1. Derivation of Equation (5.68). There are many ways to derive the result. A direct approach is to subtract $z_{t-1}$ from both sides of the model

$$z_t = \sum_{i=1}^{p} \Phi_i z_{t-i} + c(t) + \Theta(B)a_t$$

and then use a recursive argument:

$$\Delta z_t = (\Phi_1 - I_k)z_{t-1} + \sum_{i=2}^{p} \Phi_i z_{t-i} + c(t) + \Theta(B)a_t$$

$$= (\Phi_1 - I_k)\Delta z_{t-1} + (\Phi_1 - I_k)z_{t-2} + \sum_{i=2}^{p} \Phi_i z_{t-i} + c(t) + \Theta(B)a_t$$

$$= \phi_1^* \Delta z_{t-1} + (\Phi_1 + \Phi_2 - I_k)z_{t-2} + \sum_{i=3}^{p} \Phi_i z_{t-i} + c(t) + \Theta(B)a_t$$

$$= \phi_1^* \Delta z_{t-1} + \phi_2^* \Delta z_{t-2} + (\Phi_1 + \Phi_2 - I_k)z_{t-3} + \sum_{i=3}^{p} \Phi_i z_{t-i} + c(t) + \Theta(B)a_t$$

$$= \sum_{i=1}^{3} \phi_i^* \Delta z_{t-i} + \sum_{j=1}^{3} (\Phi_i - I_k)z_{t-j} + \sum_{i=4}^{p} \Phi_i z_{t-i} + c(t) + \Theta(B)a_t$$

$$\vdots$$

$$= \sum_{i=1}^{p-1} \phi_i^* \Delta z_{t-i} + \sum_{i=1}^{p} \Phi_i z_{t-i} + c(t) + \Theta(B)a_t$$

$$= \Pi z_{t-p} + \sum_{i=1}^{p-1} \phi_i^* \Delta z_{t-i} + c(t) + \Theta(B)a_t,$$

which is Equation (5.68).

2. Log(GDP) of UK, CA, and US.

(a) The Johansen test of co-integration with $K = 3$ and constant term shows that the hypothesis of $r = 1$ cannot be rejected, even at the 10% level, but $r = 0$ is rejected. Thus, the three series are co-integrated at the 5% level.

(b) Based on the test result, there is one co-integrating vector, given by $(1, -0.7202, -0.2228)'$. 
(c) The ADF test with lag = 3 and drift shows that $t = -2.866$, which is close to the 5% critical value $-2.88$. As a matter of fact, the $p$-value is 0.054 so that the unit-root hypothesis is rejected at the 10% level, even though, it is not at the 5% level. This provides some weak justification that the co-integrated series is stationary. **Note:** simple time plot shows that the co-integrated series changed markedly at the end of data span. This explains partially the $p$-value is 0.054. If one uses the first 100 observations, then the $p$-value drops to 0.01.

(d) The fitted ECM after removing some insignificant parameters with threshold 1.0 is

\[
\Delta z_t = \begin{bmatrix} 0 \\ 0.058 \\ 0.033 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.053 \\ 0.148 \end{bmatrix} w_t + \begin{bmatrix} 0.461 \\ 0.269 \\ 0.386 \end{bmatrix} \Delta z_{t-1} + \begin{bmatrix} 0 \\ 0.269 \\ 0.386 \\ 0.131 \end{bmatrix} \Delta z_{t-2} + a_t, \quad \Sigma_a = \begin{bmatrix} 2.97 & 0.33 & 0.81 \\ 0.33 & 2.98 & 1.30 \\ 0.81 & 1.30 & 3.14 \end{bmatrix} \times 10^{-5},
\]

where $w_t$ denotes the co-integrated series. Model checking indicates that the fitted model is adequate. See Figure 1.

3. Daily log prices of BHP and VALE stocks.

(a) Yes, Johansen co-integration with $K = 2$ (selected by AIC) and constant rejects $r = 0$, but fails to reject $r = 1$ at the 5% level.

(b) The co-integrating vector is $(1, -0.718)$, i.e. $w_t = z_{1t} - 0.718z_{2t}$, where $z_t$ denotes the log stock prices.

(c) The ECM model is

\[
\Delta z_t = \begin{bmatrix} 0.114 \\ -0.059 \end{bmatrix} + \begin{bmatrix} -0.062 \\ 0.033 \end{bmatrix} w_t + \begin{bmatrix} -0.115 \\ 0.053 \\ 0.045 \end{bmatrix} \Delta z_{t-1} + a_t
\]

where

\[
\Sigma_a = \begin{bmatrix} 3.70 & 2.08 \\ 2.08 & 4.96 \end{bmatrix} \times 10^{-4}.
\]

4. Components of U.S. monthly industrial production index from December 1963 to December 2012. The two components are (a) nondurable consumer goods and (b) materials.

