

**Lecture 6: Structural Specification of VARMA Models continued**

### 4.3 Order Specification

Turn to data analysis. For a given data set  $\{z_t | t = 1, \dots, T\}$ , we discuss methods to identify the Kronecker indices or the orders of scalar component models. It turns out that the technique of canonical correlation analysis is applicable for both approaches of structural specification. For details of canonical correlation analysis, readers are referred to any multivariate statistical analysis textbook, e.g. Johnson and Wichern (2002, Ch.10).

In the literature, canonical correlation analysis has been used to identify Kronecker indices by Akaike (1976), Cooper and Wood (1982) and Tsay (1989). A canonical correlation  $\rho$  between two random vectors  $\mathbf{P}$  and  $\mathbf{F}$  can be obtained from the eigenvalue-eigenvector problem:

$$\Sigma_{pp}^{-1} \Sigma_{pf} \Sigma_{ff}^{-1} \Sigma_{fp} \mathbf{v}_p = \rho^2 \mathbf{v}_p, \quad \Sigma_{ff}^{-1} \Sigma_{fp} \Sigma_{pp}^{-1} \Sigma_{pf} \mathbf{v}_f = \rho^2 \mathbf{v}_f, \quad (4.37)$$

where  $\Sigma_{fp} = \text{Cov}(\mathbf{F}, \mathbf{P})$  and  $\mathbf{v}_f$  and  $\mathbf{v}_p$  are eigenvectors associated with the eigenvalue  $\rho^2$ . The variable  $X = \mathbf{v}'_f \mathbf{F}$  and  $Y = \mathbf{v}'_p \mathbf{P}$  are the corresponding canonical variates. The canonical correlation  $\rho$  is the cross-correlation between  $X$  and  $Y$ , i.e.  $\rho = \text{corr}(X, Y)$ . In practice, sample covariance matrices of  $\mathbf{F}$  and  $\mathbf{P}$  are used to perform canonical correlation analysis.

#### 4.3.1 Reduced Rank Tests

Let  $p$  and  $f$  be the dimension of  $\mathbf{P}$  and  $\mathbf{F}$ , respectively. Without loss of generality, we assume that  $f \leq p$ . Suppose that  $\mathbf{X} = (\mathbf{F}', \mathbf{P}')'$  is  $(p + f)$ -dimensional normal random vector and we are interested in testing the null hypothesis  $H_0 : \Sigma_{fp} = \mathbf{0}$ . As discussed in Johansen's co-integration test, this is equivalent to testing for  $\beta = \mathbf{0}$  in the multivariate linear regression

$$\mathbf{F}'_i = \mathbf{P}'_i \beta' + \mathbf{E}'_i, \quad i = 1, \dots, T,$$

and we can employ the likelihood ratio statistic to perform the test. Furthermore, the likelihood ratio test can be written as

$$\text{LR} = -[T - 1 - \frac{1}{2}(p + f + 1)] \sum_{i=1}^f \ln(1 - \hat{\rho}_i^2)$$

where  $T$  is the sample size and  $\hat{\rho}_1^2 \geq \hat{\rho}_2^2 \geq \dots \geq \hat{\rho}_f^2$  are the ordered squared sample canonical correlations between  $\mathbf{F}$  and  $\mathbf{P}$ . Under the assumption of independent random sample, the LR statistic has an asymptotic Chi-squared distribution with  $p \times f$  degrees of freedom.

Suppose that the null hypothesis  $H_0 : \Sigma_{fp} = \mathbf{0}$  is rejected. It is then natural to examine the magnitude of the individual canonical correlations. Since the canonical correlations are ordered from

the largest to the smallest, we can begin by assuming that only the smallest canonical correlation, in modulus, is zero and the remaining  $(f - 1)$  canonical correlations are non-zero. That is, we are interested in the hypothesis

$$H_0 : \rho_{f-1}^2 > 0 \quad \text{but} \quad \rho_f^2 = 0 \quad \text{vs} \quad H_a : \rho_f^2 > 0. \quad (4.38)$$

Bartlett (1939) has argued that the hypotheses in Eq. (4.38) can be tested by the likelihood ratio criterion using the test statistic

$$C^* = -[T - 1 - \frac{1}{2}(p + f - 1)] \ln(1 - \hat{\rho}_f^2) \quad (4.39)$$

which is asymptotically a Chi-squared distribution with  $(p - f + 1)$  degrees of freedom. We shall use this type of likelihood ratio statistic in this chapter.

## 4.4 Finding Kronecker Indices

As discussed before, Kronecker indices are closely related to the rank and the row dependence of the Hankel matrix of  $\mathbf{z}_t$ . In practice, however, we can only entertain a finite-dimensional Hankel matrix. To this end, we approximate the past vector  $\mathbf{P}_{t-1}$  by a truncated subset. Let  $\mathbf{P}_{r,t-1} = (\mathbf{z}'_{t-1}, \dots, \mathbf{z}'_{t-r})'$  be a subset of  $\mathbf{P}_{t-1}$ , where  $r$  is a properly chosen positive integer. In practice,  $r$  is often the order of a VAR model for  $\mathbf{z}_t$  selected by an information criterion, e.g. AIC or BIC. This choice of  $r$  can be justified because under the assumption that  $\mathbf{z}_t$  follows a VARMA model the rank  $m$  of the Hankel Matrix  $\mathbf{H}_\infty$  is finite.

