Bus 41910: Time Series Analysis

Solutions to Final Exam, Autumn Quarter 2005

The solutions are brief. In some cases, there are multiple solutions and I can give a few that are most common. The notation in lectures and lecture notes is used throughout.

Problem A: (60 pts) Let $\{a_t\}$ be a sequence of independent and identically distributed random variables with mean zero and variance σ_a^2, $\{Z_t\}$ be a time series, and B the back-shift operator such that $BZ_t = Z_{t-1}$. Also all forecasts are based on the minimum mean squared error criterion. Briefly answer the following questions.

1. Suppose that $Z_t = Z_{t-1} + a_t$. Define an innovational outlier (IO) for the series at time index t_0.

$$Y_t = \frac{1}{1-B}(a_t + \omega I_t^{(t_0)}) = Z_t + \frac{\omega}{1-B} I_t^{(t_0)}.$$

2. What is the impact of the IO on the Z_t process of Problem 1?

   A: A level shift starting at time index t_0.

3. State two differences between intervention analysis and outlier analysis in a time series model.

   A: (1) The time index o fintervention is known. (2) The limiting distributions of the test statistics used are different.

4. Consider the ARMA(2,1) model $(1 - B + .2B^2)Z_t = (1 - 0.5B)a_t$. Put the model into a state-space form.

   A: I give three forms, but there are other possibilities:

   (a) Akaike's approach

$$\begin{bmatrix} Z_{t+1} \\ Z_{t+2|t+1} \\ Z_{t+3|t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -.2 & 1 \end{bmatrix} \begin{bmatrix} Z_t \\ Z_{t+1|t} \\ Z_{t+2|t} \end{bmatrix} + \begin{bmatrix} 1 \\ .5 \\ .3 \end{bmatrix} a_{t+1}$$

$$Z_t = [1, 0, 0]S_t.$$

   (b) Aoki's approach:

$$\begin{bmatrix} Z_t \\ Z_{t-1} \\ a_t \end{bmatrix} = \begin{bmatrix} 1 & -.2 & -.5 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_{t-1} \\ z_{t-2} \\ a_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} a_t,$$

$$Z_t = [1, -.2, -.5]S_t + a_t.$$

   (c) Harvey's approach:

$$S_{t+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -.2 & 1 \end{bmatrix} S_t + \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} a_t$$

$$Z_t = [1, 0, 0]S_t + a_t.$$

5. Consider the model $Z_t = T_t + R_t$, where $T_t = T_{t-1} + e_t$ and $(1 - \phi B)R_t = (1 - \theta B)a_t$, where $\phi \neq \theta$ and $\{e_t\}$ and $\{a_t\}$ are two independent white noise processes. Put the model in a state-space form.

   A: Again, I give two forms, but there are other possibilities:

   (a) Akaike's approach:

$$\begin{bmatrix} T_{t+1} \\ R_{t+1} \\ R_{t+2|t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0\phi & \end{bmatrix} \begin{bmatrix} T_t \\ R_t \\ R_{t+1|t} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \phi - \theta \end{bmatrix} \begin{bmatrix} e_{t+1} \\ a_{t+1} \end{bmatrix},$$

$$Z_t = [1, 1, 0]S_t.$$

   (b) Aoki's approach:

$$\begin{bmatrix} T_t \\ R_t \\ a_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi & -\theta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_t \\ R_t \\ a_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_t \\ a_t \end{bmatrix},$$

$$Z_t = [1, 1, -\theta]S_t + [1, 1] \begin{bmatrix} e_t \\ a_t \end{bmatrix}.$$

6. Suppose that Z_t follows a stationary AR(2) model $(1 - \phi_1 B - \phi_2 B^2)Z_t = a_t$ and that Z_{100} is missing. Based on the model and the data $\{Z_1, \ldots, Z_{150}\}$. Provide a least squares estimate of Z_{100}.

