

Credit Crises, Precautionary Savings, and the Liquidity Trap*

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First draft: July 2010
This draft: October 2011

Abstract

We study the effects of a credit crunch on consumer spending in a heterogeneous-agent incomplete-market model. After an unexpected permanent tightening in consumers' borrowing capacity, some consumers are forced to deleverage and others increase their precautionary savings. This depresses interest rates, especially in the short run, and generates an output drop, even with flexible prices. The output drop is larger with nominal rigidities, if the zero lower bound prevents the interest rate from adjusting downwards. Adding durable goods to the model, households take larger debt positions and the output response may be larger.

*Email addresses: vguerrie@chicagobooth.edu; glorenzo@mit.edu. We thank Bob Hall and John Leahy for their useful discussions and for numerous exchanges. For helpful comments, we are grateful to Andy Atkeson, Chris Carroll, V. V. Chari, Vasco Curdia, Greg Kaplan, Juan Pablo Nicolini, Thomas Philippon, Valery Ramey, Rob Shimer, Nancy Stokey, Amir Sufi, Gianluca Violante, Iván Werning, Mike Woodford, Pierre Yared, and numerous seminar participants. Adrien Auclert and Amir Kermani provided outstanding research assistance. Guerrieri thanks the Sloan Foundation for financial support and the Federal Reserve Bank of Minneapolis for its hospitality. Lorenzoni thanks the NSF for financial support and Chicago Booth and the Becker Friedman Institute for their hospitality.

1 Introduction

How does an economy adjust from a regime of easy credit to one of tight credit? Suppose it is relatively easy for consumers to borrow and the economy is in a stationary state with a stable distribution of borrowing and lending positions. An unexpected shock hits the financial system and borrowing gets harder in terms of tighter borrowing limits and/or in terms of higher credit spreads. The most indebted consumers need to readjust towards lower levels of debt (deleveraging). Since the debtor position of one agent is the creditor position of another, this also means that lenders have to reduce their holdings of financial claims. How are the spending decisions of borrowers and lenders affected by this economy-wide financial adjustment? What happens to aggregate activity? How long does the adjustment last?

In this paper, we address these questions, focusing on the response of the household sector, using a workhorse Bewley (1977) model in which households borrow and lend to smooth transitory income fluctuations. Since the model cannot be solved analytically, our approach is to obtain numerical results under plausible parametrizations and then to explore the mechanism behind them. The model captures two channels in the consumers' response to a reduction in their borrowing capacity. First, a direct channel, by which constrained borrowers are forced to reduce their indebtedness. Second, a precautionary channel, by which unconstrained agents increase their savings as a buffer against future shocks. Both channels increase net lending in the economy, so the equilibrium interest rate has to fall in equilibrium.

Our analysis leads to two sets of results. First, we look at interest rate dynamics and show that they are characterized by a sharp initial fall followed by a gradual adjustment to a new, lower steady state. The reason for this interest rate overshooting is that, at the initial asset distribution, the agents at the lower end of the distribution try to adjust faster towards a higher savings target. So the initial increase in net lending is stronger. To keep the asset market in equilibrium, interest rates have to fall sharply. As the asset distribution converges to the new steady state the net lending pressure subsides and the interest rate moves gradually up.

Second, we look at the responses of aggregate activity. Overly indebted agents can adjust in two ways: by spending less and by working more. In our model they do both,

so, for a given interest rate, the credit shock would lead to a reduction in consumer spending and to an increase in labor supply. Whether a recession follows depends on the relative strength of these two forces and on the interest rate elasticity of consumption and labor supply. In our baseline calibration, the consumption side dominates and output declines. As for the case of interest rates, the contraction is stronger in the short run, when the distribution of asset holdings is far from its new steady state and some agents are far below their savings target. A tightening of the credit limit that reduces household debt-to-GDP by 10 percentage points generates a 1% drop in output on impact.

