

An Analysis of Covariance Risk and Pricing Anomalies

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This article examines the link between several well-known asset pricing “anomalies” and the covariance structure of returns. I find size, book-to-market, and momentum strategies exhibit a strong, weak, and negligible relation to covariance risk, respectively. A size factor helps predict future volatility and covariation, improving the efficiency of investment strategies. Moreover, its premium rises following increases in both its volatility and covariation with other assets. These effects are amplified in recessions. No such relations exist for book-to-market or momentum. These findings may shed light on explanations for these premia and present a challenging set of facts for future theory.

While an extensive theoretical and empirical literature has analyzed the relation between various firm characteristics and the cross section of expected returns, very little is known about their relation to the second moment of returns. This article attempts to answer two important and related questions. First, to what extent are premia associated with firm characteristics related to the covariance matrix of returns? Second, how “risky” (in terms of covariance risk) are investment strategies designed from these characteristics?

There is considerable debate over why firm characteristics seem to be related to the cross section of average returns. Much of this debate centers on whether this relation constitutes evidence for or against market efficiency. The joint test of market efficiency and the underlying equilibrium return model makes resolution of this debate difficult. Under the capital asset pricing model (CAPM) of Sharpe (1964), Lintner (1965), and Black (1972), market β 's (or the covariation of a security's return with the return on the market portfolio) should be sufficient to describe the cross section of expected returns. Hence the relation between other firm characteristics and average returns were termed “anomalies” with respect to the CAPM. However, other return-generating models, such as the multifactor models of Merton (1973)

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and Ross (1976), may be consistent with these characteristics reflecting multifactor risk premia.¹

The literature primarily focuses on the first moment of returns. However, the second moment of returns also plays a vital role in asset pricing theory. Analyzing the first two moments of the return distribution simultaneously is paramount to our understanding of asset prices. I focus explicitly on the second moment of returns and their relation to well-known predictors of mean returns.

The goal of this article is to establish an economic link between the CAPM anomalies and covariance risk. This analysis is important for several reasons. First, if covariance risk is priced, then establishing such a link can aid in determining whether the premia associated with firm characteristics are due to risk or mispricing. For example, this could be a test of the exact pricing relation of an equilibrium version of Ross' arbitrage pricing theory (APT; 1976).²

Second, even if covariance risk is not priced, the results may still shed light on the market efficiency debate. For example, the intertemporal CAPM of Merton (1973) does not require a link between return premia and the covariance matrix. Yet understanding the dynamics of the covariance matrix may aid in identifying the changing investment opportunities facing investors.

Third, the second moment of returns is analyzed in relation to the business cycle, which may capture consumption states investors care about. This has asset pricing implications in general and may yield further insights into the source of the return premia.

Finally, even outside of the rational/mispricing debate, this study documents a useful set of facts on time-varying volatility and covariances, and their relation to the business cycle. This augments existing evidence on time-varying second moments [e.g., French, Schwert, and Stambaugh (1986) and Schwert (1989)] by focusing on the changing *covariances* of returns, which has not been extensively studied. Therefore, at the very least, the results in this article present a challenge for theory to accommodate the stylized facts of conditional return second moments.

I employ a unique multivariate GARCH model to characterize the time-varying covariance structure of returns. A set of well-diversified industry portfolios and portfolios formed from various firm characteristics are used as the basis assets. The analysis focuses on the three most notable and strongest return premia associated with firm size, book-to-market equity, and past-year

¹ On the other hand, tests based on Sharpe ratio bounds [or Hansen and Jagannathan (1991) bounds] show that Sharpe ratios generated from trading strategies based on these characteristics are often difficult to justify [e.g., MacKinlay (1995) and Chen (2001)]. Consequently, other theories have sought to explain the source of these returns from models of investor irrationality. While this article does not attempt to (nor can it) reconcile these explanations, the results may shed light on this debate.

² As Shanken (1982) notes, to test the APT empirically requires obtaining an exact rather than approximate factor pricing relation, which in turn requires an assumption about market equilibrium.

returns (momentum).³ Fama and French (1996) find that these three characteristics are the most robust for describing the cross section of average returns and capture the returns associated with a host of other firm characteristics. In addition, recent research has shown that these premia are present in international equity markets with low correlation to U.S. data,⁴ and in subsequent time periods,⁵ providing out-of-sample evidence that diminishes data mining concerns.

I find that a size factor has significant explanatory power for both contemporaneous and future return second moments. However, the explanatory power of book-to-market for the covariance matrix of returns (both in and out of sample) is weaker, and that of momentum is negligible. The market portfolio, on the other hand, is the single most important factor contributing to the covariance matrix of asset returns. These results have important statistical and economic implications. Furthermore, in out-of-sample tests, the market portfolio in combination with a portfolio based on size and book-to-market describes future return second moments better than factors derived statistically from principal components analysis. All of these results are magnified during recessions, when both conditional correlations and volatility are highest and when investors may care most about these forecasts.

Finally, this study shows that volatility increases during low consumption states (e.g., recessions). Moreover, the size premium rises in response to this. No such relation exists, however, for book-to-market or momentum. In addition, when a size factor covaries more strongly with other assets in the economy, its future premium also rises (controlling for the volatility effect). Again, no such relation exists for the other characteristics. This may point to a risk-based interpretation of the size effect.

In addition to the “anomalies” debate, the analysis also provides other important economic implications. First, investment strategies that account for the changing covariance structure of returns are shown to be more efficient than portfolios that ignore second moment dynamics [consistent with Graham and Harvey (1996) and Fleming, Kirby, and Ostdiek (2000)]. The connection between the return premia associated with firm characteristics and the dynamics of the covariance matrix may provide useful portfolio and asset allocation guidance.

³ Banz (1981) finds that size (market capitalization) has additional explanatory power over market β for describing the cross section of expected stock returns. Stattman (1980) and Rosenberg, Reid, and Lanstein (1985) find that the ratio of book value of equity to market value of equity has additional explanatory power over both β and size. Jegadeesh and Titman (1993) document that past 6- to 12-month returns predict future average returns beyond the effects of size and book-to-market.

⁴ See Fama and French (1998) for evidence of the book-to-market effect in international markets, Rouwenhorst (1998) for evidence of momentum in 12 European countries, Liew and Vassalou (1999) for evidence of size, book-to-market, and momentum in 10 international economies, and Hawawini and Keim (1995) and references therein for evidence of a host of anomalies in international markets.

⁵ See Davis, Fama, and French (2000) for evidence of book-to-market effects in the 1929–1967 time period, and Jegadeesh and Titman (2000) and Grinblatt and Moskowitz (2001) for evidence of momentum in the 1990–1998 time period.

Second, the use of time-varying conditional covariance estimates provides a setting for evaluating the *predictive* ability of firm characteristics and other proposed factors for *future* covariances. Identifying factors that best describe the second moment of returns has both academic and practical appeal. This may support the use of firm characteristics as instruments for conditional covariation, similar in spirit to the work of Shanken (1990), Ferson and Harvey (1997, 1999), and Lewellen (1999). In addition, the changes in market volatility and correlations across assets and the business cycle are consistent with recent theory by Veronesi (1999) and Ribeiro and Veronesi (2001).

The rest of the article is organized as follows. Section 1 describes the data and the portfolios formed from firm characteristics. Section 2 describes the multivariate GARCH model of time-varying covariances and its estimates. Section 3 documents the link between firm characteristics and the contemporaneous and future covariance structure of returns. The strength of this link across the business cycle is examined. Section 4 evaluates the economic impact of forecasting covariances on minimum variance portfolios. Section 5 examines the relation between return premia associated with the characteristics and their conditional covariation and volatility, as well as how this relation changes over the business cycle. Section 6 concludes.

1. Data and Portfolio Formation

The sample employs all listed equities on the Center for Research in Security Prices daily files, covering New York Stock Exchange-, American Stock Exchange-, and NASDAQ-traded securities, over the period August 1963 to December 1997, that had a book value of common equity from the previous fiscal year available on COMPUSTAT. Trading strategies or factor-mimicking portfolios based on firm size, book-to-market equity, and momentum are generated each week and weekly returns (from Wednesday to Wednesday closing prices) are calculated on these strategies.⁶ Since one of the trading strategies requires a year of prior return history, the analysis begins in August 1964. Given the high dimensional multivariate GARCH model discussed in the next section, higher frequency data is useful for improving estimation accuracy and generating reliable inferences. However, the cost of using daily or intraday data in terms of confounding microstructure influences (such as bid-ask bounce and nonsynchronous trading) can be large. As a compromise, weekly returns are employed.

Conditional covariance matrices are estimated in the next section for 32 “representative” assets: 20 value-weighted industry portfolios [formed from two-digit SIC codes, and identical to those in Moskowitz and Grinblatt

⁶ Wednesday to Wednesday closing prices are used to compute weekly returns since Chordia and Swaminathan (2000), Hou (2001), and others document high autocorrelations using Friday to Friday prices and low autocorrelations using Monday to Monday prices. Wednesday seems like a natural compromise.

(1999)], six size and book-to-market value-weighted portfolios formed from intersecting the 50% biggest and smallest stocks (based on market capitalization in the previous year) with the 30% largest, 40% middle, and 30% smallest book-to-market equity stocks (based on book-to-market equity in the most recent fiscal year), the Center for Research in Security Prices value-weighted index (in excess of the Treasury bill rate), and five past one-year return-sorted portfolios (where individual stocks are sorted into quintiles based on their cumulative past 12-month returns, excluding the most recent month, and value-weighted within the quintile groups).

The order of the assets corresponds to the order in which they will be listed in the covariance matrix. Hence the 27th row (or column) of the matrix represents the covariances of the market portfolio with the other assets. The six size and book-to-market portfolios are ordered as follows: small size, high book-to-market (*SH*); small size, medium book-to-market (*SM*); small size, low book-to-market (*SL*); big size, high book-to-market (*BH*); big size, medium book-to-market (*BM*); and big size, low book-to-market (*BL*). The five individual stock past return portfolios are ordered from low to high past one-year returns.

The size strategy/factor entails going long the 50% smallest stocks and short the biggest 50% of firms, which is $(SH + SM + SL)/3 - (BH + BM + BL)/3$. This small minus big (*SMB*) portfolio is identical to the factor-mimicking portfolio constructed by Fama and French (1993). Likewise, a high minus low (*HML*) book-to-market portfolio is formed, identical to Fama and French (1993), which is long the 30% highest book-to-market firms and short the lowest 30%, which is $(SH + BH)/2 - (SL + BL)/2$. Finally, a winners (high past one-year return) minus losers (low past one-year return) momentum portfolio similar to Carhart's (1997) *PR1YR* portfolio, and an industry momentum portfolio (*IM*) similar to Moskowitz and Grinblatt (1999) are formed. *PR1YR* is a value-weighted portfolio of the highest quintile of the past one-year return stocks portfolio minus the lowest, where stocks are ranked on their cumulative returns over the past 12 months, skipping the most recent month. This portfolio is constructed slightly differently from Carhart (1997) who forms equal-weighted portfolios and uses 30% and 70% breakpoints. *IM* is constructed as in Moskowitz and Grinblatt (1999); it is long the three highest past one-year return industries and short the three lowest, where industries are also sorted on their cumulative returns over the prior 12 months, skipping the nearest month.⁷

These "representative" assets are chosen to reflect the cross-sectional dispersion in ex ante expected returns in the economy. The use of well-diversified portfolios drastically reduces estimation error, allowing for reliable inferences

⁷ Two momentum portfolios are employed since Moskowitz and Grinblatt (1999) and Grundy and Martin (2001) document both industry and individual stock momentum at the one-year horizon, neither of which subsumes the other.

about return second moments to be made. This, however, comes at the expense of not fully characterizing the true cross section of expected returns in the economy.⁸ The finance literature documents, however, that a substantial portion of the cross-sectional variation in expected returns can be captured by dispersion in size, book-to-market, and past return-sorted portfolios [see Fama and French (1996)]. The 20 industry portfolios also provide a grouping of stocks that do not depend on a previously discovered firm characteristic related to average returns. Grouping firms in similar lines of business also increases the dispersion in covariation across portfolios since firms within an industry covary more strongly than firms across industries.⁹

2. Time-Varying Second Moments

2.1 Motivating the GARCH methodology

Across a broad range of assets and econometric specifications, the evidence suggests that volatility is predictable. [For reviews of this literature see Bollerslev, Engle, and Nelson (1994), Diebold and Lopez (1995), and Palm (1996).] Officer (1973) and Bollerslev, Chou, and Kroner (1992) document substantial time variation in stock return variances and French, Schwert, and Stambaugh (1986) and Campbell and Hentschel (1992) find stock return volatility to be serially correlated. A GARCH model seems to accommodate these stylized facts. Andersen and Bollerslev (1998) find that GARCH models explain about 50% of the variation in ex post volatility. In addition to capturing the empirical features of stock return volatility, a multivariate GARCH model allows for out-of-sample tests to be conducted on forecasting the covariance structure of returns.

I employ the methodology of Ledoit and Santa-Clara (1998), whose multivariate GARCH(1,1) parameterization is the following:

$$H_t = C + A * \epsilon_{t-1} \epsilon'_{t-1} + B * H_{t-1} \tag{1}$$

$$\epsilon_t = R_t - E_{t-1}[R_t] \tag{2}$$

where R_t is an $(N \times 1)$ vector of asset returns, and $E_{t-1}[\cdot]$ is its expectation conditional on information revealed through time $t - 1$. The model is estimated via maximum likelihood. The GARCH parameters are used to reconstruct a time series of covariance matrices, where the identity matrix is used as a starting point for H_0 .¹⁰

⁸ In a previous version of this article, the return covariance matrix of random sets of individual stocks was examined in an attempt to more fully characterize ex ante expected return dispersion in the economy. However, due to the noise in individual stock returns, large estimation errors were generated that destroyed the ability to make reliable inferences about common variations in returns.

