Bus 41910: Time Series Analysis

Solution to Mid-term, Autumn Quarter 2005

The solution is brief.

Problem A: (72 pts) Let $\{a_t\}$ be a sequence of independent and identically distributed random variables with mean zero and variance σ_a^2, $\{Z_t\}$ be a time series, and B the back-shift operator such that $BZ_t = Z_{t-1}$. Also all forecasts are based on the minimum mean squared error criterion. Briefly answer the following questions.

1. What are the conditions that the series $(1-\phi_1 B - \phi_2 B^2)Z_t = \phi_0 + a_t$ is weakly stationary?

   Answer: All zeros of $\Phi(B)$ lie outside the unit circle, where $\Phi(B) \equiv (1-\phi_1 B - \phi_2 B^2)$.

2. What are the mean and variance of the process $(1-0.75B)Z_t = 0.25 + a_t$, if $\sigma_a^2 = 1.0$?

   Answer: mean=0.25/(1-0.75)=1; Variance=$\sigma_a^2/(1-0.75^2) = 2.29$

3. Assume that $\sigma_a^2 = 3.0$. What are the variance and covariance of the process $Z_t = 1.0 + a_t - 0.4a_{t-1}$?

   Answer: $\gamma_0 = (1+0.4^2)*3 = 3.48$, $\gamma_1 = -0.4*3 = -1.2$, and $\gamma_\ell = 0$ if $\ell > 1$.

4. Assume that $(1-1.3B+0.4B^2)Z_t = a_t$, where the variance of a_t is 1.0. What are the first two lags of the autocorrelation function of Z_t?

   Answer: (lecture 3, page 3). $\rho_0 = 1$, $\rho_1 = 1/(1+0.4) = 0.9286$, $\rho_2 = \phi_1\rho_1 + \phi_2\rho_0 = 0.807$.

5. Suppose that $(1-B)Z_t = 1 + a_t$, where $\sigma_a^2 = 2.0$, and at the forecast origin $T = 100$, we have 1-step ahead forecast $Z_{100}(1) = 20$. Compute the 2-step and 3-step ahead forecasts $Z_{100}(2)$ and $Z_{100}(3)$.

   Answer: $Z_t = Z_{t-1} + 1 + a_t$, therefore, $Z_{100}(2) = Z_{100}(1)+1 = 21$; $Z_{100}(3) = Z_{100}(2)+1 = 22$.

6. Suppose that $(1-\phi B)Z_t = (1-\theta B)a_t$, where $\phi \neq \theta$, and Z_t is stationary and invertible. What is the model of y_ℓ, where $y_\ell = Z_{2\ell} + Z_{2\ell-1}$? It suffices to write down the order.

   Answer: an ARMA(1,1) model. [More specifically, it is in the form $(1-\phi^2 B)y_\ell = (1-\theta B)b_\ell$.]

7. Suppose $Z_t = X_t + Y_t$, where $X_t = 0.8X_{t-1} + a_t$ and $Y_t = 0.5Y_{t-1} + b_t$ with $\{b_t\}$ being a white noise series independent of $\{a_t\}$ and $\text{Var}(b_t) = \sigma_b^2$. What is the model of Z_t? It suffices to write down the order of the model.

   Answer: an ARMA(2,1) model. [More specifically,

$$(1-0.8B)(1-0.5B)Z_t = (1-0.5B)a_t + (1-0.8B)b_t = a_t - 0.5a_{t-1} + b_t - 0.8b_{t-1},$$

with the right hand side being an MA(1) structure.]

8. Suppose that $Z_t = 1.0 + (1 - 0.87B + 0.27B^2)a_t$ and $a_{99} = 0.3$ and $a_{100} = -0.5$. Also, a_t is Gaussian with $\sigma_a^2 = 1.0$. Compute the 1-step ahead 95% interval forecast of Z_{101} at the forecast origin $T = 100$..

   Answer: $Z_{100}(1) = 1.0 - 0.87a_{100} + 0.27a_{99} = 1.516$; Var($e_{100}(1)$)= $\sigma_a^2 = 1$. 95% interval forecast= (1.516-1.96, 1.516+1.96)= (-0.444, 3.476).

9. Suppose that $(1 - \phi B)Z_t = (1 - \theta B)a_t$ is a stationary and invertible series. At the forecast origin T, what is the 2-step ahead forecast error? What is the variance of the 2-step ahead forecast error?

   Answer: $e_T(2) = \sum_{i=0}^{1} \psi_i a_{T+\ell-i} = a_{T+2} + (\phi - \theta)a_{T+1}$. Var($e_T(2)$)= $[1 + (\phi - \theta)^2]\sigma_a^2$.

10. Obtain all non-zero ACF of the model $Z_t = (1 - 0.5B - 0.8B^{12})a_t$, where $\text{Var}(a_t) = 2.0$.

    Answer: $\rho_1 = -0.5/(1 + 0.5^2 + 0.8^2) = -0.2646$, $\rho_{11} = 0.5 * 0.8/(1 + 0.5^2 + 0.8^2) = 0.212$, and $\rho_{12} = -0.8/(1 + 0.5^2 + 0.8^2) = -0.423$.

11. Consider the difference equation $(1 - 1.3B + 0.4B^2)Z_t = 0$. If $Z_0 = 2.0$ and $Z_1 = 1.3$, what is the value of Z_{15}?

