Graduate School of Business, University of Chicago
Business 41914, Spring Quarter 2007, Mr. Ruey S. Tsay

Solutions to Midterm

Problem A: (30 points; 2 points per question) Answer briefly the following questions.

1. **(For problems 1 to 5).** Consider the following 2-dimensional linear time series $\boldsymbol{z}_t$

$$\boldsymbol{z}_t = \boldsymbol{\mu} + \sum_{i=0}^{\infty} \boldsymbol{\psi}_i \boldsymbol{a}_{t-i},$$

where $\boldsymbol{\psi}_0 = \boldsymbol{I}$, the 2×2 identity matrix. What is the necessary and sufficient condition that $\boldsymbol{z}_t$ is weakly stationary?

Answer: $\sum_{i=0}^{\infty} \|\boldsymbol{\psi}_i\|^2 < \infty$, where $\|\boldsymbol{A}\|$ denotes a matrix norm of $\boldsymbol{A}$.

2. Assume that $\boldsymbol{z}_t$ is stationary, derive the lag-ℓ autocovariance matrix of $\boldsymbol{z}_t$ with $\ell \geq 0$.

Answer: $\boldsymbol{\Gamma}_\ell = \sum_{i=0}^{\infty} \boldsymbol{\psi}_{\ell+i} \boldsymbol{\Sigma} \boldsymbol{\psi}_i'$.

3. Suppose that $\boldsymbol{z}_t = (z_{1t}, z_{2t})'$. Give a sufficient condition that z_{1t} does not depend on any past values of z_{2t}, but z_{2t} depends on some past values of z_{1t}.

Answer: $\psi_{12}^{(i)} = 0$ for all $i > 0$ and $\psi_{21}^{(j)} \neq 0$ for some $j > 0$, where $\psi_{uv}^{(j)}$ is the (u, v)-th element of the matrix $\boldsymbol{\psi}_j$.

4. Given the model, what is the 3-step ahead forecast error of $\boldsymbol{z}_t$ at the forecast origin $t = n$?

Answer: $\boldsymbol{e}_n(3) = \boldsymbol{a}_{n+3} + \boldsymbol{\psi}_1 \boldsymbol{a}_{n+2} + \boldsymbol{\psi}_2 \boldsymbol{a}_{n+1}$, because $\boldsymbol{z}_n(3) = \boldsymbol{\psi}_3 \boldsymbol{a}_n + \boldsymbol{\psi}_4 \boldsymbol{a}_{n-1} + \ldots$.

5. Given the model, what is the covariance matrix of 3-step ahead forecast errors of the series?

Answer: $\text{Cov}(\boldsymbol{e}_n(3)) = \boldsymbol{\Sigma} + \boldsymbol{\psi}_1 \boldsymbol{\Sigma} \boldsymbol{\psi}_1' + \boldsymbol{\psi}_2 \boldsymbol{\Sigma} \boldsymbol{\psi}_2'$.

6. **(For problems 6 to 7).** Suppose that the k-dimensional process $\boldsymbol{z}_t$ follows the VARMA(2,1) model

$$(\boldsymbol{I} - \boldsymbol{\phi}_1 B - \boldsymbol{\phi}_2 B^2) \boldsymbol{z}_t = \boldsymbol{c} + (\boldsymbol{I} - \boldsymbol{\theta}_1 B) \boldsymbol{a}_t,$$

where $\boldsymbol{c}$ is a constant vector. What is the necessary and sufficient condition that $\boldsymbol{z}_t$ is weakly stationary?

Answer: All zeros of the polynomial $|\boldsymbol{I} - \boldsymbol{\phi}_1 B - \boldsymbol{\phi}_2 B^2|$ are outside the unit circle, e.g. their absolute values are greater than 1.

