Graduate School of Business
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Bus 41910, Univariate Time Series Analysis, Mr. R. Tsay
Solutions to Homework Assignment #1

1. Using the backshift operator and factorization, we have $(1-\phi_1 B-\phi_2 B^2)Y_t = (1-\lambda_1 B)(1-\lambda_2 B)Y_t = 0$. From the factorization, we have

$$\begin{aligned}\lambda_i &= \frac{\phi_1 \pm \sqrt{\phi_1^2+4\phi_2}}{2} && (1)\\ \phi_1 &= \lambda_1+\lambda_2 && (2)\\ \phi_2 &= -\lambda_1\lambda_2. && (3)\end{aligned}$$

For Y_t to be stable, we require $|\lambda_i|<1$ or equivalently $-1<\lambda_i<1$. Consequently, by Eq.(3),

$$-1<\phi_2<1. \quad (4)$$

By Eq. (2),

$$-2<\phi_1<2. \quad (5)$$

Next, if $\phi_1^2+4\phi_2<0$, then $\phi_2<-\frac{\phi_1^2}{4}$. This condition gives an inside regime of a parabola determined by three points $(0,0),(2,-1)$, and $(-2,-1)$. Next, consider $\phi_1^2+4\phi_2\geq 0$. In this case, $\sqrt{\phi_1^2+4\phi_2}\geq 0$. Furthermore, we have $-1<\frac{\phi_1}{2}\pm\frac{\sqrt{\phi_1^2+4\phi_2}}{2}<1$. Based on the 2nd inequality,

$$\begin{aligned}\frac{\phi_1}{2} &+ \frac{\sqrt{\phi_1^2+4\phi_2}}{2}<1\\ &\Rightarrow \sqrt{\phi_1^2+4\phi_2}<2-\phi_1\\ &\Rightarrow \phi_1^2+4\phi_2<\phi_1^2-4\phi_1+4\\ &\Rightarrow \phi_2<1-\phi_1. && (6)\end{aligned}$$

Based on the first inequality,

$$\begin{aligned}-1 &< \frac{\phi_1}{2}-\frac{\sqrt{\phi_1^2+4\phi_2}}{2}\\ &\Rightarrow -2-\phi_1<-\sqrt{\phi_1^2+4\phi_2}\\ &\Rightarrow \phi_1^2+4\phi_1+4>\phi_1^2+4\phi_2\\ &\Rightarrow \phi_1+1>\phi_2. && (7)\end{aligned}$$

Combining Eqs. (4)-(7), we obtain a triangular with vertices at $(0,1),(2,-1)$ and $(-2,-1)$. The figure can be seen in p.17 of Hamilton (1994) or Figure 3.2 on page 61 of Box, Jenkins, and Reinsel (1994).

2. Part (a). From $(1-1.2B+0.5B^2)Y_t = 0$, we have $\lambda_j = 0.6 \pm \frac{\sqrt{0.56}i}{2} \approx 0.6 \pm 0.374i$, where $j = 1, 2$. The homogeneous solution (see lecture note 1) is $Y_t = \gamma_1 \lambda_1^t + \gamma_2 \lambda_2^t$, where the coefficient γ_1 and γ_2 are determined by the initial conditions. Specifically, we have

$$\begin{aligned} Y_0 &= \gamma_1 + \gamma_2 = 8, \\ Y_1 &= \gamma_1 \lambda_1 + \gamma_2 \lambda_2 = 8.54 \end{aligned}$$

The 2nd equation gives $0.6(\gamma_1+\gamma_2) - 0.374i(\gamma_1-\gamma_2) = 8.54$. Thus, $\gamma_1 - \gamma_2 = -10i$. Therefore, $\gamma_1 = 4-5i$ and $\gamma_2 = 4+5i$.

Part (b). First, using the result for the first-order difference equation and $X_0 = 2$, we have $X_t = 2(-0.5)^t$. Next, using the result on page 3 of Lecture 1,

$$Y_t = \frac{1-0.5^t}{0.5} + 0.4 \sum_{i=0}^{t-1} 0.5^i X_{t-i} + 0.5^t.$$

3. First, the lag-k autocovariance of X_t is $\gamma_k = E(X_t X_{t+k}) = E[(\sum_{i=0}^{\infty} \psi_i a_{t-i})(\sum_{j=0}^{\infty} \psi_j a_{t+k-j})]$ $= E(\sum_{i=0}^{\infty} \psi_{i+k}\psi_i a_{t-i}^2) = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}$.

Next, $\Gamma(z) = \sigma^2(\sum_{i=0}^{\infty} \psi_i z^i)(\sum_{i=0}^{\infty} \psi_i z^{-i})$. Work out the coefficient for the z^k (or z^{-k}) term of $\Gamma(z)$. You can verify that the coefficient is exactly γ_k.

4. See the file hw1-4.doc.