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Solutions to Homework Assignment 3

1. Assume that $\{a_t\}$ is a sequence of martingale difference such that $E(a_t|F_{t-1}) = 0$, $E(a_t^2|F_{t-1}) = \sigma_a^2$, and $E(|a_t|^\delta|F_{t-1}) < \infty$ for some $\delta > 2$, where F_{t-1} is the σ-field generated by $\{a_{t-1}, a_{t-2}, \ldots\}$. Further, assume that $z_t = z_{t-1} + a_t$ with z_0 being a finite real number. In addition, let T be the sample size, $w(u)$ be a standard Brownian motion on the unit interval [0,1], and $\to_d$ denotes convergence in distribution. Derive the following results as $T \to \infty$.

   - $(\sigma_a T)^{-2} \sum_{t=1}^T z_{t-1}^2 \to_d \int_0^1 w(u)^2 du$.
   - $(\sigma_a T)^{-1} \sum_{t=1}^T z_{t-1} a_t \to_d \frac{1}{2}[w(1)^2 - 1]$.
   - Let $\hat{\phi} = \sum_{t=1}^T z_t z_{t-1} / \sum_{t=1}^T z_{t-1}^2$ be the ordinary least square estimate of AR(1) coefficient, and t_ϕ be the t-ratio for testing the null hypothesis $H_o : \phi = 1$. Show that $t_\phi \to_d \frac{(1/2)[w(1)^2-1]}{[\int_0^1 w(u)^2 du]^{1/2}}$.

   Proof. Let $S_t = \sum_{i=1}^t a_i$ be the partial sum of $\{a_i\}$. From the model, $z_t = z_0 + S_t$. Also, let $[x]$ denotes the integer part of x. By Functional Central Limit theorm (see Lemma 2.2 of Phillip (1987)),

$$X_T(u) \equiv \frac{1}{\sqrt{T}\sigma_a} S_{[Tu]} \to_d w(u), \quad 0 < u \le 1.$$

   Now, for part (a), we have

$$\begin{aligned}
(\sigma_a T)^{-2} \sum_{t=1}^T z_{t-1}^2 &= (\sigma_a T)^{-2} \sum_{t=1}^T (S_{t-1} + z_0)^2 \\
&= (\sigma_a T)^{-2} \sum_{t=1}^T (S_{t-1}^2 + 2 z_0 S_{t-1} + z_0^2) \\
&= \sum_{t=1}^T \int_{(t-1)/T}^{t/T} (1/(\sigma_a^2 T)) S_{[Tu]}^2 du + 2 z_0 \sigma_a^{-1} T^{-1/2} \sum_{t=1}^T \int_{(t-1)/T}^{t/T} (1/(\sigma_a \sqrt{T})) S_{[Tu]} du + z_0^2/(\sigma_a^2 \\
&= \int_0^1 X_T^2(u) du + 2 z_0 \sqrt{T} \int_0^1 X_T(u) du + z_0^2/(\sigma_a^2 T) \\
&\to_d \int_0^1 w(u)^2 du, \quad T \to \infty.
\end{aligned}$$

   For part (b), we have

$$(\sigma_a^2 T)^{-1} \sum_{t=1}^T z_{t-1} a_t = (\sigma_a^2 T)^{-1} \sum_{t=1}^T (S_{t-1} + z_0) a_t$$

$$
\begin{aligned}
&= (\sigma_a^2 T)^{-1} \sum_{t=1}^{T} S_{t-1} a_t + z_0 \bar{a}/\sigma_a^2 \\
&= (\sigma_a^2 T)^{-1} \sum_{t=1}^{T} \frac{1}{2}(S_t^2 - S_{t-1}^2 - a_t^2) + z_0 \bar{a}/\sigma_a^2 \\
&= (2\sigma_a^2 T)^{-1} S_T^2 - (2\sigma_a^2 T)^{-1} \sum_{t=1}^{T} a_t^2 + z_0 \bar{a}/\sigma_a^2 \\
&= \frac{1}{2} X_T(1)^2 - \frac{1}{2\sigma_a^2} T^{-1} \sum_{t=1}^{T} a_t^2 + z_0 \bar{a} \sigma_a^2 \\
&\to_d \frac{1}{2}(w(1)^2 - 1),
\end{aligned}
$$

because $\bar{a} \to 0$ and $T^{-1}\sum_{t=1}^{T} a_t^2 \to \sigma_a^2$ almost surely as $T \to \infty$.