⁹ Recent evidence supporting this claim can be found in Chan, Karceski, and Lakonishok (1999) and Campbell et al. (2001). This is intuitive since firms within an industry face similar regulatory, supply, demand, and production shocks.

¹⁰ Results are robust to other initial matrix starting points, such as the unconditional sample covariance matrix.

The number of parameters to be estimated in this model is $\frac{3N(N+1)}{2}$. With only a relatively short history of security returns, many GARCH models place restrictions on the parameter matrices C , A , and B [i.e., constant conditional correlation as in Ding and Engle (1994), Engle and Kroner (1995), and Engle, Ng, and Rothschild (1990)], particularly when they wish to estimate a system for at least five assets. The Ledoit and Santa-Clara (1998) methodology, however, avoids placing additional structure on the model yet allows large-dimensional matrices to be estimated. Their algorithm estimates the parameters C , A , and B for each pair of assets ($N = 2$) independently and then minimizes the “distance” (Frobenius norm) between each of these bivariate estimates and the space of potential values for C , A , and B that satisfy conditions ensuring positive definite and covariance stationary matrices. The advantage of this approach is that it does not impose further restrictions on the parameter matrices and estimates C simultaneously with the other parameters. Their procedure essentially resamples the data to estimate the model. This, in some sense, imposes some structure on the model, hence the approach is not without its drawbacks. However, simulations of their procedure’s small-sample properties did not yield any substantial biases. Furthermore, the estimation errors of the parameters are shown to be small when using weekly data.

2.2 The precision and robustness of the GARCH estimates

Throughout the article the GARCH-estimated covariance matrices are treated as the true covariance matrix of returns at each point in time. Obviously model uncertainty and estimation error will add noise to the analysis. Appendix A discusses the precision and robustness of the GARCH estimates in an attempt to mitigate such concerns.

2.3 Conditional second moments of returns

Figure 1 plots various measures of conditional second moment estimates over time, using the GARCH-constructed conditional covariance matrices. Figure 1A plots the average weekly standard deviation over time, where each week the equal-weighted conditional standard deviation across the 32 portfolios is plotted. Figure 1B plots the average weekly correlation by taking the average correlation of each portfolio with the other 31 portfolios at each point in time, and then taking the (equal-weighted) average of these averages across the 32 portfolios. For comparison, the *unconditional* average standard deviation and average correlation are presented on the graphs. Both the conditional standard deviations and conditional correlations exhibit significant time variation.

Recessionary periods, as defined by the National Bureau of Economic Research, are also depicted on the graphs. The average standard deviations, and to some extent correlations, appear higher during recessions. Both volatility and correlation blow up during the market crash of October 1987.

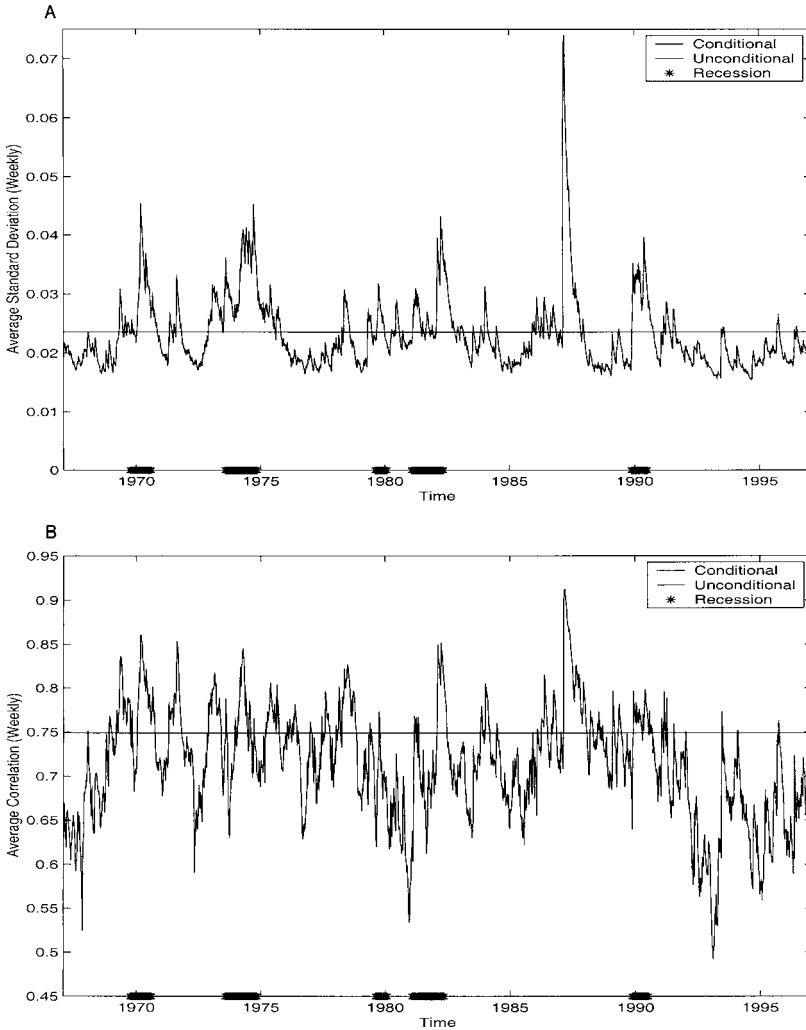


Figure 1
Time-varying second moments of returns

The figure plots various measures of conditional second moments averaged across 32 portfolios (20 industries, 6 size and book-to-market equity portfolios, the market index, and 5 past-year return portfolios) using the Ledoit and Santa-Clara (1998) multivariate GARCH-estimated conditional covariance matrices. The average standard deviation (A), average correlation (B), sum of all eigenvalues of the covariance matrix (C), and standard deviation of the equal- and value-weighted index portfolio (D) are plotted over time from August 1964 to December 1997. For reference, the unconditional estimates of these measures are also plotted, obtained from the unconditional sample covariance matrix. Contractionary periods (recessions) defined by the NBER are highlighted on the graph.

Combining the information from the conditional standard deviation and correlation plots, Figure 1C plots the sum of the eigenvalues of the conditional covariance matrix over time. This represents the total amount of covariation among the assets at each point in time. From Figure 1C, it is evident

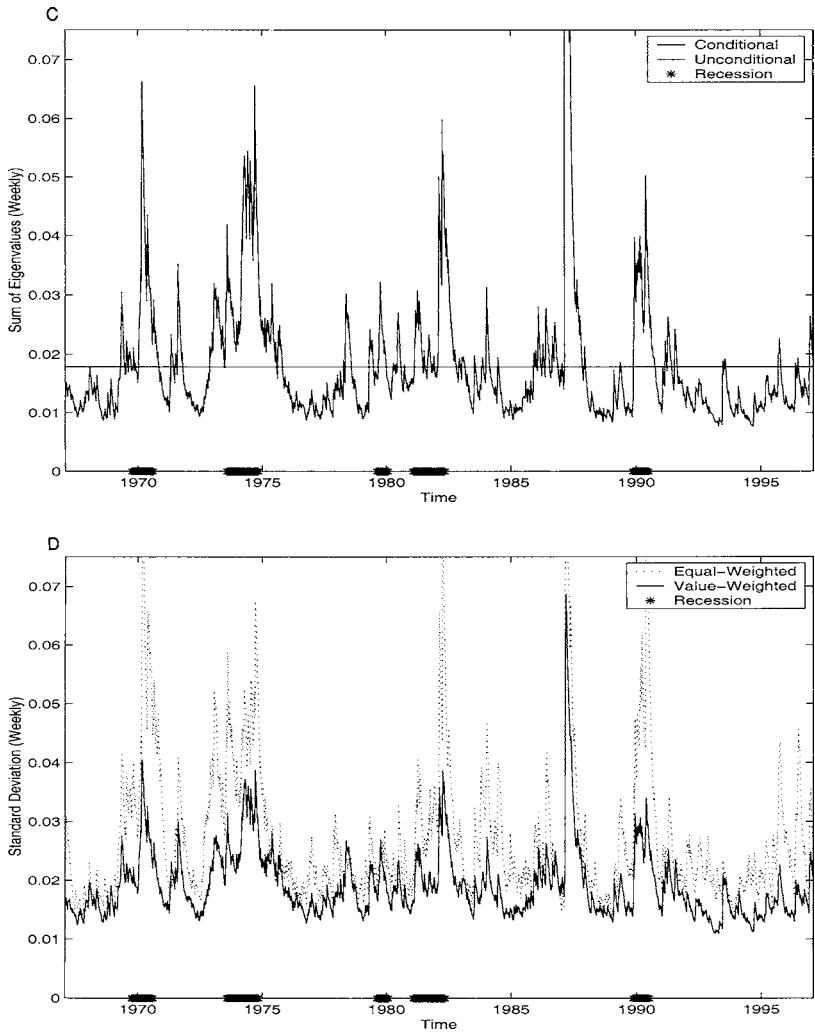


Figure 1
(continued)

that conditional covariation is highest during recessions and was extremely large during the October 1987 stock market crash. This is consistent with evidence from Roll (1988) as well as Ang and Chen (2001) on market downturns. These features of the data would not be captured by a more restrictive conditional covariance model, such as constant correlation models, further motivating the use of the Ledoit and Santa-Clara (1998) methodology.

Finally, Figure 1D plots the conditional volatility of the equal- and value-weighted index for comparison. The figure demonstrates the same patterns

exhibited in the previous three plots: aggregate volatility changes substantially through time, appears somewhat autocorrelated, and is higher during recessions. The abilities of the firm characteristics or proposed factors to capture these features will be tested in the data.

2.4 Conditional investment strategies

Figure 2 examines the role of time-varying covariances from an investment perspective. Using the conditional covariance matrix at each point in time, minimum variance portfolio (MVP) weights are formed every week. Without additional constraints, such a procedure will often generate extreme portfolio positions in certain assets, often involving large short positions. Figure 2A plots the minimum, maximum, and mean portfolio weight at each point in time. As readily seen from the figure, the MVP weights fluctuate wildly and often exhibit enormous positions. The scale on the vertical axis expresses the weights in terms of decimals, not percents. Hence a maximum (minimum) weight of 80 (-80) represents an 8000% long (short) position in an asset! This is clearly extreme and not very practical. The mean MVP weights, however, are generally between -2 and 2 .

To generate more reasonable portfolio weights, MVPs are also computed subject to the constraint that no short positions be allowed and therefore no weight on any single asset exceed 100% (assuming no borrowing at the risk-free rate). This is a realistic assumption since many money managers are either constrained from taking, or are unwilling to take, short positions, and many other investors find shorting to be too costly, cannot borrow the stock, or are simply unwilling to short [see D'Avolio (2001) and Géczy, Musto, and Reed (2001)]. Figure 2B plots the minimum, maximum, and mean portfolio weights at each point in time. The weights are much more reasonable and stable.

Of course, the cost of adding constraints is that the portfolio will be less efficient in sample. Figure 2C plots the in-sample minimized standard deviation of the unconstrained and constrained MVPs. Obviously the unconstrained minimized variance is smaller at every point in time. However, two important points are worth noting. First, the unconstrained minimized variance is only about one-third smaller on average than the constrained minimized variance. In some periods, the two are nearly identical, even during extreme volatility swings. Second, both minimized variances are highly correlated. The constrained MVP appears to capture much of the dynamics of aggregate volatility. Thus the loss in efficiency appears small in sample.

From an investment perspective, the out-of-sample or ex post volatility of the MVPs is of interest. Although less efficient in sample, the constrained MVP may perform better than the unconstrained portfolio out of sample. To gauge this, the constrained and unconstrained MVP weights at each time t are applied to returns in the next period, $t + 1$. The time-series annualized standard deviation of the returns from the constrained MVP is 10.3%

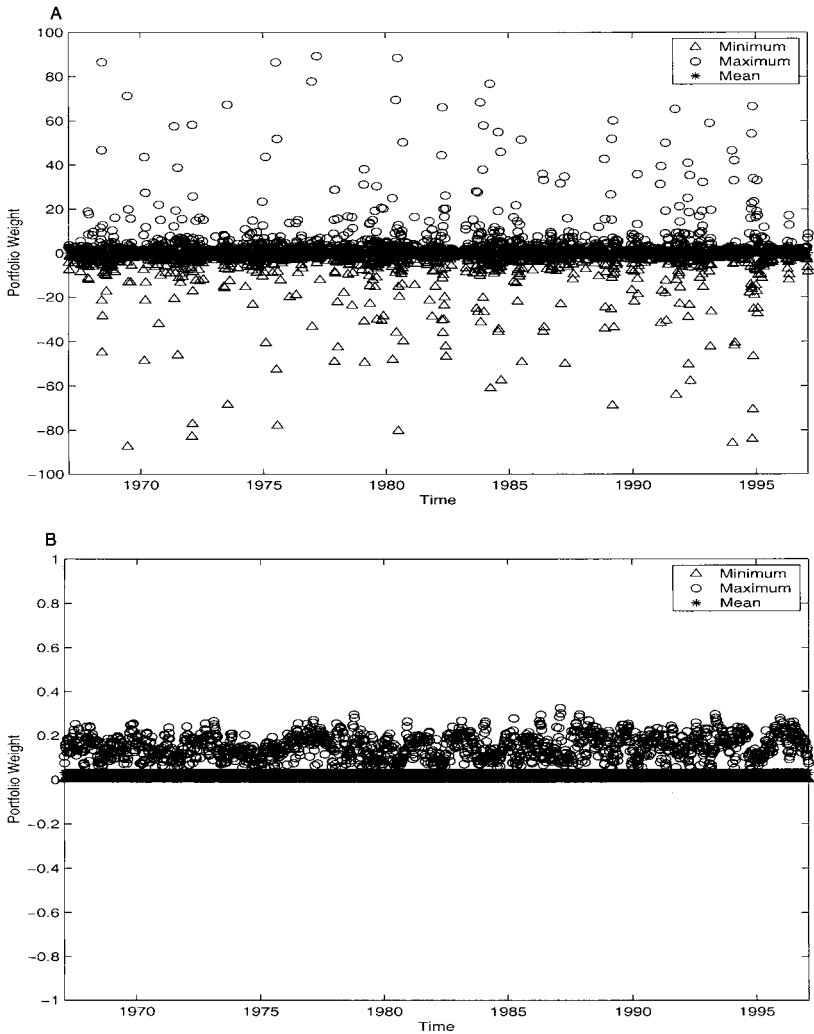


Figure 2
Minimum variance investment strategies

(A) Plots the minimum, maximum, and mean portfolio weights of the unconstrained minimum variance portfolio (MVP), derived from the Ledoit and Santa-Clara (1998) multivariate GARCH-estimated conditional covariance matrix, weekly from August 1964 to December 1997. (B) Plots the minimum, maximum, and mean portfolio weights of the constrained minimum variance portfolio, where no short positions are allowed and no weights on individual assets can exceed 1 (100%). (C) Plots the minimized standard deviation (in sample) of both the unconstrained and constrained MVPs, and highlights recessionary periods defined by the NBER. Finally, (D) plots the ex post conditional volatility (out of sample) of the unconstrained and constrained MVPs, as well as a third constrained MVP that allows for short positions no greater than 50% and long (levered) positions no greater than 150% in each asset. Recessionary periods defined by the NBER are also highlighted on the graphs.