Using the Toeplitz property of the Hankel matrix, we can search the Kronecker indices by examining the row dependence one by one starting from the first row. Specifically, we construct a subvector  $\mathbf{F}_t^*$  of  $\mathbf{F}_t$  by moving elements one by one from  $\mathbf{F}_t$  into  $\mathbf{F}_t^*$ , starting with  $\mathbf{F}_t^* = \mathbf{z}_{1t}$ . To check the row dependence of  $\mathbf{H}_\infty$ , we employ  $\mathbf{P}_{r,t-1}$  and  $\mathbf{F}_t^*$  and use the following procedure:

1. Suppose that the last element of  $\mathbf{F}_t^*$  is  $z_{i,t+h}$  with  $h \geq 0$ . Perform the canonical correlation analysis between  $\mathbf{P}_{r,t-1}$  and  $\mathbf{F}_t^*$ . Let  $\hat{\rho}$  be the smallest sample canonical correlation in modulus between  $\mathbf{P}_{r,t-1}$  and  $\mathbf{F}_t^*$ . Let  $x_{t+h} = \mathbf{v}'_f \mathbf{F}_t^*$  and  $y_{t-1} = \mathbf{v}'_p \mathbf{P}_{r,t-1}$  be the corresponding canonical variates. The use of subscripts  $t+h$  and  $t-1$  will be explained later.
2. Consider the null hypothesis  $H_0 : \rho = 0$  versus the alternative hypothesis  $H_a : \rho \neq 0$ . Here it is understood that by the nature of the procedure, the second smallest canonical correlation between  $\mathbf{P}_{r,t-1}$  and  $\mathbf{F}_t^*$  is non-zero. Consequently, we employ a modified test statistic of Eq. (4.39), namely

$$C = -(T - r) \ln\left(1 - \frac{\hat{\rho}^2}{\hat{d}}\right) \quad (4.40)$$

where  $T$  is the sample size,  $r$  is the VAR order for  $\mathbf{z}_t$ , and  $\hat{d} = 1 + 2 \sum_{j=1}^h \hat{\rho}_{xx}(j) \hat{\rho}_{yy}(j)$ . In Eq. (4.40), it is understood that  $\hat{d} = 1$  if  $h = 0$  and  $\hat{\rho}_{xx}(j)$  and  $\hat{\rho}_{yy}(j)$  are the lag- $j$  sample autocorrelation coefficients of  $\{x_t\}$  and  $\{y_t\}$  series, respectively.

3. Compare the test statistic  $C$  of Eq. (4.40) with a Chi-squared distribution with  $kr - f + 1$  degrees of freedom, where  $kr$  and  $f$  are the dimensions of  $\mathbf{P}_{r,t-1}$  and  $\mathbf{F}_t^*$ , respectively.

- (a) If the test statistic  $C$  is statistically significant, then there is no linearly dependent row found. Go to Step 4.
  - (b) If the test statistic  $C$  is not significant, then  $z_{i,t+h}$  gives rise to a linearly dependent row of the Hankel matrix. In this case, we find the Kronecker index  $k_i = h$  for  $z_{it}$  and remove all elements  $z_{i,t+s}$  with  $s \geq h$  from  $\mathbf{F}_t$ .
4. If  $\mathbf{F}_t$  reduces to an empty set, stop. Otherwise, augment  $\mathbf{F}_t^*$  by the next available element of  $\mathbf{F}_t$  and go to Step 1.

The asymptotic limiting distribution

$$C = -(T - r) \ln\left(1 - \frac{\hat{\rho}^2}{\hat{d}}\right) \sim \chi_{kr-f+1}^2$$

was shown in Tsay (1989). It is a modification of Barlett test statistic in Eq. (4.39). The basic idea is as follows. Since the last element of  $\mathbf{F}_t^*$  is  $z_{i,t+h}$ , the canonical variate  $x_{t+h}$  is a linear function of  $\{\mathbf{z}_{t+h}, \dots, \mathbf{z}_t\}$ . On the other hand, the other canonical variate  $y_{t-1}$  is a linear function of  $\{\mathbf{z}_{t-1}, \dots, \mathbf{z}_{t-r}\}$ . Therefore, the time lag between  $x_{t+h}$  and  $y_{t-1}$  is  $h + 1$ . Since the canonical correlation coefficient  $\rho$  is the cross-correlation between the canonical variates  $x_{t+h}$  and  $y_{t-1}$ , we can think of  $\rho$  as the lag- $(h + 1)$  cross-correlation between  $\{x_t\}$  and  $\{y_t\}$  series.

Let  $\rho_{xy}(j)$  be the lag- $j$  cross-correlation between  $x_t$  and  $y_t$ . Under the null hypothesis  $H_0 : \rho_{xy}(\ell) = 0$ , the asymptotic variance of the sample cross-correlation  $\hat{\rho}_{xy}(\ell)$  is (Box and Jenkins (1976, p.736))

$$\text{Var}[\hat{\rho}_{xy}(\ell)] \approx T^{-1} \sum_{v=-\infty}^{\infty} \{\rho_{xx}(v)\rho_{yy}(v) + \rho_{xy}(\ell + v)\rho_{yx}(\ell - v)\}. \quad (4.41)$$