For part (c), the t-ratio is

$$
\begin{aligned}
t_\phi &= \frac{\hat{\phi} - 1}{se(\hat{\phi})} \\
&= \frac{\sum_{t=1}^{T} z_{t-1} a_t}{\sum_{t=1}^{T} z_{t-1}^2} \times \frac{(\sum_{t=1}^{2} z_{t-1}^2)^{0.5}}{s} \\
&= \frac{\sum_{t=1}^{T} z_{t-1} a_t}{s(\sum_{t=1}^{T} z_{t-1}^2)^{0.5}},
\end{aligned}
$$

where s is the residual standard error giving by $s^2 = \frac{1}{T-1}\sum_{t=1}^{T}(z_t - \hat{\phi} z_{t-1})^2$. Since $(\hat{\phi} - 1) = O_P(T^{-1})$, s^2 is asymptotically equivalent to $\sigma^2 = \frac{1}{T}\sum_{t=1}^{T}(z_t - z_{t-1})^2 = frac1T \sum_{t=1}^{T} a_t^2$, which converges to σ_a^2 as $T \to \infty$. Consequently, we have

$$
\begin{aligned}
t_\phi &= \frac{\sum_{t=1}^{T} z_{t-1} a_t}{\sigma_a(\sum_{t=1}^{T} z_{t-1}^2)^0.5} \\
&= \frac{\sigma_a^{-2} T^{-1} \sum_{t=1}^{T} z_{t-1} a_t}{(\sigma_a T)^{-1}(\sum_{t=1}^{T} z_{t-1}^2)^0.5} \\
&= \frac{\sigma_a^{-2} T^{-1} \sum_{t=1}^{T} z_{t-1} a_t}{\sqrt{(\sigma_a T)^{-2} \sum_{t=1}^{T} z_{t-1}^2}}.
\end{aligned}
$$

Using the results of parts (a) and (b), and continuous mapping theorm (Lemma 2.3 of Phillips (1987)), we have

$$
t_\phi \to_d \frac{(1/2)[w(1)^2 - 1]}{[\int_0^1 w(u)^2 du]^{1/2}}.
$$

2. For simplicity, assume that $z_t = z_{t-1} + a_t - \theta a_{t-1}$, where $\{a_t\}$ is an *iid* sequence of $N(0, \sigma_a^2)$, $|\theta| < 1$ and $\sigma_a^2 > 0$. Derive the distribution of $T^{-2}\sum_{t=1}^{T} z_{t-1}^2$ as $T \to \infty$.

Proof: Let $u_t = a_t - \theta a_{t-1}$. It is easy to see that u_t is a stationary MA(1) process so that its partial sum $S_T = \sum_{t=1}^T u_t$ gives

$$\sigma^2 = \lim_{T\to\infty} T^{-1}E(S_T^2) = \lim_{T\to\infty}[\text{var}(u_t) + 2\frac{T-1}{T}\text{cov}(u_t, u_{t-1}] = (1-\theta)^2\sigma_a^2.$$

Consequently, from part(a) of Question 1, we have

$$T^{-2}\sum_{t=1}^T z_{t-1}^2 \to_d \sigma^2 \int_0^1 w(u)^2 du = (1-\theta)^2\sigma_a^2 \int_0^1 w(u)^2 du.$$

3. Solve the same problem as Problem 2, but assuming that z_t follow the ARIMA(1,1,0) model $(1-B)(1-\phi B)z_t = a_t$, where $|\phi| < 1$.

   Proof: From the model, we have

$$z_t = z_{t-1} + \phi(z_{t-1} - z_{t-2}) + a_t = z_{t-1} + \phi x_{t-1} + a_t,$$

where $x_t = z_t - z_{t-1}$. Also, it follows that $x_t = \phi x_{t-1} + a_t$, which is stationary. Let $S_t = \sum_{t=1}^T x_t$ be the partial sum of x_t. Then,

$$\begin{aligned}
\sigma^2 &= \lim_{T\to\infty} T^{-1}E(S_T^2) \\
&= \lim_{T\to\infty}[\gamma_0 + 2\frac{T-1}{T}\gamma_1 + 2\frac{T-2}{T}\gamma_2 + \cdots + 2\frac{1}{T}\gamma_{T-1}] \\
&= \lim_{T\to\infty}\gamma_0(1 + 2\frac{T-1}{T}\phi + 2\frac{T-2}{T}\phi^2 + \ldots + 2\frac{1}{T}\phi^{T-1}) \\
&\to \gamma_0\frac{1+\phi}{1-\phi},
\end{aligned}$$

where γ_i is the lag-i autovariance of the AR(1) process x_t. Since $\text{var}(x_t) = \frac{\sigma_a^2}{1-\phi^2}$, we have $\sigma^2 = \frac{\sigma_a^2}{(1-\phi)^2}$. Consequently,

$$T^{-2}\sum_{t=1}^T z_{t-1}^2 \to_d \frac{\sigma_a^2}{(1-\phi)^2}\int_0^1 w(u)^2 du.$$

4. Consider the U.S. monthly series of 30-year fixed mortgage rate from June 1976 to March 2007. Take the log transformation. Is there a unit root in the series? Test the hypothesis using 5% significance level.

   Answer: Yes, there is a unit root. The augmented Dicky-Fuller test with 9 lags gives test statistic -0.88 with p-value 0.79.

5. Consider the prior mortgage rate series and the U.S. monthly series of treasury interest rate (constant maturity 2 years) of the same period. Again, take the log transformation of both series. Is there a co-integration between them? If yes, what is the co-integrating vector?

   Answer: No, Johansen's tests fail to reject the null hypothesis of rank zero for the error-correction term. The test statistics for the trace are 7.91 and 1.21 for $H(0)$ and $H(1)$, respectively when $p = 3$ is used.