Abstract

We propose a general equilibrium model where investors hire fund managers to invest their capital either in a risky bond or in a riskless asset. There is a small fraction of informed managers with superior information on the default probability. Looking at the past performance, investors update their beliefs on the information of their managers and make hiring and firing decisions. This leads to career concerns which affect the investment decision of uninformed managers, generating a “reputational premium”. When the default probability is high enough, uninformed managers prefer to invest in the riskless asset to reduce the probability of being fired. On the contrary, if the probability of default is low enough, investing in the risky bonds has a reputational advantage and the premium is negative. As the economic and financial conditions change, the reputational premium can switch sign, amplifying the reaction of prices and capital flows. We then propose an extended version of the model where the default decision is endogenous, generating a feedback effect from asset pricing to the real economy, which further magnifies the reaction of prices to shocks.
1 Introduction

The financial turmoil started in the summer of 2007 and exploded in 2008 has costly testified that the incentives of financial intermediaries and the performance of the economy are closely related. In this paper, we provide a general equilibrium model of the interaction between investors, financial intermediaries and the users of capital. In particular, we focus on the career concerns of fund managers who take into account the effect of their portfolio choices on their reputation. We show that career concerns can magnify the reaction of asset prices and capital flows to both real and financial shocks.

Although our model can be applied to any risky asset, in this paper we focus on debt instruments. In the few years before the recent subprime crises, many market observers were concerned about a growing “overenthusiasm” for risky investments in debt instruments, including high-yield corporate bonds, mortgage-backed assets and emerging market bonds. One observer notices:

Bonds issued by Ecuador, which is politically very unstable, are among the riskiest bets in the emerging markets. It is hard to predict what will happen there next month, let alone in 10 years time. Yet buyers appear to be ready and willing to line up for a sale by the government of up to Dollars 750m in 10-years bonds, the first international bond offer since the country defaulted in 1999. The issue, [...] is the latest example that the prolonged love affair with emerging market debt is far from over. (December 9, 2005, Financial Times).

A similar observation related to the role of financial intermediaries in the leveraged buy-out deals follows:

The head of one of the biggest commercial lenders in the US describes the amount of leverage on some buy-out deals as “nutty”. Much of the wildest lending is being done by hedge funds awash with cash, he says. “Some funds believe they have to invest the money even if it’s not a smart investment. They think the people that gave them the money expect them to invest it. But it’s madness.” (March 14, 2005, Financial Times)

Figure 1 shows the pattern of the yield spreads of a sample of emerging market bonds, the AAA and the B-graded corporate bonds, and the BBB-graded commercial mortgage-backed assets, between October 1994 and February 2008. The figure shows at least two periods in
which all spreads shrunk to very low levels, close to the AAA corporate spreads: in 1996-1997 and then again from 2005 to the summer of 2007. Observers describe these periods as periods of overenthusiasm, which typically occur right before the emergence of a crisis (e.g. Kamin and von Kleist, 1999, IMF, 1999b, Duffe et al., 2003). The figure also shows three episodes of high turbulence in which the spreads of many high-risk bonds jump up and capital tends to flow out of these markets, a phenomenon dubbed as flight-to-liquidity or flight-to-quality. Our model is able to rationalize both episodes of overenthusiasm and episodes of flight-to-liquidity.

We propose a model where investors delegate their portfolio decision to risk-neutral fund managers. We assume that fund managers can invest either in risk-less assets or in risky bonds, and differ in the degree of information about the default risk. The core of our model builds on the career concerns of the fund managers. Every period, each investor has a manager working for him. We consider an extreme environment, where a small portion of informed managers

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1As a columnist of the Wall Street Journal observes, the 5-year credit default swap spreads for Brazil, Peru, Columbia were at the record-tight levels of 0.70, 0.65 and 0.80 percentage point at the time when, for example, the Boston Scientific Corp, an investment grade company traded at 0.78 percentage point. (April 24, 2007, Tight spreads are emerging, WSJ).
can perfectly predict if the risky bond is going to default, and a mass of uninformed managers do not have any superior information and know only the ex ante default probability. At the end of the period, the investor updates his belief based on his manager’s performance, and decides whether to keep him or to hire a new one. The firing decision of the investors distorts the investment decision of uninformed managers who would like to be perceived as informed managers.

One of our main result is that managers’ career concerns impose a premium on the spread of risky bonds that depends on the default probability. Uninformed fund managers try to time the market in order to behave as if they were informed and knew in advance if there would be default or not. Default hurts the reputation of uninformed managers who invest in the risky bond, and no default hurts the reputation of uninformed managers who invest in the riskless asset. Thus, when the probability of default is high, the premium is positive to compensate for the foregone reputation. When instead the default probability is low, the risky bond will trade at a negative premium. A shock either to the ex ante default probability or to the financial market, such as a change in the risk-free rate, may generate a switch in the sign of the premium. This tends to amplify the reaction of the bond price to the shock, in comparison to an economy with no career concerns.

We then consider an extended version of the model where we endogenize the supply of risky bonds and the default rule. In particular, we introduce a continuum of entrepreneurs who have access to a risky project and can issue risky bonds. They take the bond price as given and choose how much to borrow in order to cover their consumption and finance their project. After observing the realization of the project’s productivity, they can decide whether to pay back the outstanding debt or to default and suffer a loss. The interaction between managers and entrepreneurs determines the price of the risky bond, the probability of default and the volume of debt in the economy. The investment choice of the managers determines the required bond price for a given probability of default. The default rule of the entrepreneurs determines the ex ante probability of default on the bond for a given price. Hence, the equilibrium bond price and the default probability are jointly determined by the financial market and the fundamentals of the risky project.

We show that, once we endogenize the supply side of the model, there is a feedback effect from asset pricing to the real economy, which magnifies further the reaction of prices and capital flows to financial and real shocks. On the financial side, a higher default probability leads to
a lower bond price, also because of a larger premium. On the real side, when borrowing is more expensive, entrepreneurs have to make larger repayments and, hence, default with higher probability. In general equilibrium, these two mechanisms reinforce each other and generate excess volatility in bond prices.

**Literature review.** To our knowledge, this is the first paper to address the asset pricing consequences of the interaction between the real economy and the agency problem between fund managers and investors. Our work is related to several areas of macroeconomics and finance.

First, our paper is related to herding models, such as Scharfstein and Stein (1990), Zweibel (1995), and Ottaviani and Sorensen (2006), where, as in our paper, decision makers with career concerns make inefficient decisions to convince their clients that they are informed. However, there are two main points of departure from our work. On the one hand, this literature traditionally concentrates on partial equilibrium models while our focus is on the interaction of career concerns and asset prices. On the other hand, these papers present mechanisms in which each decision maker herds on others’ decision because going against the average action is a bad signal about his ability. In our model, at the equilibrium prices, fund managers choose the inefficient action regardless of other managers’ decision. That is, there are no strategic complementarities. The closest paper to ours is Rajan (1994), who shows that herding might motivate bank executives to overextend credit in good times by amplifying real shocks. In contrast to our model, Rajan (1994) predicts that in bad times banks provide the right amount of credit while we argue that in bad times managers underinvest in the risky bonds.

Second, there is a growing literature which analyzes the effect of delegated portfolio management on asset prices, including Allen and Gorton (1993), Shleifer and Vishny (1997), Vayanos (2003), Dasgupta and Prat (2005, 2007), Cuoco and Kaniel (2007), He and Krishnamurthy (2007), and Dasgupta, Prat and Verardo (2008). This literature is silent about the real effects of the agency frictions in financial markets. Unlike our work, most of this literature takes managers’ distorted incentives as given. A notable exception is Dasgupta and Prat (2006, 2007) and Prat and Verardo (2008) who introduce herding into a Glosten-Milgrom type of sequential trading model. They show that reputational concerns can lead to excessive trading, slow revelation of information and (if the market maker has market power) biased prices. This series of work is a predecessor to our model in the sense that they are the first to use the term
premium and to point out that the potential trade-off between reputation and trading profits might lead managers to choose bets with negative net present value. However, our context is different as we are interested in the way effects amplify the price response of financial and real shocks depending on the state of the economy. Furthermore, we build a standard, competitive, asset pricing model to show that effects have systematic effects on prices.

Our paper is also related to a large literature on the propagation and amplification of fundamental shocks due to the interaction between asset values and collateralized lending. Seminal papers in this area are Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) on the macro-side, and Gromb and Vayanos (2002) on the finance side\(^2\). The main difference with our mechanism is that these papers have typically an asymmetric distortion, given that collateral constraints build into the model an external finance premium, usually generating underinvestment. In our model, instead, we microfound the financial distortion and we generate a premium that can be either positive or negative.

Finally, our model can be interpreted also as a model of sovereign debt. In this sense is related to the vast literature on financial crisis in emerging economies and reversal of capital flows, including Atkenson (1991), Cole and Kehoe (2000), Calvo and Mendoza (2000), Caballero and Krishnamurthy (2003), Benczur and Ilut (2005), Aguiar and Gopinath (2006), Arellano (2006), Uribe and Yue (2006), and Kovrijnykh and Szentes (2007). This strand of literature abstracts from the effects of intermediation in financial markets, and could be interestingly complemented with our mechanism.

The rest of the paper is organized as follows. In Section 2, we introduce an example to illustrate the main mechanism of our model. In Section 3, we describe the model, and define and characterize an equilibrium. In Section 4, we analyze an extended version of the model, where the supply of risky bonds and the default decision are endogenous. In Section 5, we analyze an economy where productivity is persistent and we propose some numerical exercises. Finally, Section 6 concludes. Appendix A includes all the proofs which are not in the text, and Appendix B presents a parametric example.

2 An Example

In this section, we introduce a simple example to show the main mechanism of the model: incentive schemes can distort fund managers’ investment decisions and, hence, affect equilibrium prices and capital flows.

Assume that a large group of risk-neutral fund managers have to decide whether to invest a unit of capital in a risky asset or in a riskless asset. The risky asset has price $p$ and pays 1 if the good state realizes and 0 if the bad state realizes. The probability of the bad state is equal to $q$. The riskless asset pays the safe return $R < 1/p$. Just for this example, assume that a manager obtains a bonus $W$ if he succeeds in his investment, that is, if he invests in the risky bond when there is no default or in the riskless asset otherwise. The riskless asset is in infinite supply, while the supply of the risky bond is fixed and smaller than the total capital invested by the managers.

It is straightforward that the bond market clears if and only if managers are indifferent between investing in the risky bond and in the riskless asset. Hence, the equilibrium price of the risky bond has to satisfy the following indifference condition

\[
(1 - q) \left( \frac{1}{p} + W \right) = R + qW.
\]  

(1)

The left-hand side of equation (1) represents the expected payoff of a manager who invests in the risky bond. With probability $1 - q$ there is no default and the manager gets a return $1/p$ and the bonus $W$. If instead there is default, the manager gets zero revenues and no bonus. Similarly, the right-hand side of equation (1) represents the expected payoff of a manager who invests in the riskless asset. He gets always a return $R$, but he obtains the bonus only if there is default.\(^3\)

In order to characterize the price distortion generated by the bonus $W$, we define the premium $\Pi$ as the difference between the expected return and the risk free rate

\[
\Pi = \frac{1 - q}{p} - R.
\]

The indifference condition (1) immediately implies that $\Pi = 0$ when there is no bonus scheme, that is, $W = 0$. In this case, fund managers care only about the expected returns of the bond and the premium is zero. When instead $W > 0$, the premium can be negative or positive.\(^3\) The equilibrium price is consistent with the assumption that $1/p > R$ if $R > W (1 - 2q)/q$. 

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In particular, if \( q > 1/2 \), the payoff of the risky bond is skewed to the left as the probability of default is larger than the probability of no default. In this case, investing in the riskless asset has an advantage over the risky bond as this ensures the bonus payment with larger probability. If the expected return of the two assets were equal, all managers would prefer the riskless one, because of this advantage. Thus, in equilibrium there must be a positive premium on the risky bond to induce managers to hold it. Similarly, if \( q < 1/2 \) the payoff of the risky bond is skewed to the right. In this case, the risky bond has an advantage and the premium is negative.

This simple example is suggestive, but it clearly calls for some microfundations behind the bonus scheme. Why investors should ex post reward managers who make successful investment? The story we have in mind is a story of reputation, which needs both a dynamic environment and some form of heterogeneous information. In the rest of the paper we build a dynamic general equilibrium model of delegated portfolio management with informed and uninformed managers. Investors rationally learn about the type of the fund managers based on their past performance. Uninformed managers’ career concerns generate a reputational premium similar to the one described in this example. In particular, \( W \) becomes an equilibrium object equal to the expected utility of an employed uninformed manager. We show that small shocks to the financial market or to the fundamentals of the risky project may lead to large changes in asset prices and capital flows, due to the presence of the reputational premium. In Section 4, we also endogenize the default probability \( q \) and the supply of the risky bond. We introduce entrepreneurs who issue the risky bond and can decide to default on their loans. This generates a feed-back effect from asset pricing to the real economy, which further magnifies the reaction of prices to shocks.