7. Assuming weak stationarity, write down the moment equations of the $\boldsymbol{z}_t$ series.

Answer: $\boldsymbol{\Gamma}_\ell - \boldsymbol{\phi}_1 \boldsymbol{\Gamma}_{\ell-1} - \boldsymbol{\phi}_2 \boldsymbol{\Gamma}_{\ell-2} = \begin{cases} \boldsymbol{\Sigma} - \boldsymbol{\theta}_1 \boldsymbol{\Sigma} \boldsymbol{\psi}_1' & \text{if } \ell = 0 \\ -\boldsymbol{\theta}_1 \boldsymbol{\Sigma} & \text{if } \ell = 1 \\ \boldsymbol{0}, & \text{for } \ell > 1. \end{cases}$.

8. Consider the transfer function model $y_t = 1.0 + \frac{2.0B^2+3.0B^3-0.5B^4}{1-0.5B}x_t + \frac{1}{1-0.6B}a_t$, where x_t is the input variable satisfying $(1-0.8B)x_t = (1+0.4B)b_t$, and $\{a_t\}$ and $\{b_t\}$ are two independent white noise series with mean zero and variance σ_a^2 and σ_b^2, respectively. Derive the first 4 lags of the impulse response function of the model.

   Answer: Using the long division, we have $v_0 = v_1 = 0$, $v_2 = 2$, $v_4 = 4$, and $v_5 = 1.5$, etc.

9. Consider the unit-root time series $(1-B)(1-\phi B)z_t = a_t$, where $|\phi| < 1$. Let $\hat{\rho}_\ell$ be the lag-ℓ sample autocorrelation of z_t based on T data points. What is the limiting value of $\hat{\rho}_\ell$ as $T \to \infty$, where ℓ is a fixed integer. Briefly outline a proof to justify your statement.

   Answer: For a fixed ℓ, $\hat{\rho}_\ell$ converges to 1 as $T \to \infty$. Justtification: $\hat{\rho}_\ell - 1 \approx \frac{\sum_{t=\ell+1} z_{t-\ell}w_t}{\sum_{t=\ell+1}^{T} z_{t-\ell}^2}$, where $w_t = z_t - z_{t-1}$ is the differenced series. Using the unit-root results, $T(\hat{\rho}_\ell - 1)$ is asymptotically a ratio of functions of standard Brownian motion so that $\hat{\rho}_\ell \to_p 1$ as $T \to \infty$,.

10. Consider the seasonal time series

$$\boldsymbol{z}_t = \boldsymbol{\mu} + (\boldsymbol{I} - \boldsymbol{\theta}B)(\boldsymbol{I} - \boldsymbol{\Theta}B^{12})\boldsymbol{a}_t.$$

   Derive all non-zero autocovariance matrices $\boldsymbol{\Gamma}_\ell$ of $\boldsymbol{z}_t$ for $\ell > 0$.

   Answer: Write the model as $\boldsymbol{z}_t - \boldsymbol{\mu} = \boldsymbol{a}_t - \boldsymbol{\theta}\boldsymbol{a}_{t-1} - \boldsymbol{\Theta}\boldsymbol{a}_{t-12} + \boldsymbol{\theta}\boldsymbol{\Theta}\boldsymbol{a}_{t-13}$. It is easy to obtain that $\boldsymbol{\Gamma}_0 = \boldsymbol{\Sigma} + \boldsymbol{\theta}\boldsymbol{\Sigma}\boldsymbol{\theta}' + \boldsymbol{\Theta}\boldsymbol{\Sigma}\boldsymbol{\Theta}' + \boldsymbol{\theta}\boldsymbol{\Theta}\boldsymbol{\Sigma}\boldsymbol{\Theta}'\boldsymbol{\theta}'$. $\boldsymbol{\Gamma}_1 = -\boldsymbol{\theta}\boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\Theta}\boldsymbol{\Sigma}\boldsymbol{\Theta}'$. $\boldsymbol{\Gamma}_{11} = \boldsymbol{\Theta}\boldsymbol{\Sigma}\boldsymbol{\theta}'$. $\boldsymbol{\Gamma}_{12} = -\boldsymbol{\Theta}\boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\Theta}\boldsymbol{\Sigma}\boldsymbol{\theta}'$, and $\boldsymbol{\Gamma}_{13} = \boldsymbol{\theta}\boldsymbol{\Theta}\boldsymbol{\Sigma}$.