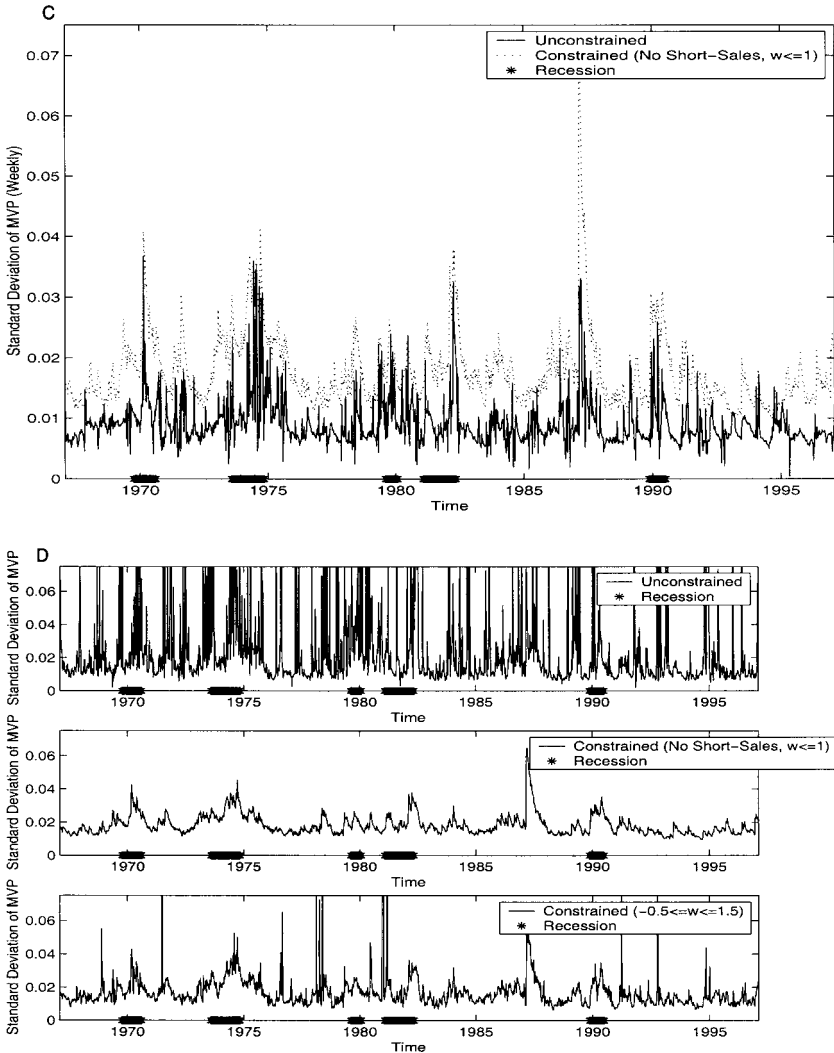


Figure 2
(continued)

compared to 14.8% from the unconstrained MVP. For reference, the standard deviation of the unconditional MVP, obtained from the unconditional sample covariance matrix over the entire period is 12.1%. This suggests that there is valuable information in changing covariances that can reduce portfolio volatility, consistent with the findings of Fleming, Kirby, and Ostdiek (2000), and that GARCH models provide valuable information for predicting volatility.

To examine the differences in conditional volatility out of sample, the MVP weights are applied each period to the following period's conditional covariance matrix and the conditional standard deviation of the MVP is computed "ex post." Figure 2D plots these ex post volatility measures. The unconstrained MVP produces erratic ex post volatilities, which is not surprising given the extreme weights this portfolio contains. The constrained MVP, however, generates much smoother and stable volatilities that mirror the patterns of aggregate volatility in Figure 1.¹¹

Finally, Figure 2D also plots the ex post conditional standard deviation of another constrained MVP that allows for short positions no larger than 50% on any asset and thus long positions no greater than 150%. This provides an interesting comparison for how short positions affect the ex post efficiency of the portfolio. As evident in the figure, the conditional volatility estimates are relatively stable, though slightly less so than the constrained MVP that prohibits short sales.

Clearly, from Figure 2, it is apparent that the unconstrained MVP does not provide useful or meaningful investment implications. Hence, in subsequent analysis, the MVP weights subject to the constraints above are employed.

2.5 Conditional asset pricing factors

Many studies employ factor-mimicking portfolios derived from firm characteristics to proxy for underlying-state variables representing the changing investment opportunity set [as in Merton (1973)] or factors in an APT setting. This article examines the link between these "factor-mimicking" portfolios and the covariance matrix. The sets of factors or "pricing models" used are represented by

$$\tilde{R}_t - r_{ft}\iota = \beta_t(w_t\tilde{R}_t) + \tilde{\epsilon}_t, \tag{3}$$

$$\beta_t = H_t w_t (w_t' H_t w_t)^{-1}, \quad \forall t, \tag{4}$$

where R_t is the (32×1) vector of asset returns, r_{ft} is the risk-free rate at time t (Treasury bill rate), ι is a (32×1) vector of ones, ϵ_t are the residuals at time t , β_t is the $(32 \times K)$ coefficient matrix of the K factors, H_t is the GARCH-estimated conditional covariance matrix at time t , and w_t is a $(32 \times K)$ matrix of the weights on the 32 portfolios used to form the K factors.

¹¹ Even though the conditional covariance matrices are highly correlated from week to week due to the GARCH specification, it is interesting how erratic the one-week-ahead out-of-sample, MVP volatilities are in Figure 2D. This is because of the extreme positions taken when the MVP weights are unconstrained. Employing longer lags between the portfolio weights and the subsequent covariance matrix for computing conditional volatility amplifies these results (available upon request).

Five pricing models are considered that differ in the weighting matrix w_t :

1. Fama and French (1993) three-factor model, which includes the market, *SMB*, and *HML* as factors. Here $w_t = w$ is a (32×3) matrix of zeros with a 1 in the 27th row of the first column; $1/3$ in the 21st, 22nd, and 23rd rows and $-1/3$ in the 24th, 25th, and 26th rows of the second column; and $1/2$ in the 21st and 24th rows and $-1/2$ in the 23rd and 26th rows of the third column.
2. Carhart (1997) four-factor model, which adds the momentum factor, *PR1YR*, to the Fama and French (1993) model. Here $w_t = w$ is a (32×4) matrix whose first three columns are identical to the Fama and French (1993) model, and whose fourth column is a vector of zeros with a 1 in the 32nd row and a -1 in the 28th row.
3. Industry momentum four-factor model, which replaces the Carhart (1997) momentum factor *PR1YR* with an industry momentum factor, *IM*, motivated by Moskowitz and Grinblatt (1999). Here w_t is a (32×4) matrix whose first three columns are the Fama and French (1993) model factor weights, and whose fourth column is a vector of zeros with $1/3$ in each of the rows corresponding to the three industries which exhibited the highest cumulative returns over the prior year and $-1/3$ in the rows corresponding to the three industries which exhibited the lowest cumulative returns over this period.

In addition to factors derived from firm characteristics, two additional models are employed that construct factors statistically to capture the maximal amount of return covariation in sample.

4. Conditional principal components model, which extracts the first four principal components from H_{t-1} . Here w_t is a (32×4) matrix corresponding to the four eigenvectors (rescaled to sum to one) of the four largest eigenvalues of H_{t-1} . The principal components pricing factor weights are computed at time $t - 1$ and applied to time t returns, in order to test how well they perform *out of sample*. By construction, they capture the greatest amount of return covariation of any four factors in sample.
5. Unconditional principal components model, which extracts the first four principal components from the *unconditional* sample covariance matrix. Here $w_t = w$ is a (32×4) matrix of the eigenvectors associated with the four largest eigenvalues (rescaled to sum to one) of the unconditional sample covariance matrix. These factors are similar in spirit to the Connor and Korajczyk (1988) factors. Obviously this model cannot be tested out of sample, but provides an interesting benchmark from which to compare the other models.

3. Contributing to the Covariance Structure of Returns

The literature has focused predominantly on explaining the mean return of assets. However, the second moment of returns also plays a vital role in asset pricing theory. Analyzing the first two moments of the return distribution simultaneously is paramount to our understanding of asset prices. I focus on the second moment of returns and their relation to well-known predictors of mean returns.

Numerous studies have documented the explanatory power of size, book-to-market, and momentum for predicting the cross section of expected returns, but few have documented their explanatory power for describing and predicting second moments.¹² How much of the conditional covariance matrix of asset returns do each of the trading strategies/factors capture?

3.1 Capturing conditional covariance in sample

Factors are examined both individually (single-factor models) and in combination (multifactor models) to assess how they might interact to improve conditional covariance estimates. The conditional covariance matrix, denoted V_t , is decomposed into a systematic portion and a residual portion implied by each set of factors as follows,

$$V_t = \beta_t \Sigma_t \beta_t' + \Omega_t, \quad (5)$$

where β_t is the $(32 \times K)$ matrix of factor loadings, Σ_t is the $(K \times K)$ time t covariance matrix of the factors, and Ω_t is the (32×32) residual covariance matrix implied by the factor model at time t . The fraction of the total or “true” covariance matrix V_t that the systematic portion $(\beta_t \Sigma_t \beta_t')$ captures is then measured at each point in time.

The “true” covariance matrix, V_t , is estimated in two ways:

1. H_t —the multivariate GARCH(1,1)-estimated matrix, which employs the rolling sample mean return on each asset from time $t - 37$ to time $t - 2$ to compute residual returns in Equation (2);
2. H_t^{full} —a GARCH(1,1)-estimated matrix using past information from all available factors in Section 2.5 and Equation (3) to estimate ex ante mean returns and hence provide an estimate of the residual vector ϵ_t .

¹² A recent exception is Chan, Karceski, and Lakonishok (1999), who evaluate how various pricing models predict the covariance between individual stock returns. Because of the enormous estimation error in individual stock return sample covariances, they assume constant covariation between all stocks in order to generate meaningful portfolio implications. The GARCH methodology and the use of well-diversified portfolios reduces estimation error considerably, allowing more reliable inferences, fewer restrictions, and more flexible investment analysis. In addition, the motivation in Chan, Karceski, and Lakonishok (1999) is different from that here. They compare how pricing models forecast the average covariance among stocks and use this information to compute portfolios that minimize total variance and minimize tracking error relative to some benchmark.

Two conditional covariance matrices are employed to assess whether mismeasurement in expected returns, and hence the residuals, alters the measurement of the contribution each factor makes to the covariance matrix.

The following three measures are employed to quantify the ability of each factor model to capture the covariance structure of asset returns. The first is inspired by the multivariate statistics literature and is designed to capture the similarity between two matrices. Because the matrices are multidimensional, a metric is needed to summarize all of the information contained in the matrix. The sum of the eigenvalues of the matrix is chosen as this metric. The first measure is

$$\%Eig_t = \frac{\sqrt{\text{trace}\{(\hat{\beta}_t \widehat{\Sigma}_t \hat{\beta}_t')'(\hat{\beta}_t \widehat{\Sigma}_t \hat{\beta}_t')\}}}{\sqrt{\text{trace}\{V_t' V_t\}}} \tag{6}$$

Since the trace of a matrix equals the sum of its eigenvalues, $\%Eig_t$ represents the sum of the eigenvalues of the systematic portion of the covariance matrix implied by each factor model as a fraction of the sum of the eigenvalues of the true matrix, V_t . The matrices are squared to capture the absolute amount of covariation in the economy. The eigenvalues of the covariance matrix transform the information from its $N \times N$ elements into N orthogonal components. Summing the eigenvalues provides a measure of total covariation represented by the matrix at time t .

The time-series average of $\%Eig_t$ is calculated and reported in Table 1, along with its time-series standard error, adjusted for autocorrelation up to 12-week lags. Since little is known about the distributional properties of this measure, standard errors were also computed via bootstrapping, but deviating only slightly from the time-series standard errors, are omitted for brevity.

Two additional measures of covariance contribution are used to evaluate the factors. The first metric, $Magnitude_t$, measures how accurately the factor-implied covariance matrix describes the “true” covariance matrix in terms of the magnitude of the covariances. This measure simply sums the absolute value of all of the elements of the difference between the two matrices and scales this sum by the sum of the absolute value of all elements of the true covariance matrix.

$$Magnitude_t = \frac{v'(|V_t - \hat{\beta}_t \widehat{\Sigma}_t \hat{\beta}_t'|)t}{v'(|V_t|)t}, \tag{7}$$

which measures the percentage error in capturing the size of the covariances in the economy.