3 Baseline Model

3.1 Set up

Consider an infinite horizon economy with discrete time. There is a mass \( \Gamma \) of risk-neutral investors and two investment possibilities: a riskless asset, in infinite supply, which pays \( R \) units of consumption per unit of capital invested, and a risky bond with fixed supply \( b \) and price in terms of capital \( p_t \), which pays 1 unit of consumption if there is no default and 0 otherwise. Let \( \chi_{t+1} \) denote the default indicator such that \( \chi_{t+1} = 1 \) if there is default and
\( \chi_{t+1} = 0 \) otherwise. Assume that the ex-ante probability of default is equal to \( q \).

At any time \( t \), each investor has one unit of capital and needs a fund manager to invest it. There are two types of risk-neutral fund managers: informed and uninformed.\(^4\) Informed managers know in advance \( \chi_{t+1} \), that is, they can perfectly predict if there is going to be default.\(^5\) Uninformed managers, instead, expect the bond to default with probability \( q \). There is a mass \( M^I \) of informed managers and a larger continuum of uninformed managers. Fund managers do not have any capital, and need to be employed by an investor to make any investment decision. Each investor can employ only one fund manager and a fund manager can work only for a single investor. All the managers have to pay a cost \( \kappa \leq \gamma R \) to search for an investor.\(^6\) Moreover, investors do not know the managers’ type.

At time \( t \) there is a mass \( \Gamma^s_t \) of employed managers of type \( s \), and all investors have a manager working for them, so that \( \Gamma^I_t + \Gamma^U_t = \Gamma \). Then, employed managers choose how to invest the unit of capital they manage. Next, investors observe the return of their manager’s investment and decide whether to fire him or not. Moreover, they receive a signal that reveals the type of an uninformed manager with probability \( 1 - \omega \).\(^7\) Each manager has a probability \( 1 - \delta \) to die, in which case a new manager of the same type is born.\(^8\) Finally, all investors who do not have a manager, either because they have fired him or because he is dead, search for a new one. At the same time, the unemployed managers decide whether to pay a cost \( \kappa \) to search for a job. Let \( N^s_t \) be the mass of managers of type \( s \), for \( s = I, U \), who decide to look for a job. The probability for a manager to find a job, denoted by \( \mu_t \), is equal to the ratio of the mass of investors searching for a manager to the mass of managers searching for an investor.

We look for a stationary equilibrium where the mass of informed and uninformed employed managers, \( \Gamma^I_t \) and \( \Gamma^U_t \), and the matching probability, \( \mu_t \), are constant over time. Hence, from now on we can drop the time dependence for these objects. Moreover, for simplicity, we fix the contract between investors and fund managers: fund managers keep a share \( \gamma \) of the revenues

\(^4\)Risk neutrality ensures that the demand for the risky bond is perfectly inelastic, making the model more tractable. Moreover, it allows us to focus on the price distortion coming exclusively from our mechanism. For a more quantitative exercise it would be useful to enrich our model with risk aversion.

\(^5\)The extreme assumption that informed managers have perfect information is not crucial for the analysis. However, it allows not to keep track of the history of beliefs, making the model more tractable. In a more general version of the model, the two types of managers could get signals with different degrees of precision.

\(^6\)The assumption that \( \kappa \leq \gamma R \) is sufficient to ensure that it is profitable for an informed manager to search for a job.

\(^7\)As we will discuss later on, this exogenous signal makes the analysis more tractable because it guarantees that we can focus on equilibria where a manager who does succeed in his investment is never fired.

\(^8\)This ensures that the mass of informed unemployed managers stay positive over time.
and leave the rest to the investors. Both investors and managers fully consume their net revenues in each period.

Any employed manager selects a demand schedule \( d^I_t (p_t, x_{t+1}) \in [0, 1] \) if he is informed and \( d^U_t (p_t) \in [0, 1] \) if he is uninformed. At the same time, there is a mass \( y_t \) of noise traders, with one dollar each, who demand the bond, where \( y_t \) is a random variable uniformly distributed on the support \([0, \bar{y}]\), with \( \bar{y} \in [M^I, b] \). After collecting all the demand schedules, an auctioneer selects the equilibrium price \( p_t \) and assigns the risky bonds to the managers and the noise traders. The selected price and bond allocation must be consistent with the submitted demand schedules and with the bond market clearing. Let us denote by \( x^I_t \) and \( x^U_t \) the equilibrium fraction of informed and uninformed managers who obtain the bond. The market clearing condition is then

\[
\Gamma^I x^I_t + \Gamma^U x^U_t + y_t = p_t b. \tag{2}
\]

The right-hand side of the market clearing condition represents the value of the supply of bonds. The left-hand side, instead, represents the demand of bonds, which comes from three different sources: 1) a proportion \( x^I_t \) of informed employed managers, 2) a proportion \( x^U_t \) of uninformed employed managers, and 3) a mass \( y_t \) of noise traders.

At the beginning of time \( t \), each investor has a manager \( j \) working for him that he believes is informed with probability \( \eta^I_t \). Let \( \theta^I_t \) be an indicator variable, which is equal to 1 if manager \( j \) is allocated a unit of the risky bond, and to 0 otherwise. A law of large number ensures that \( x^I_t \) represents also the probability for an agent of type \( s \) of receiving the risky bond. At the end of time \( t \), the investor gets his share of the realized returns, observes \( \theta^I_t \) and whether there has been default or not. Moreover, if his manager is uninformed, he discovers his type with probability \( 1 - \omega \). Then, according to the Bayes’ Rule, he updates his belief about the type of his manager to \( \eta^I_{t+1} \). Conditional on his posterior, he chooses his firing strategy \( \phi^I_t \), that is, whether to keep his manager for next period (\( \phi^I_t = 0 \)), or to fire him and hire a new one (\( \phi^I_t = 1 \)). Clearly, the investor’s firing decision is affected by the probability that a new
hire is informed. The key feature of our model is that manager $j$ knows that his investment decision will affect the investors’ firing decision by changing his posterior belief. This generates career concerns affecting the investment strategy that are at the core of our model.

### 3.2 Equilibrium

Let us first introduce the definition of a stationary equilibrium for any $M^I > 0$. Next, we will propose the type of stationary equilibrium we are interested in, an *interior equilibrium*, we will characterize it, and we will show under which assumptions it exists.

**Definition 1** For a given $M^I > 0$, a stationary equilibrium is a demand schedule for informed managers, $d^I(p_t, x_{t+1})$, a demand schedule for uninformed managers, $d^U(p_t)$, a firing strategy for investors, $\phi(\eta_{t+1})$, bond allocations for the informed and uninformed managers, $x^I(x_{t+1}, y_t)$ and $x^U(x_{t+1}, y_t)$, a price $p(x_{t+1}, y_t)$, a constant mass of employed informed and uninformed managers, $\Gamma^I$ and $\Gamma^U$, a constant matching probability $\mu$, and an updating rule $\eta_{t+1} = \zeta(\eta_t, \theta_t, p_t, x_{t+1})$, such that

1. investors maximize their expected utility, taking as given equilibrium price, allocation and strategies of other agents;

2. fund managers maximize their expected utility, taking as given equilibrium price, allocation and strategies of other agents;

3. the bond price and allocations are consistent with the managers’ demand schedules and with market clearing;

4. $\Gamma^I, \Gamma^U$ and $\mu$ are consistent with free entry in the labor market for fund managers;

5. investors’ beliefs are consistent with the Bayes’ rule.

### 3.3 Interior Equilibrium

When $M^I > 0$, employed uninformed managers face the risk of being fired and, hence, their investment decisions are affected by their expected future utility. Moreover, they can potentially extract some information about the strategy of the informed managers from the equilibrium price. We focus on equilibria where uninformed managers are typically the marginal traders, that is, are indifferent between investing in the bond and in the risk-free asset whenever prices are not fully revealing. We call this type of equilibrium, an “interior” equilibrium.
Let $z$ represent the total potential demand of informed managers and noise traders, that is, $z(\chi_{t+1}, y_t) \equiv \Gamma^I (1 - \chi_{t+1}) + y_t$.

**Definition 2** A stationary interior equilibrium is an equilibrium where

(i) prices are

$$
p(\chi_{t+1}, y_t) = \begin{cases} 
y_t/b & \text{if } z(\chi_{t+1}, y_t) \in [0, \Gamma^I) \\
p^* & \text{if } z(\chi_{t+1}, y_t) \in [\Gamma^I, \bar{y}] \\
1/R & \text{if } z(\chi_{t+1}, y_t) \in (\bar{y}, \bar{y} + \Gamma^I] 
\end{cases}
$$

with $p^* \in (\bar{y}/b, 1/R)$;

(ii) managers’ demand schedules are

$$
d^I(\chi_{t+1}, p_t) = \begin{cases} 
1 - \chi_{t+1} & \text{if } p_t < 1/R \\
0 & \text{if } p_t = 1/R
\end{cases}
$$

and

$$
d^U(p_t) = \begin{cases} 
0 & \text{if } p_t \leq \bar{y}/b \\
\{0, 1\} & \text{if } p_t \in (\bar{y}/b, 1/R] 
\end{cases}
$$

(iii) managers’ bond allocations are

$$
x^I(\chi_{t+1}, y_t) = \begin{cases} 
1 - \chi_{t+1} & \text{if } z(\chi_{t+1}, y_t) \in [0, \bar{y}] \\
\frac{b/R - y_t}{\Gamma} & \text{if } z(\chi_{t+1}, y_t) \in (\bar{y}, \bar{y} + \Gamma^I] 
\end{cases}
$$

and

$$
x^U(\chi_{t+1}, y_t) = \begin{cases} 
0 & \text{if } z(\chi_{t+1}, y_t) \in [0, \Gamma^I) \\
\frac{p^*b - z(\chi_{t+1}, y_t)}{\Gamma} & \text{if } z(\chi_{t+1}, y_t) \in [\Gamma^I, \bar{y}] \\
\frac{b/R - y_t}{\Gamma} & \text{if } z(\chi_{t+1}, y_t) \in (\bar{y}, \bar{y} + \Gamma^I] 
\end{cases}
$$

(iv) investors’ firing rule requires to fire manager $j$ if the exogenous signal reveals that $j$ is uninformed or $\theta^j_t \neq 1 - \chi_{t+1}$ and $p(\chi_{t+1}, y_t) < 1/R$, and keep him otherwise.

An interior equilibrium is characterized by a value $\Gamma^I$ for the mass of employed informed managers and an interior price $p^*$, which ensure that conditions (i)-(iv) characterize a stationary equilibrium. In particular, $\Gamma^I$ must be consistent with free entry in the labor market for fund managers, while $p^*$ must be such that the uninformed managers are indifferent between demanding the risky asset and the risk-free one when their only information is the ex-ante probability of default $q$.

In an interior equilibrium, three possible revelation regimes arise: $p(\chi_{t+1}, y_t) = 1/R$ reveals that there is going to be default, $p(\chi_{t+1}, y_t) = \bar{y}/b$ reveals that there is going to be no default,
and, thanks to the uniform distribution of $y$, $p(\chi_{t+1}, y_t) = p^*$ does not reveal any information. If $p(\chi_{t+1}, y_t) = 1/R$, the two assets are equivalent and all managers are indifferent between them. If $p(\chi_{t+1}, y_t) = \bar{y}/b$, there is revelation of default and all managers demand the riskless asset. Finally, when $p(\chi_{t+1}, y_t) = p^*$, informed managers know in advance if there is going to be default or not, and hence can time the market perfectly, by demanding the risky bond if and only if there is not going to be default. The uninformed managers, however, do not have this superior information and $p^*$ makes them indifferent between obtaining the bond or not. This is the notion of interior solution we refer to when we call this equilibrium an interior equilibrium. It follows that $\Gamma^I / \bar{y}$ represents the probability that in equilibrium the price is fully revealing. The assumption that $\bar{y} > M^I$ ensures that prices are not always fully revealing, that is, $\Gamma^I / \bar{y} < 1$. According to the demand schedules, the auctioneer determines the fraction of uninformed and informed managers who obtain the bond in order to guarantee that the bond market clears. Finally, investors fire their manager whenever they have an exogenous signal that he is uninformed and, if $p(\chi_{t+1}, y_t) < 1/R$, when their manager makes an unsuccessful investment.