For canonical correlation analysis, under  $H_0 : \rho = 0$ , we have  $\text{Cov}(x_{t+h}, \mathbf{P}_{t-1}) = \mathbf{0}$ . Thus,  $\text{Cov}(x_{t+h}, \mathbf{z}_{t-j}) = \mathbf{0}$  for all  $j > 0$ . Therefore,  $\text{Cov}(x_t, x_{t-j}) = 0$  for all  $j \geq h + 1$  because  $x_{t-j}$  is a linear function of  $\mathbf{P}_{t-1}$ . Consequently,  $\rho_{xx}(j) = 0$  for  $j \geq h + 1$  and the  $x_t$  series is an  $\text{MA}(h)$  process. Using this fact and Eq. (4.41), the asymptotic variance of the sample canonical correlation coefficient  $\hat{\rho}$  is  $\text{Var}(\hat{\rho}) = T^{-1}[1 + 2\sum_{v=1}^h \rho_{xx}(v)\rho_{yy}(v)] \equiv \hat{d}$ . The test statistic in Eq. (4.40) uses normalized squared canonical correlation instead of the ordinary canonical correlation. The normalization is to take care of the serial correlations in the data. If the data are from a random sample, then there are no serial correlations in the canonical variates and  $\hat{d}$  reduces to one.

Note that if  $\mathbf{z}_t$  has conditional heteroscedasticity, then the asymptotic variance of  $\hat{\rho}$  involves the fourth moments of the canonical variates  $\{x_t\}$  and  $\{y_t\}$ . One must modify the test statistic  $C$  accordingly to maintain the limiting Chi-squared distribution; see Min and Tsay (2005).

#### 4.4.1 Application

To illustrate, we consider the logarithms of indices of monthly flour prices in three U.S. cities, Buffalo, Minneapolis and Kansas City, over the period from August 1972 to November 1980. The data were analyzed in Tiao and Tsay (1989) and have 100 observations. Figure 4.1 show the time plot of the three series and they seem to move in unison. Based on the augmented Dickey-Fuller unit-root test with order 3, all three series have a unit root. However, Johansen's co-integration test fails to reject the null hypothesis of no-cointegration. All test results are given in Table 4.2.

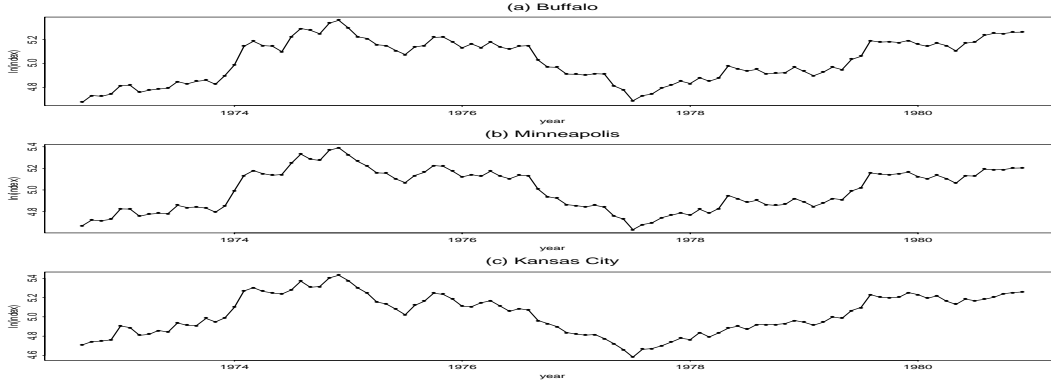


Figure 4.1: Time Plots of the logarithms of indices of monthly flour prices from August 1972 to November 1980. The three cities are Buffalo, Minneapolis and Kansas City.

Next, turn to specification of Kronecker indices of the data. Let  $\mathbf{z}_t$  be the 3-dimensional time series under study. If VAR models are entertained, a VAR(2) model is selected by either the sequential chi-squared test of Tiao and Box (1981) or the AIC criterion. Thus, one can use  $r \geq 2$  to approximate the past vector  $\mathbf{P}_{t-1}$ , and we choose  $r = 3$ , i.e.  $\mathbf{P}_{3,t-1} = (\mathbf{z}'_{t-1}, \mathbf{z}'_{t-2}, \mathbf{z}'_{t-3})'$ . Following the procedure outlined in this section, we obtain the Kronecker indices as  $\{k_1 = 1, k_2 = 1, k_3 = 1\}$ . Details of the test statistic  $C$  of Eq. 4.40) and the associated smallest squared canonical correlations are given in Table 4.3. Based on the Kronecker indices, a VARMA(1,1) model is specified for the data.

## 4.5 Specifying Scalar Component Models

We adopt the method of Tiao and Tsay (1989) to identify the SCM for  $\mathbf{z}_t$ . For the linear process  $\mathbf{z}_t$  of Eq. (4.1) we define an extended  $k(m+1)$ -dimensional vector process  $\mathbf{Y}_{m,t}$  as

$$\mathbf{Y}_{m,t} = (\mathbf{z}'_t, \mathbf{z}'_{t-1}, \dots, \mathbf{z}'_{t-m})' \quad (4.42)$$

where  $m$  is a non-negative integer. To search for SCM's of  $\mathbf{z}_t$ , Tiao and Tsay (1989) consider a two-way table of covariance matrices of the  $\mathbf{Y}_{m,t}$  series. Specifically, given  $(m, j)$ , where  $m \geq 0$

Table 4.2: Unit-root and co-integration tests for the logarithms of indices of monthly flour prices in three U.S. cities

(a) Univariate unit-root tests							
Series	Test	p-value	cons.	AR coefficients			
$z_{1t}$	-1.87	0.35	0.28	-0.054	0.159	0.048	
$z_{2t}$	-1.84	0.36	0.26	-0.051	0.258	-0.046	
$z_{3t}$	-1.70	0.43	0.24	-0.047	0.206	-0.005	

(b) Johansen co-integration tests							
Rank	Eig.Va.	Trace	95%C.V.	99% C.V.	Max.Stat	95%C.V.	99%V.V.
0	0.142	22.65	29.68	35.65	14.97	20.97	25.52
1	0.063	7.68	15.41	20.04	6.37	14.07	18.63
2	0.013	1.31	2.76	6.65	1.31	3.76	6.65

Table 4.3: Specification of Kronecker indices for the 3-dimensional series of logarithms of indices of monthly flour price from August 1972 to November 1980, where the past vector is  $\mathbf{P}_{t-1} = (z'_{t-1}, z'_{t-2}, z'_{t-3})'$ , and last element denotes the last element of the future vector.