11. Define a scalar component model of order (2,1) for the vector time series $\boldsymbol{z}_t$.

   Answer: A non-zero vector $\boldsymbol{v}_0$ is a scalar component model of order (2,1), if there exist vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ such that $\boldsymbol{w}_t = \boldsymbol{v}_0'\boldsymbol{z}_t + \boldsymbol{v}_1'\boldsymbol{z}_{t-1} + \boldsymbol{v}_2'\boldsymbol{z}_{t-2}$ is uncorrelated with $\boldsymbol{a}_{t-q}$ for all $q > 1$.

12. **(For Problems 12 to 15).** Suppose that $\boldsymbol{z}_t$ follows the model

$$\boldsymbol{z}_t = \begin{bmatrix} 1.0 & 0.5 \\ -0.3 & 1.8 \end{bmatrix} \boldsymbol{z}_{t-1} - \begin{bmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{bmatrix} \boldsymbol{z}_{t-2} + \boldsymbol{a}_t.$$

   Write down the AR polynomial matrix of the model and find the matrix $\boldsymbol{\phi}(1)$.

   Answer: The AR polynomial matrix is

$$\begin{bmatrix} 1 - B + .2B^2 & -.5B + .3B^2 \\ .3B - .6B^2 & 1 - 1.8B + 1.1B^2 \end{bmatrix}, \quad \text{with} \quad \boldsymbol{\phi}(1) = \begin{bmatrix} .2 & -.2 \\ -.3 & .3 \end{bmatrix}.$$

13. Show that z_{1t} and z_{2t} are unit-root nonstationary.

   Answer: The determinant of the AR polynomial matrix is $\phi(B) = 1 - 2.8B + 3.25B^2 - 1.85B^3 + 0.4B^4$. Since $|\phi(1)| = 0$, the determinant contains the factor $(1-B)$. Therefore, the marginal models of z_{it} is unit-root nonstationary.

14. Write down an Error-Correction Model for the system.

    Answer:
$$\Delta \boldsymbol{z}_t = \begin{bmatrix} -.2 & .2 \\ .3 & -.3 \end{bmatrix} \boldsymbol{z}_{t-1} + \begin{bmatrix} .2 & .3 \\ -.6 & 1.1 \end{bmatrix} \Delta \boldsymbol{z}_{t-1} + \boldsymbol{a}_t,$$
    where $\Delta \boldsymbol{z}_t = \boldsymbol{z}_t - \boldsymbol{z}_{t-1}$.

15. Write down the co-integrating vector for the system in which the first element is 1.

    Answer: From the error correction model,
$$\begin{bmatrix} -.2 & .2 \\ .3 & -.3 \end{bmatrix} = \begin{bmatrix} -.2 \\ .3 \end{bmatrix} [1, -1].$$
    Thus, the co-integrating vector is $(1, -1)$.

Problem B. (20 points) Consider a 2-dimensional time series $\boldsymbol{z}_t = (z_{1t}, z_{2t})'$ with 200 observations. The SCA analysis of the data is attached in which z_{1t} and z_{2t} are called "x" and "y", respectively. Use the output and answer the following questions.

1. (4 points) Identify the order of a VARMA model for the series using either AIC or the Chi-square test statistics.

   Answer: Both AIC and the Chi-square statistics identify a VAR(4) model.

2. (7 points) A VARMA(1,1) model was entertained because the residual CCM of the AR(1) fit indicates simple serial dependence. In addition, insignificant parameters are removed. Write down the final fitted model, including the residual covariance matrix.

