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Lecture 8: Vector Seasonal Time Series

5.1 Introduction

Many empirical time series exhibit strong seasonal pattern. Figure 5.1 shows the time plots of the U.S. monthly housing series from January 1965 to May 1975. The first series is the Housing-Starts and the second series Housing-Sold all measured in thousands of units. The annual seasonal pattern is clearly seen. We shall study properties of such series and discuss vector models applicable in analyzing such data.

In the literature, multiplicative seasonal models, e.g., the *airline* model, have been found to be useful in analyzing univariate seasonal time series. Suppose that the seasonality is s such that $s \geq 2$, then the airline model is

$$(1-B)(1-B^s)z_t = (1-\theta B)(1-\Theta B^s)a_t, \tag{5.1}$$

where z_t is a scalar time series and $\{a_t\}$ is a sequence of white noises with mean zero and variance σ_a^2. The model states that the seasonal effect and the regular effect operate separately. In statistical terms, the model implies that the seasonal effect and the regular effect are roughly orthogonal to each other. A key feature of the model in Eq. (5.1) is that the autocorrelation function of $w_t = (1-B)(1-B^s)z_t$ satisfies (a) $\rho_1 = -\theta/(1+\theta^2)$, (b) $\rho_s = -\Theta/(1+\Theta^2)$, (c) $\rho_{s-1} = \rho_{s+1} = \rho_1\rho_s$. Letting $y_t = (1-B)z_t/(1-\theta B)$, we have

$$(1-B^s)y_t = (1-\Theta B^s)a_t, \quad \text{and} \quad (1-B)z_t = (1-\theta B)y_t. \tag{5.2}$$

Here y_t follows a purely seasonal model such that

$$y_t = (1-\Theta)y_{t-s} + \Theta(1-\Theta)y_{t-2s} + \Theta^2(1-\Theta)y_{t-3s} + \cdots + a_t.$$

Thus, y_t is a seasonal exponential smoothing model in the sense that the prediction of y_t at the forecast origin $t-1$ is a weighted linear combination of y_{t-js} for $j = 1, \ldots$. That is,

$$y_{t-1}(1) = (1-\Theta)[y_{t-s} + \Theta y_{t-2s} + \Theta^2 y_{t-3s} + \cdots].$$

Also, from Eq. (5.2), z_t becomes

$$z_t = (1-\theta)(z_{t-1} + \theta z_{t-2} + \theta^2 z_{t-3} + \cdots) + y_t,$$

which is similar to an exponential smoothing model except that y_t is not a white noise series. Therefore, the overall airline model in Eq. (5.1) can be interpreted as an exponential smoothing model operates on top of a seasonal exponential smoothing model.

In some applications, non-multiplicative seasonal models are used. For example, the model

$$(1-B)z_t = (1-\theta_1 B - \theta_{12}B^{12})a_t,$$

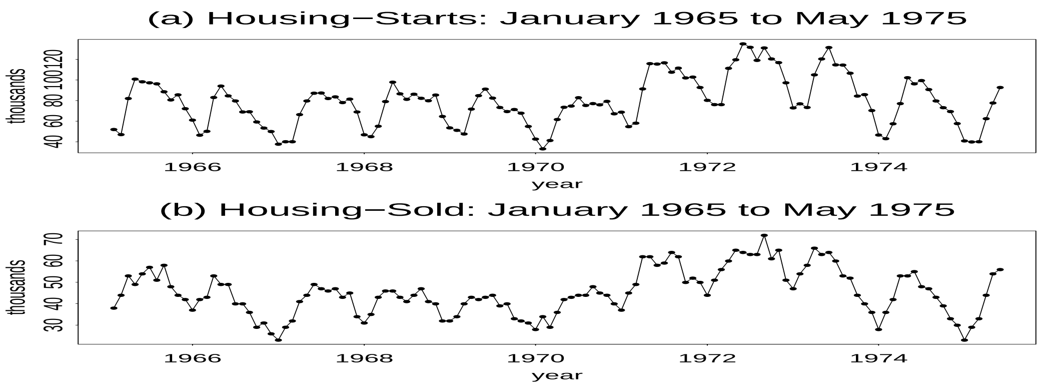

Figure 5.1: Time Plots of U.S. Monthly Housing Series from January 1965 to May 1975.

can be used to describe univariate monthly economic time series when the seasonality is not strong. In this particular case, the autocorrelation function of $w_t = (1-B)z_t$ satisfies (a) $\rho_1 = -\theta_1/(1+\theta_1^2+\theta_{12}^2)$, (b) $\rho_{11} = \theta_1\theta_{12}/(1+\theta_1^2+\theta_{12}^2)$, and (c) $\theta_{12} = -\theta_{12}/(1+\theta_1^2+\theta_{12}^2)$. Clearly, for the transformed series w_t, the major difference between this non-multiplicative model and the airline model in Eq. (5.1) is that the latter model has a nonzero serial correlation at lag 13.

