Graduate School of Business
University of Chicago
Bus 41910, Time Series Analysis, Mr. R. Tsay

Solutions to Homework Assignment #3

1. Consider the models

$$(1-B)(1-B^4)Z_t = (1-0.5B)(1-0.6B^4)a_t,$$

where $\{a_t\}$ is a Gaussian white noise series with mean zero and variance 1. Assume also that $(Z_{96}, Z_{97}, \cdots, Z_{100}) = (119.0, 118.7, 116.5, 118.6, 126.2)$ and $(a_{96}, \cdots, a_{100}) = (0.4, 0.6, 1.9, -0.8, -0.5)$. Use the minimum mean squared error criterion to compute forecasts at the forecast origin $t = 100$.

- What is the 1-step ahead forecast $Z_{100}(1)$? What is the variance of the associated forecast error?

  Answer:

$$\begin{aligned}
Z_t &= Z_{t-1} + Z_{t-4} - Z_{t-5} + a_t - 0.5a_{t-1} - 0.6a_{t-4} + 0.3a_{t-5} \\
Z_{101} &= Z_{100} + Z_{97} - Z_{96} + a_{101} - 0.5a_{100} - 0.6a_{97} + 0.3a_{96} \\
Z_{100}(1) &= Z_{100} + Z_{97} - Z_{96} - 0.5a_{100} - 0.6a_{97} + 0.3a_{96} \\
Z_{100}(1) &= 125.91
\end{aligned}$$

  The variance of forecast error is Var(a_{101})=1.0

- What is the 2-step ahead forecast $Z_{100}(2)$? What is the variance of the associated forecast error?

  Answer: $Z_{100}(2) = Z_{100}(1) + Z_{98} - Z_{97} - 0.6a_{98} + 0.3a_{97} = 122.75$. The variance of forecast error $= (1+\psi_1^2)\text{Var}(a_t) = 1.25$, where $\psi_1 = 0.5$ is obtained via $(1 + \psi_1 B + \psi_2 B^2 + \dots) = \frac{(1-0.5B)(1-0.6B^4)}{(1-B)(1-B^4)}$.

- What is the 3-step ahead forecast $Z_{100}(3)$?

  Answer: $Z_{100}(3) = Z_{100}(2) + Z_{99} - Z_{98} - 0.6a_{99} + 0.3a_{98} = 125.9$.

- What is the 4-step ahead forecast $Z_{100}(4)$?

  Answer: $Z_{100}(4) = Z_{100}(3) + Z_{100} - Z_{99} - 0.6a_{100} + 0.3a_{99} = 133.56$.

2. Suppose that Z_t follows the model

$$(1-0.7B)Z_t = a_t,$$

where $\{a_t\}$ is a Gaussian white noise series with mean zero and variance 1. Consider the non-overlapping 3 aggregates, i.e. $Y_\ell = Z_{3\ell} + Z_{3\ell-1} + Z_{3\ell-2}$. What is the order of the ARMA model for Y_ℓ? Can you write down the specific model for Y_ℓ?
Answer: It is an ARMA(1,1) model. Specifically,

$$
\begin{aligned}
&(1 - 0.7^3 B)Y_\ell \\
&= (Z_{3\ell} + Z_{3\ell-1} + Z_{3\ell-2}) - 0.7^3(Z_{3\ell-3} + Z_{3\ell-4} + Z_{3\ell-5}) \\
&= (Z_{3\ell} - 0.7^3 Z_{3\ell-3}) + (Z_{3\ell-1} - 0.7^3 Z_{3\ell-4}) + (Z_{3\ell-2} - 0.7^3 Z_{3\ell-5}) \\
&= 0.7^2 a_{3\ell-2} + 0.7 a_{3\ell-1} + a_{3\ell} \\
&+ 0.7^2 a_{3\ell-3} + 0.7 a_{3\ell-2} + a_{3\ell-1} \\
&+ 0.7^2 a_{3\ell-4} + 0.7 a_{3\ell-3} + a_{3\ell-2} \\
&= a_{3\ell} + 1.7 a_{3\ell-1} + 2.19 a_{3\ell-2} + 1.19 a_{3\ell-3} + 0.49 a_{3\ell-4} \\
&\equiv (1 - \theta B) b_\ell
\end{aligned}
$$