In equilibrium, the values of $\Gamma^I$ and $p^*$ are jointly determined with the values of the expected utility of an employed uninformed manager, $W$, and the probability that an unemployed manager finds a job, $\mu$. In particular, $\Gamma^I$ must be such that the mass of employed informed managers stay constant after $\delta$ of them die and $\mu$ of the unemployed ones are hired, that is, it satisfies

$$\Gamma^I = \delta \Gamma^I + \mu (M^I - \delta \Gamma^I).$$

Due to free entry, uninformed managers looking for a job get zero ex ante expected utility, that is,

$$\mu W - \kappa = 0,$$

while the expected utility of an employed uninformed manager $W$ satisfies

$$W = \gamma R + \delta \omega \left[ \Gamma^I / \bar{y} + q (1 - \Gamma^I / \bar{y}) \right] W.$$
signal. When instead the price does not reveal any information, with probability $1 - \Gamma I/y$, uninformed managers are indifferent in each point in time between investing in the risk-free asset or in the risky bond, and hence their expected utility can be calculated as the value of always investing in the risk-free asset. Again, they will always gain $\gamma R$, but, in this case, they will get the continuation utility $W$ only if there is default, no exogenous signal, and no death.

Finally, the indifference condition for an uninformed manager who believes that there is going to be default with probability $q$ is given by

$$
(1 - q) \left( \frac{\gamma}{p^*} + \delta \omega W \right) = \gamma R + q \delta \omega W, \tag{8}
$$

This condition is the analogous to condition (1) in the example in Section 2. The left-hand side of equation (8) represents the expected payoff of a manager who invests in the risky bond. With probability $1 - q$, there is no default and the manager gets a return $\gamma/p^*$. If he survives and there is no exogenous signal, he is not fired and gets expected continuation utility $W$. If instead there is default, the manager gets zero revenues, is fired, and gets 0 continuation utility, due to free entry. Similarly, the right-hand side of equation (8) represents the expected payoff of a manager who invests in the risk-free bond. He gets always a return $\gamma R$, but only if he survives, there is default, and there is no exogenous signal, he is not fired and gets expected continuation utility $W$. Otherwise, the investor learns that he was not informed and fires him.

The system of equations (5)-(8) determine the equilibrium values of $I$, $p^*$, $W$, and $\mu$ for a given $q$, so defining a map $P(\cdot)$ from $q$ to $p^*$.

The following assumptions are sufficient to ensure that such a stationary interior equilibrium exists:

$$
\frac{M^I + \bar{y}}{b} < P(q) < \min \left\{ \frac{\Gamma - M^I}{b}, \frac{1}{R} \right\}, \tag{A1}
$$

and

$$
\omega < \frac{1}{1 + \delta}. \tag{A2}
$$

Assumption A1 guarantees that there are always some uninformed managers investing in both the risky bond and the risk-less asset. Assumption A2 ensures that the proportion of informed managers among those who are searching for a job is sufficiently small that if an uninformed manager does not make a mistake, he is not fired. This last assumption is not crucial, but it makes the analysis simpler.

**Proposition 1** Suppose A1 and A2 hold. Then an interior equilibrium exists.
3.4 Limit Equilibrium

Let us now focus on the model with $M^I \to 0$. This limit case is very tractable and insightful at the same time. We show that as $M^I \to 0$, the sequence of interior equilibria converges to a limit interior equilibrium where the bond price never reveals any information, and is constant over time. Intuitively, this can be the case because as the fraction of informed manager is infinitesimal, the uninformed managers will have to demand all the bonds supplied and hence won’t learn any information from the equilibrium price.

**Definition 3** When $M^I \to 0$, a limit interior equilibrium is an equilibrium where the price $p^* \in (\bar{y}/b, 1/R)$ is determined by the indifference condition of the uninformed managers; the informed managers’ demand schedule is $d^I(\chi_{t+1}) = 1 - \chi_{t+1}$ and their bond allocation is $x^I(\chi_{t+1}) = d^I(\chi_{t+1})$; the uninformed managers’ demand schedule is $d^U = \{0, 1\}$ and their bond allocation is $x^U(y_t) = (p^*b - y_t)/\Gamma$; the investors’ strategy is to fire manager $j$ if he receives a negative exogenous signal or if $\theta^I_t \neq 1 - \chi(a_{t+1})$, and keep him otherwise.

In a limit interior equilibrium, career concerns always affect the bond price by generating a reputational premium. The equilibrium behaves in a similar way to the general one, in the case in which the price does not reveal any information. Investors fire the managers who reveal to be uninformed, while informed managers never make a mistake and are never fired. The market clearing condition for the risky bond determines the fraction $x^U(y_t)$ of uninformed managers who invest in the risky bond. Assumption A1 guarantees that $x^U(y_t) \in (0, 1)$ for any $y_t$, so that there are always some uninformed managers investing in the risky bond and some investing in the risk-free asset. Hence, it must be that the uninformed managers are indifferent between the two investment possibilities. For a given default probability $q$, the equilibrium price $p^*$ is determined by the same indifference condition (8), where the expected continuation utility of an employed uninformed manager $W$ satisfies condition (7) with $\Gamma^I \to 0$, which yields

$$W = \frac{\gamma R}{1 - \delta \omega q}.$$  \hspace{1cm} (9)

Given that in a limit equilibrium managers are always indifferent between investing in the risk-free asset and in the risky bond, their expected utility can be calculated as the value of always investing in the risk-free asset. Combining (8) and (9) implies that the limit equilibrium is characterized by

$$p^* = P^L(q) \equiv \frac{1 - q}{R} \left[ \frac{1 - \delta \omega q}{1 - \delta \omega (1 - q)} \right].$$
Similarly to section 2, let the *reputational premium* $\Pi$ be the difference between the expected repayment and the risk free rate $R$, that is,

$$
\Pi = \frac{1-q}{p} - R.
$$

This premium characterizes the price distortion generated by the career concerns of the uninformed managers.

As a point of comparison, consider a model with $M^I = 0$. In this case, all managers are uninformed, so investors will be indifferent between keeping the manager they started with and hiring a new one. Then, there exists an equilibrium where managers are never fired and maximize their period by period profit. In this case, the bond price is determined by the standard no-arbitrage condition

$$
\frac{1-q}{p} = R.
$$

We call this equilibrium the *benchmark equilibrium*.

In the benchmark equilibrium, the standard arbitrage condition (11) immediately implies that $\Pi$ is equal to zero. When instead there is a positive measure of informed managers, $M^I > 0$, the reputational premium can be negative or positive. Typically, it is positive when $q$ is sufficiently large and negative when $q$ is sufficiently small. Betting on large probability events is especially attractive for an uninformed manager with career concerns, because it increases the chance that he will not make an unsuccessful decision and will not be fired. The equilibrium price reflects this preference for large probability events. Fund managers are willing to get a lower expected return in exchange for a large probability of not being fired. It is interesting to point out that there is a discontinuity at $M^I = 0$, given that the benchmark equilibrium disappears as $M^I > 0$, even if $M^I \to 0$.

In the benchmark equilibrium the pricing rule is given by the standard no-arbitrage condition (11), that is, $p^B = P^B(q) = (1-q)/R$. Notice that the reputational premium is zero iff $q = 1/2$ and both $P^L(q)$ and $P^H(q)$ are decreasing in $q$. This implies that $\Pi > 0$ iff $q > 1/2$ and $\Pi < 0$ otherwise.

Proposition 1 guarantees that under assumptions A1 and A2, a limit interior equilibrium exists. The equilibrium regime is determined jointly by the fundamentals of the risky project and the state of the financial market. Figure 2 represents graphically both the limit equilibrium for an economy with career concerns (L) and the benchmark equilibrium (B).
In this numerical example we assume $q < 1/2$ so that $P_L(q) > P_B(q)$ and the reputational premium is positive. It is immediate to see that if $q > 1/2$ then $P_L(q) < P_B(q)$ and the reputational premium is negative. The figure shows that in a limit interior equilibrium prices react more to changes in the fundamentals of the economy, both on the supply and on the demand side. This is simply due to the fact that the pricing rule $P_L(q)$ is steeper than the benchmark pricing rule $P_B(q)$ around the equilibrium. For example if there is an exogenous shock either to the default probability $q$ or to the return of the safe asset $R$, it is easy to see that prices react more in a limit equilibrium than in the benchmark equilibrium. This shows the essence of the amplification result that we will discuss in more detail in Section 4.4, after introducing the extended model with endogenous supply.

4 Endogenous Bond Supply

In this section, we extend the model in order to endogenize the supply of risky bonds and their default probability. In particular, we introduce entrepreneurs who issue the risky bond
described so far in order to finance a risky project they have access to. We assume that they can decide to default ex post on their loans at a given cost. In our baseline model we have established that the equilibrium bond price is affected by the default probability. This extended model generates a feedback effect from the bond price to the borrowing and default choice. We will see that this feedback effect magnifies the amplification generated by the presence of career concerns.

4.1 Entrepreneurs

The set up of the economy is identical to the one presented in Section 3.1 except that now $b$ and $q$ are endogenous objects. There are overlapping generations of entrepreneurs who live for two periods. A generation is represented by a continuum of measure 1 of entrepreneurs. In each period a new generation is born. Consider an entrepreneur born at time $t$. When she is young, she can choose to pay a cost $k > 0$ to invest in a risky project with return $a_{t+1}$, distributed according to the cumulative distribution function $F(a_{t+1})$ with $a_{t+1} \in [0, \infty)$, or to enjoy an outside option that gives her utility $V$. Assume that $k \geq \tilde{y}$ to ensure that the supply of the risky bond is always big enough to cover the demand of the noise traders. If she decides to undertake the risky project, she can borrow by issuing one-period discount bonds. Define $p_t$ the price of the bonds issued at time $t$. The entrepreneur chooses how much to borrow and how much to consume, taking $p_t$ as given. When she is young, her budget constraint is

$$c_t + k \leq p_t b_{t+1},$$

where $c_t$ represents consumption at time $t$ and $b_{t+1}$ represents the one-period discount bonds issued at time $t$. There is an upper bound $\bar{b}$ on how much entrepreneurs can borrow.

When she is old, she collects the project pay-off $a_t$ and has the option to default on her debt $b_{t+1}$. If she defaults she does not repay the debt, but she suffers an output loss of $(1 - \theta) a_t$, that is, she keeps only $\theta a_t$ of the return on the project. If she does not default, she has to repay her debt and consume the rest. Her budget constraint when old is

$$c_{t+1} \leq a_{t+1} - (1 - \chi_{t+1}) b_{t+1} - \chi_{t+1} (1 - \theta) a_{t+1},$$

where $\chi_{t+1} : [p, 1/R] \times \mathbb{R}_+ \mapsto \{0, 1\}$ denotes the default decision that the agent is making at time $t + 1$ taking as given the price $p_t$ and after observing the realization of $a_{t+1}$. Hence, the default probability $q$ which was exogenous in the previous section is now endogenous with $q_t \equiv E_t[\chi_{t+1}]$. 18
The problem for an active entrepreneur born at $t$ is to maximize her utility

$$u(c_t) + \beta \mathbb{E} [v(c_{t+1}) | p_t],$$

subject to (12) and (13), taking $p_t$ as given. We assume that $u(\cdot)$ and $v(\cdot)$ are increasing and strictly concave and have continuous first and second derivative. Moreover,

$$-cu''(c) / u'(c) \geq 1. \quad \text{(A3)}$$

The problem can be rewritten as

$$V(p_t) = \max_{b_{t+1} \leq a_{t+1}} u(a_{t+1} - k) + \beta E[v\{a_{t+1} - (1 - \chi_{t+1}) b_{t+1} - \chi_{t+1} (1 - \theta) a_{t+1}\} | p_t] \quad \text{(14)}$$

Ex ante, an entrepreneur will choose to undertake the risky project if and only if $V(p_t) \geq V$. We denote the aggregate supply of bonds issued by entrepreneurs at a given price by $B(p_t)$, which corresponds to $b$ in the previous section.

In order to have a well-behaved problem we also need to make the following assumption.

**Assumption A4.** For any $p \in (\underline{p}, 1/R)$, assume that the function

$$\Phi(p, b) \equiv u(pb - k) + \beta \int_0^{b \chi_{t+1}} v(\theta a) dF(a) + \beta \int_{b \chi_{t+1}}^{\infty} v(a - b) dF(a)$$

is quasi-concave in $b$ for $b \in [0, \bar{b}]$ and there exists an optimum $V(p) = \max_{b \in [0, \bar{b}]} \Phi(p, b)$.

Assumption A5 is satisfied for many different parametric assumptions. In Appendix B we show that it is satisfied for the example illustrated in Figure 3. Intuitively, it rules out the possibility that the marginal cost of default is not large enough compared to the advantage of additional borrowing. In that case, the entrepreneur would always like to borrow more and default more often, so that problem (14) could not have a finite solution.

### 4.2 Interior Equilibrium

In this section we characterize the natural extension of a stationary interior equilibrium for our baseline model as defined in Definition 2. In the extended model, informed managers have superior information about the project productivity $a_{t+1}$, from which they can infer the entrepreneurs’ default decision $\chi_{t+1} = \chi(p_t, a_{t+1})$. Again, there are going to be three regimes, two with full revelation with default and no default respectively, and one where no information is revealed. The last regime is the more interesting one, where uninformed managers are the
marginal traders, that is, are indifferent between investing in the bond and in the risk-free asset. A sufficient assumption to ensure that prices are not always fully revealing is now $\tilde{y} \geq 2M^I$. In the extended model, the equilibrium prices, the managers’ demand schedules, the bond allocation, and the investors’ firing rule are the same as in conditions (i)-(iv) in Definition 2, with the caveat that borrowing and default are now endogenous, that is, $b = B(p_t)$ and $\chi_{t+1} = \chi(p_t, a_{t+1})$, and $\bar{p}$ substitutes $y_t/b$ for any $y_t$, where $\bar{p}$ is such that $V(\bar{p}) = \overline{V}$.