(a) Compute the percentage growth rate series $z_t$. See Figure 2.

(b) If VAR models are entertained, then a VAR(3) model is suggested by AIC and HQ criteria. If VARMA models are entertained, then the **Ecmm** command suggests an VARMA(1,1) model. See the R output below:

```r
> dd=log(da[,c(3,5)])
> zt=diffM(dd)*100
> dim(zt)
[1] 588  2
```
> tdx=(c(1:588)/12)+1964
> MTSplot(zt,tdx)
> VARorder(zt)
selected order: aic = 3
selected order: bic = 1
selected order: hq = 3
Summary table:

<table>
<thead>
<tr>
<th>p</th>
<th>AIC</th>
<th>BIC</th>
<th>HQ</th>
<th>M(p)</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1,]</td>
<td>-0.7941</td>
<td>-0.7941</td>
<td>-0.7941</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>[2,]</td>
<td>-0.9649</td>
<td>-0.9351</td>
<td>-0.9533</td>
<td>105.3947</td>
<td>0.0000</td>
</tr>
<tr>
<td>[3,]</td>
<td>-0.9737</td>
<td>-0.9142</td>
<td>-0.9505</td>
<td>12.7680</td>
<td>0.0125</td>
</tr>
<tr>
<td>[4,]</td>
<td>-0.9958</td>
<td>-0.9065</td>
<td>-0.9610</td>
<td>20.2554</td>
<td>0.0004</td>
</tr>
<tr>
<td>[5,]</td>
<td>-0.9919</td>
<td>-0.8728</td>
<td>-0.9455</td>
<td>5.4655</td>
<td>0.2428</td>
</tr>
<tr>
<td>[6,]</td>
<td>-0.9854</td>
<td>-0.8366</td>
<td>-0.9274</td>
<td>4.0535</td>
<td>0.3988</td>
</tr>
<tr>
<td>[7,]</td>
<td>-0.9725</td>
<td>-0.7939</td>
<td>-0.9029</td>
<td>0.3753</td>
<td>0.9845</td>
</tr>
<tr>
<td>[8,]</td>
<td>-0.9598</td>
<td>-0.7514</td>
<td>-0.8786</td>
<td>0.5035</td>
<td>0.9732</td>
</tr>
<tr>
<td>[9,]</td>
<td>-0.9477</td>
<td>-0.7095</td>
<td>-0.8549</td>
<td>0.8191</td>
<td>0.9359</td>
</tr>
<tr>
<td>[10,]</td>
<td>-0.9471</td>
<td>-0.6791</td>
<td>-0.8427</td>
<td>7.2298</td>
<td>0.1242</td>
</tr>
<tr>
<td>[11,]</td>
<td>-0.9429</td>
<td>-0.6452</td>
<td>-0.8269</td>
<td>5.2162</td>
<td>0.2658</td>
</tr>
<tr>
<td>[12,]</td>
<td>-0.9424</td>
<td>-0.6149</td>
<td>-0.8148</td>
<td>7.2382</td>
<td>0.1238</td>
</tr>
<tr>
<td>[13,]</td>
<td>-0.9441</td>
<td>-0.5868</td>
<td>-0.8049</td>
<td>8.4003</td>
<td>0.0780</td>
</tr>
<tr>
<td>[14,]</td>
<td>-0.9321</td>
<td>-0.5451</td>
<td>-0.7813</td>
<td>0.8876</td>
<td>0.9263</td>
</tr>
</tbody>
</table>

> Eccm(zt)
p-values table of Extended Cross-correlation Matrices:
Column: MA order
Row : AR order

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0002</td>
<td>0.2704</td>
<td>0.2917</td>
<td>0.2953</td>
</tr>
<tr>
<td>1</td>
<td>0.0147</td>
<td>0.4064</td>
<td>0.9763</td>
<td>0.8202</td>
<td>0.9773</td>
<td>0.9667</td>
<td>0.9795</td>
</tr>
<tr>
<td>2</td>
<td>0.0618</td>
<td>0.9541</td>
<td>0.2369</td>
<td>0.9932</td>
<td>0.9770</td>
<td>0.9488</td>
<td>0.7670</td>
</tr>
<tr>
<td>3</td>
<td>0.9537</td>
<td>0.9999</td>
<td>0.9886</td>
<td>0.9976</td>
<td>0.9492</td>
<td>0.9561</td>
<td>0.7508</td>
</tr>
<tr>
<td>4</td>
<td>0.9990</td>
<td>0.9996</td>
<td>0.9813</td>
<td>0.9981</td>
<td>0.9996</td>
<td>0.9795</td>
<td>0.7196</td>
</tr>
<tr>
<td>5</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9988</td>
<td>0.9975</td>
<td>0.8427</td>
</tr>
</tbody>
</table>