Last element	Small.Eig.Va	Test	Deg.Fre.	p-value	remark
$z_{1t}$	0.940	266.3	9	0	
$z_{2t}$	0.810	156.3	8	0	
$z_{3t}$	0.761	134.0	7	0	
$z_{1,t+1}$	0.045	4.28	6	0.64	$k_1 = 1$
$z_{2,t+1}$	0.034	3.04	6	0.80	$k_2 = 1$
$z_{3,t+1}$	0.027	2.44	6	0.88	$k_3 = 1$

and  $j \geq 0$ , consider the covariance matrix

$$\mathbf{\Gamma}(m, j) = \text{Cov}(\mathbf{Y}_{m,t}, \mathbf{Y}_{m,t-j-1}) = \begin{bmatrix} \mathbf{\Gamma}_{j+1} & \mathbf{\Gamma}_{j+2} & \mathbf{\Gamma}_{j+3} & \cdots & \mathbf{\Gamma}_{j+1+m} \\ \mathbf{\Gamma}_j & \mathbf{\Gamma}_{j+1} & \mathbf{\Gamma}_{j+2} & \cdots & \mathbf{\Gamma}_{j+m} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{\Gamma}_{j+1-m} & \mathbf{\Gamma}_{j+2-m} & \mathbf{\Gamma}_{j+3-m} & \cdots & \mathbf{\Gamma}_{j+1} \end{bmatrix}, \quad (4.43)$$

which is a  $k(m+1) \times k(m+1)$  matrix of auto-covariance matrices of  $\mathbf{z}_t$ . The key to understand scalar component models is to study the impact of a SCM on the singularity of the matrices  $\mathbf{\Gamma}(m, j)$  for  $m \geq 0$  and  $j \geq 0$ .

We define  $\mathbf{\Gamma}(m, j)$  as a square matrix. This is purely for simplicity. In fact, Tiao and Tsay (1989) define a general matrix

$$\mathbf{\Gamma}(m, h, j) = \text{Cov}(\mathbf{Y}_{m,t}, \mathbf{Y}_{h,t-j-1}) = E(\mathbf{Y}_{m,t} \mathbf{Y}'_{h,t-j-1}),$$

where  $h \geq m$ . All the results discussed below continue to hold if  $\mathbf{\Gamma}(m, j)$  of Eq. (4.43) is replaced by  $\mathbf{\Gamma}(m, h, j)$  with  $h \geq m$ .

#### 4.5.1 Implication of scalar-component models

Let  $\mathbf{u}$  be a  $k(m+1)$ -dimensional row vector of real numbers. We said that  $\mathbf{u}$  is a *singular left vector* of  $\mathbf{\Gamma}(m, j)$  if  $\mathbf{u}\mathbf{\Gamma}(m, j) = \mathbf{0}$ .

Suppose that  $y_t = \mathbf{v}_0 \mathbf{z}_t$  is a  $\text{SCM}(r, s)$  of  $\mathbf{z}_t$ . By definition, there exist  $r$   $k$ -dimensional row vector  $\mathbf{v}_i$  ( $i = 1, \dots, r$ ) such that  $w_t = \sum_{i=0}^r \mathbf{v}_i \mathbf{z}_{t-i}$  is uncorrelated with  $\mathbf{a}_{t-j}$  for  $j > s$ . Consequently, post-multiplying the SCM structure in Eq. (4.21) by  $\mathbf{z}'_{t-j}$  and taking expectation, we have

$$\sum_{i=0}^r \mathbf{v}_i \mathbf{\Gamma}_{j-i} = \mathbf{0}, \quad \text{for } j > s. \quad (4.44)$$

Let  $\mathbf{v} = (\mathbf{v}_0, \dots, \mathbf{v}_r)$  be the  $k(r+1)$ -dimensional row vector consisting of all  $\mathbf{v}_i$ 's. Then, we have

$$w_t = \mathbf{v} \mathbf{Y}_{r,t}. \quad (4.45)$$

By Eq. (4.44), the existence of  $y_t \sim \text{SCM}(r, s)$  implies that the matrix  $\mathbf{\Gamma}(r, j)$  is singular for  $j \geq s$  and  $\mathbf{v}$  is the corresponding singular left vector. Furthermore, assume that the order  $(r, s)$  satisfies the condition that  $r + s$  is as small as possible. Then, the row vector  $\mathbf{v}$  of  $y_t$  is not a singular left vector of  $\mathbf{\Gamma}(r, s-1)$ ; otherwise the SCM order can be reduced to  $(r, s-1)$ .