Moreover, now the entrepreneurs’ borrowing and default decisions are also equilibrium objects. In particular, the equilibrium borrowing strategy is

$$b(p_t, y_t) = \begin{cases} \bar{b} \text{ with pr. } y_t/(p\bar{b}) \text{ and } 0 \text{ otherwise} & \text{if } p_t = \bar{p} \\ b^* & \text{if } p_t = p^* \\ \bar{b} & \text{if } p_t = 1/R \end{cases} \tag{15}$$

where $b^* > \bar{b}$ in $[0, \bar{b}]$, and the equilibrium default rule is $\chi(p_t, a_{t+1}) = 1$ if and only if $a_{t+1} < \hat{a}(p_t, y_t)$, where $\hat{a}(p_t, y_t) = b(p_t, y_t) / (1 - \theta)$.

An interior equilibrium for the extended model is characterized by equilibrium values for $\Gamma^I$, $p$, $p^*$, $b^*$, $\bar{b}$, which ensure that the adapted versions of conditions (i)-(iv) in Definition 2, condition 15, and the default rule $\hat{a}(p_t, y_t) = b(p_t, y_t) / (1 - \theta)$ characterize a stationary equilibrium.

Entrepreneurs do not have superior information about $a_{t+1}$ and also learn information from the equilibrium price only when it is fully revealing. If $p_t = p$, entrepreneurs know that all managers think that there is going to be default and that they are going to invest in the risk-free asset. Hence, they do not have any incentive not to default and if they are able to issue some bonds, they choose to borrow as much as they can, $\bar{b}$, and always to default. However, in this case, the demand for bonds comes only from noise traders and is equal to $y_t$. Hence, in order to have an equilibrium it must be that the entrepreneurs are indifferent between borrowing $\bar{b}$ and defaulting with probability 1 and get their outside option $\overline{V}$, that is, $\bar{p}$ must be exactly such that $V(\bar{p}) = \overline{V}$. At that price a fraction $y_t/(p\bar{b})$ of entrepreneurs becomes active and borrows from noise traders, while the others stay out of the market and get their outside option. If $p_t = p^*$, entrepreneurs do not have any additional information on $a_{t+1}$ and choose $b^*$ to maximize their expected utility. Finally, if $p_t = 1/R$, they know that all the managers believe that there is not going to be default and choose to borrow $\bar{b}$ and default whenever $a_{t+1} < \hat{b}/(1 - \theta)$, which is never the case. Hence, in equilibrium it must be that $\bar{b} < b^*$. 
The equilibrium values of $\Gamma^I$ and $p^*$ are characterized, for a given $q$, in a similar way to the baseline benchmark. The value of $\Gamma^I$ is jointly determined with $W$ and $\mu$ by equations (5)-(7), where (7) is modified to take into account that the probability of full revelation is now endogenous. Then, equation (8) defines a pricing map $P(\cdot)$ that gives a price for any given default probability $q$. Next, we can define a repayment map $Q(\cdot)$ which assigns a default probability to any given price $p$. This map is given by $Q(p) \equiv F(B(p) / (1 - \theta))$ where $B(p)$ is implicitly defined by the optimality condition of problem 14 in the case of no information revelation, that is,

$$ pu' (pB(p) - k) - \beta \int_{\frac{R(\mu)}{1 - \theta}}^{\infty} v'(a_{t+1} - B(p)) dF(a_{t+1}) = 0. \quad (16) $$

Hence, an interior equilibrium, in the regime of no revelation, is characterized by a price $p^*$ and a default probability $q^*$ that solve $p^* = P(q^*)$ and $q^* = Q(p^*)$.

In order to ensure that an interior equilibrium exists, we still need assumption A2 together with a modified version of A1, that is,

$$ M^I \bar{y} < pB(\bar{y}) \text{ and } \Gamma - M^I > \frac{1}{R} B \left( \frac{1}{R} \right), \quad (A1') $$

which ensures that there are always some uninformed managers investing in both the risky bond and the riskless asset. In the next proposition, we state sufficient conditions for the existence of a stationary equilibrium. Let us define

$$ q_1 = F \left( \frac{\bar{y}}{1 - \theta} \right) \text{ and } q_2 = F \left( \frac{\bar{y}}{1 - \theta} \right). $$

**Proposition 2** For a given $M^I > 0$, such that assumptions A1’, A2, A3, and A4 hold and

$$ P(q_1) < 1/R \text{ and } P(q_2) > p, \quad (17) $$

an interior equilibrium exists.

### 4.3 Limit Equilibrium

As in the baseline case, we focus on a limit interior stationary equilibrium with $M^I \to 0$, which is the natural extension of the equilibrium described in section 3.4.

**Definition 4** A limit interior equilibrium with $M^I \to 0$, is an equilibrium where $p^* \in (\bar{y}/b, 1/R)$ is determined by the indifference condition of the uninformed managers; the informed managers’ demand function is $d^I(x_{t+1}) = 1 - x_{t+1}$ and their bond allocation is $x^I(\chi_{t+1}) = d^I(\chi_{t+1})$.
the uninformed managers’ demand correspondence is \( d^U = \{0, 1\} \) and their bond allocation is \( x^U (y_t) = (p^* b - y_t) / \Gamma \); the investors’ strategy is to fire manager \( j \) if he receives a negative exogenous signal or \( \theta^j_t \neq 1 - \chi (a_{t+1}) \), and keep him otherwise; the borrowers’ strategy is \( b^* \in [0, \bar{b}] \) and \( \chi (p_t, a_{t+1}) = 0 \) if \( a_{t+1} \geq \hat{a}^* \) and \( \chi (p_t, a_{t+1}) = 1 \) otherwise, with \( \hat{a}^* = b^* / (1 - \theta) \).

As in the baseline model, the limit equilibrium behaves in a similar way to an interior equilibrium with \( M^I > 0 \), in the case in which the price does not reveal any information. In particular, it can be characterized by the constant bond price and default probability, \( p^* \) and \( q^* \), which solve the fixed point problem defined by \( p^* = P^L (q^*) \) and \( q^* = Q (p^*) \). On the one hand, the pricing rule \( P^L (\cdot) \) is exactly the same as in the baseline model, that is,

\[
P^L (q) = \frac{(1 - q) (1 - \delta q)}{[1 - \delta (1 - q)] R}.
\]

On the other hand the repayment rule is the same as described in the general model for \( M^I > 0 \), that is, \( Q (p) = F (B (p) / (1 - \theta)) \) with \( B (p) \) solving condition (16).

Next lemma establishes some important properties of entrepreneurs’ optimal behavior.\(^{12}\)

**Lemma 1** In a limit equilibrium, as \( p \) increases, (i) the face value of debt \( B (p) \) decreases, (ii) the probability of default \( Q (p) \) decreases, and (iii) the value of the bonds \( p B (p) \) increases.

As borrowing becomes cheaper, entrepreneurs need to borrow less and, hence, there is higher chance that they can repay their debt. This implies that the probability of default decreases as a function of the bond price, that is, \( q \) is downward sloping in the space \((p, q)\) as shown in Figure 3. Moreover, thanks to assumption A3, the value of the borrowing \( p b \) increases, because entrepreneurs want to smooth consumption between the two periods of their life and, hence, they decrease \( b \) less than proportionally with respect to the initial increase of \( p \).

Once again, as a point of comparison, consider the benchmark equilibrium, where entrepreneurs behave exactly as described above, but all managers are uninformed, \( M^I = 0 \), investors never fire any manager, and managers maximize their period by period profit. Such an equilibrium can also be characterized by a fixed point \((\hat{a}^B, p^B)\), where \( q^B = Q (p^B) \), but the pricing rule is given by the standard no-arbitrage condition (11), that is, \( p^B = P^B (q^B) \equiv (1 - q^B) / R \).

Proposition 2 guarantees that, under assumptions A1-A4, a limit equilibrium exists. The equilibrium regime is determined jointly by the fundamentals of the risky project and the

\(^{12}\)Note that the same Lemma applies to the general equilibrium when the economy is in a not revealing regime.
state of the financial market. Figure 3 represents graphically both the limit equilibrium for an economy with career concerns (L) and the benchmark equilibrium (B). The prices in the two equilibria, respectively, $p^*$ and $p^B$, correspond to the intersections of the repayment rule $Q(p)$ and the corresponding pricing rule, that is, $P^L(q)$ and $P^B(q)$, graphed in the space $(p, q)$.

Figure 3: The red line represents the repayment rule and the blue curve and the green curve represent the pricing rule in the economy with career concerns and in the benchmark economy, respectively. Points L and B denote respectively the limit interior equilibrium when $M^I \to 0$ and the benchmark equilibrium when $M^I = 0$.

Notice that the premium is zero iff $q^* = q^B = 1/2$ and $p^* = p^B = 1/2R$. Moreover, both $P^L(q)$ and $P^B(q)$ are decreasing in $q$. This proves the following proposition.

**Proposition 3** In equilibrium, one of the following regimes arises: (i) if $q^* = 1/2$, then $\Pi^* = 0$, (ii) if $q^* < 1/2$, then $\Pi^* < 0$; (iii) if $q^* > 1/2$, then $\Pi^* > 0$.

In the baseline numerical exercise we assume that $u(c) = \log c$, $v(c) = c$ and $F(a) = 1 - a^\gamma a^{-\gamma}$. We work out this example in detail in Appendix B. In Figure 3, we choose $a = 1.5$, $\gamma = .6$, and $k = .45$. Such parameters implies that $q^* < 1/2$ and then $p^* > p^B$, that is, that the premium is positive. Changing the parameters, we can easily obtain the analogous figure.
where \( p^* < p^B \) and the premium is negative. For example, this is the case if we decrease the lower bound for the productivity process \( \varpi \) to 1.

### 4.4 Amplification

Next, we analyze some interesting properties of the limit equilibrium. In particular, we are interested in the reaction of the economy to shocks both to financial markets and to the fundamentals of the risky project. The first type of shocks affect the pricing rule and we refer to them as demand-side shocks; the second type affect the repayment rule and we label them supply-side shocks. Our main result is that there is an amplification effect that magnifies the reaction of our equilibrium to both types of shocks, in comparison to the benchmark model. The mechanism behind this result is that both types of shocks can move the economy from one regime to the other, generating a natural amplification in the price and in the default probability.

Let us focus on the vector of parameters \( \sigma = \{R, \alpha\} \), where \( R \) is the return on the risk-free asset and \( \alpha \) represents a parameter affecting the distribution of the productivity shock \( a \), such that if \( \alpha'' > \alpha' \), then \( F(a|\alpha'') < F(a|\alpha') \). A change in \( R \) represents a typical demand-side shock, that is, a change in the return of alternative investment opportunities. A change in \( \alpha \), instead, represents a first-order stochastic shift of the productivity distribution of the risky project, that is, a typical supply-side shock.

With a slight abuse of notation, let us denote by \((q^* (\sigma), p^* (\sigma))\) the default probability and price in the limit equilibrium when \( M^I \rightarrow 0 \) and the parameters are \( \sigma \), and \((q^B(\sigma), p^B(\sigma))\) the default probability and price in the benchmark equilibrium when \( M^I = 0 \) and the parameters are \( \sigma \). When there are multiple equilibria, let us focus on the equilibrium with the highest bond price.

Next proposition states our main amplification result.

**Proposition 4** Suppose there exists a pair \((\sigma', \sigma'')\) such that \( q^B(\sigma') < 1/2, q^B(\sigma'') > 1/2, \) and \( q^*(\sigma'') > 1/2 \). Then, there is amplification, that is, \( q^*(\sigma'') - q^*(\sigma') > q^B(\sigma'') - q^B(\sigma') \) and \( p^*(\sigma'') - p^*(\sigma') > p^B(\sigma'') - p^B(\sigma') \).

Proposition 4 shows that if there is a change in the parameters such that the equilibrium switches regime from a positive premium to a negative premium, then both prices and default probabilities respond more than in the benchmark model. Suppose, for example, we start from
a regime where the premium is negative. As the outside investment opportunities improve, that is, \( R \) increases, the bond price decreases making borrowing more expensive and default happening more often. If the shock is big enough, it can generates a shift in the sign of the premium and a switch of regime. Alternatively, the economy can move from a regime to another because of a change in the parameters on the supply-side of the model. For example, a big enough decrease in \( \alpha \) can increase the default probability enough to make the premium negative. The effect on both prices and quantities is amplified in comparison to the benchmark model.