(c) Perform the estimation by iterations of removing insignificant parameters. We used two iterations. The first iteration uses threshold 1.5 and the second iteration uses threshold 1.96 as requested. The resulting model is

\[
 z_t = \begin{bmatrix} 0.11 \\ 0.05 \end{bmatrix} + \begin{bmatrix} 0 & 0.11 \\ 0 & 0.79 \end{bmatrix} z_{t-1} + a_t - \begin{bmatrix} 0.28 & 0 \\ 0 & 0.55 \end{bmatrix} a_{t-1}, \quad \Sigma_a = \begin{bmatrix} 0.54 & 0.17 \\ 0.17 & 0.77 \end{bmatrix}
\]

where all parameters are significant at the 5% level.

(d) Model checking indicates that the fitted VARMA(1,1) model is adequate. Figure 3 shows the p-value plot of the multivariate Ljung-Box statistics for the residuals, after adjusting for 6 degrees of freedom. Figure 4 shows the time plots of the residuals. From the plots, it seems that the growth rate of the material component may contain some outliers.
Figure 1: Plot of p-values of Q(m) statistics for residuals of the fitted ECM-VAR model.

(e) The impulse response functions of the fitted VARMA(1,1) model are shown in Figure 5. The cumulative impulse response functions are in Figure 6.


(a) Figure 7 shows the time plots of the percentage growth rate series.

(b) For VAR models, AIC selects VAR(7). For VARMA models, the Ecm selects a VARMA(2,1) model. In our analysis, we employ a VARMA(2,1) model. The fitted model, after refinement, is

\[
\mathbf{z}_t = \begin{bmatrix} 0.03 \\ 0.05 \end{bmatrix} + \begin{bmatrix} 0.70 & 0.29 \\ -0.11 & 0.95 \end{bmatrix} \mathbf{z}_{t-1} + \begin{bmatrix} 0.11 & -0.11 \\ 0 & 0 \end{bmatrix} \mathbf{z}_{t-2} + \begin{bmatrix} 0.74 & 0 \\ -0.20 & 0.80 \end{bmatrix} \mathbf{a}_{t-1},
\]

where all estimates have t-ratio greater than 1.645 and the residual covariance matrix is

\[
\Sigma_a = \begin{bmatrix} 1.23 & 0.52 \\ 0.52 & 0.74 \end{bmatrix}.
\]

Model checking shows that the fitted model is reasonable except for a few possible outliers.

(c) The fitted model does not imply Granger causality between the two series because the two series are dynamically dependent of each other; see \(\phi_1\).

(d) The 1-step to 4-step ahead predictions of the model are (0.40, 0.32), (0.53, 0.31), (0.50, 0.29), and (0.49, 0.28), respectively. The corresponding standard errors are (1.11, 0.86), (1.13, 0.89), (1.15, 0.91), and (1.17, 0.92), respectively.
Figure 2: Time plots of the percentage growth rates of the nondurable consumer goods and materials of the monthly U.S. industrial production index from January 1964 to December 2012.

Figure 3: The $p$-values of multivariate $Q(m)$ statistics of the residuals of a fitted VARMA(1,1) model to the percentage growth rates of the nondurable consumer goods and materials of the monthly U.S. industrial production index from January 1964 to December 2012.
Figure 4: Time plots of the residuals of VARMA(1,1) model for the percentage growth rates of the nondurable consumer goods and materials of the monthly U.S. industrial production index from January 1964 to December 2012.

Figure 5: Impulse response functions of the fitted VARMA(1,1) model for the percentage growth rates of two U.S. monthly industrial production index: (a) nondurable consumer goods and (b) materials
Figure 6: Cumulative impulse response functions of the fitted VARMA(1,1) model for the percentage growth rates of two U.S. monthly industrial production index: (a) nondurable consumer goods and (b) materials.

Figure 7: Time plots of components of U.S. monthly industrial production index: (a) business equipments and (b) materials.