   A: Three observations involve Z_{100}. They are

$$\begin{aligned} Z_{100} &= \phi_1 Z_{99} + \phi_2 Z_{98} + a_{100} \\ Z_{101} &= \phi_1 Z_{100} + \phi_2 Z_{99} + a_{101} \\ Z_{102} &= \phi_1 Z_{101} + \phi_2 Z_{100} + a_{102}. \end{aligned}$$

   Consequently, we obtain

$$\begin{aligned} y_1 &= 1Z_{100} + b_1, \quad y_1 = \phi_1 Z_{99} + \phi_2 Z_{98}, \quad b_1 = -a_{100} \\ y_2 &= \phi_1 Z_{100} + b_2, \quad y_2 = Z_{101} - \phi_2 Z_{99}, \quad b_2 = -a_{101} \\ y_3 &= \phi_2 Z_{100} + b_3, \quad y_3 = Z_{102} - \phi_1 Z_{101}, \quad b_3 = a_{102}. \end{aligned}$$

   The OLS estimate of Z_{100} is

$$\hat{Z}_{100} = \frac{y_1 + \phi_1 y_2 + \phi_2 y_3}{1 + \phi_1^2 + \phi_2^2}.$$

7. Suppose $Z_t = X_t + Y_t$, where X_t is an AR(4) process and Y_t is an ARMA(1,1) process. Assume further that X_t and Y_t are independent. What is the model for Z_t? It suffices to provide the maximum order.

   A: An ARMA(5,5) model.

8. For an ARIMA process Z_t, what is the relationship between the two forecasts $Z_T(\ell)$ and $Z_{T+1}(\ell-1)$, where $Z_h(j)$ denotes the j-step ahead forecast of Z_{h+j} at the forecast origin h.

$$Z_{T+1}(\ell - 1) = Z_T(\ell) + \psi_{\ell-1} a_{T+1},$$

where ψ_i is the ith ψ-weight of Z_t.

9. Suppose that Z_t follows the random-walk model

$$Z_t = Z_{t-1} + a_t, \qquad Z_0 = 0$$

where, for simplicity, $\{a_t\}$ is a sequence of independent and identically distributed normal random variables with mean zero and variance 1. Consider the linear regression

$$\Delta Z_t = \beta Z_{t-1} + e_t,$$

with the ordinary least squares estimate

$$\hat{\beta} = \frac{\sum_{t=1}^{T} Z_{t-1} \Delta Z_t}{\sum_{t=1}^{T} Z_{t-1}},$$

where $\Delta Z_t = Z_t - Z_{t-1}$. What is the limiting distribution of $T^{-1} \sum_{t=1}^{T} Z_{t-1} \Delta Z_t$ as $T \to \infty$? (No proof is needed, but keep it as simple as possible.)

$$T^{-1} \sum_{t=1}^{T} Z_{t-1} \Delta Z_t \to_D \frac{1}{2} \left[W(1)^2 - 1 \right],$$

where $W(r)$ is a standard Brownian motion (or Wiener process).

10. Consider an intervention model

$$Y_t = \left(\omega_1 B + \frac{\omega_2 B^2}{1 - \delta B} \right) I_t^{(h)} + Z_t,$$

where $0 < \delta < 1$ and h is the time of intervention. What is the meaning of the parameter ω_1? Does the intervention have a permanent impact?

A: (1) ω_1 denotes a delay (by 1 time period) impact of the intervention.

(2) No, there is no permanent impact, because the second part is just a temporary change.

11. Consider an MA(1) model $Z_t = a_t - \theta a_{t-1}$, where $\{a_t\}$ is a Gaussian white noise with mean zero and variance σ_a^2. Write down the conditional log likelihood function for the data $\{Z_1, Z_2, \ldots, Z_T\}$.

A: The log likelihood function is

$$\ell \propto -\frac{T}{2} \ln(\sigma_a^2) - \frac{1}{2\sigma_a^2} \sum_{t=1}^{T} a_t^2,$$

where $a_0 = 0$ and $a_t = Z_t + \theta a_{t-1}$.

12. Describe two methods to check a fitted time series model.

A: Any two of (a) residual plot, (b) Ljung-Box statistics, (c) outlier detection.

13. Suppose that Z_t is a stationary AR(1) process $Z_t = \phi Z_{t-1} + a_t$. Suppose also that Z_{100} and Z_{101} are two consecutive missing values. What are the least squares estimates Z_{100} and Z_{101} based on the sample $\{Z_1, Z_2, \ldots, Z_{150}\}$ and the model.