We then add nominal rigidities to the model. In presence of nominal rigidities, the zero lower bound on the nominal interest rate means that the central bank may be unable to achieve the real interest rate that replicates the flexible price allocation. Moreover, with nominal rigidities, aggregate activity is purely driven by the response of consumer demand. Therefore, when the zero lower bound is binding the households' net saving pressure translates into a larger output drop.

Finally, we generalize the model to include durable consumption goods, which can be used as collateral. In this extension, households face a richer portfolio choice as they can invest in liquid bonds or in durable goods. To make bonds and durables imperfect substitutes, we assume a proportional cost of re-selling durables, so that durables are less liquid. After a credit crunch, net borrowers are forced to deleverage and have to reduce consumption of durable and non-durable goods. On the other hand, the precautionary motive induces net lenders to save more by accumulating both bonds and durables. Durable purchases may increase or decrease, depending on the strength of these two effects. In our calibration, the net effect depends on the nature of the shock. A pure shock to the credit limit affects only borrowers close to the limit, so the lenders' side dominates and durable purchases increase. A shock to credit spreads, on the other hand, affects a larger fraction of borrowers, leading to a contraction in durable purchases. Here the output effects can be large, leading to a 4% drop in consumption following a transitory shock that raises the spread on a one year loan from 1% to 3.8%. The consumption drop can be as large as 10% if prices are fixed and the zero lower bound is binding.

Our paper focuses on households' balance sheets adjustment and consumer spending and is complementary to a growing literature that looks at the effects of credit shocks

on firms' balance sheets and investment spending.¹ Hall (2011a, 2011b) argues that the response of the household sector to the credit tightening is an essential ingredient to account for the recent U.S. recession. Mian and Sufi (2011a, 2011b) use cross-state evidence to argue that the contraction in households' borrowing capacity, mainly driven by a decline in house prices, was responsible for the fall in consumer spending and, eventually, for the increase in unemployment. Our model aims to capture the effects of a similar contraction in households' borrowing capacity in general equilibrium.

In modeling the household sector, we follow the vast literature on consumption and saving in incomplete market economies with idiosyncratic income uncertainty, going back to Bewley (1977), Deaton (1991), Huggett (1993), Aiyagari (1994), Carroll (1997).² Our approach is to compute the economy's transitional dynamics after a one-time, unexpected aggregate shock. This relates our paper to recent contributions that look at transitional dynamics after different types of shocks.³ Much work on business cycles in economies with heterogeneous agents and incomplete markets, follows Krusell and Smith (1998) and looks at approximate equilibria in which prices evolve as functions of a finite set of moments of the wealth distribution. Here, we prefer to keep the entire wealth distribution as a state variable at the cost of focusing on a one time shock, because our shock affects very differently agents in different regions of the distribution.⁴ Midrigan and Philippon (2011) take a different (and complementary) approach to modeling the effects of a credit crunch on the household sector. They use a cash-in-advance model to explore the idea that credit access, as money, is needed to facilitate transactions. Finally, our model with durables is related to Carroll and Dunn (1997), an early paper that uses an heterogeneous agent, incomplete market model with durable and non-durable goods to look at the dynamics of consumer debt and spending following a shock

¹Classic models of the role of firms' balance sheets are Kiyotaki and Moore (1997) and Bernanke, Gertler, and Gilchrist (1999). Recent contributions include Jermann and Quadrini (2009), Brunnermeier and Sannikov (2010), Gertler and Kiyotaki (2010), Khan and Thomas (2010), Buera and Moll (2011), Del Negro, Eggertsson, Ferrero, and Kiyotaki (2011), Cagetti, De Nardi, and Bassetto (2011). Goldberg (2011) is a model that combines financial frictions on both the firms' and the households' side, but focuses on steady states.

²Heathcote, Storesletten, and Violante (2009) offer an excellent review.

³For example, Mendoza, Rios Rull and Quadrini (2010) look at the response of an economy opening up to international asset trade.

⁴Heathcote, Storesletten, and Violante (2009) point out that the nature of the shock is important in determining whether or not an heterogeneous agent economy behaves approximately as its representative agent counterpart.

to unemployment risk.

