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Bus 41910, Time Series Analysis, Mr. R. Tsay
Solutions to Homework Assignment #1 (Autumn 2008)

For computation, I used **R** to obtain the solutions. You may use any package. 1. The commands

I used are given below (the plot is omitted):

```
> setwd("C:/teaching/uts")
> da=read.table("gdp.txt")
> gdp=log(da[,4])
> grwgdp=diff(gdp)
> m1=acf(grwgdp,lag=12)
> names(m1)
[1] "acf"    "type"   "n.used" "lag"    "series" "snames"
> print(m1$acf,digits=1)
      [,1]
 [1,]  1.00
 [2,]  0.46
 [3,]  0.31
 [4,]  0.06
 [5,] -0.02
 [6,] -0.14
 [7,] -0.04
 [8,]  0.03
 [9,]  0.05
[10,]  0.19
[11,]  0.22
[12,]  0.19
[13,]  0.05
```

2. Part (a). From $(1-1.2B+0.5B^2)Y_t = 0$, we have $\lambda_j = 0.6 \pm \frac{\sqrt{0.56}i}{2} \approx 0.6 \pm 0.374i$, where $j = 1, 2$. The general solution (see lecture note 1) is $Y_t = \gamma_1\lambda_1^t + \gamma_2\lambda_2^t$, where the coefficient γ_1 and γ_2 are determined by the initial conditions. Specifically, we have

$$\begin{aligned} Y_0 &= \gamma_1 + \gamma_2 = 5, \\ Y_1 &= \gamma_1\lambda_1 + \gamma_2\lambda_2 = 6.74. \end{aligned}$$

The 2nd equation gives $0.6(\gamma_1+\gamma_2) - 0.374i(\gamma_1-\gamma_2) = 6.74$. Thus, $\gamma_1 - \gamma_2 = -10i$. Therefore, $\gamma_1 = 2.5 - 5i$ and $\gamma_2 = 2.5 + 5i$.

Part (b). First, using the result for the first-order difference equation and $X_0 = 2$, we have $X_t = 2(-0.5)^t$. Next, using the result given in Lecture 1,

$$Y_t = \frac{1-0.5^t}{0.5} + 0.4\sum_{i=0}^{t-1} 0.5^i X_{t-i} + 0.5^t.$$

3. First, the lag-k autoconvariance of X_t is $\gamma_k = E(X_t X_{t-k}) = E[(\sum_{i=0}^{\infty} \psi_i a_{t-i})(\sum_{j=0}^{\infty} \psi_j a_{t-k-j})]$ $= E(\sum_{j=0}^{\infty} \psi_{k+j}\psi_j a_{t+k-j}^2) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi j + k$.
Next, work out the coefficient for the z^k or z^{-k} term of $\Gamma(z)$. You can verify that the coefficient is exactly γ_k.
4. Using the backshift operator and factorization, we have $(1 - \phi_1 B - \phi_2 B^2)Y_t = (1 - \lambda_1 B)(1 - \lambda_2 B)Y_t = 0$. From the factorization, we have

$$\lambda_i = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} \quad (1)$$

$$\phi_1 = \lambda_1 + \lambda_2 \quad (2)$$

$$\phi_2 = -\lambda_1 \lambda_2. \quad (3)$$

For Y_t to be stable, we require $|\lambda_i| < 1$ or equivalently $-1 < \lambda_i < 1$. Consequently, by Eq.(3),

$$-1 < \phi_2 < 1. \quad (4)$$

By Eq. (2),

$$-2 < \phi_1 < 2. \quad (5)$$

Next, if $\phi_1^2 + 4\phi_2 < 0$, then $\phi_2 < -\frac{\phi_1^2}{4}$. This condition gives an inside regime of a parabola determined by three points $(0, 0), (2, -1)$, and $(-2, -1)$. Next, consider $\phi_1^2 + 4\phi_2 \geq 0$. In this case, $\sqrt{\phi_1^2 + 4\phi_2} \geq 0$. Furthermore, we have $-1 < \frac{\phi_1}{2} \pm \frac{\sqrt{\phi_1^2+4\phi_2}}{2} < 1$. Based on the 2nd inequality,

$$\begin{aligned} \frac{\phi_1}{2} &+ \frac{\sqrt{\phi_1^2 + 4\phi_2}}{2} < 1 \\ &\Rightarrow \sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1 \\ &\Rightarrow \phi_1^2 + 4\phi_2 < \phi_1^2 - 4\phi_1 + 4 \\ &\Rightarrow \phi_2 < 1 - \phi_1. \end{aligned} \quad (6)$$

Based on the first inequality,

$$\begin{aligned} -1 &< \frac{\phi_1}{2} - \frac{\sqrt{\phi_1^2 + 4\phi_2}}{2} \\ &\Rightarrow -2 - \phi_1 < -\sqrt{\phi_1^2 + 4\phi_2} \\ &\Rightarrow \phi_1^2 + 4\phi_1 + 4 > \phi_1^2 + 4\phi_2 \\ &\Rightarrow \phi_1 + 1 > \phi_2. \end{aligned} \quad (7)$$

Combining Eqs. (4)-(7), we obtain a triangular with vertices at $(0, 1), (2, -1)$ and $(-2, -1)$.

5. The commands used and summary statistics of generated series are given below:

```
(a) > x=rnorm(300)
> mean(x)
[1] -0.002530567
```

```
> sqrt(var(x))
[1] 0.9832138
> m1=acf(x,lag=5)
> print(m1$acf,digits=1)
[1,]  1.00
[2,]  0.02
[3,] -0.06
[4,]  0.09
[5,]  0.03
[6,]  0.01

> z=arima.sim(400,model=list(ma=c(-1.1,.4)),sd=1)
> z=z+0.4
> mean(z)
[1] 0.3970292
> sqrt(var(z))
[1] 1.585545
> m1=acf(z,lag=5)
> print(m1$acf,digits=1)
[1,]  1.00
[2,] -0.69
[3,]  0.26
[4,] -0.09
[5,]  0.06
[6,] -0.03
> z=arima.sim(300,model=list(ar=c(0.8)),sd=1)
> z=z+1/(1-0.8)
> mean(z)
[1] 5.062337
> sqrt(var(z))
[1] 1.597056
> m1=acf(z,lag=5)
> print(m1$acf,digits=1)
[1,]  1.0
[2,]  0.8
[3,]  0.5
[4,]  0.4
[5,]  0.2
[6,]  0.1
> z=arima.sim(600,model=list(ar=c(.9),ma=c(0.4)),sd=1)
> z=z+1/(1-0.9)
> mean(z)
[1] 9.609824
> sqrt(var(z))
[1] 2.999586
```

```
> m1=acf(z,lag=5)
> print(m1$acf,digits=2)
[1,] 1.00
[2,] 0.93
[3,] 0.84
[4,] 0.76
[5,] 0.69
[6,] 0.64
```