Assume that \( \lim_{\alpha \to -\infty} F(a_t | \alpha) = 1 \). Next proposition shows that when there exists a unique interior equilibrium for a given set of parameters, it is possible to change \( R \) or \( \alpha \) enough that the regime shifts.

**Proposition 5** Suppose that \( \lim_{\alpha \to -\infty} F(a_t | \alpha) = 1 \) and that there exists a unique interior equilibrium with \( \sigma' = \{R', \alpha'\} \) such that \( Q(p^* | \alpha') < 1/2 \), where \( \alpha' \in [a, \bar{a}] \) for some \( a \) and \( \bar{a} \). Then,

1. if \( Q(p | \alpha') > 1/2 \), there is an \( \hat{R} > R' \) such that for any \( R'' > \hat{R} \), \( Q(p^* | \alpha') > 1/2 \);

2. there is an \( \hat{\alpha} \) with \( \underline{\alpha} < \hat{\alpha} < \alpha' \) such that for any \( \alpha'' \) with \( \underline{\alpha} < \alpha'' < \hat{\alpha} \), \( Q(p^* | \alpha'') > 1/2 \).

In the example illustrated in figure 3 and worked out in Appendix B, there is a unique equilibrium.\(^{13}\)

Propositions 4 and 5 show that as the financial environment or the fundamentals of the risky project change, the economy can switch from a regime with low bond spreads (high \( p \)) and high level of capital invested in the risky bond market (high \( pb \)) to a regime with high bond spreads (low \( p \)) and low level of capital invested (low \( pb \)). The first type of regimes are frequently described as regimes of **abundant liquidity** or with **traders reaching for yield**. To describe phenomena where the economy switch to the second type of regime, common terms are **flight-to-quality**, **flight-to-liquidity**, **disappeared liquidity**, or **drop in risk appetite**. In our model, phenomena of this type can arise even if fund managers are risk-neutral and their aggregate funds are constant. We argue that abrupt changes in prices can be caused by managers’ career concerns. In good times, when the default probability of credit instruments is low, it is very

\(^{13}\)In particular, it is possible to show that if \( u(c) = \log c \), \( v(c) = c \) and \( F(a) = 1 - a^\gamma \), there exists a unique equilibrium if \( k \) is small enough.
attractive for uninformed fund managers to invest in these instruments because if they prefer less risky investment opportunities, they are likely to produce lower returns, lose reputation, and, hence, funds. If suddenly a negative shock hits either the demand or the supply side of the market, the probability of default increases, and investing in the risky asset increases the probability of losing their reputation. Hence, prices increase not only because of the higher probability of default, but also because of an additional premium coming from career concerns. This generates the amplification result we have discussed.

Our main result is that the impact of shocks can be amplified by the sign change in the premium, leading to excess volatility of the bond price, default probability, and capital flows. This is consistent with the empirical evidence that shows that emerging market bond prices fluctuate more than what is accounted for by changes in probability of default. On the one hand, Broner, Lorenzoni and Schmukler (2007) argue that the premium over the expected repayment on emerging market bonds is especially high during crises times. On the other hand, Duffie et al. (2003) document that the implied short spread of Russian bonds was very low during the first 10 months of 1997. Moreover, their estimation shows that in one short interval in 1997, bond prices were so high that the implied default adjusted short spread was negative. Although this observation is model specific, it is still interesting to point out that this is consistent with our model, while inconsistent with most risk-aversion based explanation. Note also that the result that demand-side shocks can be important determinants of bond prices is broadly consistent with the empirical evidence that a large proportion of the variation in prices of both corporate bonds and emerging market bonds cannot be explained by the variation of fundamentals, and that a large part of this unexplained component is common across bonds (see Collin-Dufresne at al., 2000, Gruber et al., 2001, Westphalen, 2001).

5 Persistent Productivity Shock

In this section, we introduce persistency in the productivity process of the risky project. In particular, assume that \( a_{t+1} \) is distributed according to a first-order Markov process with cumulative density function \( G(a_{t+1}|a_t) \). The environment is a natural generalization of the one with \( i.i.d. \) shock, where \( a_t \) represents an additional state variable. We look for Markovian equilibria.
5.1 Equilibrium characterization

The equilibrium we focus on is a natural generalization of the interior limit equilibrium described in definition 3.

Definition 5 A Markovian interior equilibrium with $M^I \to 0$ is an equilibrium where $p^* (a_t) \in (0, 1/R)$ is determined by the indifference condition of the uninformed managers; the informed managers’ bond allocation is $x^I (y_t, a_{t+1}) = 1 - \chi (y_t, a_{t+1})$; the uninformed managers’ allocation is $x^U (y_t, a_t) = (p^* (a_t) b^* (a_t) - y_t) / \Gamma$; the investors’ strategy is to fire manager $j$ if he receives a negative exogenous signal or $\theta^j_t \neq 1 - \chi (y_t, a_{t+1})$, and keep him otherwise; the borrowers’ strategy is $b^* (a_t) > 0$ and $\chi (y_t, a_{t+1}) = 0$ if either $a_{t+1} \geq \hat{a}^* (a_t)$, and $\chi (y_t, a_{t+1}) = 1$, otherwise.

When the process for $a_t$ is not i.i.d., the expected utility of the uninformed managers at time $t$ depends on the realization of $a_t$, because they can use the past information to update the distribution of $a_{t+1}$, that is,

$$W (a_t) = \gamma R + \delta \int_0^{\hat{a} (a_t)} W (a_{t+1}) dF (a_{t+1} | a_t).$$

Moreover, their indifference condition becomes

$$(1 - q (a_t)) \frac{\gamma}{p (a_t)} + \delta \int_{\hat{a} (a_t)}^{\infty} W (a_{t+1}) dF (a_{t+1} | a_t) = \gamma R + \delta \int_0^{\hat{a} (a_t)} W (a_{t+1}) dF (a_{t+1} | a_t).$$

This condition implicitly defines the equilibrium price as a function of the state $a_t$, $P (a_t, q (a_t)) = p (a_t)$, and the default rule $\hat{a} (a_t)$ where $\hat{a} (a_t) = F^{-1} (q (a_t))$.

Also the borrowers update their expectation of the distribution of $a_{t+1}$, conditional on $a_t$. Their default rule is $Q (a_t, p (a_t)) = F (B (a_t, p (a_t)) / (1 - \theta) | a_t)$, where $B (a_t, p (a_t)) = b (a_t)$ is implicitly defined by

$$p (a_t) u' (p (a_t) b (a_t) - k) - \beta \int_{b (a_t)}^{\infty} v' (a_{t+1} - b (a_t)) dF (a_{t+1} | a_t) = 0.$$ 

Hence, a Markovian interior equilibrium is characterized by a fixed point such that $p^* (a_t) = P (a_t, q^* (a_t))$ and $q^* (a_t) = Q (a_t, p^* (a_t))$.

When $M^I = 0$ and the shock $a_t$ is persistent, the benchmark equilibrium is defined as a fixed point such that $q^B (a_t) = Q (a_t, p^B (a_t))$ and

$$p^B (a_t) = \frac{1 - q^B (a_t)}{R}.$$
5.2 Numerical example

Here we present some numerical exercises to illustrate the dynamic properties of our equilibrium when productivity shocks are persistent. In particular, we show how career concerns can magnify the reaction of the economy to shocks, hence, increasing the volatility of prices.

First, we show how the default probability, the bond price, the amount of capital borrowed by entrepreneurs, and the premium vary with the realization of the productivity shock. Let us start with the equilibrium behavior in the benchmark economy. As a bad shock hits, the financial market will realize that, even for a given default rule, the probability of default will be higher and will require a lower bond price. As borrowing becomes more expensive, borrowers will then increase their default cut-off, magnifying the reduction in the bond price. A lower bond price also decreases the amount of capital entrepreneurs will borrow, so capital flows out from the market of risky bonds. Hence, for low realizations of productivity, the default cut-off will be higher and the bond price and the dollar value of outstanding bonds lower. However, the change in the bond prices is limited by the fact that the expected pay-off from holding the bonds will remain constant. Now, consider the economy with career concerns. Suppose the default probability is high enough that the premium is positive. In this case, the financial market will require a bond price even lower than the benchmark economy because of the premium. Given that productivity is persistent, a bad realization of the shock will further increase the probability of default, increasing the fear of the uninformed managers of being fired and pushing the bond price further down. This implies that the premium itself is higher after bad shocks. Moreover, if the economy starts from a regime where the premium is negative, a bad shock not only increases the premium, but can even make it switch sign. Thus, the effect of the productivity shock on the bond price, the probability of default and the capital flows is amplified by the career concerns of managers.

We report the numerical results for an example similar to the one illustrated in Figure 3 with \( u(c) = \log c \) and \( v(c) = c \). However, now \( \log(a_t) \) follows an AR(1) process.\(^{14}\) Figures 4 and 5 show how the premium, the bond price, and the default probability vary in equilibrium with the different realizations of the productivity shocks.

Now, consider an economy that at time zero is hit by a shock. Figure 6 shows how the

\(^{14}\)Figures (4), (5) and (6) use the following process for \( a_t \): \( \log(a_{t+1}) = (1 - \rho) \mu + \rho \log(a_t) + \epsilon_t \) with \( \rho = .7 \), \( \mu = 2.8 \) and \( \epsilon_t \sim N(0, 2) \). Moreover, \( \beta = .75 \), \( \delta = .5 \), \( \gamma = 1 \), \( k = .4 \), and \( \theta = .1 \).
Figure 4: The figure shows the reputational premium as a function of the realization of log(a). The blue line is the premium with career concerns and the green line shows the premium in the benchmark case.

equilibrium prices react in expected terms to a bad and to a good shock, both with and without career concerns. The figure shows our amplification result: the economy with career concerns reacts more to the shocks than the benchmark economy. Moreover, notice that in the economy considered, the premium would be negative in expected terms and a bad shock can actually make the economy shift regime.

6 Conclusion

In this paper, we have proposed a general equilibrium model of delegated portfolio management where career concerns distort asset prices. Investors hire fund managers to invest their capital either in a risky bond or in a riskless asset and a small fraction of managers have superior information on the default risk. Looking at their performance, investors update their beliefs on the information of fund managers and, based on that, they make firing decisions. This leads to career concerns that affect the managers' investment decisions, generating a “reputational premium”. When the probability of default is sufficiently high, fund managers prefer to invest in safe bonds even at a lower expected return to reduce the probability of being fired. On the contrary, if the probability of default is low enough, investing in the risky bond has a
advantage. The premium can switch sign in response to shocks, both to the financial market and to the fundamentals of the risky project that requires financing. This can generate an overreaction of the market leading to excess volatility of spreads and capital flows.

For future research, it would be interesting to introduce alternative risky assets in the portfolio choice of the managers. In this case, our mechanism would generate contagion. Imagine that there are two risky bonds and a risk-less asset. The cost of investing in the risk-less asset depends on the default probability of both the risky bonds. If none of them defaults, the manager who invests in the risk-less asset looses his reputation. Thus, if the probability of default of any of the risky bonds decreases, the risk-less asset will be less attractive, and the prices of both bonds will have to increase in order to make uninformed managers indifferent between different investment opportunities.

Finally, an interesting application of our model is to emerging market bonds. A large literature on business cycle characteristics of emerging markets \(^{15}\) highlights that emerging market bond spreads are very volatile. In particular, the magnitude of volatility of interest

Figure 6: The two panels show the reaction of the equilibrium prices to a bad and a good shock, respectively. The blue line represents the price in the benchmark economy, and the green line the price in the economy with career concerns. At time zero productivity drops to the lowest possible realization in the first case and rises to the highest possible one in the second case.

rates is hard to reconcile with models where bond prices are determined by the standard no-arbitrage condition. Our model provides an appealing framework to think about this excess volatility. It would be interesting to calibrate our model to quantify how much of the volatility of specific emerging markets bonds can be explained with our mechanism.

Appendix A

Proof of Proposition 1

Here we show that there exists a unique interior stationary equilibrium, as characterized in definition 2, where $\Gamma^I$, $p^*$, $W$, and $\mu$ solve the system of equations (5)-(8). The proof proceeds in four steps. We first show that the managers’ flows are consistent with the labor market for fund managers and that there exist unique values for $\Gamma^I$, $W$, and $\mu$ that solve (5)-(7); second we show that the demand schedules chosen by the managers, (ii) in definition 2, are optimal; third we prove that the bond’s allocation, (iii), ensures that the bond’s market clears and that
there exists a unique \( p^* \) that satisfies (8); and finally we show that the firing decision of the investors, (iv), is optimal.