The second metric, $Direction_t$, measures whether the model can capture the *direction* of the covariances. This metric simply determines what fraction

Table 1
Capturing and forecasting the conditional covariance structure of returns

	Panel A: In sample				Panel B: Out of sample			
	H_t	H_t^{full}			H_{t+4}	H_{t+4}^{full}		
	%Eig	%Eig	Mag.	Dir.	%Eig	%Eig	Mag.	Dir.
<i>Single factors</i>								
$Mkt - r_f$	0.646* (0.022)	0.464* (0.018)	0.542* (0.023)	0.974* (0.022)	0.644* (0.024)	0.440* (0.017)	0.578* (0.023)	0.969* (0.022)
<i>SMB</i>	0.238* (0.009)	0.128* (0.004)	0.806* (0.020)	0.895* (0.046)	0.251* (0.011)	0.124* (0.012)	0.819* (0.020)	0.887* (0.045)
<i>HML</i>	0.155* (0.015)	0.114 (0.004)	0.836 (0.021)	0.744 (0.031)	0.161* (0.017)	0.116* (0.008)	0.849 (0.021)	0.734 (0.032)
<i>IM</i>	0.030 (0.008)	0.085 (0.008)	0.884 (0.023)	0.709 (0.032)	0.035 (0.010)	0.084 (0.011)	0.891 (0.024)	0.702 (0.033)
<i>PR1YR</i>	0.041 (0.010)	0.021 (0.002)	0.959 (0.022)	0.678 (0.022)	0.052 (0.012)	0.022 (0.004)	0.960 (0.022)	0.675 (0.022)
<i>Multiple factors</i>								
<i>FF</i>	0.779* (0.024)	0.797* (0.027)	0.110* (0.027)	0.926* (0.026)	0.716* (0.024)	0.740* (0.027)	0.202* (0.031)	0.945* (0.025)
<i>FF + IM</i>	0.796* (0.023)	0.812* (0.025)	0.096* (0.026)	0.928* (0.027)	0.723* (0.023)	0.736* (0.027)	0.190* (0.029)	0.936* (0.025)
<i>Carhart</i>	0.800* (0.024)	0.799* (0.026)	0.109* (0.027)	0.934* (0.027)	0.723* (0.023)	0.741* (0.027)	0.200* (0.031)	0.942* (0.025)
<i>PC</i>	0.971* (0.022)	0.961* (0.024)	0.022* (0.015)	0.924* (0.027)	0.581* (0.022)	0.553* (0.022)	0.484* (0.022)	0.924* (0.025)
<i>UPC</i>	0.503* (0.014)	0.231* (0.015)	0.613* (0.019)	0.955* (0.025)	0.491* (0.016)	0.215* (0.016)	0.658* (0.020)	0.959* (0.024)
<i>Random</i>	0.128 (0.043)	0.101 (0.064)	0.869 (0.120)	0.686 (0.182)	0.136 (0.049)	0.085 (0.047)	0.880 (0.124)	0.676 (0.177)

Panel C: out-of-sample expansions versus recessions

	Expansions				Recessions			
	H_t	H_t^{full}			H_{t+4}	H_{t+4}^{full}		
	%Eig	%Eig	Mag.	Dir.	%Eig	%Eig	Mag.	Dir.
<i>Single factors</i>								
$Mkt - r_f$	0.638* (0.026)	0.459* (0.019)	0.549* (0.025)	0.965* (0.026)	0.680* (0.033)	0.325* (0.023)	0.754* (0.023)	0.998* (0.006)
<i>SMB</i>	0.236* (0.012)	0.114* (0.011)	0.811 (0.023)	0.886* (0.050)	0.344* (0.025)	0.185* (0.023)	0.872 (0.013)	0.894* (0.041)
<i>HML</i>	0.158* (0.014)	0.117* (0.004)	0.841 (0.024)	0.728 (0.034)	0.172 (0.036)	0.114* (0.019)	0.903 (0.008)	0.773* (0.044)
<i>IM</i>	0.031 (0.011)	0.086 (0.011)	0.886 (0.026)	0.708 (0.033)	0.059 (0.012)	0.072 (0.013)	0.924 (0.031)	0.668 (0.057)
<i>PR1YR</i>	0.044 (0.012)	0.022 (0.003)	0.956 (0.025)	0.680 (0.023)	0.100 (0.018)	0.021 (0.004)	0.990 (0.001)	0.643 (0.034)
<i>Multiple factors</i>								
<i>FF</i>	0.715* (0.026)	0.749* (0.029)	0.180* (0.032)	0.947* (0.026)	0.723* (0.030)	0.684* (0.042)	0.336* (0.055)	0.932* (0.033)
<i>FF + IM</i>	0.722* (0.026)	0.749* (0.029)	0.172* (0.031)	0.937* (0.026)	0.727* (0.030)	0.658* (0.048)	0.299* (0.053)	0.927* (0.034)
<i>Carhart</i>	0.723* (0.026)	0.750* (0.029)	0.179* (0.032)	0.944* (0.026)	0.726* (0.030)	0.682* (0.042)	0.332* (0.055)	0.932* (0.033)

Table 1
(continued)

Panel C: out-of-sample expansions versus recessions

	H_t %Eig	Expansions				Recessions			
		H_t^{full}				H_{t+4} %Eig	H_{t+4}^{full}		
		%Eig	Mag.	Dir.	%Eig		Mag.	Dir.	
PC	0.587* (0.024)	0.583* (0.022)	0.460* (0.022)	0.929* (0.026)	0.548* (0.033)	0.370* (0.029)	0.633* (0.035)	0.894* (0.038)	
UPC	0.494* (0.018)	0.227* (0.018)	0.636* (0.021)	0.958* (0.026)	0.469* (0.018)	0.143* (0.016)	0.795 (0.013)	0.968* (0.015)	
Random	0.133 (0.049)	0.087 (0.032)	0.879 (0.124)	0.692 (0.188)	0.154 (0.075)	0.072 (0.110)	0.883 (0.180)	0.582 (0.285)	

Measures of the fraction of the conditional covariance structure of asset returns that are captured by various factor pricing models are reported. The conditional covariance matrices are estimated via Ledoit and Santa-Clara's (1998) multivariate GARCH(1,1) model using weekly portfolio returns from August 1964 to December 1997. Two estimates are employed: (1) H_t —using the sample mean from $t - 37$ to $t - 2$ to compute residuals and (2) H_t^{full} —using all available factors to estimate residuals used in the GARCH model. The systematic portion of the covariance matrix is estimated under each factor model. Three metrics for quantifying the amount of covariation captured by the factor models are employed: the %Eig_{*t*}, which is the fraction of the sum of eigenvalues captured by each factor model, as well as the magnitude and direction of the covariance matrix, described in detail in Section 3. These measures are calculated every week, and the time-series mean and standard errors that account for autocorrelation up to 12-week lags (in parentheses) are reported. Panel A reports the in-sample explanatory power of the factor models. Panel B reports their out-of-sample (four-week) forecastability and panel C repeats the out-of-sample results in expansionary and recessionary periods separately, as defined by the National Bureau of Economic Research.

*Performs better than a set of randomly generated factors at the 5% significance level.

of the covariances are estimated with the right sign,

$$Direction_t = \frac{\iota' \text{sign}(V_t(\hat{\beta}_t, \widehat{\Sigma}_t \hat{\beta}_t)) \iota}{[rank(V_t)]^2}. \tag{8}$$

If the factor-implied covariance matrix captures the true covariances accurately, then *Magnitude_t* should be close to zero, and *Direction_t* should be close to one.

Panel A of Table 1 reports the time-series means and standard errors of these three measures for each of the factors. The magnitudes of the measures provide a ranking for how well each of the factors captures the covariance structure of returns. The market portfolio captures the largest fraction of the covariance structure of asset returns of any single factor, explaining more than 64% of total covariation under H_t (which employs the rolling sample average return to estimate mean returns, and hence, residuals in Equation (2)), and more than 46% with H_t^{full} (which uses past information from all factors to estimate mean returns). This is highly statistically significant and appears economically important.

Examining the other factors, the size factor, *SMB*, seems to contribute significantly to total covariation, capturing about 24% and 13% of the covariation represented by H_t and H_t^{full} , respectively. The factor associated with book-to-market equity (*HML*) captures about 15% and 11%, respectively. However, the momentum factors (*PR1YR* and *IM*) do not appear to contribute

significantly to total covariation. Neither of these factors explains more than 8.5% of the covariance matrix.

For comparison, “random” factors are formed by assigning random portfolio weights to assets to form zero-investment strategies. These factors provide an interesting benchmark of how much covariation can be captured by simply choosing factors randomly. Since there are only 32 assets to choose from, these random weights will sometimes produce a portfolio that is highly correlated with one of the factors. Hence the random factors manage to produce a statistically significant, though economically small, effect. Note, however, that the standard errors on the contribution of the random factors are much larger than those from any other set of factors. The fact that these standard errors are small relative to the mean estimate is a testament to the power of weekly data. Nevertheless, the random factors do not capture a substantial portion of covariation and neither the book-to-market nor momentum factors consistently outperform this set of randomly generated factors. Conversely, both the market and *SMB* consistently outperform the random factors.

Combining the factors into multifactor sets, the Fama–French factors capture almost 78% and 80% of the conditional covariance matrices H_t and H_t^{full} , respectively (both highly significant). Moreover, the Fama–French factors consistently reject the null hypothesis that the contribution to total covariation is the same as a set of randomly generated factors. This substantial improvement from employing the market factor alone suggests that *SMB* and *HML* in combination with the market provide greater explanatory power for describing covariance risk.

I reiterate that this is an *in-sample* exercise. Therefore adding more factors to the model will increase explanatory power. However, the addition of the two momentum factors (*IM* and *PR1YR*) to the Fama–French factors negligibly increases the contribution to covariance risk. Thus, whether in isolation or in combination with the market, size, and book-to-market factors, the proposed momentum factors do not appear to describe covariation among the assets.

The final two factor models, those constructed from principal components analysis, are designed to capture the most covariation in sample (by statistical construction) of any four factors. These factors therefore provide a benchmark on the maximal amount of covariation any four factors could capture. The conditional principal components factors capture 97% and 96% of the conditional covariance matrices, respectively, in sample. While larger than any other set of factors by construction in sample, in the next subsection, the out-of-sample performance of these factors will be evaluated.

The unconditional principal components factors are designed to capture the maximum amount of covariation of any four factors relative to the *unconditional* sample covariance matrix. However, the performance of these factors on the conditional covariance matrices is far less impressive. The unconditional principal components factors fail to outperform the market factor

alone and significantly underperform all other multifactor models, although they successfully outperform the random factors. One implication of this result is that significant time variation in the covariance structure of asset returns distorts the ability of these time-invariant factors to capture second moments, suggesting that unconditional factors miss important dynamics in return volatility.¹³

Finally, consistent with evidence for %*Eig*, only the market and *SMB* capture the covariance structure of asset returns better than a randomly generated set of factors under both the *Magnitude* and *Direction* measures. Neither *HML* nor the momentum factors capture the magnitude or direction of covariances significantly better than the random factors, and, in fact, the random model slightly outperforms the momentum factors. Not surprisingly, the multifactor models that employ the market and *SMB* as factors describe the direction and magnitude of conditional covariances better than the random factors. The conditional principal components factors also capture the covariance matrix well under these two metrics, in sample.¹⁴

3.2 Forecasting conditional covariation out of sample

Analyzing the out-of-sample predictability of the factor models is useful for enhancing models of conditional volatility as well as practically for portfolio optimization and implementation. In the next section, the predictive power of the various factors for conditional second moments is evaluated from an investment perspective to determine the economic significance of the volatility forecasts.

To gauge how effective the factor models are in forecasting future covariances, the tests from the previous section are repeated on a future covariance matrix, where the factors are extracted from a previous period's covariance matrix. Since the conditional covariance matrices are estimated under a GARCH model, they are highly autocorrelated from week to week.¹⁵ To avoid having nearly identical in- and out-of-sample results, a longer lag between the in- and out-of-sample periods is employed. As the distance in time increases between the two periods, the correlation between the covariance matrices declines. The drawback, however, is that because there is substantial time

¹³ These results are consistent with Jones (2001), who finds that the unconditional Connor and Korajczyk (1988) factors perform poorly when substantial time variation and heteroscedasticity in asset return second moments exists.

¹⁴ Since the direction measure simply quantifies whether the sign of the correlations is captured correctly, some of the factors are able to do this more than 97% of the time (e.g., the market), as this is not too stringent a task. Capturing the magnitudes of the covariances is much more difficult, as indicated in Table 1.

¹⁵ The GARCH parameter matrix B in Equation (1), which weights the lagged covariance matrix H_{t-1} , has an average weight of 0.85. The weight on residual innovations is only 0.15 on average. Hence, with only one week separating the in-sample and out-of-sample matrices, the results will be very similar since the matrices are nearly identical. This will particularly be the case for factors that do not change much from week to week, such as the Fama–French factors. A previous draft of this article demonstrated this (results available upon request).

variation in return second moments, the salience of the factor-implied covariance matrices also diminishes. Hence there is a trade-off. As a compromise, a four-week lag between the in- and out-of-sample tests is used by replacing V_t with V_{t+4} (corresponding to H_{t+4} and H_{t+4}^{full}) in Equations (6)–(8). Even with a four-week lag, the in- and out-of-sample results are similar, though not identical.¹⁶

As panel B of Table 1 indicates, the market portfolio comprises the largest component of covariance risk, out of sample, of any single factor, capturing 64% and 44% of total covariation, respectively, under H_{t+4} and H_{t+4}^{full} . These measures are highly statistically significant and are statistically different from those generated by the random factors. In addition, *SMB* also appears to forecast total covariation out of sample, capturing 25% of total covariation relative to H_{t+4} , and 12% relative to H_{t+4}^{full} , both of which are statistically significant and significantly outperform the random factors. Similarly *HML* captures a significant proportion of out-of-sample covariation and outperforms the random factors, though its contribution to covariance risk is weaker than the market or *SMB*. However, consistent with their failure to describe conditional second moments in sample, *PR1YR* and *IM* contribute negligibly to total out-of-sample covariation as well.