The modern monetary policy literature has pointed out that at the roots of a liquidity trap there must be a shock that sharply reduces the “natural” interest rate, that is, the interest rate that would arise in a flexible price economy (Krugman, 1998; Woodford and Eggertsson, 2001). In representative agent models, the literature typically generates a liquidity trap by introducing a shock to intertemporal preferences, which mechanically increase the consumer’s willingness to save (e.g., Christiano, Eichenbaum, and Rebelo, 2011). Our model shows that in a heterogenous agent environment, shocks to the agents’ borrowing capacity can be the underlying force that pushes down the natural rate, by reducing the demand for loans by borrowers and by increasing the supply of loans by lenders. This is consistent with the fact that, historically, liquidity trap episodes have always followed disruptions in credit markets. Two independent recent papers, Curdia and Woodford (2010) and Eggertsson and Krugman (2011), draw related connections between credit crises and the liquidity trap. The main difference is that they work with a representative borrower and a representative lender and mute wealth dynamics to aim for analytical tractability.⁵ This implies that there is no precautionary effect on the lenders’ side and that there is no internal dynamics associated to the wealth distribution. As we shall see, in our model the dynamics of the wealth distribution play an important role in generating large swings in the natural interest rates.

Two papers that explore the effects of precautionary behavior on business cycle fluctuations are Guerrieri and Lorenzoni (2009) and Challe and Ragot (2011). Both papers, derive analytical results under simplifying assumptions that eliminate the wealth distribution from the problem’s state variables. In this paper we take a computational approach, to study how the adjustment mechanism works when the wealth distribution evolves endogenously. Another related paper is Chamley (2010), a theoretical paper which explores the role of the precautionary motive in a monetary environment and focuses on the possibility of multiple equilibria.

The paper is organized as follows. In Section 2, we present our model and characterize the steady state. In Section 3, we perform our main exercise, that is, we analyze

⁵Iacoviello (2005) is an early paper that studies monetary policy in a two-types model where households borrow to finance housing purchases, facing a collateral constraint similar to that in our durable section.

the equilibrium transitional dynamics after an unexpected permanent tightening of the borrowing limit. Section 4 explores the role of nominal rigidities. Section 5 studies the effects of fiscal policy. Section 6 presents the results of the extended model with durable consumption goods. Section 7 concludes. In an online appendix, we describe our computational strategy.

2 Model

We consider a model of households facing idiosyncratic income uncertainty, who smooth consumption by borrowing and lending. The model is a standard incomplete markets model in the tradition of Bewley (1977), with endogenous labor supply and no capital. The only asset traded is a one-period risk-free bond. Households can borrow up to an exogenous limit, which is tighter than the natural borrowing limit. We first analyze the steady state equilibrium for a given borrowing limit. Then, we study transitional dynamics following an unexpected, one time shock that reduces this limit.

There is a continuum of infinitely lived households with preferences represented by the utility function

$$E \left[\sum_{t=0}^{\infty} \beta^t U(c_{it}, n_{it}) \right],$$

where c_{it} is consumption and n_{it} is labor effort of household i . Each household produces consumption goods using the linear technology

$$y_{it} = \theta_{it} n_{it},$$

where θ_{it} is an idiosyncratic shock to the labor productivity of household i , which follows a Markov chain on the space $\{\theta^1, \dots, \theta^S\}$. We assume $\theta^1 = 0$ and interpret this realization of the shock as unemployment. For the moment, there are no aggregate shocks.

The household budget constraint is

$$q_t b_{it+1} + c_{it} + \tilde{\tau}_{it} \leq b_{it} + y_{it},$$

where b_{it} are bond holdings, q_t is the bond price and $\tilde{\tau}_{it}$ are taxes. Tax payments are as follows: all households pay a lump sum tax τ_t and the unemployed receive the unem-

ployment benefit v_t . That is, $\tilde{\tau}_{it} = \tau_t$ if $\theta_{it} > 0$ and $\tilde{\tau}_{it} = \tau_t - v_t$ if $\theta_{it} = 0$.⁶ Household debt is bounded below by the exogenous limit ϕ , that is, bond holdings must satisfy

$$b_{it+1} \geq -\phi. \quad (1)$$

The interest rate implicit in the bond price is $r_t = 1/q_t - 1$.