To aid further discussion, we define an extended  $k(\ell+1)$ -dimensional row vector of  $\mathbf{v}$  as

$$\mathbf{v}(\ell, r, g) = (\mathbf{0}_{1g}, \mathbf{v}, \mathbf{0}_{2g}), \quad (4.46)$$

where  $\ell$  and  $g$  are integers such that  $\ell \geq r$  and  $g > 0$ ,  $r$  is associated with the original row vector  $\mathbf{v}$ , and  $\mathbf{0}_{1g}$  and  $\mathbf{0}_{2g}$  are respectively  $k(g-1)$ -dimensional and  $k(\ell+1-r-g)$ -dimensional row vector of zeros. For instance,  $\mathbf{v}(r+1, r, 1) = (\mathbf{v}, 0, \dots, 0)$  is a  $k(r+2)$ -dimensional row vector and  $\mathbf{v}(r, r, 1) = \mathbf{v}$ .

Next, consider the matrices  $\mathbf{\Gamma}(m, s)$  with  $m > r$ . Using the definition of Eq. (4.46), we have

$$w_t = \mathbf{v} \mathbf{Y}_{r,t} = \mathbf{v}(m, r, 1) \mathbf{Y}_{m,t}.$$

Consequently, the existence of  $y_t \sim \text{SCM}(r, s)$  also implies that the matrix  $\mathbf{\Gamma}(m, s)$  is singular for  $m > r$ . Furthermore, it is easy to see that  $y_t$  does not imply any singularity of  $\mathbf{\Gamma}(r-1, s)$ ; otherwise,  $y_t$  would be an SCM of order  $(r-1, s)$ .

Finally, consider the matrices  $\mathbf{\Gamma}(m, j)$  with  $m > r$  and  $j > s$ . First, let us focus on the matrix  $\mathbf{\Gamma}(r+1, s+1)$  given by

$$\mathbf{\Gamma}(r+1, s+1) = E(\mathbf{Y}_{r+1,t} \mathbf{Y}'_{r+1,t-s-2}).$$

In this particular instance, the scalar component  $y_t$  introduces two singular left vectors because

- (a)  $w_t = \mathbf{v}(r+1, r, 1) \mathbf{Y}_{r+1,t}$  is uncorrelated with  $\mathbf{a}_{t-j}$  for  $j > s$ ;
- (b)  $w_{t-1} = \mathbf{v}(r+1, r, 2) \mathbf{Y}_{r+1,t}$  is uncorrelated with  $\mathbf{a}_{t-j}$  for  $j > s+1$ .

Thus, a  $\text{SCM}(r, s)$   $y_t$  gives rise to two singular left vectors of the matrix  $\mathbf{\Gamma}(r+1, s+1)$ . In other words, moving from  $\mathbf{\Gamma}(m, s)$  to  $\mathbf{\Gamma}(r+1, s+1)$  increases the number of singular left vectors by one. By the same argument,  $y_t$  also introduces two singular left vectors for  $\mathbf{\Gamma}(r+1, j)$  with  $j \geq s+1$  and for  $\mathbf{\Gamma}(m, s+1)$  with  $m \geq r+1$ . In general, for the matrix  $\mathbf{\Gamma}(m, j)$  with  $m > r$  and  $j > s$ , the SCM  $y_t$  introduces  $\tau = \min\{m-r+1, j-s+1\}$  singular left vectors.

We summarize the results of previous discussions into a theorem.

**Theorem 1.** Suppose that  $\mathbf{z}_t$  is a stationary linear process of Eq. (4.1) and its Hankel matrix is of finite dimension. Suppose also that  $y_t = \mathbf{v}_0 \mathbf{z}_t$  follows an  $\text{SCM}(r, s)$  structure with the associated row vector  $\mathbf{v}$ . Let  $\mathbf{v}(\ell, r, g)$  be the extended row vector of  $\mathbf{v}$  defined in Eq. (4.46). Then,

- (a) for  $j \geq s$ ,  $\mathbf{v}$  is a singular left vector of  $\mathbf{\Gamma}(r, j)$ ;
- (b) for  $m > r$ ,  $\mathbf{v}(m, r, 1)$  is a singular left vector of  $\mathbf{\Gamma}(m, s)$ ;
- (c) for  $m > r$  and  $j > s$ ,  $\mathbf{\Gamma}(m, j)$  has  $h = \min\{m-r+1, j-s+1\}$  singular left vectors, namely  $\mathbf{v}(m, r, g)$  with  $g = 1, \dots, h$  and
- (d) for  $m < r$  and  $j < s$ , the vectors  $\mathbf{v}(m, r, g)$  are not singular left vectors of  $\mathbf{\Gamma}(m, j)$ .

**Example.** Suppose  $\mathbf{z}_t$  is a linear vector process of Eq. (4.1) and  $y_{1t}$  is a  $\text{SCM}(1,0)$  of  $\mathbf{z}_t$  and  $y_{2t}$  is a  $\text{SCM}(0,1)$  of  $\mathbf{z}_t$ . Also, suppose that  $y_{it}$  are not a  $\text{SCM}(0,0)$ . Then, the number of singular left vectors of  $\mathbf{\Gamma}(m, j)$  induced by  $y_{it}$  is given in Parts (a) and (b) of Table 4.4, respectively. The diagonal increasing pattern is clearly seen from the table. In addition, the coordinates  $(m, j)$  of the upper-left vertex of non-zero entries correspond exactly to the order of the SCM.