   Answer: The fitted model is
$$\boldsymbol{z}_t - \begin{bmatrix} 0.97 & 0 \\ 0 & 0.99 \end{bmatrix} \boldsymbol{z}_{t-1} = \boldsymbol{a}_t - \begin{bmatrix} 0.45 & 0.10 \\ -0.69 & 0.76 \end{bmatrix} \boldsymbol{a}_{t-1}, \quad \boldsymbol{\Sigma}_1 = \begin{bmatrix} 1.36 & -0.21 \\ -0.21 & 2.24 \end{bmatrix}.$$
   Note that the value $\theta_{12} = 0.1$ is caused by the fact that I forgot to set the parameter to zero in the etimation.

3. (4 points) Based on the fitted model, what is the marginal model for z_{1t}?

   Answer: The model for z_{1t} is
$$z_{1t} = 0.97 z_{1,t-1} + a_{1t} - 0.45 a_{1,t-1} - 0.1 a_{2,t-1}.$$
   Ignoring $0.1 a_{2,t-1}$, the model is approximately $(1 - B) z_{1t} = (1 - 0.45B) a_{1t}$.

4. (5 points) Based on the fitted model, discuss the relationship between the two series and write down the transfer function model, if it exists.

Answer: Ignoring the error of fixing $\theta_{12} = 0.1$, the model shows a transfer function system with z_{1t} being the input variable. To obtain a proper specification of a transfer function model, consider the linear regression

$$a_{2t} = \beta a_{1t} + \epsilon_t.$$

Based on the output, $\beta = -0.212/1.359 \approx -0.16$. From the fitted model, we have

$$(1-B)z_{2t} = 0.69Ba_{1t} + (1-0.76B)a_{2t}.$$

Therefore,

$$\begin{aligned}(1-B)z_{2t} &= 0.69Ba_{1t} + (1-0.76B)[-0.16a_{1t} + \epsilon_t] \\ &= (-0.16+0.81B)a_{1t} + (1-0.76B)\epsilon_t \\ &= (-0.16+0.81B)\frac{(1-B)}{1-0.45B}z_{1t} + (1-0.76B)\epsilon_t.\end{aligned}$$

Consequently, the transfer function model is

$$z_{2t} = \frac{-0.16+0.81B}{1-0.45B}z_{1t} + \frac{1-0.76B}{1-B}\epsilon_t.$$

[If one tak 0.1 seriously, then there is a feedback relationship between z_{1t} an dz_{2t}. There is no transfer function model.]

Problem C. (10 points) Consider the quarterly U.S. exports and imports of Goods, Services and Income from 1960 to 2006 for 188 observations. The data are in billions of dollars and from the Federal Reserve Banks of St Louis. In addition, the data are seasonally adjusted. We employ the log series of the two time series. S-Plus output is attached.

1. (6 points) Are the two individual log series unit-root nonstationary? Why?

   Answer: Yes, because the augmented Dickey-Fuller test statistics fail to reject the null hypothesis of unit root for both series.

2. (4 points) Are the two time series co-integrated? If yes, what is the number of co-integrating vectors? If not, why?

   Answer: No, there is no co-integration. Johansen's test fails to reject the null hypothesis of rank zero for the error correction matrix.

Problem D. (20 points) Suppose that $\boldsymbol{z}_t$ is a linear stationary process with Kronecker indices $\{k_1 = 2, k_2 = 1, k_3 = 1\}$. Write down the implied VARMA model for $\boldsymbol{z}_t$. How many parameters in the AR and MA polynomials that require estimation?

Answer: From the Kronecker indices, we have $p = \max\{2, 1, 1\} = 2$, and

$$z_{2,t+1} + \alpha_{2,1}^{(2)} z_{1,t+1} + \alpha_{1,1}^{(2)} z_{1t} + \alpha_{1,2}^{(2)} z_{2t} + \alpha_{1,3}^{(2)} z_{3t} = a_{2,t+1} + \alpha_{2,1}^{(2)} a_{1,t+1} + \boldsymbol{u}_2^{(2)} \boldsymbol{a}_t,$$

where $\boldsymbol{u}_2^{(2)}$ is a row vector.