Examining combinations of the factors in the multifactor models, the Fama–French factors capture as much as 74% of the out-of-sample conditional covariance matrices. This suggests that *SMB* and *HML* in combination with the market provide substantially more explanatory power for predicting future covariance risk. The results in panel B of Table 1 suggest that the use of conditioning variables such as the market return and firm size enhance covariance forecasts, possibly supporting the methods employed by Ferson and Harvey (1997, 1999) and others.

The momentum factors, on the other hand, do not add much predictive power for future covariances, and in fact can slightly reduce the forecasting accuracy of the Fama–French factors out of sample. These results mirror those from the in-sample tests.

The final two factor models, those constructed from principal components analysis, perform far worse when attempting to forecast covariances out of sample. The conditional principal components model (PC), which performs the best of any four factors in sample, significantly underperforms the Fama–French factors out of sample. These statistically constructed factors appear to perform worse out of sample than predetermined economic variables based on the market and firm size. Likewise, the unconditional principal components factors perform relatively poorly in capturing future covariances.

¹⁶ Tests were also conducted using a 12-week lag, which generated even greater differences between the in- and out-of-sample results. However, the qualitative results and conclusions remained unchanged. The same relative rankings in performance of the factors are maintained whether 1, 4, or 12 lags are employed.

3.3 Forecasting covariation in expansions and recessions

Finally, panel C of Table 1 reports the out-of-sample results during expansionary and recessionary periods separately, as defined by the National Bureau of Economic Research. Since recessions are periods when investors might care most about volatility and perhaps value covariance forecasts the most, it is interesting to see whether the various factors deliver useful forecasts during these bad times. Moreover, characterizing the dynamics of conditional volatility over the business cycle may aid in identifying the changing investment opportunity set. Finally, as Figure 1 indicated, volatility and correlations tend to rise during recessions, and so it is also interesting to examine whether the factors can capture this asymmetry as well.

As panel C of Table 1 demonstrates, the findings are similar to those previously. The market, *SMB*, and, to a lesser extent, *HML* seem to forecast future second moments, while the momentum factors do not. However, more interesting is the increased forecasting power of these factors during recessions. The market, and particularly *SMB*, capture a greater fraction of the covariance matrix in recessions, which is precisely when investors would find such forecasts most valuable. In addition, these factors seem to capture and forecast the asymmetric rise in volatility and correlations during recessionary periods. On the other hand, although the covariance forecasts from the momentum factors are more accurate in recessions, neither of the momentum factors exhibit much relation to future covariance risk, even during recessions.¹⁷

4. Economic Implications for Efficient Investment

This section quantifies economically the link between the factors and conditional covariation by examining the implications for efficient portfolios. In addition to providing an economic benchmark from which to compare factors, this analysis may provide insights for optimal investment.

4.1 Minimum variance portfolios

The out-of-sample performance of minimum variance portfolios derived from the estimated covariance matrix implied by each factor model is analyzed. Minimum variance portfolio (MVP) weights are computed from the conditional covariance matrix implied by each set of factors at time t , $\hat{\beta}_t \hat{\Sigma}_t \hat{\beta}_t'$.

¹⁷ I have also attempted to analyze the covariance structure of *individual* stock returns using rolling sample covariance estimates and simulations. The noise in individual stock returns overwhelmed the ability of the factors to describe common variation. However, the Fama and French (1993) factors still managed to capture almost 10% of the covariation in individual stock returns, and both the market and small minus big captured about 6% by themselves. The book-to-market and momentum factors failed to reliably capture covariation. Based on these tests, it appears too difficult to accurately examine individual stock returns, even at a weekly frequency, since individual return variation is likely dominated by idiosyncratic shocks, making common components in returns difficult to identify. Nevertheless, what little inference could be made seemed to support the findings in the article.

Since the factor-implied covariance matrix is just a function of the factor variances and covariances, to ensure nonsingularity, a constant (across models) diagonal matrix of residual volatility is added to each factor-implied covariance matrix. The matrix employed is Ω_t^{full} , the residual covariance matrix from the full model that employs all available factors ($\epsilon_t^{full} \epsilon_t^{full'}$). Adding this matrix to every factor-implied matrix will not distort the influence of the factors since Ω_t^{full} is a diagonal matrix containing only residual volatility estimates orthogonal to all factors by design. Thus only $\hat{\beta}_t \widehat{\Sigma}_t \hat{\beta}_t'$ will drive the investment differences across models.

To avoid obtaining extreme (and impractical) portfolio positions, the constraint that the weights on any asset must be between -0.50 and 1.50 is added, as was done previously in Section 2.4 and Figure 2. These weights are then applied to time $t + 4$ returns out of sample and a time series of MVP returns are generated. Specifically the return on this portfolio at time $t + 4$ is

$$\tilde{R}_{mvp,t+4} = \hat{w}'_{mvp,t} \tilde{R}_{t+4}, \tag{9}$$

$$\text{where } \hat{w}_{mvp,t} \text{ solves } \min(w_t) w_t' (\hat{\beta}_t \widehat{\Sigma}_t \hat{\beta}_t' + \Omega_t^{full}) w_t \tag{10}$$

$$\text{s.t. } -0.50 \leq w_t \leq 1.50 \tag{11}$$

where $\hat{\beta}_t \widehat{\Sigma}_t \hat{\beta}_t'$ is the factor model-implied covariance matrix, Ω_t^{full} is the residual covariance matrix using all available factors at time t , and R_{t+4} is the vector of returns on the 32 portfolios at time $t + 4$.¹⁸

A separate time series of returns are generated under each factor model and the time-series mean and t -statistic of these returns are reported in Table 2. In addition to the previous factor models employed, results are also reported using the full covariance matrix H_t^{full} , the diagonal matrix of H_t^{full} (i.e., variances only), and the identity matrix in place of $\hat{\beta}_t \widehat{\Sigma}_t \hat{\beta}_t'$. These provide interesting benchmarks. For instance, MacKinlay and Pastor (2000) find that the identity matrix performs at least as well as other covariance estimates in both out-of-sample tests and simulations when forming optimal portfolios that require estimates of the assets' mean returns. Here, since there is no estimate of mean returns required, the implied MVP from the identity matrix is simply an equal-weighted portfolio.

The average returns across the MVPs implied by the factor models are very similar. However, the volatility of these returns varies substantially across the models, as indicated by the t -statistics. Relative to both the identity matrix-implied MVP and random factor model, the market, *SMB*, and *HML* factors each imply a less-volatile MVP, out of sample, indicating that these factors

¹⁸ Again, a four-week lag between the in- and out-of-sample periods is employed. Because the GARCH-estimated covariance matrices are highly autocorrelated, using shorter lags between the MVP weights and returns "overfits" the data. At the same time, the longer the lag, the less salient the MVP weights. Results for 1- and 12-week lags are available upon request.

Table 2
Minimum variance efficient portfolios

	Total covariance -0.50 ≤ w ≤ 1.5		Total covariance No Short Sales		Correlation only -0.50 ≤ w ≤ 1.5		Residual covariance -0.50 ≤ w ≤ 1.5	
	Return	Cond. st. dev.	Return	Cond. st. dev.	Return	Cond. st. dev.	Return	Cond. st. dev.
No factors	0.0029 (4.11)	0.0186	0.0024 (5.38)	0.0179	0.0020 (4.08)	0.0195	0.0029 (4.11)	0.0171
Identity matrix	0.0024 (5.00)	0.0190	0.0024 (5.00)	0.0190	0.0024 (5.00)	0.0190	0.0024 (5.00)	0.0190
Diagonal matrix	0.0024 (5.32)	0.0178	0.0024 (5.32)	0.0178	0.0024 (5.00)	0.0190	0.0024 (5.32)	0.0178
<i>Mkt</i> - <i>r_f</i>	0.0028 (6.87)	0.0163	0.0025 (5.92)	0.0164	0.0019 (3.14)	0.0271	0.0010 (1.26)	0.0105
<i>SMB</i>	0.0025 (5.80)	0.0171	0.0025 (5.59)	0.0174	0.0024 (5.11)	0.0190	0.0022 (2.99)	0.0136
<i>HML</i>	0.0025 (6.13)	0.0171	0.0025 (5.85)	0.0176	0.0019 (3.84)	0.0207	0.0029 (3.99)	0.0156
<i>IM</i>	0.0028 (5.50)	0.0184	0.0026 (4.97)	0.0174	0.0028 (5.67)	0.0207	0.0027 (4.60)	0.0241
<i>PR1YR</i>	0.0022 (5.01)	0.0185	0.0023 (5.10)	0.0185	0.0021 (4.41)	0.0219	0.0030 (3.42)	0.0246
FF	0.0029 (7.08)	0.0145	0.0024 (5.76)	0.0168	0.0023 (3.76)	0.0262	0.0017 (2.39)	0.0085
FF + IM	0.0027 (6.67)	0.0147	0.0024 (5.59)	0.0169	0.0026 (3.63)	0.0303	0.0021 (3.11)	0.0078
Carhart	0.0027 (6.76)	0.0144	0.0024 (5.72)	0.0167	0.0021 (3.60)	0.0257	0.0012 (1.75)	0.0071
PC	0.0022 (5.24)	0.0159	0.0024 (5.67)	0.0164	0.0022 (3.14)	0.0298	0.0024 (5.02)	0.0136
UPC	0.0019 (4.55)	0.0185	0.0024 (5.61)	0.0177	0.0012 (1.38)	0.0376	0.0024 (4.91)	0.0144
Random	0.0024 (5.50)	0.0183	0.0024 (5.48)	0.0184	0.0021 (4.32)	0.0200	0.0023 (2.76)	0.0184

The covariance matrix implied by each factor model is used to form minimum variance portfolios at each point in time, which are then applied out of sample to returns four periods ahead. Four sets of minimum variance portfolio weights are computed. The first set places constraints on the weights, *w*, assigned to the individual assets such that $-0.50 \leq w \leq 1.5$ for all assets. The second set only allows positive portfolio weights (i.e., no short positions) and does not allow any weight on an asset to exceed 100% (i.e., no borrowing). The third set of MVP weights are derived solely from the correlation matrix of returns implied by each factor model with the constraint $-0.50 \leq w \leq 1.5$. Finally, the *residual* covariance matrix implied by each factor model is also used to form minimum variance portfolio weights, with the constraint that $-0.50 \leq w \leq 1.5$. The time-series means and *t*-statistics (in parentheses) of the weekly returns from these minimum variance efficient portfolios are reported over the sample period August 1964 to December 1997. In addition, the time-series average of the out-of-sample conditional standard deviation of these minimum variance portfolios implied by the GARCH-estimated conditional covariance matrices are reported.

provide useful information about the future covariance matrix that aids in creating more efficient portfolios *ex post*. The two momentum factors do not produce more efficient portfolios.

When combining factors, the Fama–French model produces a slightly higher MVP mean return and substantially lower time-series variance than any of the other factors, indicating that the combination of the market, *SMB*, and *HML* provides even better portfolio weights for efficient investment. Adding a momentum factor to the Fama–French model, however, does not improve the efficiency of the portfolio. These results echo the key findings in the article. Finally, neither of the principal components models outperform

the Fama–French factors out of sample, and are not any more efficient than the random model or identity matrix benchmarks.¹⁹

Finally, the time-series average of the weekly conditional standard deviation of the implied MVPs are reported. The time t conditional standard deviation of the MVP is defined as

$$\hat{\sigma}_{mvp,t} = \sqrt{\hat{w}'_{mvp,t} H_{t+4}^{full} \hat{w}_{mvp,t}}, \quad (12)$$

where the portfolio weights implied by each factor model are applied to the same covariance matrix across models, H_{t+4}^{full} . In this way, the impact on conditional volatility can be evaluated from differences across factors in terms of their implied efficient weights, while abstracting from their predictions about the time $t + 4$ covariance matrix (which is examined in Table 1).

As Table 2 demonstrates, the average conditional standard deviations mirror the previous patterns. The average conditional standard deviation of the MVP implied by the market, *SMB*, *HML*, and the multifactor models that incorporate them, produce lower conditional volatility than either the random model or the MVP implied by the identity matrix (i.e., an equal-weighted portfolio). However, the momentum factors fail to generate lower out-of-sample MVP volatility.

For robustness, MVPs formed from the factors subject to the constraint that no short sales be allowed (i.e., $0 \leq w_{mvp,t} \leq 1$) are also examined. As Table 2 shows, the same general patterns and conclusions can be drawn from these short-sales-constrained MVPs. However, the differences across models are less stark, indicating that short positions add more scope for efficient investment.

4.2 Correlations only

It is also interesting to examine whether the ability (or inability) of these factors in describing covariances is due to their ability to explain correlations among the assets, or whether they simply capture the variances. To examine the importance of the correlation forecasts, MVPs are computed using only information from the correlation matrix implied by each set of factors. The implied correlation matrix is given by

$$\hat{\Gamma}_t = (\hat{\beta}'_t \hat{\Sigma}_t \hat{\beta}_t) [\text{diag}(\hat{\beta}'_t \hat{\Sigma}_t \hat{\beta}_t) \text{diag}(\hat{\beta}'_t \hat{\Sigma}_t \hat{\beta}_t)']^{-1/2}. \quad (13)$$

MVP weights are then formed from $\hat{\Gamma}_t$ subject to the constraint that weights remain within -0.50 and 1.50 on all individual assets. As shown in Table 2,

¹⁹ Note that the MVPs formed here abstract from mean return estimates. This is in contrast to the portfolios formed by MacKinlay and Pastor (2000), who only examine tangency portfolios and the link between mean return estimates and the covariance matrix. This is why the results differ slightly from theirs, since I find that several of the factors provide more useful information for investment than the identity matrix. However, when estimating mean returns and forming tangency portfolios (in unreported results, available upon request), I too find that the factors do not provide much more useful information than the identity matrix, consistent with MacKinlay and Pastor (2000). This is because the influence of the mean estimates on the portfolio weights is substantial.

the out-of-sample volatilities of these portfolios do not exhibit as consistent a pattern across factor models as previously. This suggests that the magnitude of the variance terms is an important element in forming efficient portfolios, and a significant part of the ability of the factor models to predict future volatility.