5.2 Vector Seasonal Models

In the multivariate case, a straightforeward generalization of the airline model is

$$(1-B)(1-B^s)\boldsymbol{z}_t = (\boldsymbol{I}-\boldsymbol{\theta}B)(\boldsymbol{I}-\boldsymbol{\Theta}B^s)\boldsymbol{a}_t, \tag{5.3}$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$ are $k\times k$ matrices with all eigenvalues less than 1 in modulus. Let $\boldsymbol{w}_t = (1-B)(1-B^s)\boldsymbol{z}_t$. The moment equations of $\boldsymbol{w}_t$ are

$$\begin{aligned}
\boldsymbol{\Gamma}_0(w) &= \boldsymbol{\Sigma} + \boldsymbol{\Theta\Sigma\Theta}' + \boldsymbol{\theta}(\boldsymbol{\Sigma}+\boldsymbol{\Theta\Sigma\Theta}')\boldsymbol{\theta}',\\
\boldsymbol{\Gamma}_1(w) &= -\boldsymbol{\theta}(\boldsymbol{\Sigma}+\boldsymbol{\Theta\Sigma\Theta}'),\\
\boldsymbol{\Gamma}_{s-1}(w) &= \boldsymbol{\Theta\Sigma\theta}',\\
\boldsymbol{\Gamma}_s(w) &= -\boldsymbol{\Theta\Sigma} - \boldsymbol{\theta\Theta\Sigma\theta},\\
\boldsymbol{\Gamma}_{s+1}(w) &= \boldsymbol{\theta\Theta\Sigma},\\
\boldsymbol{\Gamma}_i(w) &= \boldsymbol{0}, \quad \text{otherwise}.
\end{aligned}$$

Thus, similarly to the univariate case, the nonzero cross-correlation matrices occur at lags $1, s-1, s$, and $s+1$. However, the multiplicative property of autocorrelation function no longer exists. Furthermore, because matrix multiplication is, in general, not communicative, the model in Eq. (5.3) differs from the model

$$(1-B)(1-B^s)\boldsymbol{z}_t = (\boldsymbol{I} - \boldsymbol{\Theta} B^s)(\boldsymbol{I} - \boldsymbol{\theta} B)\boldsymbol{a}_t, \tag{5.4}$$

even if the same MA coefficient matrices and $\boldsymbol{\Sigma}$ are used. No simple method is available before estimation to distinguish between the two multivariate airline models in Eqs. (5.3) and (5.4). In some cases, it might be useful to compare both models.

A general multiplicative vector model can be written as

$$\boldsymbol{\phi}(B)\boldsymbol{\Phi}(B^s)\boldsymbol{z}_t = \boldsymbol{c}_0 + \boldsymbol{\theta}(B)\boldsymbol{\Theta}(B^s)\boldsymbol{a}_t, \tag{5.5}$$

where $\boldsymbol{\Phi}(B^s) = \boldsymbol{I} - \sum_{i=1}^{P} \boldsymbol{\Phi}_i B^{is}$, $\boldsymbol{\Theta}(B^s) = \boldsymbol{I} - \sum_{i=1}^{Q} \boldsymbol{\Theta} B^{is}$, $\boldsymbol{\phi}(B) = \boldsymbol{I} - \sum_{i=1}^{p} \boldsymbol{\phi}_i B^i$, and $\boldsymbol{\theta}(B) = \boldsymbol{I} - \sum_{i=1}^{q} \boldsymbol{\theta}_i B^i$ are matrix polynomials of non-negative degrees Ps, Qs, p, and q, respectively, and $\boldsymbol{c}_0$ is a k-dimensional constant. We refer to $\boldsymbol{\Phi}(B^s)$ and $\boldsymbol{\Theta}(B^s)$ as the seasonal factors of $\boldsymbol{z}_t$ and $\boldsymbol{\phi}(B)$ and $\boldsymbol{\theta}(B)$ as the regular factors of $\boldsymbol{z}_t$.

For the vector seasonal model in Eq.(5.5), the usual stationarity and invertibility conditions continue to apply. Similarly, the identifiability condition is also applicable, e.g., the matrix polynomial $\boldsymbol{\phi}(B)$ and $\boldsymbol{\theta}(B)$ are left-coprime. Obviously, one can permute the ordering of the seasonal and regular AR and MA matrix polynomials in Eq.(5.5) to obtain other vector seasonal models. We use Eq. (5.5) as a representative model rather than the only model for a seasonal vector time series.

5.3 Empirical Analysis

In theory, the methods of structural specification discussed before can be generalized to handle vector seasonal time series. However, selection must be made about what lagged values to use in model specification. For example, consider the Kronecker index approach. One must take into consideration the seasonal structure in selecting the past vector $\boldsymbol{P}_{t-1}$. This is particularly so when the seasonality s is relatively large, because the dimension of $\boldsymbol{P}_{t-1}$ would become large when many high lagged values are included in order to handle the seasonal dependence. For instance, if $s = 12$, then a possible choice is $\boldsymbol{P}_{t-1} = (\boldsymbol{z}'_{t-1}, \boldsymbol{z}'_{t-2}, \boldsymbol{z}'_{t-12}, \boldsymbol{z}'_{t-13}, \boldsymbol{z}'_{t-14}, \boldsymbol{z}'_{t-24}, \boldsymbol{z}'_{t-25}, \boldsymbol{z}'_{t-26})'$, which is of much lower dimensional than $\boldsymbol{P}_{t-1} = (\boldsymbol{z}_{t-1}, \ldots, \boldsymbol{z}'_{t-36})'$.

