>0.5265</td>
<td>0.4681</td>
</tr>
<tr>
<td>Lag[2*(p+q)+(p+q)-1][2]</td>
<td>0.5304</td>
<td>0.6795</td>
</tr>
<tr>
<td>Lag[4*(p+q)+(p+q)-1][5]</td>
<td>2.5233</td>
<td>0.5009</td>
</tr>
<tr>
<td>d.o.f=0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
<table>
<thead>
<tr>
<th></th>
<th>statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lag[1]</td>
<td>1.166</td>
<td>0.2803</td>
</tr>
<tr>
<td>Lag[2*(p+q)+(p+q)-1][5]</td>
<td>2.672</td>
<td>0.4702</td>
</tr>
<tr>
<td>Lag[4*(p+q)+(p+q)-1][9]</td>
<td>4.506</td>
<td>0.5054</td>
</tr>
<tr>
<td>d.o.f=2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
.....