The government chooses the aggregate supply of bonds B_t , the unemployment benefit v_t and the lump sum tax τ_t so as to satisfy the budget constraint:

$$B_t + v_t u = q_t B_{t+1} + \tau_t,$$

where $u = \Pr(\theta_{it} = 0)$ is the fraction of unemployed agents in the population. For now, we assume that the supply of government bonds and the unemployment benefit are kept constant at B and v , while the tax τ_t adjusts to ensure government budget balance. In Section 5, we consider alternative fiscal policies.

In the model, the only supply of bonds outside the household sector comes from the government. When we calibrate the model, we interpret the bond supply B broadly as the sum of all liquid assets held by the household sector. The main deviation from Aiyagari (1994) and the following general equilibrium literature is the absence of capital in our model. The standard assumption in models with capital is that firms can issue claims to physical capital that are perfect substitutes for government bonds and other safe and liquid stores of value. This would not be a satisfactory assumption here, since we are trying to capture the effects of a credit crisis. A more general model of a credit crisis would have to include the effects of the crisis on the ability of firms to issue financial claims and on their accumulation of precautionary reserves, and it would have to allow for imperfect substitutability between different assets.⁷ Here, we choose to focus on the household sector and we close the model by taking as given the net supply of liquid assets coming from the rest of the economy, B . In Section 6, we enrich the household portfolio choice by allowing households to accumulate both bonds and durable goods, which are a form of capital directly employed by the households. In that setup, we will introduce imperfect substitutability between the two assets.

In our baseline model, the only motive for borrowing and lending comes from income uncertainty. In particular, we abstract from life-cycle considerations and from

⁶The presence of the unemployment benefit ensures that the natural borrowing limit is strictly positive.

⁷Along the lines of models such as those mentioned in footnote 1.

other important drivers of household borrowing and lending dynamics, like durable purchases, health expenses, educational expenses, etc. Moreover, we assume that there is a single interest rate r_t , which applies both to positive and negative bond holdings, so that household can borrow or lend at the same rate. In Section 6, we address some of these limitations, by modeling durable purchases and introducing a spread between borrowing and lending rates.

2.1 Equilibrium

Given a sequence of interest rates $\{r_t\}$ and taxes $\{\tau_t\}$, let $C_t(b, \theta)$ and $N_t(b, \theta)$ denote the optimal consumption and labor supply at time t of a household with bond holdings $b_{it} = b$ and productivity $\theta_{it} = \theta$. Given consumption and labor supply, next period bond holdings are derived from the budget constraint. Therefore, the transition for bond holdings is fully determined by the functions $C_t(b, \theta)$ and $N_t(b, \theta)$.

Let $\Psi_t(b, \theta)$ denote the joint distribution of bond holdings and current productivity levels in the population. The household's optimal transition for bond holdings together with the Markov process for productivity yield a transition probability for the individual states (b, θ) . This transition probability determines the distribution Ψ_{t+1} , given the distribution Ψ_t . We are now ready to define an equilibrium.

