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Bus 41910, Time Series Analysis, Mr. R. Tsay

Solutions to Homework Assignment #6

1. (a) Akaike's approach: The dimension of state vector is $m = \max\{2, 1+1\} = 2$ so that $S_t = (z_t, z_{t+1|t})'$ and, from the model, $\psi_1 = 1.8$. Thus,

$$\begin{aligned}\begin{bmatrix} z_{t+1} \\ z_{t+2|t+1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -0.4 & 1.3 \end{bmatrix} \begin{bmatrix} z_t \\ z_{t+1|t} \end{bmatrix} + \begin{bmatrix} 1 \\ 1.8 \end{bmatrix} a_{t+1} \\ z_t &= [1, 0] S_t.\end{aligned}$$

Aoki's approach: The state vector is $S_t = (z_{t-1}, z_{t-2}, a_{t-1})'$ and the model is

$$\begin{aligned}\begin{bmatrix} z_t \\ z_{t-1} \\ a_t \end{bmatrix} &= \begin{bmatrix} 1.3 & -0.4 & 0.5 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_{t-1} \\ z_{t-2} \\ a_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} a_t \\ z_t &= [1.3, -0.4, 0.5] S_t + a_t.\end{aligned}$$

Harvey's approach: The dimension of the state vector is $m = \max\{2, 1\} = 2$ and the state vector is $S_t(z_{t|t-1}, z_{t+1|t-1})'$. The model becomes

$$\begin{aligned}\begin{bmatrix} z_{t+1|t} \\ z_{t+2|t} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -0.4 & 1.3 \end{bmatrix} \begin{bmatrix} z_{t|t-1} \\ z_{t+1|t-1} \end{bmatrix} + \begin{bmatrix} 1.8 \\ 1.94 \end{bmatrix} a_t \\ z_t &= [1, 0] S_t + a_t.\end{aligned}$$

2. There are several ways to derive the result that the distribution of $\boldsymbol{X}$ given $\boldsymbol{Y} = \boldsymbol{y}$ is multivariate normal with mean $\boldsymbol{\mu}_x + \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}(\boldsymbol{y} - \boldsymbol{\mu}_y)$ and covariance matrix $\boldsymbol{\Sigma}_{xx} - \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}$. See, for instance, the multivariate statistics book by Johnson and Wichern (2007, 6th ed.) One approach is to use density function. Here one makes use of the identities: (a) $|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma}_{yy}||\boldsymbol{\Sigma}_{xx} - \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}|$, where $\boldsymbol{\Sigma}$ is the covariance matrix of $(\boldsymbol{X}', \boldsymbol{Y}')'$. (b) $[(\boldsymbol{X} - \boldsymbol{\mu}_x)', (\boldsymbol{Y} - \boldsymbol{\mu}_y)']\boldsymbol{\Sigma}^{-1}[(\boldsymbol{X} - \boldsymbol{\mu}_x)', (\boldsymbol{Y} - \boldsymbol{\mu}_y)']' =$

$$[\boldsymbol{X} - \boldsymbol{\mu}_x - \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}(\boldsymbol{Y} - \boldsymbol{\mu}_y)]'[\boldsymbol{\Sigma}_{xx} - \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}]^{-1}[\boldsymbol{X} - \boldsymbol{\mu}_x - \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}(\boldsymbol{Y} - \boldsymbol{\mu}_y)]$$

$$+(\boldsymbol{Y} - \boldsymbol{\mu}_y)'\boldsymbol{\Sigma}_{yy}^{-1}(\boldsymbol{Y} - \boldsymbol{\mu}_y).$$

These two identities allow us to partition the joint density function of $\boldsymbol{X}$ and $\boldsymbol{Y}$ into the marginal density of $\boldsymbol{Y}$ and the conditional density of $\boldsymbol{X}$ given $\boldsymbol{Y} = \boldsymbol{y}$.

3. The conditional distribution of $\boldsymbol{X}$ given $\boldsymbol{Y} = \boldsymbol{y}$ and $\boldsymbol{Z} = \boldsymbol{z}$ is multivariate normal with mean $\boldsymbol{\mu}_x + \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}(\boldsymbol{y} - \boldsymbol{\mu}_y) + \boldsymbol{\Sigma}_{xz}\boldsymbol{\Sigma}_{zz}^{-1}(\boldsymbol{z} - \boldsymbol{\mu}_z)$ and covariance matrix $\boldsymbol{\Sigma}_{xx} - \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx} - \boldsymbol{\Sigma}_{xz}\boldsymbol{\Sigma}_{zz}^{-1}\boldsymbol{\Sigma}_{zx}$. This result can be obtained by using result of Problem 2 and the independence between $\boldsymbol{Y}$ and $\boldsymbol{Z}$ under normality.

4. Among the equations (6) to (10), the only one that needs modification is (10), which becomes

$$C_{t+1|t} = H\boldsymbol{P}_{t+1|t} + W.$$

Eq. (11) and (12) becomes

$$\begin{aligned} S_{t+1|t+1} &= S_{t+1|t} + [P_{t+1|t}H' + W][HP_{t+1|t}H' + R]^{-1}(Z_{t+1} - Z_{t+1|t}) \\ P_{t+1|t+1} &= P_{t+1|t} - [P_{t+1|t}H' + W][HP_{t+1|t}H' + R]^{-1}(HP_{t+1|t} + W). \end{aligned}$$

5. Again, there arr multiple solutions. One of them is given below.

$$\begin{bmatrix} \beta \\ z_t \\ z_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.24 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ z_{t-1} \\ z_{t-2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} a_t.$$

$$y_t = [x_t, 1, 0]S_t.$$