4.3 Minimum residual variance portfolios

Finally, rather than examine the systematic portion of the covariance matrix implied by each factor model, an alternative is to examine the residual covariance matrix implied by the factors. From Equation (5), the implied residual covariance matrix, Ω_t , is computed under each set of factors at a point in time and minimum variance portfolio weights are formed from the residual matrix Ω_t (subject to the constraint that $-0.50 \leq w_t \leq 1.50$). These minimum residual variance portfolio (MRVP) weights are then applied to returns out of sample. The returns of the MRVP are

$$\tilde{R}_{mrvp,t+4} = \hat{w}'_{mrvp,t} \tilde{R}_{t+4} \tag{14}$$

where $\hat{w}_{mrvp,t}$ solves $\min(w_t) w_t' \hat{\Omega}_t w_t$ (15)

s.t. $-0.50 \leq w_t \leq 1.50$ (16)

where $\hat{\Omega}_t$ is the residual covariance matrix implied by each factor model.²⁰

If the factor model captures a substantial portion of the covariance matrix of asset returns, then the return of the minimum residual variance portfolio implied by the model should be small and close to zero. Hence the time-series mean and standard deviation of these returns should be close to zero. Table 2 reports the time-series means and *t*-statistics of the raw MRVP returns implied by each set of factors.

In addition, the time-series average conditional residual standard deviation of the MRVP is reported. This conditional residual standard deviation is calculated out of sample with a four-week lag,

$$\hat{\sigma}_{mrvp,t+4} = \sqrt{\hat{w}'_{mrvp,t} \hat{\Omega}_{t+4} \hat{w}_{mrvp,t}} \tag{17}$$

Once again, the MRVP implied by the factors are compared to a portfolio implied by the identity matrix (an equal-weighted portfolio) and the residual MVP implied by a set of randomly generated factors. Both the identity matrix and random factors produce significant MRVP profits, which exhibit substantial volatility. This indicates that they do not explain a substantial portion of the covariance matrix out of sample.

²⁰ The residual covariance matrix implied by each factor model is almost always nonsingular. For the rare 0.2% of the time when one of the residual covariance matrices is near singular, this observation is eliminated from the calculations.

Conversely, the market and size factors capture a large fraction of the out-of-sample covariance matrix, generating MRVPs with small mean and little volatility. For instance, the average conditional residual standard deviation on the market-implied MRVP is only 105 basis points and its mean return is not statistically different from zero. This indicates that there is not much volatility left after covariation with the market index is taken out. *SMB* also generates relatively low residual volatility in its MRVP. *HML* generates higher volatility, though lower than the random model or identity benchmarks, and the momentum factors leave substantially more variation in their residual MVPs and produce large positive returns.

Combining factors, the Fama–French model captures even more out-of-sample covariation, producing even lower conditional volatility and time-series average returns. Adding the momentum factors only reduces volatility further by a negligible amount. These findings are consistent with those previously. Finally, the principal components factors seem to account for a sizable portion of return covariation when compared to the identity matrix and random factors, but account for less covariance risk than the Fama–French factors.²¹

5. The Relation Between Return Premia, Volatility, and Conditional Covariation

From both a statistical and economic perspective, there appears to be a strong link between the market and size factor and the covariance structure of asset returns, a weaker link between book-to-market equity and covariation, and almost no link between the momentum factors and covariance risk. However, the relation between the return premia on size, book-to-market equity, and momentum and their conditional volatility and covariation has yet to be examined. While establishing such a link may shed light on possible interpretations of these “anomalies,” this article remains agnostic on whether behavioral or risk-based explanations seem more plausible.²²

Figure 3 plots the cumulative return premia and various measures of conditional second moments over time (from August 1964 to December 1997) for each of the four zero-cost portfolios *SMB*, *HML*, *PR1YR*, and *IM*. The conditional second moment estimates are the conditional standard deviation of each strategy, the average conditional correlation of each strategy with

²¹ For robustness, adjusted returns on the strategies were also computed in a previous draft. The adjusted returns are the raw returns minus the conditional mean return predicted by each factor model. Since weekly return premia are close to zero, employing the adjusted returns generally only reduced the volatility of the strategies without affecting their mean. The results and conclusions were unaltered.

²² Indeed, as Brav and Heaton (2001) point out, it may be impossible to empirically distinguish between these competing sets of hypotheses. Nevertheless, documenting the link between the return premia and conditional covariance structure of asset returns will provide a set of stylized facts that should be accommodated by theory.

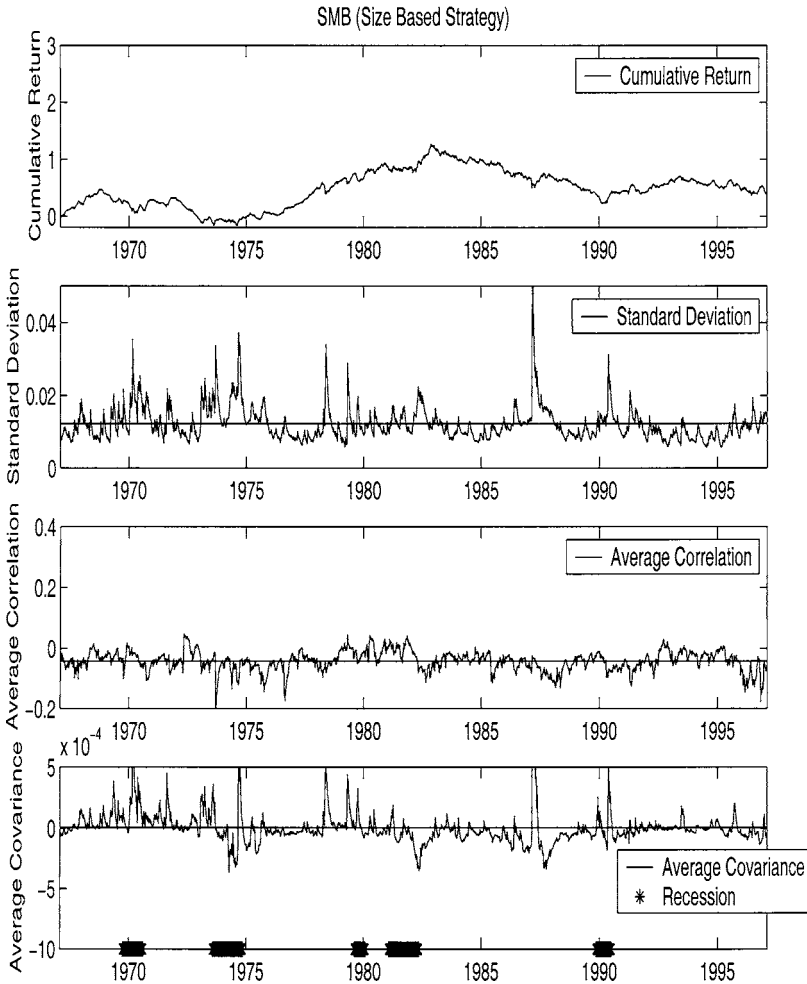


Figure 3
Trading strategy returns and conditional second moments

The figure plots various measures of conditional second moments of four trading strategies or factors: *SMB*, *HML*, *PRIYR*, and *IM*, associated with firm size, book-to-market equity, individual stock momentum, and industry momentum, respectively. Conditional second moments are estimated via the Ledoit and Santa-Clara (1998) multivariate GARCH conditional covariance matrices. The cumulative return, conditional standard deviation, average conditional correlation of each factor with the other assets in the covariance matrix, and average covariance of the factor with the other basis assets are plotted. The conditional second moment estimates are formed from the conditional covariance matrix at each point in time and are plotted weekly from August 1964 to December 1997. For reference, the unconditional estimates of these measures are also plotted, obtained from the unconditional sample covariance matrix. Recessionary periods defined by the NBER are also highlighted.

the other assets, and the average covariance of each strategy, all of which are computed from the time t conditional covariance matrix, H_t , estimated via Ledoit and Santa-Clara (1998). For reference, the unconditional second moment estimates are also plotted.

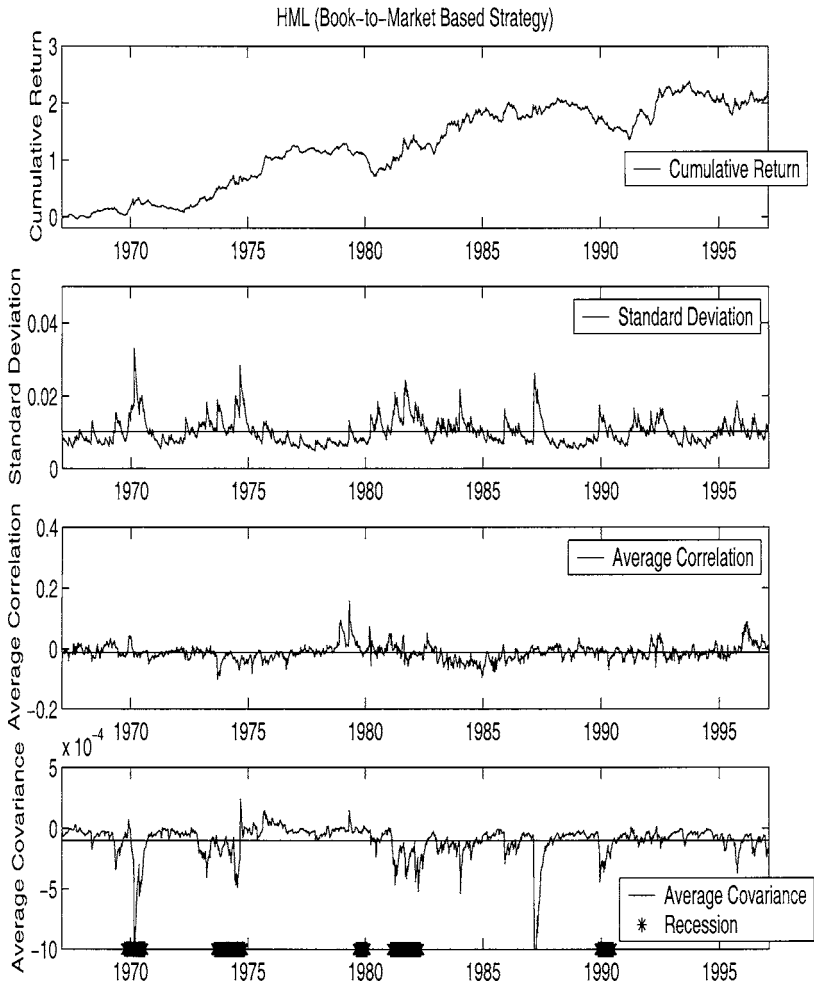


Figure 3
(continued)

Table 3 reports summary statistics on the four trading strategies including average conditional and unconditional estimates of their second moments. The average conditional second moment estimates are similar to the unconditional estimates, but as Figure 3 signifies, there is substantial and important time variation in these second moments that should be accounted for, particularly during recessions, where the link between the factors and the covariance matrix is stronger.

Figure 3 indicates a broad association between the return premia on the strategies and conditional second moments, although it is difficult to discern

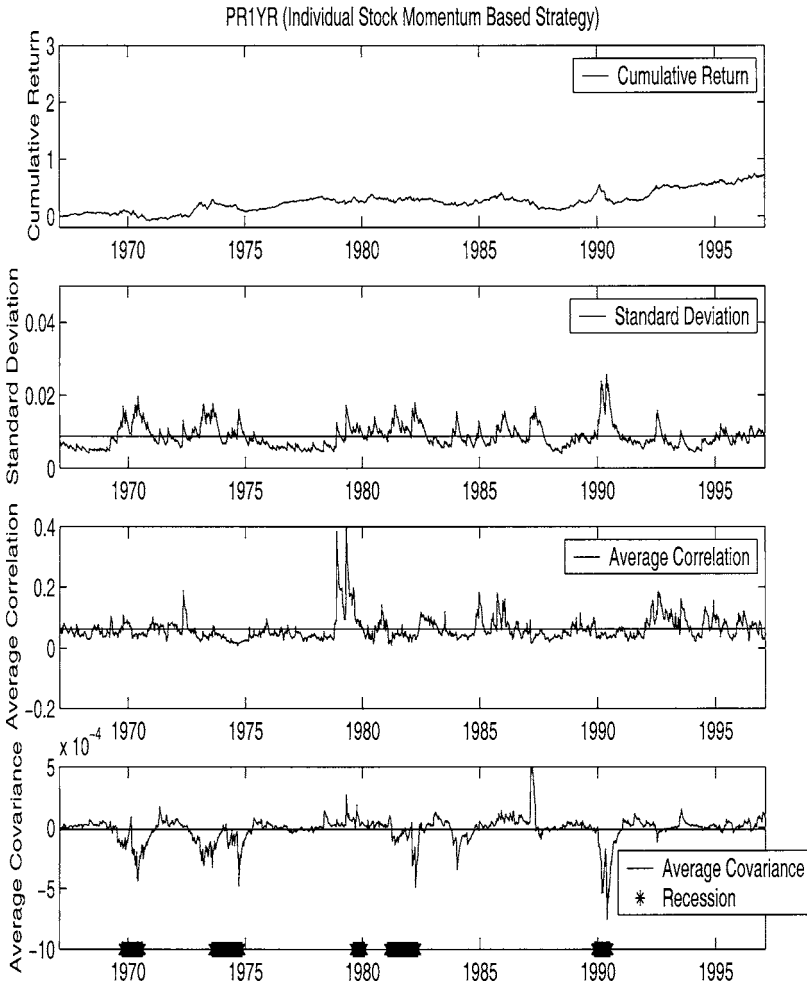


Figure 3
(continued)

information from the graphs. To better analyze and quantify these potential links, a time-series regression is run for each strategy's conditional return (Ret_t), conditional standard deviation (σ_t), and average conditional covariance (\overline{cov}_t) with the other assets, on past measures of the strategy's return, conditional volatility, as well as measures of the covariance structure of returns. The return regressors are controls that include the lagged one-week return of the strategy (Ret_{t-1}) and the contemporaneous return on the market ($Ret_{mkt,t}$) to control for the market component of returns. The conditional volatility regressors include the lagged average conditional standard