In this chapter, we shall use an approximate, but simpler, method to identify a multivariate seasonal model. This simple approach does not eliminate the need to select seasonal structure, but it is computationally easy. It also works reasonably well in practice. The basic idea is to fit sequentially some seasonal VAR models and examine the residual cross-correlation matrices of the fitted VAR models. When the residual cross-correlation matrices exhibit some simple pure VMA pattern, we obtain a model specification for the data.

Let s be the seasonality of the data. We assume that the regular VAR matrix polynomial is of order at most 2, but the seasonal VAR matrix polynomial may assume higher order. Thus, we consider the following sequential VAR fittings:

$$\begin{aligned}
\boldsymbol{z}_t &= \boldsymbol{\phi}_{0,1} + \boldsymbol{\phi}_{1,1}\boldsymbol{z}_t + \boldsymbol{a}_{1,t}, \\
\boldsymbol{z}_t &= \boldsymbol{\phi}_{0,2} + \boldsymbol{\phi}_{1,2}\boldsymbol{z}_t + \boldsymbol{\phi}_{2,2}\boldsymbol{z}_{t-2} + \boldsymbol{a}_{2,t},
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{z}_t &= \boldsymbol{\phi}_{0,s} + \boldsymbol{\phi}_{1,s}\boldsymbol{z}_t + \boldsymbol{\phi}_{2,s}\boldsymbol{z}_{t-2} + \boldsymbol{\phi}_{s,s}\boldsymbol{z}_{t-s} + \boldsymbol{a}_{s,t}, \qquad (5.6)\\
\boldsymbol{z}_t &= \boldsymbol{\phi}_{0,s+1} + \boldsymbol{\phi}_{1,s+1}\boldsymbol{z}_t + \boldsymbol{\phi}_{2,s+1}\boldsymbol{z}_{t-2} + \boldsymbol{\phi}_{s,s+1}\boldsymbol{z}_{t-s} + \boldsymbol{\phi}_{s+1,s+1}\boldsymbol{z}_{t-s-1} + \boldsymbol{a}_{s+1,t},\\
\boldsymbol{z}_t &= \boldsymbol{\phi}_{0,s+2} + \boldsymbol{\phi}_{1,s+2}\boldsymbol{z}_t + \boldsymbol{\phi}_{2,s+2}\boldsymbol{z}_{t-2} + \boldsymbol{\phi}_{s,s+2}\boldsymbol{z}_{t-s} + \boldsymbol{\phi}_{s+1,s+2}\boldsymbol{z}_{t-s-1} + \boldsymbol{\phi}_{s+2,s+2} + \boldsymbol{a}_{s+2,t},\\
\boldsymbol{z}_t &= \boldsymbol{\phi}_{0,2s} + \boldsymbol{\phi}_{1,2s}\boldsymbol{z}_t + \boldsymbol{\phi}_{2,2s}\boldsymbol{z}_{t-2} + \boldsymbol{\phi}_{s,2s}\boldsymbol{z}_{t-s} + \boldsymbol{\phi}_{s+1,2s}\boldsymbol{z}_{t-s-1} + \boldsymbol{\phi}_{s+2,2s} + \boldsymbol{\phi}_{2s,2s}\boldsymbol{z}_{t-2s} + \boldsymbol{a}_{2s,t},\\
\vdots &\quad \vdots
\end{aligned}
$$

Let $\widehat{\boldsymbol{\phi}}_{i,j}$ be the ordinary least squares estimate of $\boldsymbol{\phi}_{i,j}$ in the VAR(j) fit of Eq. (5.6). Then, the residual series is

$$\hat{\boldsymbol{a}}_{j,t} = \boldsymbol{z}_t - \widehat{\boldsymbol{\phi}}_{1,j}\boldsymbol{z}_{t-1} - \widehat{\boldsymbol{\phi}}_{2,j}\boldsymbol{z}_{t-2} - \cdots - \widehat{\boldsymbol{\phi}}_{j,j}\boldsymbol{z}_{t-j}, \qquad (5.7)$$

where it is understood that $\widehat{\boldsymbol{\phi}}_{i,j} = \mathbf{0}$ if $i > j$. We compute the sample cross-correlation matrices of $\widehat{\boldsymbol{a}}_t$ and specify the VMA order using these cross-correlation matrices.
Note that this approach only provides approximate orders of a seasonal VARMA model because the ordinary least squares estimates $\widehat{\boldsymbol{\phi}}_{i,j}$ is not a consistent estimate of $\boldsymbol{\phi}_i$ when the MA order is positive and $\boldsymbol{z}_t$ is weakly stationary.

5.4 An Example

Consider the U.S. monthly housing series of Figure 5.1. The sample ACFs of the individual component series are shown in Figure 5.2. As expected, the seasonal pattern is clearly shown. Hence, we entertain the seasonally differenced data $\boldsymbol{w}_t = (1 - B^{12})\boldsymbol{z}_t$. We applied the sequential VAR fitting of the previous section to $\boldsymbol{w}_t$ and obtained the results below.