where θ and $\mathrm{Var}(b_\ell)$ are determined by the variance and the lag-1 autocovariance of the process $b_\ell = a_{3\ell} + 1.7a_{3\ell-1} + 2.19a_{3\ell-2} + 1.19a_{3\ell-3} + 0.49a_{3\ell-4}$. That is, $(1+\theta^2)\sigma_b^2 = \mathrm{Var}(b_\ell)$=10.3423 and $-\theta\sigma_b^2 = \mathrm{Cov}(b_\ell, b_{\ell-1})$=2.023. Finally, we get $\theta = -0.2037$ (Note that $|\theta| < 1$) and $\sigma_b^2 = 9.9302$.

3. Consider the structural model

$$Z_t = T_t + S_t + \epsilon_{0t},$$

where T_t and S_t satisfy

$$(1 - B^2)T_t = \epsilon_{1t}, \quad (1 + B + B^2 + B^3)S_t = \epsilon_{2t},$$

where $\{\epsilon_{it}\}$ $(i = 0, 1, 2)$ are independent Gaussian white noise series with mean zero and variance 1. What is the ARMA model for Z_t?
Answer: The least common multiple of $(1-B^2)$ and $(1+B+B^2+B^3)$ is $(1-B^4)$. Therefore, applying $(1 - B^4)$ to Z_t, we have

$$
\begin{aligned}
(1 - B^4)Z_t &= (1 + B^2)\epsilon_{1t} + (1 - B)\epsilon_{2t} + (1 - B^4)\epsilon_{0t} \\
(1 - B^4)Z_t &= (\epsilon_{0t} + \epsilon_{1t} + \epsilon_{2t}) - \epsilon_{2,t-1} + \epsilon_{1,t-2} - \epsilon_{0,t-4}.
\end{aligned}
$$

Thus, Z_t follows an ARMA(4,4) process with AR polynomial $(1 - B^4)$ and MA polynomial $\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \theta_4 B^4$. [No lag-3 in MA.]

4. Derive the autocorrelation function of the following models, where $\{a_t\}$ is a Gaussian white noise series with mean zero and variance 1 and $s > 1$ is a positive integer.

- $Z_t = (1 - \theta B)(1 - \Theta B^s)a_t$.
- $Z_t = (1 - \theta B - \Theta B^s)a_t$.

Answer: Part 1. See lecture 5, page 3.

$$\rho_\ell = \begin{cases} 1 & \text{for } \ell = 0 \\ -\frac{\theta}{(1+\theta^2)} & \text{for } \ell = 1 \\ \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)} & \text{for } \ell = s-1 \\ -\frac{\Theta}{(1+\Theta^2)} & \text{for } \ell = s \\ \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)} & \text{for } \ell = s+1 \\ 0 & \text{otherwise} \end{cases}$$

Part 2.

$$\rho_\ell = \begin{cases} 1 & \text{for } \ell = 0 \\ -\frac{\theta}{(1+\theta^2+\Theta^2)} & \text{for } \ell = 1 \\ \frac{\Theta\theta}{(1+\theta^2+\Theta^2)} & \text{for } \ell = s-1 \\ -\frac{\Theta}{(1+\theta^2+\Theta^2)} & \text{for } \ell = s \\ 0 & \text{otherwise} \end{cases}$$

5. Consider the simple exponential smoothing method for forecasting. Suppose that the discount rate is 0.94 and the 1-step ahead forecast at origin $t = 100$ is $Z_{100}(1) = 50$. If $Z_{101} = 55$, then what is the 1-step ahead forecast at $t = 101$?
Answer: $Z_{101}(1) = (1-\delta)Z_{101} + \delta Z_{100}(1) = 0.06 \times 55 + 0.94 \times 50 = 50.3$.