A: The model gives three equations as below

$$\begin{aligned} Z_{100} &= \phi Z_{99} + a_{100} \\ Z_{101} &= \phi Z_{100} + a_{101} \\ Z_{102} &= \phi Z_{101} + a_{102}. \end{aligned}$$

From which, we obtain a multiple linear regression as

$$\begin{bmatrix} \phi Z_{99} \\ 0 \\ Z_{102} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \phi & -1 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} Z_{100} \\ Z_{101} \end{bmatrix} + \begin{bmatrix} -a_{100} \\ a_{101} \\ a_{102} \end{bmatrix}.$$

Therefore, the least squares estimates of Z_{100} and Z_{101} are

$$\begin{bmatrix} \widehat{Z}_{100} \\ \widehat{Z}_{101} \end{bmatrix} = \begin{bmatrix} 1+\phi^2 & -\phi \\ -\phi & 1+\phi^2 \end{bmatrix}^{-1} \begin{bmatrix} \phi Z_{99} \\ \phi Z_{102} \end{bmatrix}.$$

14. Consider an ARIMA model $\pi(B)Z_t = a_t$, where $\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \ldots$. Suppose that there are two consecutive additive outliers in the observed series

$$Y_t = \omega_1 I_t^{(100)} + \omega_2 I_t^{(101)} + Z_t.$$

Describe an estimate of the parameter vector $(\omega_1, \omega_2)'$ that is unbiased.

A: Let $e_t = \pi(B)Y_t$, $x_{1t} = \pi(B)I_t^{(100)}$, and $x_{2t} = \pi(B)I_t^{(101)}$. In addition, let $\omega = (\omega_1, \omega_2)'$ and $x_t = (x_{1t}, x_{2t})'$. Then, using multiple linear regression, we have

$$\hat{\omega} = \left(\sum_{t=100}^{T} x_t x_t' \right)^{-1} \left(\sum_{t=100}^{T} x_t e_t \right).$$

15. Define the lag-2 first extended autocorrelation function (EACF) for a time series Z_t.

A: Lag-2 ACF of $W_t = Z_t - \frac{\rho_3}{\rho_2} Z_{t-1}$, where ρ_i is the lag-i ACF of Z_t.

Problem B. (10 pts) Consider a state-space model

$$\begin{aligned} S_{t+1} &= FS_t + Ge_t \\ Z_t &= HS_t + \epsilon_t \end{aligned}$$

where $\{e_t\}$ and $\{\epsilon_t\}$ are serially uncorrelated with mean zero, and $\text{Cov}(e_t) = Q$, $\text{Cov}(\epsilon_t) = R$, and $\text{Cov}(e_t, \epsilon_t) = \Omega$. Derive a Kalman filter algorithm for the model.
A: From the model, we obtain

$$\begin{aligned} S_{t+1|t} &= FS_{t|t} \\ P_{t+1|t} &= FP_{t|t}F' + GQG' \\ Z_{t+1|t} &= HS_{t+1|t} \\ V_{t+1|t} &= HP_{t+1|t}H' + R \\ C_{t+1|t} &= HP_{t+1|t} + \Omega G', \quad C_{t+1|t} = \text{Cov}(Z_{t+1}, S_{t+1}|F_t). \end{aligned}$$

Using the normality of $(S_{t+1}, Z_{t+1})'$ given F_t, we obtain

$$\begin{aligned} S_{t+1|t+1} &= S_{t+1|t} + C'_{t+1|t}V^{-1}_{t+1|t}(Z_{t+1} - Z_{t+1|t}) \\ P_{t+1|t+1} &= P_{t+1|t} - C'_{t+1|t}V^{-1}_{t+1|t}C_{t+1|t}. \end{aligned}$$

Consequently, a version of Kalman filter is as below: Given $S_{1|0}$ and $P_{1|0}$,

$$\begin{aligned} Z_{t+1|t} &= HS_{t+1|t} \\ r_{t+1} &= Z_{t+1} - Z_{t+1|t} \\ V_{t+1|t} &= HP_{t+1|t}H' + R \\ C_{t+1|t} &= HP_{t+1|t} + \Omega G' \\ S_{t+2|t+1} &= FS_{t+1|t+1} = FS_{t+1|t} + FC'_{t+1|t}V^{-1}_{t+1|t}r_{t+1} \\ P_{t+2|t+1} &= FP_{t+1|t+1}F' + GQG'. \end{aligned}$$