**Step 1.** First we show that, given conditions (i)-(iv) in definition 2, there exist three constant values \( \mu, \Gamma^I, \) and \( \Gamma^U \) consistent with free entry in the job market for fund managers. Thanks to free entry and the assumption that there is a large continuum of uninformed managers, condition (6) holds and an uninformed managers looking for a job gets zero ex ante utility. On the contrary, there is a constant mass \( M^I \) of informed managers, who are never fired in equilibrium, and get positive expected utility when searching for a job. It follows that they all search for a job at any \( t \), and hence \( N^I_t = M^I - \delta \Gamma^I_t \), where \( \Gamma^I \) satisfies condition (5). Conditions (5) and (6) together with the condition for \( W \) (7) determine unique values for \( \mu, \Gamma^I, \) and \( W \). By combining the three equations we obtain an equation in \( \mu \) only \( g(\mu) = 0 \), where

\[
g(\mu) = \frac{\kappa}{\mu} - \gamma R \left( 1 - \delta \omega \left[ q + \frac{M^I (1 - q)}{\bar{y} (\delta + \frac{1 - \delta}{\mu})} \right] \right)^{-1}.
\]

Notice that \( \lim_{\mu \to 0} g(\mu) = \infty \), \( \lim_{\mu \to 1} g(\mu) < 0 \) thanks to the assumption that \( \kappa < \gamma R \), and \( g'(\mu) < 0 \) by inspection. It immediately follows that there exists a unique \( \mu \in (0, 1) \) such that \( g(\mu) = 0 \). Given \( \mu \), one can use equation (5) to solve for a unique \( \Gamma^I < M^I \), and equation (7) to solve for a unique \( W \). Finally, \( \Gamma^U \) must be such that the mass of employed uninformed managers stay constant after \( 1 - \delta \) of them die, \( 1 - \omega \xi_t \) of them are fired and \( \mu \) of the unemployed uninformed are hired, that is,

\[
\Gamma^U = \delta \Gamma^U + (1 - \omega \xi_t) \delta \Gamma^U + \mu N^U_t,
\]

where \( \xi_t \) denotes the proportion of uninformed managers who are successful, that is,

\[
\xi_t = \begin{cases} 
(1 - x^U_t) \chi_{t+1} + x^U_t (1 - \chi_{t+1}) & \text{if } z_t \in [\Gamma^U, \bar{y}] \\
1 & \text{if } z_t \notin [\Gamma^U, \bar{y}]
\end{cases}
\]

Hence, conditions (18) and (19) together with \( \Gamma^U = \Gamma - \Gamma^I \) determine the value of \( N^U_t \) for any \( t \), changing over time together with \( \xi_t \), that guarantees that \( \Gamma^U \) is uniquely determined and the equilibrium is stationary.

**Step 2.** Next, we show that the managers’ demand schedules characterized in condition (ii) of definition 2 are optimal, taking as given conditions (i), (ii) and (iv). First, notice that at any \( t \) there are three possible regimes. Recall that \( y_t \) is uniformly distributed on \([0, \bar{y}]\), and
hence $z_t$ must be in $[0, \bar{y} + \Gamma']$. From the equilibrium pricing condition (i), if $p_t \leq \bar{y}/b$, then $z_t \in [0, \Gamma')$ and the uninformed managers learn that $\chi_{t+1} = 1$. If, instead, $p_t = 1/R$, then $z_t \in (\bar{y}, \bar{y} + \Gamma']$ and they learn that $\chi_{t+1} = 0$. In both these cases, the price is fully revealing. When instead $p_t = p^*$, then $z_t \in [\Gamma', \bar{y}]$ and the uninformed managers update their beliefs about the probability of default as follows:

$$\Pr(\chi_{t+1} = 1|p_t = p^*) = \frac{\Pr(\chi_{t+1} = 1, z_t \in [\Gamma', \bar{y}])}{\Pr(\chi_{t+1} = 1, z_t \in [\Gamma', \bar{y}]) + \Pr(\chi_{t+1} = 0, z_t \in [\Gamma', \bar{y}])}.$$  

Notice that the assumption that $\bar{y} \geq \Gamma'$ ensures that this regime exists, that is, that $z_t$ can be in $[\Gamma', \bar{y}]$. This happens in two cases: when $\chi_{t+1} = 1$ and $y_t \in [\Gamma', \bar{y}]$ and when $\chi_{t+1} = 0$ and $y_t \in [0, \bar{y} - \Gamma']$. Given that $y_t$ is uniformly distributed, the first case arises with probability $q(\bar{y} - \Gamma')/\bar{y}$ and the second with probability $(1 - q)(\bar{y} - \Gamma')/\bar{y}$. It follows that $\Pr[\chi_{t+1} = 1|p = p^*] = q$. This shows that, thanks to the uniform distribution of $y_t$, the price $p^*$ does not reveal any information.

When $p_t = 1/R$ and there is full revelation of no default, the risky bond pays for sure $1/R$ and is equivalent to the risk-free asset. Hence, both the informed and the uninformed managers are indifferent between the two bonds and $d^I(p_t, \chi_{t+1}) = d^U(p_t) = \{0, 1\}$ if $p_t = 1/R$. When instead $p_t < 1/R$, informed managers will always choose to invest in the risk-free asset if there is going to be default, and in the bond if there is not going to be default, that is, $d^I(p_t, \chi_{t+1}) = 1 - \chi_{t+1}$. This is optimal for them given that investors fire managers who do not do that. Also, they do not have any incentive to deviate, given that the bond price is smaller than $1/R$. However, the uninformed managers cannot follow the same strategy because they do not know $\chi_{t+1}$. When $p_t \leq \bar{y}/b$, there is full revelation and they learn that there is going to be default. Hence, it is optimal to demand $d^U(p_t) = 0$ to avoid to be fired. If, instead, $p_t = p^*$, there is no information revelation and they are indifferent between the two assets, that is, $d^U(p_t) = \{0, 1\}.$

**Step 3.** Given the equilibrium prices (i) and the equilibrium conditions (ii) and (iii) in Definition 2, it is easy to show that the bond allocation defined in (iii) is consistent with the managers’ demand and ensures that the bond market clears. If $z_t \in [0, \Gamma')$, all managers demand the riskless asset and the auctioneer must allocate risky bonds only to the noise traders $y_t$. Given that in this case the price is equal to $y_t/b$, the market clearing condition is automatically satisfied. If $z_t \in (\bar{y}, \bar{y} + \Gamma']$, all managers are indifferent between the two assets,
the price is \(1/R\) and the auctioneer can randomly allocate the risky bond to the managers. In particular, the allocation \(x^I(\chi_{t+1}, y_t) = x^U(\chi_{t+1}, y_t) = (b/R - y_t)/\Gamma\) guarantees that the bond market clears. If instead \(z_t \in [\Gamma^I, \bar{y}]\), there is no information revelation. Given that the auctioneer’s allocation must be consistent with the managers’ demand schedule, the informed managers obtain the risky bond if and only if they demand it, that is, \(x^I(\chi_{t+1}, y_t) = 1 - \chi_{t+1}\), while the uninformed managers are indifferent between the two assets, and the allocation \(x^U(\chi_{t+1}, y_t) = (bp^* - z_t)/\Gamma^U\) guarantees that the market clearing condition is satisfied. By combining equations (7) and (8) we immediately obtain the not revealing price as a function of the default probability \(q\), that is, we obtain the following pricing rule:

\[
p^* = P(q) = \frac{(1 - q) \left(1 - \delta \left[\Gamma^I/\bar{y} + q \left(1 - \Gamma^I/\bar{y}\right)\right]\right)}{R \left[1 - \delta \left(1 - q\right) \left(1 + \Gamma^I/\bar{y}\right)\right]},
\]

where \(\Gamma^I\) is jointly determined by equations (5), (6), and (7). Given that in step 1 we have shown that there exists a unique triple \((\Gamma^I, W, \mu)\) that solve (5), (6), and (7), it is immediate to see that for any \(q\), there exists a unique \(p^*\) that satisfies equation (20). Moreover, assumption A1 guarantees that \(p^* \in (\bar{y}/b, 1/R)\) and ensures that \((bp^* - z_t)/\Gamma^U \in (0, 1)\), so that the allocation is internally consistent.

**Step 4.** Finally, we show that the investors’ firing rule (iv) in definition (2) is optimal, given conditions (i)-(iii). First, notice that the investors are indifferent between an informed and an uninformed manager when \(p_t = 1/R\) or \(p_t \leq \bar{y}/b\), and all managers choose the same demand schedule. However, when \(p_t = p^*\), the expected return of an informed manager is higher. Hence, investors would like to have informed managers investing their capital, and in order to make their firing decisions, they assess the probability that employed managers are informed. At the end of time \(t\), each investor observes the investment realization of his manager \(\theta^I_t\) and the default realization \(\chi_{t+1}\). Then if \(p_t < 1/R\) and either \(\chi_{t+1} = 0\) and \(\theta^I_t = 1\), or \(\chi_{t+1} = 1\) and \(\theta^I_t = 0\), he realizes that his manager is not informed, that is, \(\eta^I_{t+1} = 0\). In this case, the investor fires him, since there is always a positive probability that a new hire is informed, denoted by \(\varepsilon_t\), and the probability of finding a new manager is always equal to 1. By definition \(\varepsilon_t\) satisfies

\[
\varepsilon_t = \frac{M^I - \delta \Gamma^I}{M^I - \delta \Gamma^I + N^U_t} > 0.
\]

For the same reason, the investor also fires an uninformed manager whose type is revealed by an exogenous signal with probability \(1 - \omega\). On the other hand, if the manager does not make
a mistake, that is, if $\theta^j_t = \chi_{t+1}$ and/or $p_t = 1/R$, so that he does not reveal to be uninformed, then he is not fired if and only if his updated belief $\eta^j_{t+1}$ is higher than $\varepsilon_{t+1}$. When manager $j$ realizes $\theta^j_t = 1 - \chi_{t+1}$ and/or $p_t = 1/R$, the investor’s belief is updated as follows:

$$\eta^j_{t+1} = \frac{\eta^j_t}{\eta^j_t + \omega \xi_t (1 - \eta^j_t)}$$

where $\xi_t$, defined in (19), represents the proportion of uninformed managers who make the same investment decision of the informed managers. Next, we show that assumption A2 is sufficient to make sure that in equilibrium $\eta^j_{t+1} \geq \varepsilon_{t+1}$ for any $\xi_t$ and $\eta^j_t > 0$.

First, consider an investor who has just hired manager $j$ and hence, by definition, has prior belief $\eta^j_t = \varepsilon_t$. In this case, if $\theta^j_t = 1 - \chi_{t+1}$, then

$$\eta^j_{t+1} = \frac{\varepsilon_t}{\varepsilon_t + \omega \xi_t (1 - \varepsilon_t)}.$$  

Next, we want to show that $\eta^j_{t+1} \geq \varepsilon_{t+1}$. This condition can be rewritten as

$$\frac{1 - \varepsilon_{t+1}}{\varepsilon_{t+1}} \geq \left( \frac{1 - \varepsilon_t}{\varepsilon_t} \right) \omega \xi_t,$$  

(22)

Using the expression (21) for $\varepsilon_t$, we have that $(1 - \varepsilon_{t+1}) \varepsilon_{t+1} = N^U_t / (M^I - \delta \Gamma^I)$, and, hence, condition (22) can be rewritten as $N^U_t / N^U_{t-1} \geq \omega \xi_t$, where $N^U_t = (1 - \omega \xi_t \delta) \Gamma^U / \mu$. Hence, in order for (22) to be satisfied it must be that $1 - \delta \omega \xi_t \geq (1 - \delta \omega \xi_{t-1}) \omega \xi_t$, which is ensured by assumption A2.

Let us now consider managers who were working for an investor for longer than 1 period. First, notice that the investors’ beliefs about any manager who is still working but was hired at time $t - \tau$ with $\tau \in [0, t]$ must be higher than the initial belief $\varepsilon_{t-\tau}$, given that if he was not fired he never made any mistake, that is, $\eta^j_t \geq \varepsilon_{t-\tau}$. Hence, the belief about a manager who was hired at time $t - \tau$ and did not make a mistake at time $t$ is

$$\eta^j_{t+1} = \frac{\eta^j_t}{\eta^j_t + \omega \xi_t (1 - \eta^j_t)} \geq \frac{\varepsilon_{t-\tau}}{\varepsilon_{t-\tau} + \omega \xi_{t-\tau} (1 - \varepsilon_{t-\tau})}.$$  

It follows that a sufficient condition for this manager not being fired is

$$\frac{1 - \varepsilon_{t+1}}{\varepsilon_{t+1}} \geq \left( \frac{1 - \varepsilon_{t-\tau}}{\varepsilon_{t-\tau}} \right) \omega \xi_{t-\tau},$$  

which, by the same argument, is satisfied when assumption A2 holds. For the same reason, when $p_t = 1/R$ no manager is fired, given that there is no information in their action, completing the proof.
Proof of Proposition 2

Here we show that there exists a unique interior stationary equilibrium, as characterized in section 4.2. The proof proceeds in two steps: first we show that the proof of existence for the baseline model goes through for the extended model to show that managers investment decision and investors’ firing rule are optimal, and that the bond’s allocation ensures that the bond’s market clears, moreover we show that entrepreneurs decision are optimal as well; second we show that there exists a unique fixed point that determines the pair \((p^*, q^*)\).