SCA Demonstration
Output edited to simplify the results.

```
 --
input z1,z2. file 'hous.dat'
 Z1      , A  125  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
 Z2      , A  125  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
 --
miden z1,z2. dfor 12. arfits 1,2,12,13,14. rccm 1,2,12,13,14. @
output level(deta).
                                                        12
 DIFFERENCE ORDERS. . . . . . . . . . . . . . . (1-B  )
 TIME PERIOD ANALYZED . . . . . . . . . . . . .  1  TO   125
 EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .            113

 SERIES   NAME           MEAN       STD. ERROR
    1    Z1             -1.2230     17.8072
    2    Z2             -0.8496     10.0099
 SAMPLE CORRELATION MATRIX OF THE SERIES
   1.00
   0.86  1.00

 SAMPLE CROSS CORRELATION MATRICES FOR THE ORIGINAL SERIES.
 CROSS CORRELATION MATRICES IN TERMS OF +,-,.
```

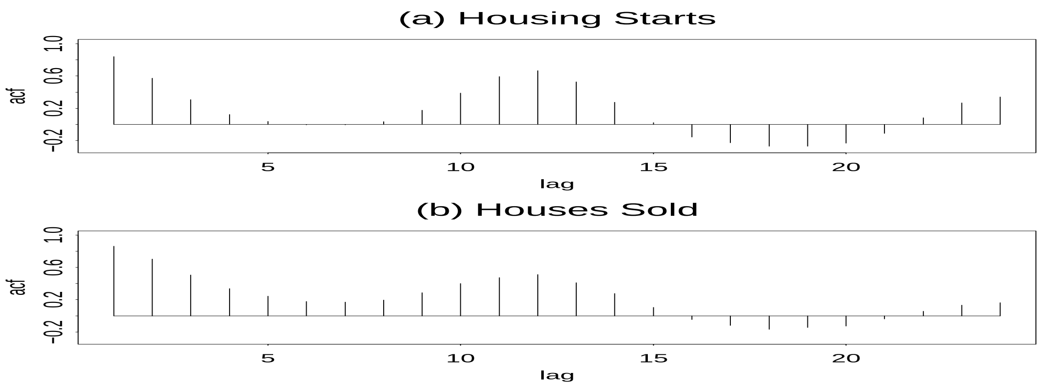

Figure 5.2: Sample ACFs of the U.S. Monthly Housing Series from January 1965 to May 1975: (a) Housing Starts and (b Houses Sold.

```
LAGS  1 THROUGH  6
     + +      + +      + +      + +      + +      + +
     + +      + +      + +      + +      + +      + +
LAGS  7 THROUGH 12
     + +      + +      + +      + +      . .      . .
     + +      + +      . .      . .      . .      . .
LAGS 13 THROUGH 18
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .
LAGS 19 THROUGH 24
     - .      - .      - .      - -      - -      - -
     . .      . .      . .      - .      - .      - .

DETERMINANT OF S(0) = 0.740885E+04

AUTOREGRESSIVE FITTING ON LAG(S)  1
=== PHI( 1) ===
     .378    1.037      + +
     .081     .744      . +
STANDARD ERRORS
     .083     .153
     .051     .093
RESIDUAL COVARIANCE MATRIX S( 1)
 0.543E+02
```

```
 0.115E+02 0.203E+02
RESIDUAL CORRELATION MATRIX RS( 1)
  1.00
  0.35  1.00
EIGENVALUES AND EIGENVECTORS OF S( 1)
EIGENVALUES
    16.783    57.823
EIGENVECTORS
    -0.293     0.956
     0.956     0.293
DETERMINANT OF S(J) = 0.970443E+03
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =  194.12

SAMPLE CROSS CORRELATION MATRICES FOR THE RESIDUAL SERIES.

CROSS CORRELATION MATRICES IN TERMS OF +,-,.
LAGS  1 THROUGH  6
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .
LAGS  7 THROUGH 12
     . .      . .      . .      . .      . .      - .  <== VMA(12)
     . .      . .      . .      . .      . .      . -
LAGS 13 THROUGH 18
     . .      . .      . +      . .      . .      . +
     . .      . .      . .      . .      . .      . .
LAGS 19 THROUGH 24
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

AUTOREGRESSIVE FITTING ON LAG(S)  1  2
=== PHI( 1) ===
     .286    1.021      + +
     .092     .766      . +
STANDARD ERRORS
     .105     .172
     .065     .107
=== PHI( 2) ===
     .185    -.122      . .
     .006    -.053      . .  <== phi(2) = 0?
STANDARD ERRORS
     .093     .202
     .058     .126
RESIDUAL COVARIANCE MATRIX S( 2)
 0.522E+02
 0.115E+02 0.203E+02
RESIDUAL CORRELATION MATRIX RS( 2)
  1.00
  0.35  1.00
DETERMINANT OF S(J) = 0.926763E+03
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =    4.31
```

```
SAMPLE CROSS CORRELATION MATRICES FOR THE RESIDUAL SERIES.

CROSS CORRELATION MATRICES IN TERMS OF +,-,.
LAGS  1 THROUGH  6
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .
LAGS  7 THROUGH 12
     . .      . .      . .      . .      . .      - .
     . .      . .      . .      . .      . .      . -
LAGS 13 THROUGH 18
     . .      . .      . .      . .      . .      . +
     . .      . .      . .      . .      . .      . .
LAGS 19 THROUGH 24
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

AUTOREGRESSIVE FITTING ON LAG(S)  1  2 12
=== PHI( 1) ===
     .289     .996      + +
     .124     .670      . +
STANDARD ERRORS
     .103     .175
     .063     .107
=== PHI( 2) ===
     .203    -.134      + .
     .061    -.091      . .
STANDARD ERRORS
     .095     .196
     .058     .120
=== PHI(12) ===
    -.216     .335      - +
     .054    -.227      . -
STANDARD ERRORS
     .081     .141
     .049     .087
RESIDUAL COVARIANCE MATRIX S( 3)
 0.486E+02
 0.126E+02 0.182E+02
RESIDUAL CORRELATION MATRIX RS( 3)
  1.00
  0.42  1.00
DETERMINANT OF S(J) = 0.730089E+03
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =   21.83

SAMPLE CROSS CORRELATION MATRICES FOR THE RESIDUAL SERIES.
CROSS CORRELATION MATRICES IN TERMS OF +,-,.
LAGS  1 THROUGH  6
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .
LAGS  7 THROUGH 12
     . .      . .      . .      . .      . .      - -
```

```
     . .      . .      . .      . .      . .      . -
LAGS 13 THROUGH 18
     . .      . .      . .      . .      . .      . +
     . .      . .      . .      . .      . .      . .
LAGS 19 THROUGH 24
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

AUTOREGRESSIVE FITTING ON LAG(S)  1  2 12 13
=== PHI( 1) ===
     .303     .976      + +
     .120     .677      . +
STANDARD ERRORS
     .100     .169
     .064     .107
=== PHI( 2) ===
     .198    -.054      + .
     .058    -.051      . .
STANDARD ERRORS
     .087     .180
     .055     .114
=== PHI(12) ===
    -.458     .006      - .
    -.055    -.387      . -
STANDARD ERRORS
     .096     .149
     .061     .095
=== PHI(13) ===
     .127     .575      . +
     .034     .312      . +
STANDARD ERRORS
     .093     .175
     .059     .111
RESIDUAL COVARIANCE MATRIX S( 4)
 0.407E+02
 0.873E+01 0.164E+02
RESIDUAL CORRELATION MATRIX RS( 4)
  1.00
  0.34  1.00
DETERMINANT OF S(J) = 0.590056E+03
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =   19.06

SAMPLE CROSS CORRELATION MATRICES FOR THE RESIDUAL SERIES.
CROSS CORRELATION MATRICES IN TERMS OF +,-,.
LAGS  1 THROUGH  6
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .
LAGS  7 THROUGH 12
     . .      . .      . .      . .      . .      . .
     + .      . .      . .      . .      . .      . .
LAGS 13 THROUGH 18
```

```
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .
LAGS 19 THROUGH 24
     . .      . .      . .      . .      . .      - .
     . .      . .      . .      . .      . .      - -

AUTOREGRESSIVE FITTING ON LAG(S)  1  2 12 13 14
=== PHI( 1) ===
     .281     .951      + +
     .093     .654      . +
STANDARD ERRORS
     .104     .171
     .065     .107
=== PHI( 2) ===
     .250    -.063      + .
     .041     .054      . .
STANDARD ERRORS
     .097     .199
     .061     .125
=== PHI(12) ===
    -.479     .004      - .
    -.041    -.384      . -
STANDARD ERRORS
     .097     .148
     .061     .093
=== PHI(13) ===
     .062     .537      . +
    -.004     .236      . .
STANDARD ERRORS
     .114     .189
     .071     .119
=== PHI(14) ===
     .104     .010      . .
    -.056     .222      . .
STANDARD ERRORS
     .095     .186
     .060     .117
RESIDUAL COVARIANCE MATRIX S( 5)
 0.401E+02
 0.880E+01 0.158E+02
RESIDUAL CORRELATION MATRIX RS( 5)
  1.00
  0.35  1.00
DETERMINANT OF S(J) = 0.555966E+03
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =    5.21

SAMPLE CROSS CORRELATION MATRICES FOR THE RESIDUAL SERIES.
CROSS CORRELATION MATRICES IN TERMS OF +,-,.
LAGS  1 THROUGH  6
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .
```

```
LAGS  7 THROUGH 12
     . .      . .      . .      . .      . .      . .
     + .      . .      . .      . .      . .      . .
LAGS 13 THROUGH 18
     . .      . .      . .      . .      . .      . +
     . .      . .      . .      . .      . .      . .
LAGS 19 THROUGH 24
     . .      . .      . .      . .      . .      - .
     . .      . .      . .      . .      . .      - -

==========  STEPWISE AUTOREGRESSION SUMMARY  ==========
-----------------------------------------------------------------------
    I RESIDUAL I EIGENVAL.I CHI-SQ  I          I SIGNIFICANCE
LAG I VARIANCESI OF SIGMA I  TEST   I   AIC    I OF PARTIAL AR COEFF.
----+----------+----------+---------+----------+----------------------
  1 I .543E+02 I .168E+02 I  194.12 I    6.942 I + +
    I .203E+02 I .578E+02 I         I          I . +
----+----------+----------+---------+----------+----------------------
  2 I .522E+02 I .166E+02 I    4.31 I    6.960 I . .
    I .203E+02 I .559E+02 I         I          I . .
----+----------+----------+---------+----------+----------------------
 12 I .486E+02 I .137E+02 I   21.83 I    6.785 I - +
    I .182E+02 I .532E+02 I         I          I . -
----+----------+----------+---------+----------+----------------------
 13 I .407E+02 I .136E+02 I   19.06 I    6.636 I . +
    I .164E+02 I .435E+02 I         I          I . +
----+----------+----------+---------+----------+----------------------
 14 I .401E+02 I .129E+02 I    5.21 I    6.641 I . .
    I .158E+02 I .430E+02 I         I          I . .
-----------------------------------------------------------------------
NOTE:  CHI-SQUARED CRITICAL VALUES WITH   4  DEGREES OF FREEDOM ARE
             5 PERCENT:   9.5     1 PERCENT:  13.3
```