#### 4.5.2 Exchangeable scalar-component models

There are cases in which a SCM of  $\mathbf{z}_t$  has two different orders  $(p_1, q_1)$  and  $(p_2, q_2)$  such that  $p_1 + q_1 = p_2 + q_2$ . For example, consider the VAR(1) and VMA(1) model

$$\mathbf{z}_t - \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \mathbf{z}_{t-1} = \mathbf{a}_t \Leftrightarrow \mathbf{z}_t = \mathbf{a}_t - \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \mathbf{a}_{t-1}. \quad (4.47)$$

Table 4.4: Numbers of singular left vectors of the  $\Gamma(m, j)$  matrices induced by (a) a scalar-component model  $y_{1t} \sim \text{SCM}(1,0)$  and (b) a scalar-component model  $y_{2t} \sim \text{SCM}(0,1)$ .

(a) $m$	$j$							(b) $m$	$j$						
	0	1	2	3	4	5	$\dots$		0	1	2	3	4	5	$\dots$
0	0	0	0	0	0	0	$\dots$	0	0	1	1	1	1	1	$\dots$
1	1	1	1	1	1	1	$\dots$	1	0	1	2	2	2	2	$\dots$
2	1	2	2	2	2	2	$\dots$	2	0	1	2	3	3	3	$\dots$
3	1	2	3	3	3	3	$\dots$	3	0	1	2	3	4	4	$\dots$
4	1	2	3	4	4	4	$\dots$	4	0	1	2	3	4	5	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\vdots$				

In this particular case, the scalar series  $y_t = (1,0)\mathbf{z}_t = z_{1t}$  is both  $\text{SCM}(1,0)$  and  $\text{SCM}(0,1)$ . This type of SCM orders are referred to as *exchangeable orders* in Tiao and Tsay (1989). SCMs with exchangeable orders have some special properties.

**Lemma 3.** Suppose that  $\mathbf{z}_t$  is a  $k$ -dimensional linear process of Eq. (4.1) and  $y_t$  is both  $\text{SCM}(p_1, q_1)$  and  $\text{SCM}(p_2, q_2)$  of  $\mathbf{z}_t$ , where  $p_1 + q_1 = p_2 + q_2$ . Then, there exists a  $\text{SCM}(p_3, q_3)$   $x_t$  such that  $p_3 < p_0 = \max\{p_1, p_2\}$  and  $q_3 < q_0 = \max\{q_1, q_2\}$ .

**Proof.** From  $y_t \sim \text{SCM}(p_1, q_1)$ , we have the structure

$$w_t = \mathbf{v}_0 \mathbf{z}_t + \sum_{i=1}^{p_1} \mathbf{v}_i \mathbf{z}_{t-i} = \mathbf{v}_0 \mathbf{a}_t + \sum_{j=1}^{q_1} \mathbf{u}_j \mathbf{a}_{t-j}.$$

But  $y_t$  is also  $\text{SCM}(p_2, q_2)$  so that there exist  $p_2$   $k$ -dimensional row vectors  $\mathbf{v}_i^*$  ( $i = 1, \dots, p_2$ ) and  $q_2$   $k$ -dimensional row vectors  $\mathbf{u}_j^*$  ( $j = 1, \dots, q_2$ ) such that

$$w_t^* = \mathbf{v}_0 \mathbf{z}_t + \sum_{i=1}^{p_2} \mathbf{v}_i^* \mathbf{z}_{t-i} = \mathbf{v}_0 \mathbf{a}_t + \sum_{j=1}^{q_2} \mathbf{u}_j^* \mathbf{a}_{t-j}.$$

Consequently, by subtraction, we obtain

$$\sum_{i=1}^{p_0} \boldsymbol{\delta}_i \mathbf{z}_{t-i} = \sum_{j=1}^{q_0} \boldsymbol{\varpi}_j \mathbf{a}_{t-j},$$

where some  $\boldsymbol{\delta}_i$  is non-zero. Suppose that  $\boldsymbol{\delta}_d \neq \mathbf{0}$  and  $\boldsymbol{\delta}_i = \mathbf{0}$  for  $i < d$ . Then, by the linearity of  $\mathbf{z}_t$ , we also have  $\boldsymbol{\varpi}_i = \mathbf{0}$  for  $i < d$  and  $\boldsymbol{\delta}_d = \boldsymbol{\varpi}_d$ . Let  $x_t = \boldsymbol{\delta}_d \mathbf{z}_t$ . The prior equation shows that  $x_t$  is  $\text{SCM}(p_3, q_3)$  with  $p_3 < p_0$  and  $q_3 < q_0$ . The lemma then follows.  $\diamond$

As an illustration, consider the VAR(1) or VMA(1) model in Eq. (4.47). There is indeed a  $\text{SCM}(0,0)$  in the system as shown by the lemma.

Let  $\mathbf{v}$  be the row vector associated with  $y_t \sim \text{SCM}(p_1, q_1)$  and  $\mathbf{v}^*$  be the row vector associated with  $y_t \sim \text{SCM}(p_2, q_2)$ . Obviously, the first  $k$  elements of both  $\mathbf{v}$  and  $\mathbf{v}^*$  are  $\mathbf{v}_0$ . Let  $\boldsymbol{\delta}$  be the row vector associated with  $x_t \sim \text{SCM}(p_3, q_3)$ . Then, from the proof of Lemma 3, we have the following result.

Table 4.5: Numbers of singular left vectors of the  $\mathbf{\Gamma}(m, j)$  matrices induced by a scalar-component model  $y_t$  with exchangeable orders (0,1) and (1,0).