**Step 1.** First, the equilibrium prices are identical to the ones in the baseline model, except for the price in the regime with full revelation of default, which is now \(p\) instead of \(\bar{y}/b\), where \(p\) is such that \(V(p) = \bar{V}\). The proof that the managers’ demand schedule and the investors’ firing rule are actually optimal, taking as given all the other equilibrium objects is analogous to the one for the baseline model above, with the only caveat that now \(a_{t+1} = \chi(a_{t+1}, y_t)\), \(b = b(p_t, y_t)\) and \(p\) substitutes \(\bar{y}/b\). With the same modifications, the bond allocation is the same of the baseline model and follows from the bond market clearing as we have shown above. The new part of the proof is to show that the borrowing and default decisions of the entrepreneurs are actually optimal, taking as given the other equilibrium objects. That is, that, in equilibrium, condition (15) is satisfied, with \(\bar{b} < b^*\) in \([0, \bar{b}]\), and \(\chi(p_t, a_{t+1}) = 1 \{a_{t+1} < \hat{a}(p_t, y_t)\}\), where \(\hat{a}(p_t, y_t) = b(p_t, y_t) / (1 - \theta)\).

Active entrepreneurs choose their default rule and how much to borrow and to consume in order to solve problem (14), taking \(p_t\) as given. Let us first consider the default decision of an old entrepreneur. For a given realization of the shock \(a_{t+1}\), she will default if and only if \(a_{t+1} - b_{t+1} < \theta a_{t+1}\). Then \(\chi(p_t, a_{t+1}) = 1\) if \(a_{t+1} \leq \hat{a}(p_t, y_t)\) and \(\chi(p_t, a_{t+1}) = 0\), otherwise, with

\[
\hat{a}(p_t, y_t) = \frac{b(p_t, y_t)}{1 - \theta}.
\]

(23)

Given that \(y_t\) is uniformly distributed between 0 and 1, the signal of the uninformed \(z_t = \Gamma_f (1 - \chi_{t+1}) + y_t\) must be in \([0, \bar{y} + \Gamma_f]\). If \(p_t = 1/R\), then \(z_t \in (\bar{y}, \bar{y} + \Gamma_f]\), and the uninformed managers learn that \(\chi_{t+1} = 0\). If instead \(p_t = \bar{p}\), then \(z_t \in [0, \Gamma_f]\), and they learn that \(\chi_{t+1} = 1\). If instead \(p_t = p^*\), as in the baseline model, there is no information revealed.

In the first case, \(p_t = 1/R\) and there is full revelation that \(\chi_{t+1} = 0\). Define \(\hat{a}^e\) the default rule expected by the informed managers. Given that the entrepreneurs learn that \(a_{t+1} \geq \hat{a}^e\),
in this case, problem (14) can be written as

\[ b\left(\frac{1}{R}\right) = \arg \max_b u\left(\frac{b}{R} - k\right) + \frac{\beta}{1 - F(\hat{a}^e)} \int_{\hat{a}^e}^{\min\{\frac{b}{1-\theta}, \hat{a}^e\}} v(\theta a_{t+1}) \, dF(a_{t+1}) \]

\[ + \frac{\beta}{1 - F(\hat{a}^e)} \int_{\min\{\frac{b}{1-\theta}, \hat{a}^e\}}^{\infty} v'(a_{t+1} - b) \, dF(a_{t+1}). \]

It follows that \( b(1/R) = \tilde{b} \) where \( g(\tilde{b}) = 0 \) with

\[ g(b) \equiv \frac{1}{R} u'\left(\frac{b}{R} - k\right) - \beta \int_{\hat{a}^e}^{\infty} v'(a_{t+1} - b) \, dF(a_{t+1}). \]

Hence, in order for this to be an equilibrium it must be that \( \hat{a}^e = \tilde{b} / (1 - \theta) \), so that the default rule is consistent with the managers’ expectations. Moreover, we have to check that if prices are not revealing, that is, \( p_t = p^* \), entrepreneurs would default for any \( a_{t+1} < \hat{a}^e \), that is, that \( b^* \geq \tilde{b} \), where \( b^* \) solves \( \tilde{g}(b^*) = 0 \), with

\[ \tilde{g}(b) \equiv p^* u'(p^* b - k) - \beta \int_{\hat{a}^e}^{\infty} v'(a_{t+1} - b) \, dF(a_{t+1}). \]

Hence, it is enough to show that \( \tilde{g}(\tilde{b}) > 0 \). This comes straight from assumption A3.

In the second case, \( p_t = p \) and there is full revelation that \( \chi_{t+1} = 1 \). Let again denote by \( \hat{a}^e \) the default rule expected by the managers. Then, it must be that \( a_{t+1} < \hat{a}^e \) and the entrepreneurs problem (14) can be written as

\[ b(p) = \arg \max_b u(p b - k) + \frac{\beta}{F(\hat{a}^e)} \int_{0}^{\min\{\frac{b}{1-\theta}, \hat{a}^e\}} v(\theta a_{t+1}) \, dF(a_{t+1}) \]

\[ + \frac{\beta}{F(\hat{a}^e)} \int_{\min\{\frac{b}{1-\theta}, \hat{a}^e\}}^{\hat{a}^e} v(a_{t+1} - b) \, dF(a_{t+1}). \]

In order to have an equilibrium it must be that \( \hat{a}^e = \tilde{b} / (1 - \theta) \) and hence it is immediate that \( b(p) = \tilde{b} \). It follows that if prices are not revealing entrepreneurs would not default for any \( a_{t+1} \geq \hat{a}^e \) given that by construction \( b^* \leq \tilde{b} \). Moreover, \( p \) must be such that

\[ V = u(p \tilde{b} - k) + \frac{\beta}{F(\tilde{b})} \int_{0}^{\tilde{b}} v(\theta a_{t+1}) \, dF(a_{t+1}) \].

This, guarantees that they can choose a mixed strategy to ensure that the bond market clears.

When \( p_t = \underline{p} \) the demand for the risky bond is given by \( y_t \), so that to have market clearing it must be that the entrepreneurs choose to borrow with probability \( y_t / (\underline{p} \tilde{b}) \), and take their outside option and borrow 0 with probability \( 1 - y_t / (\underline{p} \tilde{b}) \).
Finally, when \( p_t = p^* \), then \( z_t \in [\Gamma^l, \tilde{y}] \) and no information is revealed. This implies that
the entrepreneurs have no additional information on the realization of \( a_{t+1} \). After substituting
the default decision \( \chi (p_t, a_{t+1}) \), for any \( p \in (\underline{p}, 1/R) \) problem (14) becomes

\[
B (p) = \arg \max_b u (pb - k) + \beta \int_0^1 v (\theta a_{t+1}) dF (a_{t+1}) + \beta \int_{b / \theta}^{\infty} v (a_{t+1} - b) dF (a_{t+1}).
\]  

(24)

Then \( B (p) \) denotes the optimal borrowing policy for a given non fully revealing price \( p \in (\underline{p}, 1/R) \) and, thanks to assumption A4, is implicitly defined by the first order condition (16).

**Step 2.** To complete the proof we need to show that there exists a pair \((p^*, q^*)\) that solves
the fixed point defined by \( q^* = Q (p^*) \) and \( p^* = P (q^*) \). Recall that \( Q (p) = F(B (p) / 1 - \theta) \),
where \( B (p) \) solves (16). Equations (5), (6), (8), and

\[
W = \gamma R + \delta \omega \left\{ q + (1 - q) \frac{\Gamma^l}{\tilde{y}} \left[ 1 - F \left( \frac{\tilde{b}}{1 - \theta} \right) + F \left( \frac{\tilde{b}}{1 - \theta} \right) \right] \right\} W,
\]

which replaces (7), can be combined to obtain

\[
P (q) = \frac{(1 - q) (1 - \delta [\nu (q) + q (1 - \nu (q))])}{[1 - \delta (1 - q) (1 + \nu (q))] R},
\]

where \( \nu (q) \) is the endogenous probability of full revelation and is implicitly defined by

\[
\nu (q) = \frac{M^l [\tilde{q} + (1 - \tilde{q})]}{\tilde{y} \left[ \delta + \frac{(1 - \delta) \gamma R}{\kappa (1 - \delta \omega [1 + (1 - q) \beta])} \right]} \equiv \psi (\nu (q) ; q),
\]

(25)

where \( \tilde{q} = F (\tilde{b} / (1 - \theta)) \) and \( \tilde{q} = F (\tilde{b} / (1 - \theta)) \) with \( \tilde{b} \) defined above.

First, assumption A4 ensures that \( B (p) \) is uniquely defined by equation (16) for any \( p \in (\underline{p}, 1/R) \) and the proof of Lemma 1 below shows that \( Q' (p) < 0 \). Then, \( Q (p) \) is decreasing on the interval \((\underline{p}, 1/R)\), given that it is immediate that \( Q' (p) \propto B'(p) \).

Next, we show that there is a unique \( \nu (q) \) that solves equation (25). First, notice that
\( \partial \psi (\nu (q) ; q) / \partial \nu (q) < 0 \) and that \( \psi (0 ; q) > 0 \). Then, we only need to show that \( \psi (1 ; q) < 1 \),
where

\[
\psi (1 ; q) = \frac{M^l [\tilde{q} + (1 - \tilde{q})]}{\tilde{y} \left[ \delta + \frac{(1 - \delta) \gamma R}{(1 - \delta \omega) \kappa} \right]}.
\]

This follows from the following chain of inequalities:

\[
\frac{M^l}{\tilde{y}} \left[ 1 + \tilde{q} - \tilde{q} \right] < 2 \frac{M^l}{\tilde{y}} < 1 < \delta + (1 - \delta) \frac{\gamma R}{(1 - \delta \omega) \kappa},
\]

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where the first one follows from the fact that $\bar{q}$ and $\bar{\tilde{q}}$ are in $[0, 1]$, the second one comes from assumption A1', and the third one follows from the assumption that $\gamma R > \kappa$.

This implies that equation (25) has a unique interior solution for $\nu(q)$. Then there exists a fixed point $(q^*, p^*)$ with $Q(1/R) < q^* < Q(p)$, given that by assumption $P(q_1) < 1/R$ and $P(q_2) > p$.

**Proof of Lemma 1**

We prove Lemma 1 for the general equilibrium when prices do not reveal any information. Let us start to prove condition (i). First, notice that

$$B(p) \equiv \arg \max_b \Phi(p, b)$$

Assumption A4 implies that there exists a unique $B(p)$ and

$$\Phi_b(p, B(p)) = 0, \quad (26)$$

$$\Phi_{bb}(p, B(p)) < 0.$$  

Then, by total differentiating (26), we obtain

$$\Phi_{bp}(p, B(p)) + \Phi_{bb}(p, B(p)) B'(p) = 0, \quad (27)$$

and then

$$B'(p) = -\frac{\Phi_{bp}(p, B(p))}{\Phi_{bb}(p, B(p))}. \quad (27)$$

Given that $\Phi_{bb}(p, B(p)) < 0$, to show that $B'(p) < 0$, it is enough to show that $\Phi_{bp}(p, B(p)) < 0$. We can calculate that

$$\Phi_{bp}(p, B(p)) = u'(p_t b_{t+1} - k) + p_t b_{t+1} u''(p_t b_{t+1} - k), \quad (28)$$

and hence $\Phi_{bp}(p, B(p)) < 0$ given A3 and the assumption that $k > 0$. From this, it is immediate that

$$\frac{dF(\hat{a})}{dp} = \frac{dF(\hat{a})}{\hat{a}} \left( \frac{1}{1 - \theta} \right) B'(p) \leq 0.$$

proving condition (ii).

Finally, we need to prove condition (iii). Notice that

$$\frac{d(pb)}{dp} = b + p B'(p),$$
where $B'(p)$ satisfies equation (27), with $\Phi_{bp}(p, B(p))$ given in equation (28) and

$$\Phi_{bb}(p, B(p)) = p^2u''(pb - k) + \beta v'\left(\frac{\theta}{1 - \theta}b\right) + \beta \int_{b}^{\infty} v''(a - b) dF(a) < 0, \quad (29)$$

given assumption A4. Then, after some algebra, we obtain

$$\frac{d(pb)}{dp} = \frac{b}{\Phi_{bb}(p, B(p))} \left\{ \beta \left[ v'\left(\frac{\theta}{1 - \theta}b\right) + \int_{b}^{\infty} v''(a - b) dF(a) \right] - \frac{p}{b} u'(pbt_1 + k) \right\}. $$

Hence $d(pb)/dp \geq 0$ iff the term in parenthesis is negative. We can show that this is the case combining condition (29) and the fact that $-cu''(c)/u'(c) \geq 1$, hence completing the proof.

