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Bus 41910, Time Series Analysis, Mr. R. Tsay

Solutions to Homework Assignment #2

Problem 1: Compute mean and ACF.

a. $E(Z_t) = 3/(1-0.7) = 10$. ACF is $\rho_i = 0.7^i$.

b. $E(Z_t) = 0.044/(1-0.33-0.13) = 0.081$. For ACF, $\rho_1 = \phi_1/(1-\phi_2) = 0.37931$. Then, use $\rho_i = \phi_1\rho_{i-1} + \phi_{i-2}\rho_{i-2}$ to obtain $\rho_2 = 0.25517$, $\rho_3 = 0.13352$, etc.

c. $E(Z_t) = 2$. For ACF, $\rho_1 = (-1.3 - 1.3 * 0.4)/(1 + 1.3^2 + 0.4^2) = -0.6386$, $\rho_2 = 0.4/(1 + 1.3^2 + .4^2) = 0.140$, and $\rho_3 = 0$.

d. $E(Z_t) = 1/(1-0.8) = 5$. For ACF, use the first two moment equations:(p.6 of lecture note 3) (i) $\gamma_0 - 0.8\gamma_1 = (1-\theta_1\psi_1)\sigma^2$ and (ii) $\gamma_1 - 0.8\gamma_0 = -\theta_1\sigma^2$, where $\theta_1 = -0.4$ and $\psi_1 = 1.2$. We have $\gamma_1 = 4.4\sigma^2$ and $\gamma_0 = 5\sigma^2$. Therefore, $\rho_1 = 0.88$, $\rho_2 = 0.8\rho_1 = 0.704$, and $\rho_3 = 0.8\rho_2 = 0.5632$.

e. $E(Z_t) = 1.0$. $\rho_1 = \rho_2 = \rho_3 = 0$ and $\rho_4 = -0.7/(1 + 0.7^2) = -0.4698$.

2. Simulation. Results depend on the particular realization. But they should show certain patterm as follows: (a) Sample ACFs are close to 1. (b) Sample ACFs are close to 1. (c) Sample ACF shows spikes at lags 4, 8, 12, etc. (d) Sample ACF shows a since and consince function. (d) In theory, $\rho_1 = 0.0725$ and $\rho_k = 0.95^{k-1}\rho_1$; the sample ACF should be close to this pattern. In this particular case, the ACFs are small and positive, and decay slowly. This feature is often regarded as an indication of *long memory* in the time series literature. Our example demonstrates that an ARMA(1,1) model can provide good approximations to such a series. [We discussed differencing in the class. There is substantial interest in considering the series $Y_t = (1-B)^d Z_t$, where $-0.5 < d < 0.5$. The operator $(1-B)^d$ is referred to as the *fractional differencing* and, if Y_t is stationary and invertiable then Z_t is said to be a fractionally integrated process. The ACFs of Z_t will be small and positive, and decay slowly at a hyperbolic rate (instead of the exponential rate).]

3. Define $\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$ and $\pi(B) = 1 - \sum_{i=1}^{\infty} \pi_i B^i$. Part (a): Using $\psi(B)\phi(B) = \theta(B)$ and by equating the coefficients, we have $\psi_1 = \phi_1 - \theta$, $\psi_2 = \psi_1\phi_1 + \phi_2$, $\psi_3 = \psi_2\phi_1 + \psi_1\phi_2$, and $\psi_4 = \psi_3\phi_1 + \psi_2\phi_2$. Using $\theta(B)\pi(B) = \phi(B)$ and equating the coefficients, we obtain $\pi_1 = \phi_1 - \theta$, $\pi_2 = \phi_2 + \pi_1\theta$, $\pi_3 = \pi_2\theta$ and $\pi_4 = \pi_3\theta$. Part (b): Using the results from Part (a), we obtain $\pi_1 = \sqrt{3}$, $\pi_2 = -1$, $\pi_3 = 0$, and $\psi_1 = \sqrt{3}$, $\psi_2 = \sqrt{3}\sqrt{3} - 1 = 2$, $\psi_3 = 2\sqrt{3} + \sqrt{3}(-1) = \sqrt{3}$.

4. Applying the operator $(1-0.5B)$ to Z_t, we have $(1-0.5B)Z_t = (1-0.5B)T_t + (1-0.5B)\epsilon_t$ $= a_t + (1-0.5B)\epsilon_t$. Thus, Z_t is an ARMA(1,1) model in the form $(1-0.5B)Z_t = (1-\theta B)b_t$, where $\{b_t\}$ is a white noise sequence. The parameters θ and σ_b^2 are determined by $(1+\theta^2)\sigma_b^2 = \sigma_a^2 + (1+0.5^2)\sigma_\epsilon^2 = 3.25$ and $\theta\sigma_b^2 = 0.5\sigma_\epsilon^2 = 0.5$. Therfore, $\theta = \frac{6.5 \pm \sqrt{38.25}}{2}$ and $\sigma_b^2 = 0.5/\theta$. Note that $|\theta| < 1$ so that there is only one solution.

5. See another word file.