$m$	$j$						
	0	1	2	3	4	5	...
0	0	1	1	1	1	1	...
1	1	1	2	2	2	2	...
2	1	2	2	3	3	3	...
3	1	2	3	3	4	4	...
4	1	2	3	4	4	5	...
5	1	2	3	4	5	5	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

**Lemma 4.** Suppose  $y_t$  is a SCM of the linear vector process  $z_t$  of Eq. (4.1) with exchangeable orders  $(p_1, q_1)$  and  $(p_2, q_2)$  such that  $p_1 + q_1 = p_2 + q_2$  and  $p_1 > p_2$ . Let  $x_t$  be the implied SCM( $p_3, q_3$ ) and denote the row vector associated with  $x_t$  by  $\delta$ . Furthermore, let  $p_0 = \max(p_1, p_2)$ . Then,

$$\mathbf{v}(p_0, p_1, 1) = \mathbf{v}^*(p_0, p_2, 1) + \sum_{j=1}^h \eta_j \mathbf{u}(p_0, p_3, j + 1)$$

where  $h = p_1 - p_2$  and  $\{\eta_j | j = 1, \dots, h\}$  are constants such that  $\eta_j \neq 0$  for some  $j > 0$ .

The result of Lemma 4 can be extended to the case where  $y_t$  has more than two exchangeable orders. Using Lemma 4, we can extend Theorem 1 to include cases in which some SCMs have exchangeable orders.

**Theorem 2.** Suppose the scalar component  $y_t$  has the exchangeable orders stated in Lemma 4. Let  $h_1 = \min\{m - p_1 + 1, j - q_1 + 1\}$  and  $h_2 = \min\{m - p_2 + 1, j - q_2 + 1\}$ . Then, the number of singular left vectors of  $\mathbf{\Gamma}(m, j)$  induced by  $y_t$  is  $\max\{h_1, h_2\}$ .

**Example.** Suppose  $z_t$  is a linear vector process of Eq. (4.1) and  $y_t$  is an SCM of  $z_t$  with exchangeable orders (0,1) and (1,0). In addition,  $y_t$  is not an SCM(0,0) of  $z_t$ . Then, the number of singular left vectors of  $\mathbf{\Gamma}(m, j)$  induced by  $y_t$  is given in Table 4.5. In this case, the diagonal increasing pattern continues to hold, and the coordinates  $(m, j)$  of the two upper-left vertices of non-zero entries correspond exactly to the orders of the SCM.

Consider Tables 4.4 and 4.5. Let  $N(m, j)$  be the number of singular left vectors of  $\mathbf{\Gamma}(m, j)$  induced by the SCM  $y_t$ . Define the diagonal difference as

$$d(m, j) = \begin{cases} N(m, j) & \text{if } m = 0 \text{ or } j = 0 \\ N(m, j) - N(m - 1, j - 1) & \text{otherwise.} \end{cases}$$

Then, it is easily seen that all three differenced tables consist of “0” and “1” with the upper-left vertex of the “1” locates precisely at the SCM order. For the case of exchangeable orders, there are two upper-left vertices of “1” in the differenced table. See Table 4.6.

Table 4.6: Diagonal differences of the numbers of singular left vectors of the  $\mathbf{\Gamma}(m, j)$  matrices induced by a scalar-component model  $y_t$  with exchangeable orders (0,1) and (1,0).

$m$	$j$						
	0	1	2	3	4	5	...
0	0	1	1	1	1	1	...
1	1	1	1	1	1	1	...
2	1	1	1	1	1	1	...
3	1	1	1	1	1	1	...
4	1	1	1	1	1	1	...
5	1	1	1	1	1	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The special feature discussed above concerning the number of singular left vectors of  $\mathbf{\Gamma}(m, j)$  holds for each SCM( $r, s$ ) of  $\mathbf{z}_t$  provided that the sum  $r + s$  is as small as possible. This feature is a consequence of Theorems 1 and 2. In fact, the converse of Theorems 1 and 2 also hold. For instance, if there exists a  $k(r + 1)$ -dimensional row vector  $\mathbf{v}$  whose first  $k$  elements are not all zero such that  $\mathbf{v}$  and its extended vector  $\mathbf{v}(\ell, r, g)$  of Eq. (4.46) have the properties (a) to (d) of Theorem 1, then  $y_t = \mathbf{v}_0 \mathbf{z}_t$  is a scalar component model of order ( $r, s$ ), where  $\mathbf{v}_0$  is the subvector of  $\mathbf{v}$  consisting of its first  $k$  elements. We shall use these results to find SCMs of  $\mathbf{z}_t$ .

### 4.5.3 Searching for SCMs

The objective here is to search for  $k$  linearly independent SCMs of  $\mathbf{z}_t$ , say SCM( $p_i, q_i$ ) ( $i = 1, \dots, k$ ), such that  $p_i + q_i$  is as small as possible. To this end, let  $K = m + j$  and we study the number of singular left vectors of  $\mathbf{\Gamma}(m, j)$  using the following sequence:

1. Start with  $K = 0$ , i.e.  $(m, j) = (0, 0)$
2. Increase  $K$  by 1 and for a fixed  $K$ ,
  - (a) Start with  $j = K$  and  $m = 0$ ,
  - (b) Reduce  $j$  by 1 until  $j = 0$ .