Based on the output, we specify a tentative model

$$(\boldsymbol{I} - \boldsymbol{\phi}B)(1 - B^{12})\boldsymbol{z}_t = (\boldsymbol{I} - \boldsymbol{\Theta}B^{12})\boldsymbol{a}_t, \tag{5.8}$$

for the housing data, and proceed to estimation. The estimation results are given below and summarized in Table 5.1. The residual cross-correlation matrices of the refined model, which contains only significant parameter estimates, show no strong serial correlations. Thus, the model appears to be adequate. From the table, the MA coefficient matrix of the final fitted model is close to the identity matrix. Thus, the model is essentially like

$$(1 - B^{12})(\boldsymbol{I} - \boldsymbol{\phi}_1 B)\boldsymbol{z}_t = (1 - B^{12})\boldsymbol{a}_t,$$

indicating that the seasonal components for both component series are close to being deterministic. In other words, the seasonal components of the two housing series satisfy the equation $s_t = s_{t-12}$.

SCA Demonstration
Output edited.

Table 5.1: Estimation Results of the model in Eq. (5.8) for the U.S. housing series via the exact likelihood method

Parameter	Initial Estimation	Final Estimation
$\boldsymbol{\phi}_1$	0.460 0.955 0.100 0.763	0.464 0.948 0.096 0.762
std. error	0.073 0.135 0.046 0.086	0.073 0.136 0.046 0.086
$\boldsymbol{\Theta}$	0.989 -0.046 0.048 0.924	0.975 0 0 0.999
std. error	0.062 0.115 0.048 0.073	0.055 – – 0.059
$\boldsymbol{\Sigma}$	28.65 5.16 5.16 11.83	28.78 5.54 5.54 11.40

```
 --
mtsm m1. series z1(12),z2(12). model (i-p1*b)series=@
(i-t12*b**12)noise.

 SUMMARY FOR MULTIVARIATE ARMA MODEL --    M1

 VARIABLE   DIFFERENCING
  Z1          12
  Z2          12

  PARAMETER     FACTOR     ORDER    CONSTRAINT
  1       P1    REG AR       1        CP1
  2      T12    REG MA      12        CT12
 --
mestim m1. method exact. hold resi(r1,r2)

 SUMMARY FOR THE MULTIVARIATE ARMA MODEL

 SERIES    NAME            MEAN         STD DEV    DIFFERENCE ORDER(S)
   1     Z1              -1.2230        17.8072      12
   2     Z2              -0.8496        10.0099      12
 NUMBER OF OBSERVATIONS =  125  (EFFECTIVE NUMBER = NOBE =  112)

 ITERATIONS TERMINATED DUE TO:
   MAXIMUM NUMBER OF ITERATIONS  10 REACHED
   TOTAL NUMBER OF ITERATIONS IS   19

 FINAL MODEL SUMMARY WITH MAXIMUM LIKELIHOOD PARAMETER ESTIMATES
 ----- PHI MATRICES -----
 ESTIMATES OF    PHI(  1 ) MATRIX AND SIGNIFICANCE
      .460     .955      + +
      .100     .763      + +
```

```
 STANDARD ERRORS
      .073     .135
      .046     .086
 ----- THETA MATRICES -----
 ESTIMATES OF  THETA( 12 ) MATRIX AND SIGNIFICANCE
      .989    -.046      + .
      .048     .924      . +
 STANDARD ERRORS
      .062     .115
      .048     .073
 -----------------------
 ERROR COVARIANCE MATRIX
 -----------------------
              1           2
   1    28.648914
   2     5.164306   11.825700

 -2*(LOG LIKELIHOOD AT FINAL ESTIMATES)  IS   0.91436615E+03
 --
t12(1,2)=0
 --
t12(2,1)=0
 --
ct12(1,2)=1
 --
ct12(2,1)=1
 --
mestim m1. method exact. hold resi(r1,r2)

 ITERATIONS TERMINATED DUE TO:
   CHANGE IN  (-2*LOG LIKELIHOOD)/NOBE  .LE. 0.100E-03
   TOTAL NUMBER OF ITERATIONS IS    7

 FINAL MODEL SUMMARY WITH MAXIMUM LIKELIHOOD PARAMETER ESTIMATES
 ----- PHI MATRICES -----
 ESTIMATES OF    PHI(  1 ) MATRIX AND SIGNIFICANCE
      .464     .948      + +
      .096     .762      + +
 STANDARD ERRORS
      .073     .136
      .046     .086
 ----- THETA MATRICES -----
 ESTIMATES OF  THETA( 12 ) MATRIX AND SIGNIFICANCE
      .975     .000      + .
      .000     .999      . +
 STANDARD ERRORS
      .055      --
       --      .059
 -----------------------
 ERROR COVARIANCE MATRIX
 -----------------------
```

```
               1           2
  1    28.779958
  2     5.535404   11.398505

 -2*(LOG LIKELIHOOD AT FINAL ESTIMATES)  IS   0.91487288E+03
 --
miden r1,r2.

 TIME PERIOD ANALYZED . . . . . . . . . . . . . 14  TO   125
 EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .          112

 SERIES   NAME           MEAN       STD. ERROR
    1    R1              0.2989      5.1296
    2    R2             -0.0876      3.2014
 SAMPLE CORRELATION MATRIX OF THE SERIES
   1.00
   0.35  1.00

 SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,.

 CROSS CORRELATION MATRICES IN TERMS OF +,-,.
 LAGS  1 THROUGH  6
      . .      . .      . .      . .      . .      . .
      . .      . .      . .      . .      . .      . .
 LAGS  7 THROUGH 12
      . .      . .      . .      . .      . .      . .
      . .      . .      . .      . .      . .      . .
 LAGS 13 THROUGH 18
      . .      . .      . .      . .      . .      . +
      . .      . .      . .      . .      . .      . .
 LAGS 19 THROUGH 24
      . .      . .      . .      . .      . .      . .
      . .      . -      . .      . .      . .      . .
```