$$z_{3,t+1} + \alpha_{2,1}^{(3)} z_{1,t+1} + \alpha_{1,1}^{(3)} z_{1t} + \alpha_{1,2}^{(3)} z_{2t} + \alpha_{1,3}^{(3)} z_{3t} = a_{3,t+1} + \alpha_{2,1}^{(3)} a_{1,t+1} + \boldsymbol{u}_3^{(3)} \boldsymbol{a}_t.$$

$$z_{1,t+2} + \alpha_{1,3}^{(1)} z_{1,t+1} + \alpha_{2,3}^{(1)} z_{1t} + \alpha_{1,2}^{(1)} z_{2t} + \alpha_{1,3}^{(1)} z_{3t} = a_{1,t+2} + \boldsymbol{u}_1 \boldsymbol{a}_{t+1} + \boldsymbol{u}_2 autwia_t.$$

Consequently, we have a VARMA(2,2) model with structure as

$$\begin{bmatrix} 1 & 0 & 0 \\ \alpha_1 & 1 & 0 \\ \alpha_2 & 0 & 1 \end{bmatrix} \boldsymbol{z}_t - \begin{bmatrix} \alpha_3 & 0 & 0 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{bmatrix} \boldsymbol{z}_{t-1} - \begin{bmatrix} \alpha_{10} & \alpha_{11} & \alpha_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{z}_{t-2} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \alpha_1 & 1 & 0 \\ \alpha_2 & 0 & 1 \end{bmatrix} \boldsymbol{a}_t - \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \\ \beta_7 & \beta_8 & \beta_9 \end{bmatrix} \boldsymbol{a}_{t-1} - \begin{bmatrix} \beta_{10} & \beta_{11} & \beta_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{a}_{t-2}.$$

The number of AR parameters is 12 and the number of MA parameters is 12.

Problem E. (20 points total, 10 points each question) Simple proofs

1. Consider a k-dimensional stationary time series $\boldsymbol{z}_t$. Let $\widehat{\boldsymbol{\Gamma}}_\ell$ and $\widehat{\boldsymbol{\rho}}_\ell$ be the lag-ℓ sample cross covariance and correlation matrix, respectively. The multivariate Ljung-Box statistics for the null hypothesis $H_o : \boldsymbol{\rho}_1 = \cdots = \boldsymbol{\rho}_m = \boldsymbol{0}$ versus $H_a : \boldsymbol{\rho}_i \neq \boldsymbol{0}$ for some $i \in \{1, \ldots, m\}$ is defined as

$$Q_k(m) = T^2 \sum_{\ell=1}^{m} \frac{1}{T-\ell} tr(\widehat{\boldsymbol{\Gamma}}_\ell' \widehat{\boldsymbol{\Gamma}}_0^{-1} \widehat{\boldsymbol{\Gamma}}_\ell \widehat{\boldsymbol{\Gamma}}_0^{-1}),$$

where T is the sample size and tr denotes the trace of a square matrix. Show that $Q_k(m)$ can be written as

$$Q_k(m) = T^2 \sum_{\ell=1}^{m} \frac{1}{T-\ell} \boldsymbol{b}_\ell' (\widehat{\boldsymbol{\rho}}_0^{-1} \otimes \widehat{\boldsymbol{\rho}}_0^{-1}) \boldsymbol{b}_\ell,$$

where $\otimes$ denotes the Kronecker product and $\boldsymbol{b}_\ell = \text{vec}(\widehat{\boldsymbol{\rho}}_\ell')$ with vec being the column-stacking operator.

