Graduate School of Business
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Bus 41910, Time Series Analysis, Mr. R. Tsay

Homework Assignment #2 (due October 9, **Before** class)

1. Taking the first difference, we have

$$(1-B)Z_t = (1-B)X_t + (1-B)b_t = a_t + b_i - b_{t-1}.$$

Let $W_t = a_t + b_1 - b_{t-1}$. It is easy to see that (a) $\text{Var}(W_t) = 3$, (b) $\text{Cov}(W_t, W_{t-1}) = -1$, and (c) $\text{Cov}(W_t, W_{t-j}) = 0$ for $j > 1$. Thus, W_t is an MA(1) series, which can be written as $W_t = e_t - \theta e_{t-1}$ with $\{e_t\}$ being an i.i.d. sequence with mean zero and variance σ_e^2. The values of θ and σ_e^2 can be determined by equating the variance and lag-1 covariance of W_t. Specifically, from the MA(1) model, w ehave $\text{Var}(W_t) = (1+\theta^2)\sigma_e^2$ and $\text{Cov}(W_t, W_{t-1})$ $= -\theta\sigma_e^2$. Consequently, we have $(1+\theta^2)\sigma_e^2 = 3$ and $\theta\sigma_e^2 = 1$. Taking the ratio, we have $\theta/(1+\theta^2) = 1/3$. The solutions for θ are approximately 0.382 and 2.618. Since we are interested in invertiable MA(1) model, θ must satisfy the condition $|\theta| < 1$. Therefore, θ $= 0.382$ and $\sigma_e^2 = 2.618$.

2. I use R to perform the simulation. For simplicity, I do not show the plto, but give the actual values of sample ACF.

```
(a)
> at=rnorm(200)
> zt=rep(0,200)
> zt[1]=at[1]
> for (i in 2:200){
+ zt[i]=zt[i-1]+at[i]
+ }
> plot(zt,type='l')
> m1=acf(zt,lag=12)
> print(m1$acf,digits=2)
      [,1]
 [1,] 1.00
 [2,] 0.95
 [3,] 0.90
 [4,] 0.86
 [5,] 0.81
 [6,] 0.76
 [7,] 0.72
 [8,] 0.68
```

```
 [9,] 0.65
[10,] 0.61
[11,] 0.59
[12,] 0.56
[13,] 0.53
Alternatively, you may use the command ‘‘cumsum’’, cumulative sum.
> yt=cumsum(at)
> m2=acf(yt,lag=12)
> print(m2$acf,digits=2)
      [,1]
 [1,] 1.00
 [2,] 0.95
 [3,] 0.90
....
(b) (Note: the slope dominates the plot.)
> at=rnorm(200)
> zt[1]=at[1]+5
> for (i in 2:200){
+ zt[i]=zt[i-1]+5+at[i]
+ }
> plot(zt,type='l')
> m2=acf(zt,lag=12)
> print(m2$acf,digits=2)
 [1,] 1.00
 [2,] 0.98
 [3,] 0.97
 [4,] 0.95
 [5,] 0.94
 [6,] 0.92
 [7,] 0.91
 [8,] 0.89
 [9,] 0.88
[10,] 0.86
[11,] 0.85
[12,] 0.83
[13,] 0.82
(c) Seasonal model
> zt=arima.sim(200,model=list(ar=c(0,0,0,0.8)),sd=1)
> plot(zt,type='l')
> m3=acf(zt,lag=12)
> print(m3$acf,digits=2)
 [1,]  1.000
```

```
 [2,] -0.095
 [3,] -0.222
 [4,] -0.110
 [5,]  0.779
 [6,] -0.074
 [7,] -0.225
 [8,] -0.093
 [9,]  0.568
[10,] -0.097
[11,] -0.182
[12,] -0.065
[13,]  0.388
```

(d) Complex unit roots (You should see the since and cosine pattern of ACFs).

```
> at=rnorm(200)
> zt[1]=at[1]
> zt[2]=sqrt(3)*zt[1]+at[2]
> for (i in 3:200){
+ zt[i]=sqrt(3)*zt[i-1]-zt[i-2]+at[i]
+ }
> plot(zt,type='l')
> m4=acf(zt,lag=12)
> print(m4$acf,digits=2)
 [1,]  1.000
 [2,]  0.855
 [3,]  0.497
 [4,]  0.029
 [5,] -0.423
 [6,] -0.745
 [7,] -0.858
 [8,] -0.745
 [9,] -0.445
[10,] -0.048
[11,]  0.339
[12,]  0.617
[13,]  0.722
```

(e) An approximate model for fractional difference time series.
You should see small, but significant, ACFs.

```
> zt=arima.sim(1000,model=list(ar=c(0.9),ma=c(-0.8)),sd=1)
> plot(zt,type='l')
> m5=acf(zt,lag=12)
> print(m5$acf,digits=2)
```

```
 [1,]  1.0000
 [2,]  0.1494
 [3,]  0.1228
 [4,]  0.1006
 [5,]  0.1104
 [6,]  0.1055
 [7,]  0.0487
 [8,]  0.0360
 [9,]  0.1009
[10,]  0.0264
[11,]  0.0686
[12,] -0.0533
[13,]  0.0083
```

3. Applying $(1-0.8B)$ to Z_t, we have

$$(1-0.8B)Z_t = (1-0.8B)X_t + (1-0.8B)Y_t = a_t - 0.3a_{t-1} + b_t.$$

Let $W_t = a_t - 0.3a_{t-1} + b_t$. Similar to Q1, W_t is an MA(1) series and can be written as $W_t = e_t - \theta e_{t-1}$ with e_t being a Gaussian white noise series with mean zeor and variance σ_e^2. Working out the algrbra, we have approximately $\theta = 0.147$ and $\sigma_e^2 = 6.82$.

4. Suppose that X_t follows the model $X_t = 0.8X_{t-1} + a_t$, where a_t is a Gaussian white noise series with mean 0 and variance 1. What is the model for the aggregated series $Y_t = X_{2t} + X_{2t-1}$?
Applying $(1-0.8^2B)$ to Y_t and noting that "B" of Y_t is B^2 of X_t, we have

$$\begin{aligned}
(1-0.64B)Y_t &= (X_{2t} + X_{2t-1}) - 0.64(X_{2t-2} + X_{2t-3}) \\
&= (X_{2t} - 0.64X_{2t-2}) + (X_{2t-1} - 0.64X_{2t-3}) \\
&= [(X_{2t} - 0.8X_{2t-1}) + 0.8(X_{2t-1} - X_{2t-2})] + [(X_{2t-1} - 0.8X_{2t-2}) + 0.8(X_{2t-2} - X_{2t-3})] \\
&= a_{2t} + 0.8a_{2t-1} + a_{2t-1} + 0.8a_{2t-2} \\
&= a_{2t} + 1.8a_{2t-1} + 0.8a_{2t-2}.
\end{aligned}$$

In the time scale of Y_t, the right hand side of the previous equation involves 1 lag so that it is an MA(1) model. Consequently, Y_t follows an ARMA(1,1) model with AR polynomial $(1-0.64B)$. The MA part can be obtained as before.

5. For part (1), write the model as $X_t = a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13}$. Simply computing the autocovariances at lags 0, 1, 11, 12, and 13, you can obtain the results given in the lecture note. For part (2), write the model as $X_t = a_t - \theta a_{t-1} - \Theta a_{t-12}$. The result given in the lecture note follows by direct computing the autocovariance of lags 0, 1, 11, and 12. All other lags of ACFs are zero.