Further Analysis
Because of the deterministic seasonality of the data, we remove the monthly sample means from the data. This is effectively equivalent to fitting 12 dummy variables (without a constant term) to the data. The resulting series should have no seasonality. Figure 5.3 shows the time plots of the two adjusted series. Indeed, the plots show no seasonal pattern. We proceed to analyze the adjusted series. Based ont he Tiao-Box chi-squared statistics, a VAR(1) model is specified for the adjusted series. In addition, the sequential VAR fits, as expected, fail to show any significant lag-12 effect. The fitted VAR(1) model is

$$\boldsymbol{w}_t - \begin{bmatrix} 0.556 & 0.761 \\ 0.093 & 0.767 \end{bmatrix} \boldsymbol{w}_{t-1} = \begin{bmatrix} 5.77 \\ -1.78 \end{bmatrix} + \boldsymbol{a}_t,$$

where $\boldsymbol{w}_t$ is the monthly-mean adjusted series of $\boldsymbol{z}_t$ and the covariance matrix of $\boldsymbol{a}_t$ is

$$\boldsymbol{\Sigma} = \begin{bmatrix} 27.50 & 4.88 \\ 4.88 & 11.38 \end{bmatrix}.$$

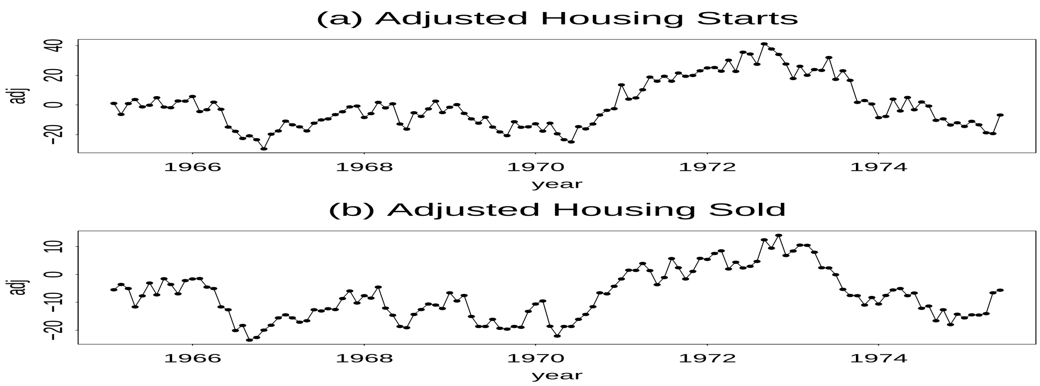

Figure 5.3: Time Plots of Seasonally Adjusted U.S. Monthly Housing Series from January 1965 to May 1975.

There is feedback relationship between the two housing series, but the dependence of the Housing Sold on the past Housing Starts is relatively weak.