Proof. Let $\boldsymbol{D}$ be the diagonal matrix of the sample standard errors of the time series.

$$\begin{aligned} tr(\widehat{\boldsymbol{\Gamma}}_\ell' \widehat{\boldsymbol{\Gamma}}_0^{-1} \widehat{\boldsymbol{\Gamma}}_\ell \widehat{\boldsymbol{\Gamma}}_0^{-1}) &= tr(\widehat{\boldsymbol{\Gamma}}_\ell' \widehat{\boldsymbol{\Gamma}}_0^{-1} \widehat{\boldsymbol{\Gamma}}_\ell \widehat{\boldsymbol{\Gamma}}_0^{-1} \boldsymbol{D}\boldsymbol{D}^{-1}) \\ &= tr(\boldsymbol{D}^{-1} \widehat{\boldsymbol{\Gamma}}_\ell' \widehat{\boldsymbol{\Gamma}}_0^{-1} \widehat{\boldsymbol{\Gamma}}_\ell \widehat{\boldsymbol{\Gamma}}_0^{-1} \boldsymbol{D}) \\ &= tr(\widehat{\boldsymbol{\rho}}_\ell' \widehat{\boldsymbol{\rho}}_0^{-1} \widehat{\boldsymbol{\rho}}_\ell \widehat{\boldsymbol{\rho}}_0^{-1}). \end{aligned}$$

The last step is obtained by properly inserting $\boldsymbol{D}\boldsymbol{D}^{-1}$ or $\boldsymbol{D}^{-1}\boldsymbol{D}$.

Next, using $vec(ABC) = vec(B')'(I \otimes C)vec(A)$ with $A = \widehat{\boldsymbol{\rho}}_\ell'$ and $B = \widehat{\boldsymbol{\rho}}_0^{-1} \widehat{\boldsymbol{\rho}}_\ell$, we have

$$\begin{aligned} tr(\widehat{\boldsymbol{\Gamma}}_\ell' \widehat{\boldsymbol{\Gamma}}_0^{-1} \widehat{\boldsymbol{\Gamma}}_\ell \widehat{\boldsymbol{\Gamma}}_0^{-1}) &= [vec(\widehat{\boldsymbol{\rho}}_\ell' \widehat{\boldsymbol{\rho}}_0^{-1})]' (\boldsymbol{I} \otimes \widehat{\boldsymbol{\rho}}_0^{-1}) \boldsymbol{b}_\ell \\ &= [(\widehat{\boldsymbol{\rho}}_0^{-1} \otimes \boldsymbol{I}) \boldsymbol{b}_\ell]' (\boldsymbol{I} \otimes \widehat{\boldsymbol{\rho}}_0^{-1}) \boldsymbol{b}_\ell \\ &= \boldsymbol{b}_\ell' (\widehat{\boldsymbol{\rho}}_0^{-1} \otimes \boldsymbol{I})(\boldsymbol{I} \otimes \widehat{\boldsymbol{\rho}}_0^{-1}) \boldsymbol{b}_\ell \\ &= \boldsymbol{b}_\ell' (\widehat{\boldsymbol{\rho}}_0^{-1} \otimes \widehat{\boldsymbol{\rho}}_0^{-1}) \boldsymbol{b}_\ell. \end{aligned}$$

In the above, we have used $(A \times B)' = A' \otimes B'$, $vec(AB) = (B' \otimes I)vec(A)$ and $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.

2. Consider the univariate process

$$z_t = z_{t-1} + a_t - \theta a_{t-2},$$

where $z_0 = 0$ and $|\theta| < 1$. Suppose that the sample $\{z_1, \ldots, z_T\}$ is available. Derive the limiting distribution of $T^{-2} \sum_{t=1}^{T} z_{t-1}^2$ as $T \to \infty$.

Proof. From the first three problems of HW#3, (see also Phillips, 1987), the limiting distribution of $T^{-2} \sum_{t=1}^{T} z_{t-1}^2$ is $\sigma^2 \int_0^1 W(r)^2 dr$, The remainding issue is to find the quantity σ^2, which is $\lim_{T\to\infty} E(T^{-1} S_T^2)$, where $S_t = u_1 + u_2 + \ldots + u_T$ with $u_t = a_t - \theta a_{t-2}$. Straightforward calculation shows that $\sigma^2 = (1-\theta)^2 \sigma_a^2$. Thus, we have $T^{-2} \sum_{t=1}^{T} z_{t-1}^2 = (1-\theta)^2 \sigma_a^2 \int_0^1 W(r)^2 dr$ as $T \to \infty$.