**Proof of Proposition 4**

With a slight abuse of notation, define

$$g^L(q; \sigma) \equiv q - Q(P^L(q; \sigma); \sigma),$$

$$g^B(q; \sigma) \equiv q - Q(P^B(q; \sigma); \sigma).$$

Then, a limit equilibrium is characterized by $q^*(\sigma)$ such that $g^L(q^*; \sigma) = 0$ and the benchmark equilibrium is characterized by $q^B(\sigma)$ such that $g^L(q; \sigma) = 0$.

First, we show that if $q^B(\sigma') < 1/2$, then $q^*(\sigma') < q^B(\sigma') < 1/2$. If $q^B(\sigma') < 1/2$, then $P^L(q^B(\sigma'); \sigma') > P^B(q^B(\sigma'); \sigma')$. Moreover, Lemma 1 implies that $Q(p; \sigma)$ is decreasing in $p$, and we obtain

$$g^L(q^B(\sigma'); \sigma') = q^B(\sigma') - Q(P^L(q^B(\sigma'); \sigma') ; \sigma') > q^B(\sigma') - Q(P^B(q^B(\sigma'); \sigma') ; \sigma') = 0.$$  

Moreover, we assumed that $Q(1/R; \sigma) > q(\sigma)$ where $q(\sigma)$ is such that $P^L(q(\sigma); \sigma) = 1/R$. Hence,

$$g^L(q(\sigma'); \sigma') = q(\sigma') - Q(P^L(q(\sigma'); \sigma') ; \sigma') < 0.$$  

It follows that $q^*(\sigma')$ is such that $q(\sigma') < q^*(\sigma') < q^B(\sigma') < 1/2$, as we wanted to show.

Second, we show that if $q^*(\sigma'') > 1/2$, then $q^B(\sigma'') > 1/2$. If $q^*(\sigma'') > 1/2$, then $P^L(q^B(\sigma''); \sigma'') < P^B(q^B(\sigma''); \sigma'')$. It follows that

$$g^B(q^*(\sigma''); \sigma'') = q^*(\sigma'') - Q(P^B(q^*(\sigma''); \sigma'') ; \sigma'') > q^*(\sigma'') - Q(P^B(q^*(\sigma''); \sigma'') ; \sigma'') = 0,$$

and $q^B(\sigma'') < q^*(\sigma'')$ as we wanted to show.
These first two steps immediately imply that $q^*(\sigma') < q^B(\sigma') < 1/2 < q^*(\sigma'') < q^B(\sigma'')$, and hence $q^*(\sigma'') - q^*(\sigma') > q^B(\sigma'') - q^B(\sigma')$. This also implies that $p^*(\sigma') > p^B(\sigma') > 1/(2R) > p^B(\sigma'') > p^*(\sigma'')$, implying that $p^*(\sigma'') - p^*(\sigma') > p^B(\sigma'') - p^B(\sigma')$. This completes the proof.

**Proof of Proposition 5**

**Claim 1.** Let us fix $\alpha'$. We know that $P^L(1; \sigma) = 0$ for any $\sigma$. Moreover we can calculate

$$
\frac{dP^L(q; \sigma)}{dq} = -\frac{1}{R} \frac{\delta \omega (1-q) (1-\delta \omega (1-q) + (1-\delta q)}{(1-\delta \omega (1-q))^2} < 0,
$$

and $\lim_{R \to 0} dP^L(q; \sigma) / dq = -\infty$. Hence, if $Q(p) > 1/2$ and we start with $R'$ such that $Q(p^* (R', \alpha') | \alpha') < 1/2$. A graphical argument available upon request implies that there exists an $\hat{R}$, such that if $R'' > \hat{R}$ then $Q(p^* (R'', \alpha') | \alpha') > 1/2$.

**Claim 2.** First, we prove that for any fixed price $p$ there is a $\alpha_1$ such that for any $\alpha < \alpha_1$, $\partial Q(p) / \partial \alpha < 0$.

From $q (b) = F(b / (1-\theta))$, with a slight abuse of notation let us define $b_\alpha(q) = (1-\theta) (F^{-1}_\alpha(q))$ be the bond holding which implies a probability of default of $q$ when the parameter is $\alpha$. Then, for a fixed $p$, we can write the first order condition of entrepreneurs as

$$
\Phi(a, b_\alpha(q)) = pu'(pb_\alpha(q) - k) - \beta \int_{b_\alpha(q)}^{\infty} v'(a_{t+1} - b_\alpha(q)) dF_\alpha(a_{t+1}).
$$

We show that there is a $\alpha_1$ such that

$$
\frac{dq}{d\alpha} = -\frac{\partial \Phi(a, b_\alpha(q))}{\partial b_\alpha(q)} \frac{\partial b_\alpha(q)}{\partial \alpha} + \frac{\partial \Phi(a, b_\alpha(q))}{\partial \alpha} \bigg|_{\alpha<\alpha_1} < 0, \quad (30)
$$

From the properties of a cumulative density function and our assumptions on $\alpha$, we know that $\partial b_\alpha(q) / \partial q > 0$ and $\partial b_\alpha(q) / \partial \alpha > 0$. Furthermore, from the result of $\Phi_{bb} (p, b) < 0$, we know that $\Phi_b(a, b_\alpha(q)) < 0$. Note that

$$
\frac{\partial \Phi(a, b_\alpha(q))}{\partial \alpha} = -\beta \frac{\partial \int_{b_\alpha(q)}^{\infty} v'(a_{t+1} - b_\alpha(q)) dF_\alpha(a_{t+1})}{\partial \alpha}.
$$

As $\lim_{\alpha \to -\infty} F_\alpha(a_{t+1}) = 1$, we can choose an $\alpha_1$ that $\partial ((F_\alpha(a_{t+1}) - F_{\alpha+\epsilon}(a_{t+1}))/\partial a_{t+1} < 0$ for any fixed $\alpha < \alpha_1$ and $a_{t+1} > b_\alpha(q) / (1-\theta)$. To show (30), it is sufficient to show that for any $\alpha < \alpha_1$

$$
\frac{\partial \Phi(a, b_\alpha(q))}{\partial \alpha} \bigg|_{\alpha<\alpha_1} < 0.
$$
This is a consequence of the following chain of inequalities for any $\alpha < \alpha_1$

$$
\int_{b_{\alpha}(q)}^{\infty} v' (a_{t+1} - b_{\alpha} (q)) d (F_\alpha (a_{t+1}) - F_{\alpha+\varepsilon} (a_{t+1})) = \\
= -v' \left( \frac{b_{\alpha} (q)}{1 - \theta} - b_{\alpha} (q) \right) \left( F_\alpha \left( \frac{b_{\alpha} (q)}{1 - \theta} \right) - F_{\alpha+\varepsilon} \left( \frac{b_{\alpha} (q)}{1 - \theta} \right) \right) \\
- \int_{b_{\alpha}(q)}^{\infty} v'' (a_{t+1} - b_{\alpha} (q)) (F_\alpha (a_{t+1}) - F_{\alpha+\varepsilon} (a_{t+1})) d a_{t+1} < \\
- v' \left( \frac{b_{\alpha} (q)}{1 - \theta} - b_{\alpha} (q) \right) \left( F_\alpha \left( \frac{b_{\alpha} (q)}{1 - \theta} \right) - F_{\alpha+\varepsilon} \left( \frac{b_{\alpha} (q)}{1 - \theta} \right) \right) \\
- \left( F_\alpha \left( \frac{b_{\alpha} (q)}{1 - \theta} \right) - F_{\alpha+\varepsilon} \left( \frac{b_{\alpha} (q)}{1 - \theta} \right) \right) \int_{b_{\alpha}(q)}^{\infty} v'' (a_{t+1} - b_{\alpha} (q)) d a_{t+1} = \\
= \left( F_\alpha \left( \frac{b_{\alpha} (q)}{1 - \theta} \right) - F_{\alpha+\varepsilon} \left( \frac{b_{\alpha} (q)}{1 - \theta} \right) \right) \left[ -v' \left( \frac{b_{\alpha} (q)}{1 - \theta} - b_{\alpha} (q) \right) \right] \\
- \lim_{a_{t+1} \to \infty} v' (a_{t+1} - b_{\alpha} (q)) + v' \left( \frac{b_{\alpha} (q)}{1 - \theta} - b_{\alpha} (q) \right) \leq 0
$$

where the first equation is by partial integration and the inequality is the consequence of

$$
\frac{\partial}{\partial \alpha} \left( (F_\alpha (a_{t+1}) - F_{\alpha+\varepsilon} (a_{t+1})) / \partial a_{t+1} \right) < 0 \text{ and } v'' (\cdot) \leq 0.
$$

Moreover, if $\lim_{\alpha \to -\infty} Q (p)$ exists, then $\lim_{\alpha \to -\infty} Q (p) = 1$. This is a simple consequence of the first order condition and $\lim_{\alpha \to -\infty} F_\alpha (a_{\alpha}) = 1$.

To summarize, so far we have shown that $\partial Q_\alpha (p) / \partial \alpha < 0$ for any $\alpha < \alpha_1$ and $\lim_{\alpha \to -\infty} Q (p) = 1$ whenever it exists. So if $\alpha$ is sufficiently low, then $Q (p^* (R', \alpha) | \alpha) < \frac{1}{2}$. Then there is an $\alpha_2 = \min \alpha$ such that $Q (p^* (R', \alpha) | \alpha) = 1/2$. The second claim of the proposition follows immediately if we set $\hat{\alpha} \equiv \min (\alpha_1, \alpha_2)$.

**Appendix B**

**Example**

Assume $u (c) = \log (c)$, $v (c) = c$ and $1 - F (a) = a^\gamma a^{-\gamma}$, with $a > 1 / (1 - \theta)$ and $\gamma < 1$. In this case we can write

$$
\Phi_b (p, b) = b^{-\gamma} \left[ \frac{p b^\gamma}{p b - k} - \beta a^\gamma (1 - \theta)^\gamma \right].
$$

For a given $p$, there is a unique solution to $\Phi_b (p, b) = 0$ which solves

$$
\frac{p b^\gamma}{p b - k} = \beta a^\gamma (1 - \theta)^\gamma.
$$

Then, it is easy to verify that the left-hand side of this condition is decreasing in $b$ whenever $\gamma < 1$, and it converges to $\infty$ for $b \to k/p$, and to $0$ for $b \to \infty$. Given that the right-hand side
is a positive constant, there must be a unique optimum $b$ that is implicitly define as follows:

\[ p = \frac{\beta a^\gamma (1 - \theta)^\gamma k}{\beta a^\gamma (1 - \theta)^\gamma b - b^\gamma}. \]

The function $\Phi(p, b)$ is quasi-concave because if $b_1 < b_2$ and $\Phi_b(p, b_1) > 0$, then $\Phi_b(p, b_2) > 0$, given that $b^{-\gamma} > 0$ for any $b > k/p$. For the same reason if $b_1 < b_2$ and $\Phi_b(p, b_2) < 0$, then $\Phi_b(p, b_1) < 0$, completing the proof.

Moreover, we can show that there always exists a limit equilibrium for $k$ small enough. The fixed point problem for $(q^*, p^*)$ can be rewritten as

\[
\begin{align*}
    p^* &= \frac{k \beta (1 - q^*)}{(1 - \theta) \beta a (1 - q^*)^{-\frac{\gamma}{1 - \gamma}} - 1}, \\
    p^* &= \frac{(1 - q^*) (1 - \delta q^*)}{[1 - \delta (1 - q^*)] R}.
\end{align*}
\]

Let us define

\[
\begin{align*}
    h_1 (q) &\equiv \frac{k \beta}{(1 - \theta) \beta a (1 - q)^{-\frac{\gamma}{1 - \gamma}} - 1}, \\
    h_2 (q) &\equiv \frac{1 - \delta q}{[1 - \delta + \delta q] R}.
\end{align*}
\]

then it is immediate that there exists a fixed point iff there exists a $q^*$ such that $h_1 (q^*) = h_2 (q^*)$. This is easy to show given that both $h_1 (q)$ and $h_2 (q)$ are decreasing in $q$. Moreover, $h_1 (0) = k \beta / [(1 - \theta) \beta a - 1] > 0$, $h_2 (0) = 1 / [(1 - \delta) R] > 0$, $h_1 (1) = 0$ and $h_2 (1) = (1 - \delta) / R > 0$. Hence, in order for a fixed point to exists, it is enough that $h_2 (0) < h_1 (0)$, that is,

\[ k < \frac{(1 - \theta) \beta a - 1}{(1 - \delta) \beta R}. \]

Moreover, if $k = 0$, then there is a unique solution, given by

\[
\begin{align*}
    q^* &= 1 - \left( \frac{1}{(1 - \theta) \beta a} \right)^{-\frac{\gamma}{1 - \gamma}}, \\
    p^* &= \frac{(1 - q^*) (1 - \omega \delta q^*)}{[1 - \omega \delta (1 - q^*)] R}.
\end{align*}
\]

By continuity if $k$ is small enough the equilibrium is also unique.

References