    Answer: $Z_t = c_1 \times 0.5^t + c_2 \times 0.8^t$ (lecture 1, page 3). From the initial conditions, $c_1 + c_2 = 2.0$ and $0.5c_1 + 0.8c_2 = 1.3$ so that $c_1 = c_2 = 1$. Consequently, $Z_{15} = 0.0352$.

12. Suppose that $(1 - 0.8B)Z_t = a_t$. Let $Y_t = Z_t - 0.6Z_{t-1}$. What is the model of Y_t?

    Answer: ARMA(1,1) because $(1-0.8B)Y_t = (1-0.8B)Z_t - 0.6(1-0.8B)Z_{t-1} = a_t - 0.6a_{t-1}$. Alternatively, $Z_t = \frac{1}{1-0.8B}a_t$ so that $Y_t = (1 - 0.6B)Z_t = \frac{1-0.6B}{1-0.8B}a_t$.

13. What is the moment generating function of the model $(1 - 0.5B)Z_t = (1 - 2B)a_t$, where $\text{Var}(a_t) = 1.0$?

    Answer: $\Gamma(z) = \frac{\theta(z)\theta(z^{-1})}{\phi(z)\phi(z^{-1})}\sigma^2 = \frac{(1-2z)(1-2z^{-1})}{(1-0.5z)(1-0.5z^{-1})}$

14. Consider a time series Z_t. What is the condition of strong stationarity for Z_t?

    Answer: The joint distribution of $(Z_t, \ldots, Z_{t+s})$ is equal to that of $(Z_{t+r}, \ldots, Z_{t+r+s})$, $\forall r, s$.

15. For any linear time series, let $e_T(\ell)$ be the ℓ-step ahead forecast error at the origin T, assuming that the model is known. True or false that $\text{Var}[e_T(\ell)] \geq \text{Var}[e_T(\ell - 1)]$ for $\ell \geq 2$?

    Answer: True.

16. Define the AIC criterion for a Gaussian ARMA(p, q) model with sample size n.

    Answer: AIC(p,q)= $n\log(\hat{\sigma}_a^2) + 2(p+q)$, where $\hat{\sigma}_a^2$ is the MLE of the variance of the innovation noise.

17. Define the lag-1 first extended autocorrelation function (EACF) for a time series Z_t.

    Answer: The lag-1 first-order EACF of Z_t is the lag-1 ACF of $W_t = Z_t - \frac{\rho_2}{\rho_1} Z_{t-1}$, where ρ_i is the lag-i ACF of Z_t.

18. Suppose that $Z_t = Z_{t-1} + a_t$. Define $Y_\ell = Z_{3\ell} + Z_{3\ell-1} + Z_{3\ell-2}$ for $\ell = 1, 2, \ldots$. What is the model for Y_ℓ?
    Answer: ARMA(1,1) in time scale ℓ.

$$\begin{aligned}(1-B)Y_\ell &= (Y_\ell - Y_{\ell-1}) \\ &= [Z_{3\ell} + Z_{3\ell-1} + Z_{3\ell-2}] - [Z_{3\ell-3} + Z_{3\ell-4} + Z_{3\ell-5}] \\ &= [Z_{3\ell} - Z_{3\ell-3}] + [Z_{3\ell-1} - Z_{3\ell-4}] + [Z_{3\ell-2} - Z_{3\ell-5}] \\ &= [a_{3\ell} + a_{3\ell-1} + a_{3\ell-2}] + [a_{3\ell-1} + a_{3\ell-2} + a_{3\ell-3}] + [a_{3\ell-2} + a_{3\ell-3} + a_{3\ell-4}] \\ &\equiv (1-\theta B) b_\ell\end{aligned}$$

Problem B. (16 pts) Consider a time series of 296 observations. Use the computer output (both SCA and R) to answer the following questions:

1. Explain why an AR(3) model is used?

   Answer: PACF cuts off after lag 3.

2. Write down the fitted AR(3) model, including the residual standard error.

   Answer: $(1 - 1.976B + 1.374B^2 - 0.343B^3)Z_t = a_t$, where a_t is a white noise with mean zero and variance 0.1887^2.

3. Is the fitted AR(3) model adequate? Why?

   Answer: Yes, the residuals have essentially no significant serial correlations. [There is a minor serial correlation at lag-4, but it is small.]

4. What are the 2-step ahead forecast and its standard error at the forecast origin T=296?

   Answer: $Z_{296}(2) = -0.2193$; Standard error = 0.4179.

Problem C. (12 pts) Consider the monthly series of demand in electricity of a manufacturing sector in the U.S. The series has 264 observations and we analyze the log series. SCA and R outputs are attached. Answer the following questions:

1. Write down the fitted model, including residual standard error.

   Answer:$(1-B)(1-B^{12})Z_t = (1-0.4869B)(1-0.9751B^{12})a_t$, and $\sigma_a = 0.0182$.

2. Why is the Airline model specified for the series?

   Answer: The sample ACF follows essentially the pattern of an Airline model with non-zero ACF at lags 1, 11, 12 and 13.

3. Discuss briefly the implication of the fitted model. Answer: The AR part of the fitted model consists of the regular and seasonal difference. It says the demand of electricity also follows a seasonal (or periodic) pattern. [In addition, the near cancellation between seasonal AR and seasonal MA factors indicates the seasonal pattern is close being deterministic.]