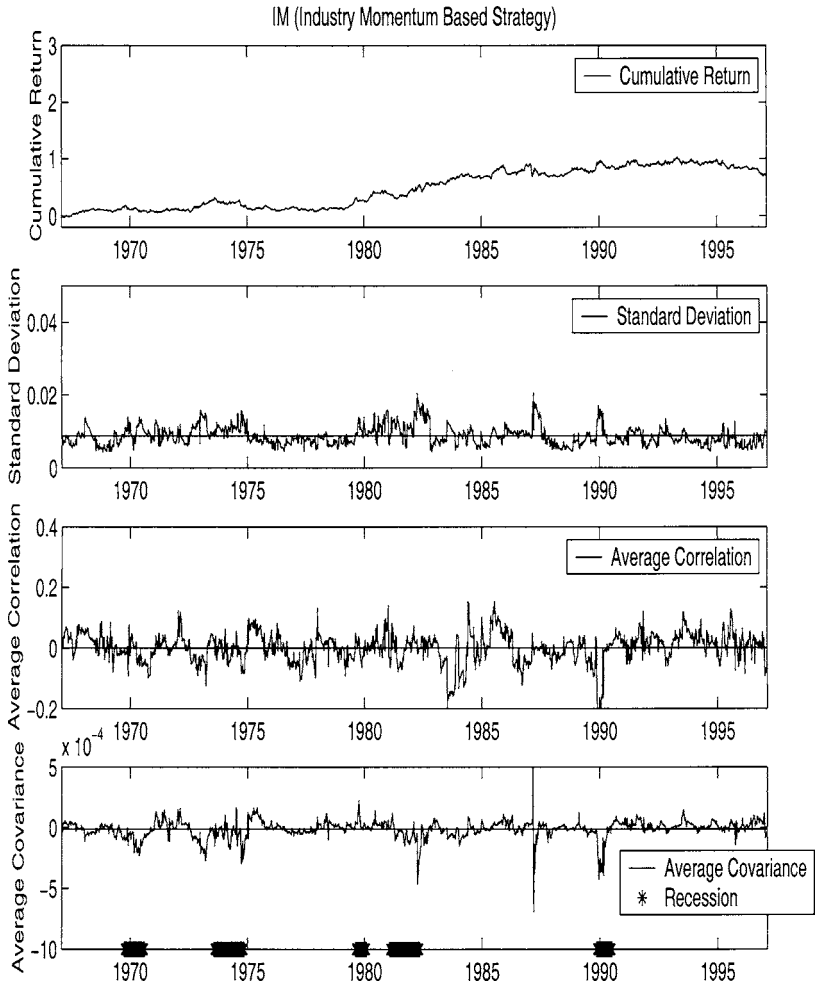


Figure 3
(continued)

deviation across the 32 assets ($avg.\sigma_{t-1}$), which captures the average volatility in the market; the conditional standard deviation of the MVP at time $t - 1$ ($\sigma_{mvp,t-1}$), a measure of total volatility that accounts for covariation as well; the lagged conditional standard deviation of the strategy itself ($own\sigma_{t-1}$), and the lagged weight of the strategy in the MVP ($w_{mvp,t-1}$). These last two measures capture the conditional volatility and covariation of the strategy itself. To capture poor consumption states in the economy, a time dummy variable is added for recessionary periods (defined by the National Bureau of Economic Research), as well as interactions between various regressors

Table 3
Conditional and unconditional return first and second moments of trading strategies

	Mean return (annual)	Conditional second moments			Unconditional second moments		
		St. dev. (annual)	Avg. corr. ^a	β	St. dev. (annual)	Avg. corr. ^a	β
<i>SMB</i>	0.0355	0.0908	-0.0386	0.1197	0.0780	-0.0291	0.0877
<i>HML</i>	0.0525	0.0705	-0.0086	-0.1067	0.0714	-0.0135	-0.1531
<i>PR1YR</i>	0.1434	0.2056	0.0604	-0.0175	0.2119	0.0594	-0.0877
<i>IM</i>	0.0628	0.0617	0.0102	-0.0157	0.0511	0.0002	-0.0115

The mean return, standard deviation, average correlation, and market beta of the trading strategies associated with firm size (*SMB*), book-to-market equity (*HML*), individual stock momentum (*PR1YR*), and industry momentum (*IM*) are reported below using weekly returns over the period August 1964 to December 1997. The reported mean returns and standard deviations are annualized. Both the unconditional and time-series average of the conditional standard deviations, average correlations, and market betas are reported, where the time series of the conditional second moments are estimated from the multivariate GARCH procedure of Ledoit and Santa-Clara (1998).

^aAverage correlation refers to the average correlation of each strategy with the other portfolios that comprise the covariance matrix. These are the 32 portfolios corresponding to 20 industries, 6 size and book-to-market equity sorted portfolios, a market index, and 5 past-year return sorted portfolios.

and the recession dummy. The lagged six-month average of some of the independent variables are also included to control for recent levels, in order to evaluate how innovations in these variables affect the dependent variable.²³ Regressions are run via ordinary least squares with White (1980)-corrected standard errors.

Table 4 reports the regression results. Panel A contains results for the size-based strategy *SMB*, panel B for the book-to-market strategy *HML*, panel C for individual stock momentum *PR1YR*, and panel D for industry momentum *IM*. Examining the first column of each panel, which corresponds to the return premium of each strategy, only the *HML* premium is significantly higher in recessions, consistent with Lakonishok, Shleifer, and Vishny (1994), but inconsistent with a risk-based story. More interestingly, the conditional standard deviation of the MVP, $\sigma_{mvp,t-1}$, which summarizes how much covariance risk exists at a point in time, increases the future premium on *SMB*, but only during recessions. This may be consistent with a risk-based interpretation of the size effect, since the *SMB* premium rises following recent aggregate covariance increases (i.e., high $\sigma_{mvp,t-1}$) and low consumption states (i.e., recession). The premia on the other strategies are unaffected by aggregate covariance risk, save for *PR1YR*, whose premium declines following risky times. In addition, the lagged weight of each strategy in the MVP, which captures the specific covariance properties of each strategy, is negatively related to the future premium on *SMB*, but does not influence the premia on the other strategies. When the MVP weight on *SMB* is high, indicating that the strategy exhibits attractively low variance and covariance properties, the subsequent premium on the size-based strategy is lower. This is also consistent with a risk-based interpretation of the size effect.

²³ First difference regressions of changes in the dependent variable on changes in the regressors produced nearly identical results. Results are available upon request.

Table 4
Return premia, conditional volatility, and covariation

Dependent variable	Panel A: <i>SMB</i> Size effect			Panel B: <i>HML</i> Book-to-market		
	Ret_t	σ_t	$10^4 \times \overline{cov}_t$	Ret_t	σ_t	$10^4 \times \overline{cov}_t$
Recession	-0.0010 (-0.38)	-0.0010 (-1.47)	-0.6491 (-1.03)	0.0048 (2.05)	-0.0007 (-1.37)	1.6270 (2.88)
$avg \cdot \sigma_{t-1}$	-0.9851 (-0.65)	3.4186 (3.24)	930.1798 (1.41)	-0.7165 (-1.02)	2.2372 (1.82)	-1228.3149 (-3.43)
$avg \cdot \sigma_{t-1} \times Rec.$	1.6308 (0.41)	-1.1995 (-1.06)	-1848.9309 (-2.11)	-2.2489 (-0.96)	-1.5179 (-1.20)	-127.9071 (-0.28)
$\sigma_{mvp,t-1}$	-0.0056 (-1.46)	0.0012 (1.63)	0.9439 (1.50)	0.0014 (0.19)	-0.0001 (-0.44)	-0.1492 (-0.65)
$\sigma_{mvp,t-1} \times Rec.$	0.0148 (2.85)	0.0070 (4.88)	2.0647 (1.28)	0.0091 (0.97)	0.0083 (4.29)	0.7255 (0.50)
$own\sigma_{t-1}$	0.1360 (1.07)	0.6137 (6.59)	107.7370 (2.42)	0.0664 (0.82)	0.7122 (5.85)	17.0806 (0.45)
$own\sigma_{t-1} \times Rec.$	-0.1072 (-0.44)	0.1425 (1.46)	117.3959 (2.11)	-0.1247 (-0.61)	0.1548 (1.34)	-150.0858 (-3.39)
$w_{mvp,t-1}$	-0.0008 (-2.05)	0.0000 (-0.22)	-0.1233 (-1.90)	-0.0004 (-1.11)	0.0000 (0.40)	-0.1630 (-3.25)
$w_{mvp,t-1} \times Rec.$	0.0007 (0.54)	-0.0001 (-0.49)	0.0342 (0.21)	-0.0005 (-0.45)	0.0001 (0.54)	-0.0193 (-0.17)
<i>Controls</i>						
Ret_{t-1}	0.1628 (4.72)	-0.0213 (-1.86)	-12.2586 (-1.06)	0.1362 (5.06)	0.0017 (0.24)	2.0937 (0.56)
$Ret_{t-1} \times Rec.$	-0.0298 (-0.38)	0.0477 (2.01)	4.4601 (0.30)	-0.0148 (-0.24)	0.0142 (0.73)	8.8511 (0.85)
$\overline{Ret}_{t-24:t-2}$	0.0688 (0.79)	0.0067 (0.43)	-47.7472 (-3.69)	-0.0144 (-0.14)	0.0464 (2.60)	-54.6785 (-5.14)
$\overline{own\sigma}_{t-24:t-2}$	0.1296 (1.48)	0.0095 (0.49)	-82.4001 (-2.76)	0.2213 (2.98)	0.0003 (0.03)	-1.4169 (-0.08)
$\overline{w}_{mvp,t-24:t-2}$	-0.0020 (-2.24)	-0.0001 (-0.93)	-0.2878 (-2.78)	-0.0016 (-2.02)	0.0003 (2.32)	-1.0770 (-12.78)
$Ret_{mkt,t}$	0.0802 (5.55)	-0.0001 (-0.05)	-2.1427 (-0.79)	-0.1542 (-14.86)	-0.0022 (-1.53)	1.4026 (1.23)
Dependent variable	Panel C: <i>PR1YR</i> Size effect			Panel D: <i>IM</i> Book-to-market		
	Ret_t	σ_t	$10^4 \times \overline{cov}_t$	Ret_t	σ_t	$10^4 \times \overline{cov}_t$
Recession	-0.0030 (-1.06)	-0.0007 (-1.62)	3.5474 (20.07)	-0.0027 (-0.79)	0.0000 (0.07)	1.4073 (5.00)
$avg \cdot \sigma_{t-1}$	-0.1461 (-0.16)	2.3501 (1.41)	882.8865 (4.36)	-0.0039 (0.00)	1.8413 (2.75)	-410.4843 (-2.06)
$avg \cdot \sigma_{t-1} \times Rec.$	2.0242 (1.05)	-1.3145 (-0.82)	-2409.2417 (-10.48)	-1.1095 (-0.54)	-1.1071 (-1.59)	-476.3168 (-2.03)
$\sigma_{mvp,t-1}$	0.0003 (0.06)	0.0003 (0.41)	0.0004 (0.00)	0.0027 (0.56)	-0.0004 (-1.09)	0.1725 (0.83)
$\sigma_{mvp,t-1} \times Rec.$	-0.0107 (-2.09)	0.0010 (0.78)	-0.4248 (-0.97)	0.0084 (1.57)	0.0003 (0.46)	0.1265 (0.43)
$own\sigma_{t-1}$	-0.0343 (-0.46)	0.7268 (5.18)	-67.6155 (-3.90)	0.0365 (0.25)	0.6392 (8.16)	4.1791 (0.20)
$own\sigma_{t-1} \times Rec.$	0.0264 (0.13)	0.1637 (1.23)	-206.7111 (-10.43)	0.2689 (0.91)	0.0869 (1.08)	-149.1223 (-4.75)

Table 4
(continued)

Dependent variable	Panel C: <i>PR1YR</i> Stock momentum			Panel D: <i>IM</i> Industry momentum		
	Ret_t	σ_t	$10^4 \times \overline{cov}_t$	Ret_t	σ_t	$10^4 \times \overline{cov}_t$
$w_{mvp,t-1}$	-0.0001 (-0.86)	0.0000 (-0.02)	0.0001 (0.02)	0.0009 (0.85)	0.0001 (0.70)	-0.2757 (-3.77)
$w_{mvp,t-1} \times Rec.$	0.0002 (0.39)	0.0000 (0.61)	-0.0040 (-0.13)	-0.0032 (-1.30)	-0.0001 (-0.16)	-0.4763 (-2.93)
<i>Controls</i>						
Ret_{t-1}	0.1045 (3.20)	-0.0145 (-1.67)	3.9259 (1.07)	0.0776 (2.17)	0.0060 (1.01)	-5.8235 (-1.31)
$Ret_{t-1} \times Rec.$	-0.0099 (-0.14)	-0.0171 (-1.13)	0.3451 (0.05)	-0.0356 (-0.46)	-0.0205 (-1.57)	3.9324 (0.49)
$\overline{Ret}_{t-24:t-2}$	-0.0202 (-0.17)	0.0807 (2.83)	-0.9056 (-0.07)	-0.1629 (-1.14)	0.1663 (4.57)	-27.1361 (-2.05)
$\overline{own\sigma}_{t-24:t-2}$	-0.0020 (-0.04)	0.0145 (1.49)	-39.5664 (-5.31)	-0.0092 (-0.08)	0.0611 (3.62)	8.6529 (0.87)
$\overline{w}_{mvp,t-24:t-2}$	0.0002 (1.01)	0.0001 (1.92)	-0.0104 (-0.63)	0.0021 (0.62)	0.0007 (1.13)	-0.9955 (-4.23)
$Ret_{mkt,t}$	-0.0847 (-7.44)	-0.0012 (-0.89)	-0.3476 (-0.44)	-0.0578 (-4.11)	-0.0007 (-0.64)	0.3258 (0.26)
Constant	0.0010 (2.01)	0.0009 (3.51)	0.5023 (7.75)	0.0002 (0.18)	0.0016 (4.52)	0.1602 (2.29)

The weekly returns, conditional standard deviations, and average conditional covariance of the trading strategies *SMB*, *HML*, *PR1YR*, and *IM* are regressed over time on past measures of the strategies' return, conditional volatility, and measures of the past covariance structure of asset returns. Three sets of dependent variables are employed for each of the four trading strategies. The first is simply the weekly return on each trading strategy. The second is the conditional standard deviation of the strategy at time t (σ_t). The third is the average conditional covariance of the strategy, which is defined as the average covariance of each strategy with the other (basis) assets in the covariance matrix. All measures of past and future conditional second moments are derived from the Ledoit and Santa-Clara (1998) multivariate GARCH estimates of the conditional covariance matrix. The regressions employ weekly estimates of returns and volatility over the period August 1964 to December 1997. Time-series regressions are run via ordinary least squares with t -statistics in parentheses that employ White (1980)-corrected standard errors.