SCA Demonstration

```
input x1,x2. file 'fort.27'
 --
miden x1,x2. arfits 1,2,3,12,13,14.

 TIME PERIOD ANALYZED . . . . . . . . . . . .  1  TO   125
 EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .           125

 SERIES   NAME           MEAN       STD. ERROR
    1    X1              0.0000     15.9481
    2    X2             -7.6171      8.6990
 SAMPLE CORRELATION MATRIX OF THE SERIES
   1.00
   0.89  1.00

 SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,.

 CROSS CORRELATION MATRICES IN TERMS OF +,-,.
```

```
 LAGS  1 THROUGH  6
      + +      + +      + +      + +      + +      + +
      + +      + +      + +      + +      + +      + +
 LAGS  7 THROUGH 12
      + +      + +      + +      + +      + +      + +
      + +      + +      + +      + +      + +      + +
 LAGS 13 THROUGH 18
      + +      + +      + +      . +      . .      . .
      + +      + +      . +      . .      . .      . .
 LAGS 19 THROUGH 24
      . .      . .      . .      . .      . .      . .
      . .      . .      . .      . .      . .      . .
 DETERMINANT OF S(0) = 0.416425E+04

 ==========  STEPWISE AUTOREGRESSION SUMMARY  ==========
 ----------------------------------------------------------------------
     I RESIDUAL I EIGENVAL.I CHI-SQ  I          I SIGNIFICANCE
 LAG I VARIANCESI OF SIGMA I  TEST   I   AIC    I OF PARTIAL AR COEFF.
 ----+----------+----------+---------+----------+----------------------
   1 I .278E+02 I .949E+01 I  289.45 I    5.706 I + +
     I .114E+02 I .297E+02 I         I          I . +
 ----+----------+----------+---------+----------+----------------------
   2 I .267E+02 I .932E+01 I    5.49 I    5.718 I + .
     I .114E+02 I .287E+02 I         I          I . .
 ----+----------+----------+---------+----------+----------------------
   3 I .262E+02 I .886E+01 I    8.28 I    5.702 I . .
     I .106E+02 I .279E+02 I         I          I . -
 ----+----------+----------+---------+----------+----------------------
  12 I .256E+02 I .877E+01 I    3.91 I    5.727 I . .
     I .103E+02 I .271E+02 I         I          I . .
 ----+----------+----------+---------+----------+----------------------
  13 I .255E+02 I .865E+01 I    1.43 I    5.777 I . .
     I .102E+02 I .271E+02 I         I          I . .
 ----+----------+----------+---------+----------+----------------------
  14 I .255E+02 I .854E+01 I    1.32 I    5.827 I . .
     I .101E+02 I .271E+02 I         I          I . .
 ----------------------------------------------------------------------
 NOTE:  CHI-SQUARED CRITICAL VALUES WITH   4  DEGREES OF FREEDOM ARE
              5 PERCENT:   9.5     1 PERCENT:  13.3
 --
mtsm m1. series x1,x2. model (i-p1*b)series=c1+noise.

 SUMMARY FOR MULTIVARIATE ARMA MODEL --    M1
 VARIABLE   DIFFERENCING
```

```
  X1
  X2
  PARAMETER     FACTOR     ORDER    CONSTRAINT
  1       C1   CONSTANT      0        CC1
  2       P1    REG AR       1        CP1
 --
mestim m1. hold resi(r1,r2).

 SUMMARY FOR THE MULTIVARIATE ARMA MODEL

 SERIES    NAME            MEAN         STD DEV    DIFFERENCE ORDER(S)
   1     X1               0.0000        15.9481
   2     X2              -7.6171         8.6990
 NUMBER OF OBSERVATIONS =  125  (EFFECTIVE NUMBER = NOBE =  124)

 ITERATIONS TERMINATED DUE TO:
   RELATIVE CHANGE IN DETERMINANT OF COVARIANCE MATRIX .LE. 0.100E-03
   TOTAL NUMBER OF ITERATIONS IS    4

 FINAL MODEL SUMMARY WITH CONDITIONAL LIKELIHOOD PARAMETER ESTIMATES
 ----- CONSTANT VECTOR (STD ERROR) -----
     5.769  (    1.032  )
    -1.782  (    0.664  )
 ----- PHI MATRICES -----
 ESTIMATES OF    PHI(  1 ) MATRIX AND SIGNIFICANCE
      .556     .761      + +
      .093     .767      + +
 STANDARD ERRORS
      .065     .120
      .042     .077
 -----------------------
 ERROR COVARIANCE MATRIX
 -----------------------
              1           2
   1    27.502793
   2     4.877264   11.378416
 --
miden r1,r2.

 TIME PERIOD ANALYZED . . . . . . . . . . . . .  2  TO   125
 EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .          124

 SERIES   NAME           MEAN       STD. ERROR
    1    R1              0.0000      5.2443
    2    R2              0.0000      3.3732
```

```
SAMPLE CORRELATION MATRIX OF THE SERIES
  1.00
  0.28  1.00

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,.
         1              2
 1   ............   ........+...
 1   ............   .....+......

 2   ............   ............
 2   ............   .......-....

CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6
     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .
LAGS  7 THROUGH 12
     . .      . .      . +      . .      . .      . .
     . .      . .      . .      . .      . .      . .
LAGS 13 THROUGH 18
     . .      . .      . .      . .      . .      . +
     . .      . .      . .      . .      . .      . .
LAGS 19 THROUGH 24
     . .      . .      . .      . .      . .      . .
     . .      . -      . .      . .      . .      . .
```