Problem C. (24 pts) This problem is concerned with analysis of the monthly U.S. producer price index (PPI): finished goods from January 1977 to October 2005. The data obtained from the Federal Reserve Bank at St. Louis are seasonally adjusted and with Index 1982 = 100. We took the logarithm to stabilize the variability. The output of Splus and SCA is attached. All tests are based on the 5% significance level. Answer the following questions:

1. Is there a unit root in the log series of PPI? Setup the null and alternative hypotheses and perform the test. Draw the conclusion.

   A: Let $Z_t = \ln(PPI_t)$. The model fitted is

$$Z_t = \phi Z_{t-1} + \sum_{i=1}^{8} \phi_i \Delta Z_{t-i} + e_t,$$

   where $\Delta Z_t = Z_t - Z_{t-1}$. The hypotheses are $H_o : \phi = 1$ versus $H_a : \phi < 1$. The test statistic is the t-ratio of $\phi - 1$, which is -2.324 with p-value 0.165. Thus, cannot reject the unit-root hypothesis.

2. Is there a double unit root in the log PPI series? Why?

   A: No, the unit-root test rejects the null for the differenced data.

3. Write down the fitted model without outlier detection.

$$(1 - .32B - .102B^3 - .123B^5 - .193B^7)(1 - B)Z_t = .0003 + a_t.$$

4. (2 pts) How many outliers are detected based on the prior model?

   A: 13 outliers.

5. (6 pts) Write down the final model with outlier detection. How do you use the model to produce forecasts?

$$Y_t = \sum_{i=1}^{13} \omega_i V_i(B) I_t^{(h_i)} + Z_t,$$

   where $v_i(B)$ depends on the type of outlier identified, and Z_t follows the model

$$(1 - .322B - .172B^3 - .153B^5 - .191B^7)(1 - B)Z_t = 0.003 + a_t,$$

   where $\sigma_a = .00137$.

   To compute forecasts, do the following:

   - Compute $\omega_i v_i(B) I_t^{(h_i)}$ for $t = T + \ell$, where $\ell > 0$.
   - Compute the forecasts of $Z_{T+\ell}$.

   Combine the two values to produce forecasts.

6. Is the final model adequate? Why?

   A: Yes, based on the Ljung-Box statistics Q(24) = 29.2, which is insignificant.

Problem D. (6 pts) Consider the model

$$Z_t = Z_{t-1} + u_t, \quad u_t = (1 - \theta_1 B - \theta_2 B^2) a_t,$$

where $Z_0 = 0$, and $\{a_t\}$ is a sequence of independent and identically distributed random variables with mean zero and variance 1. Suppose that we fit an AR(1) model

$$Z_t = \phi Z_{t-1} + e_t,$$

and let

$$\hat{\phi} = \frac{\sum_{t=1}^{T} Z_t Z_{t-1}}{\sum_{t=1}^{T} Z_{t-1}^2}$$

be the ordinary least squares estimate of ϕ, where T is the sample size. A test statistic for testing the null hypothesis $H_o : \phi = 1$ is $T(\hat{\phi} - 1)$. What is the limiting distribution of the test statistic as the sample size $T \to \infty$.

A: The keys are to obtain $\sigma^2 = \lim_{T\to\infty} T^{-1} E(S_t^2)$ and $\sigma_u^2 = p\lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} u_t^2$.

Because $u_t = (1 - \theta_1 B - \theta_2 B^2) a_t$, with $\text{Var}(a_t) = 1$, we obtain (a) $\sigma_u^2 = \text{Var}(u_t) = 1 + \theta_1^2 + \theta_2^2$, and (b) $\sigma^2 = 1 + \theta_1^2 + \theta_2^2 - 2\theta_1 - 2\theta_2 + 2\theta_1\theta_2$.

Therefore,

$$T(\hat{\phi} - 1) \to_D \frac{1}{2}\left[W(1)^2 - \frac{\sigma_u^2}{\sigma^2}\right]\left[\int_0^1 W(r)^2 dr\right]^{-1},$$

where σ_u^2 and σ^2 are given above.