The second column of each panel examines how the conditional volatility of each strategy is affected by the regressors. The conditional standard deviations of the strategies are not significantly different in recessionary periods. The lagged average conditional standard deviation of the 32 assets ($avg.\sigma_{t-1}$), a measure of aggregate volatility, is positively related to the conditional standard deviation of each strategy, but also exhibits no differences during recessions. However, the prior volatility of the MVP tends to predict higher future volatility on *SMB*, predominantly in recessions. The same holds for *HML*. This hints at a risk-based interpretation of the size and book-to-market effects. The conditional volatility of the other strategies are unaffected by the recent volatility on the MVP.

Finally, the third and last column in each panel examines how the conditional covariation of each strategy with other assets in the economy varies with the regressors. First, the average conditional covariance of *HML*, *PR1YR*, and *IM* is significantly larger during recessions, while it is smaller for *SMB* during recessions. This suggests that size-based strategies may provide a hedge to investors in bad times, since its covariance with other assets

declines on average during these times. This provides more evidence that *SMB* behaves differently than the other strategies and is most strongly linked to the covariance matrix. Second, lagged aggregate volatility is associated with lower average covariances for all strategies. However, for *SMB* and *PR1YR*, this is driven solely by recessionary periods. This asymmetry in the relation between aggregate volatility and asset covariation across the business cycle may be consistent with the model of Veronesi (1999), who shows that investors' willingness to hedge against changes in their own uncertainty leads them to overreact to economy-wide bad news in good times and underreact to good news in bad times. Ribeiro and Veronesi (2001) explore the cross-covariance implications of this model for individual assets across the business cycle. Third, the relation between the strategy's own variance and its covariation with the other assets in the economy diminishes when the volatility of *HML* and the two momentum portfolios is high, especially during recessions. On the other hand, *SMB* covaries *more strongly* when its conditional volatility is high, and this occurs primarily during recessions. Thus, in a portfolio context, the book-to-market and momentum strategies become less risky when their volatility increases, since their covariance properties become more attractive (e.g., decrease), while *SMB* becomes more risky, as both its variance and covariance rise. Furthermore, these effects are more pronounced in recessions, when investors may care most about these properties. Again, this suggests a risk-based view of the size effect. Finally, the past weight in the MVP of each strategy is negatively related to the future conditional covariance of each strategy, confirming that this is a useful measure of the strategy's contribution to covariance risk.

6. Conclusion

While much research has focused on the relation between firm characteristics and mean returns, this article examines their relation to the second moment of returns. Using both statistical and economic measures of this link, while allowing for time variation in the covariance matrix of asset returns (applying a unique GARCH methodology), a size factor is found to be most closely linked to covariance risk, both in and out of sample, book-to-market equity exhibits a weaker link, and momentum factors appear unrelated to return second moments.

The economic significance of these findings are examined from an investment perspective. Accounting for covariation with a size and book-to-market factor improves the out-of-sample performance of efficient portfolios, even more so when combined with the market portfolio. Including information from covariation with momentum factors, however, does not improve efficiency. Furthermore, these results are more acute during recessions, when investors presumably care most about the efficiency of their portfolio.

In terms of the rational-behavioral debate, the results in this study complement those of Liew and Vassalou (1999) and Chen (2001). Liew and Vassalou (1999) find that both size and book-to-market forecast future economic growth, but that momentum is unrelated to economic activity. Chen (2001) concludes that the book-to-market and momentum premia appear too high to be justified by an intertemporal CAPM framework with time-varying expected returns, but that size seems reasonably priced. I find that the premium on the size factor rises as its conditional volatility and covariation with other assets increases, particularly during recessions. The premia on the other factors, however, do not exhibit these patterns. Since the size premium is highest following times when both aggregate covariance is high and consumption is low (in recessions), this may indicate a risk-based interpretation for the size effect. Along with the evidence in Liew and Vassalou (1999) and Chen (2001), this suggests that size is more strongly linked to economic risk than either book-to-market or momentum. These results present a challenge to existing theory that may shed light on possible explanations for the relation between firm characteristics and average returns.

Finally, while this article focuses on the second moment of the return distribution, investors may also care about higher return moments. Indeed, many of the trading strategies examined in this article involve dynamic trading, which may generate significant skewness and kurtosis in their returns. Asset pricing models which incorporate higher moment factors, such as Rubinstein (1973) and Harvey and Siddique (2000), may prove useful in analyzing and understanding the profitability of these strategies. Such issues are left for future research.

Appendix

The robustness of the GARCH estimates: a brief comparison of methods

The GARCH parameters are estimated with high precision. Every element of the parameter matrix of lagged square residuals, A , and of own lagged weights, B , is highly statistically significant, ranging from 4 to more than 10 standard errors from zero. The estimate of matrix C , the constant term, is the least precise, yet more than one-third of its elements are still significantly different from zero. In addition, the average size of the elements of C is orders of magnitude below that of A or B , and does not have much effect on the time series of covariance matrices. This is confirmed by employing other matrices, such as the identity matrix, in place of C , which did not significantly alter the results. Both A and B have a substantial impact on conditional covariances, with the weight on lagged square residuals (A) ranging between 0.064 and 0.207 and on own lags (B) between 0.764 and 0.934.

The standard errors of the GARCH parameters are computed under asymptotic distributions that are valid when the number of periods, T , grows large with respect to the number of parameters. Since the Ledoit and Santa-Clara (1998) model requires estimating such a large number of parameters, the reliability of these standard errors may be questioned. However, Ledoit and Santa-Clara (1998) conduct extensive simulations of their model using weekly returns

and find that the small sample properties of their method are quite good, even for matrices of dimension 50, using only 715 weekly return observations (the current article estimates matrices of dimension 32 using 1,748 observations). Furthermore, the estimation errors computed from their model are shown to be accurate to the simulation-based errors.

To further address the reliability of the Ledoit and Santa-Clara (1998) estimates, a comparison is made to more traditional conditional covariance estimators. The first is a special case of the BEKK GARCH model of Engle and Kroner (1995). This special case is the single factor GARCH(1,1) model of Engle, Ng, and Rothschild (1990), which imposes more structure on the model and requires only $(N^2 + 5N + 2)/2$ parameters to be estimated.²⁴ For computational ease, both the Ledoit and Santa-Clara (1998) and BEKK GARCH models are estimated on a subset of 11 portfolios.

The Ledoit and Santa-Clara (1998) procedure requires estimating 198 parameters, while the BEKK model requires only 89 parameter estimates. The table reports the average covariance, average variance, and average correlation of the GARCH conditional covariance matrices. The estimates are not significantly different under the two specifications, although BEKK produces larger covariance and variance estimates and slightly smaller correlations. Both procedures produce statistically reliable estimates indicated by their standard errors. For comparison, the average covariance, variance, and correlation of the unconditional sample covariance matrix are also reported. The average Ledoit and Santa-Clara (1998) estimates are closer to the unconditional measures than the BEKK estimates, indicating perhaps that the less restrictive Ledoit and Santa-Clara (1998) procedure produces more accurate second moment estimates.

For additional comparison, the average covariance, variance, and correlation of rolling sample estimates using the past five years of weekly data are reported. At each point in time t , the sample covariance matrix of the 11 portfolios is computed from weekly returns over the past five years. These rolling estimates are similar in spirit to those used by Officer (1973), Fama and MacBeth (1973), Daniel and Titman (1997), Brennan, Chordia, and Subrahmanyam (1998), Chan, Karceski, and Lakonishok (1999), and Grundy and Martin (2001). As the table indicates, the rolling estimates are close to the unconditional and Ledoit and Santa-Clara (1998) estimates, but diverge somewhat from BEKK. This provides further comfort that the Ledoit and Santa-Clara (1998) estimates are reasonably reliable and accurate.

In addition, the comparison across covariance estimates is repeated using monthly returns data. This should disadvantage the Ledoit and Santa-Clara (1998) model the most since it requires the largest number of parameters to be estimated. Certainly, all four covariance estimates become less reliable when using monthly data. However, the Ledoit and Santa-Clara (1998) estimates still exhibit substantial statistical significance and are closer to the unconditional and rolling sample estimates than BEKK, with the exception of the average correlation.

Finally, panels B and C repeat the comparison across covariance estimation techniques for contractionary and expansionary periods, respectively, as defined by the National Bureau of Economic Research. Covariances, variances, and correlations appear higher during recessions. The Ledoit and Santa-Clara (1998) estimates pick up this asymmetry, while the BEKK estimates appear largely insensitive to the business cycle. This suggests that the Ledoit and Santa-Clara (1998) estimates may provide a richer characterization of changing return second moments.

²⁴ The GARCH model is as follows:

$$H_t = C' C + \lambda \lambda' [\beta w' H_{t-1} w + \alpha (w' \eta_t)^2],$$

where C is a lower triangular matrix, λ and w are N -vectors, and α and β are scalars. Often a normalizing restriction on w is placed so that w sums to one.

Table A.1
Comparison of second moment estimates

Covariance estimator	Weekly			Monthly		
	Avg. cov.	Avg. var.	Avg. corr.	Avg. cov.	Avg. var.	Avg. corr.
Panel A: Whole sample period (Aug. 1964–Dec. 1997)						
LS	0.000406 (0.000008)	0.000996 (0.000080)	0.686806 (0.003211)	0.001731 (0.000022)	0.002817 (0.000186)	0.498429 (0.005388)
BEKK	0.000986 (0.000003)	0.001325 (0.000002)	0.623533 (0.003386)	0.003788 (0.000058)	0.005274 (0.000039)	0.668766 (0.008952)
Rolling	0.000402 (0.000004)	0.000574 (0.000005)	0.745638 (0.001688)	0.001981 (0.000045)	0.002698 (0.000053)	0.740441 (0.003932)
Unconditional	0.000405	0.000557	0.748817	0.001879	0.002592	0.743645
Panel B: Recessions (contractionary periods)						
LS	0.000690 (0.000020)	0.000947 (0.000024)	0.744264 (0.003536)	0.002381 (0.000070)	0.002992 (0.000103)	0.534212 (0.001870)
BEKK	0.001041 (0.000004)	0.001339 (0.000004)	0.686168 (0.005157)	0.003150 (0.000138)	0.004493 (0.000089)	0.645147 (0.019732)
Rolling	0.000422 (0.000006)	0.000580 (0.000008)	0.754295 (0.002427)	0.002241 (0.000068)	0.003017 (0.000085)	0.769280 (0.004485)
Unconditional	0.000836	0.001077	0.793648	0.005221	0.006411	0.828722
Panel C: Nonrecessions (expansory periods)						
LS	0.000360 (0.000008)	0.001004 (0.000093)	0.677395 (0.003635)	0.001632 (0.000018)	0.002790 (0.000213)	0.492952 (0.006160)
BEKK	0.000977 (0.000003)	0.001323 (0.000002)	0.613274 (0.003784)	0.004130 (0.000153)	0.005336 (0.000118)	0.673185 (0.011603)
Rolling	0.000381 (0.000006)	0.000518 (0.000007)	0.672735 (0.005923)	0.001753 (0.000055)	0.002393 (0.000067)	0.665696 (0.011825)
Unconditional	0.000335	0.000472	0.734008	0.001375	0.002014	0.705802

Reported are the average covariance, average variance, and average correlation over time of various multivariate covariance estimators. The covariance matrix of 11 portfolios: oil, manufacturing, utilities, and financial sector portfolios, the small stock high book-to-market, small stock low book-to-market, big stock high book-to-market, big stock low book-to-market, the market, and the losers (worst 20% past year returns) and winners (best 20% past year returns) portfolios are computed using the multivariate GARCH estimates of Ledoit and Santa-Clara (1998) Ledoit and Santa-Clara and BEKK (using maximum likelihood), as well as the rolling sample covariance matrix using the past five years of return data. Both weekly and monthly returns data are employed over the period August 1964 to December 1997, obtained from Center for Research in Security Prices. The average covariance, average variance, and average correlation of each portfolio at each point in time are reported below, along with averages from the unconditional covariance matrix over the entire sample period for comparison. Panel A reports results over the whole sample period, panel B reports results during recessions only (contractionary periods defined by the National Bureau of Economic Research), and panel C reports results during expansionary periods only. Time-series standard errors on the averages are reported in parentheses.

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