For a given order  $(m, j)$ , we perform the canonical correlation analysis between  $\mathbf{Y}_{m,t}$  and  $\mathbf{Y}_{m,t-j-1}$  to identify the number of singular left vectors of  $\mathbf{\Gamma}(m, j)$ , which turns out to be the number of zero canonical correlations between the two extended random vectors. As in the Kronecker index case, the likelihood ratio test statistic can be used. Specifically, let  $\hat{\lambda}_i(j)$  be the  $i$  smallest squared canonical correlation between  $\mathbf{Y}_{m,t}$  and  $\mathbf{Y}_{m,t-j-1}$  where  $i = 1, \dots, k(m + 1)$ . To test that there are  $s$  zero canonical correlations, Tiao and Tsay (1989) use the test statistic

$$C(j, s) = -(T - m - j) \sum_{i=1}^s \ln\left(1 - \frac{\hat{\lambda}_i(j)}{d_i(j)}\right) \quad (4.48)$$

where  $T$  is the sample size and  $d_i(j)$  is defined as

$$d_i(j) = 1 + 2 \sum_{u=1}^j \hat{\rho}_u(w_{1t}) \hat{\rho}_u(w_{2t}),$$

where  $\hat{\rho}_u(w_t)$  is the lag- $u$  sample autocorrelation of the scalar time series  $w_t$ , and  $w_{1t}$  and  $w_{2t}$  are the two canonical variates associated with the squared canonical correlation  $\hat{\lambda}_i(j)$ . Under the null hypothesis that there are exactly  $s$  zero canonical correlations between  $\mathbf{Y}_{m,t}$  and  $\mathbf{Y}_{m,t-j-1}$ , the test statistic  $C(j, s)$  is asymptotically a chi-squared random variable with  $s^2$  degrees of freedom provided that the innovations  $\mathbf{a}_t$  of Eq. (4.1) is multivariate Gaussian. Note that if one use  $\mathbf{Y}_{m,t}$  and  $\mathbf{Y}_{h,t-j-1}$ , with  $h \geq m$ , to perform the canonical correlation analysis, then the degrees of freedom of  $C(m, j)$  become  $s[(h - m)k + s]$ .

In the searching process, once a new SCM( $p_i, q_i$ )  $y_{it}$  is found, we must use the results of Theorems 1 to 2 to remove the singular left vectors of  $\mathbf{\Gamma}(m, j)$  induced of  $y_{it}$  in any subsequent analysis. The search process is terminated when  $k$  linearly independent SCMs are found.

#### 4.5.4 Application

Again, we use the logarithms of indices of monthly flour prices in Buffalo, Minneapolis and Kansas City to demonstrate the analysis. First, we use the test statistics  $C(m, j)$  of Eq. (4.48) to check the number of zero canonical correlations between the extended vectors  $\mathbf{Y}_{m,t}$  and  $\mathbf{Y}_{m,t-j-1}$ . The results are given in Parts (a) and (b) of Table 4.7. From the table, it is seen that a VARMA(1,1) or VAR(2) is specified for the data. To gain further insight, we consider the eigenvalues and test statistics for some lower-order  $(m, j)$  positions. The results are given in Part (c) of Table 4.7. Here we use  $\mathbf{Y}_{m,t}$  and  $\mathbf{Y}_{h,t-j-1}$  to perform the canonical correlation analysis, where  $h = m + 1 - j$ . The degrees of freedom the the  $C(m, j)$  statistics of Eq. (4.48) are then  $s[(h - m)k + s]$ . From the table, we see that there are 2 SCM(2,0) components and 1 SCM(1,1) component.

Finally, the eigenvectors associated with (estimated) zero eigenvalues for the (1,0) and (1,1) positions are given in Part (d) of Table 4.7. From the table, we can obtain an estimate of the transformation matrix as

$$\mathbf{T} = \begin{bmatrix} -0.40 & 0.83 & -0.40 \\ 0.61 & -0.51 & -0.60 \\ 0.55 & 0.83 & -0.06 \end{bmatrix}.$$

One can than estimate a well specified VARMA(1,1) model for  $\mathbf{y}_t = \mathbf{Tz}_t$ .

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Table 4.7: Analysis of the 3-dimensional flour price series using the scalar component method. The 5% significance level is used.

(a) Number of singular left vectors							(b) Diagonal difference						
$m$	$j$						$m$	$j$					
	0	1	2	3	4	5		0	1	2	3	4	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	3	3	3	3	3	1	2	3	3	3	3	3
2	3	6	6	6	6	6	2	3	3	3	3	3	3
3	3	6	9	9	9	9	3	3	3	3	3	3	3
4	3	6	9	12	12	12	4	3	3	3	3	3	3
5	3	6	9	12	15	15	5	3	3	3	3	3	3

(c) Summary of eigenvalues and test statistics					
$m$	$j$	Eigenvalue	$C(m, j)$	Degrees of Freedom	p-value
0	0	0.757	140.11	4	0.000
		0.853	329.89	10	0.000
		0.943	612.71	18	0.000
0	1	0.597	90.00	1	0.00
0	1	0.795	246.98	4	0.00
0	1	0.863	443.65	9	0.00
1	0	0.026	2.57	4	0.631
1	0	0.103	13.17	10	0.214
1	0	0.259	42.53	18	0.001
1	1	0.004	0.35	1	0.555
1	1	0.031	3.40	4	0.483
1	1	0.041	7.51	9	0.585
1	1	0.785	157.91	16	0.000
1	1	0.872	359.16	25	0.000
1	1	0.944	641.22	36	0.000

(d) Eigenvectors of (estimated) zero eigenvalues					
SCM(1,0)			SCM(1,1)		
-0.403	0.614	0.070	-0.471	0.829	
0.825	-0.512	-.0588	0.849	-0.542	
-0.396	-0.601	0.806	-0.241	-0.143	
0.359	-0.535	-0.074	0.409	-0.839	
-0.726	0.278	0.681	-0.687	0.594	
0.337	0.723	-0.861	0.145	0.102	

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