Definition 1 *An equilibrium is a sequence of interest rates $\{r_t\}$, a sequence of consumption and labor supply policies $\{C_t(b, \theta), N_t(b, \theta)\}$, a sequence of taxes $\{\tau_t\}$, and a sequence of distributions for bond holdings and productivity levels $\{\Psi_t\}$ such that, given the initial distribution Ψ_0 :*

- (i) $C_t(b, \theta)$ and $N_t(b, \theta)$ are optimal given $\{r_t\}$ and $\{\tau_t\}$,
- (ii) Ψ_t is consistent with the consumption and labor supply policies,
- (iii) the tax satisfies the government budget constraint,

$$\tau_t = vu + r_t B / (1 + r_t),$$

- (iv) the bonds market clears,

$$\int b d\Psi_t(b, \theta) = B.$$

The optimal policies for consumption and labor supply are characterized by two optimality conditions. The Euler equation

$$U_c(c_{it}, n_{it}) \geq \beta (1 + r_t) E_t [U_c(c_{it+1}, n_{it+1})], \quad (2)$$

which holds with equality if the borrowing constraint $b_{it+1} \geq -\phi$ is slack. And the optimality condition for labor supply

$$\theta_{it} U_c(c_{it}, n_{it}) + U_n(c_{it}, n_{it}) = 0, \quad (3)$$

if $\theta_{it} > 0$ and $n_{it} = 0$ otherwise.

As we will see below, a tightening of the borrowing limit makes future consumption more responsive to income shocks, so that agents face higher future volatility. With prudence in preferences, this implies that the expected marginal utility on the right-hand side of (2) is higher, by Jensen's inequality. Therefore, for a given level of interest rates, consumption today falls, as if there was a negative preference shock reducing the marginal utility of consumption today. In this sense, a model with precautionary savings provides a microfoundation for models that use preference shocks to push the economy in a liquidity trap.

2.2 Calibration

We will analyze the model by numerical simulations, so we first need to specify preferences and choose parameter values. We assume the utility function is separable and isoelastic in consumption and leisure

$$U(c, n) = \frac{c^{1-\gamma}}{1-\gamma} + \psi \frac{(1-n)^{1-\eta}}{1-\eta}.$$

Our baseline parameters are reported in Table 1. The time period is a quarter. The discount factor β is chosen to yield a yearly interest rate of 2.5% in the initial steady state. The coefficient of risk aversion is $\gamma = 4$. Clearly, this coefficient is crucial in determining precautionary behavior, so we will experiment with different values. The parameter η is chosen so that the average Frisch elasticity of labor supply is 1. The parameter ψ is chosen so that average hours worked for employed workers are 40% of their time endowment, in line with the evidence in Nekarda and Ramey (2010).⁸

⁸Figure 1 in their paper shows about 39 weekly hours per worker in 2000-2008. Subtracting 70 hours per week for sleep and personal care from the time endowment, we obtain $39/98 = 0.40$.

Table 1: *Parameter Values*

Parameter	Explanation	Value	Target/Source
β	Discount factor	0.9713	Interest rate $r = 2.5\%$
γ	Coefficient of relative risk aversion	4	
η	Curvature of utility from leisure	1.88	Average Frisch elasticity = 1
ψ	Coefficient on leisure in utility	12.48	Average hours worked = 0.4 (Nekarda and Ramey, 2010)
ρ	Persistence of productivity shock	0.967	Persistence of wage process in Floden and Lindé (2001)
σ_ε	Variance of productivity shock	0.017	Variance of wage process in Floden and Lindé (2001)
$\pi_{e,u}$	Transition to unemployment	0.057	Shimer (2005)
$\pi_{u,e}$	Transition to employment	0.882	Shimer (2005)
v	Unemployment benefit	0.10	40% of average labor income
ϕ	Borrowing limit	1.04	Debt-to-GDP ratio of 0.18
B	Bond supply	1.60	Liquid-assets-to-GDP ratio of 1.78

Note: The quantities v , ϕ and B are expressed in terms of yearly aggregate output. See the text for details on the targets.

The process for θ_{it} is chosen to capture wage and employment uncertainty. We assume that, when positive, θ_{it} follows an AR1 process in logs with autocorrelation ρ and variance σ_ε^2 . The parameters ρ and σ_ε^2 are chosen to match the evidence in Floden and Lindé (2001), who use yearly panel data from the PSID to estimate the stochastic process for individual wages in the U.S. In particular, our parameters yield a coefficient of autocorrelation of 0.9136 and a conditional variance of 0.0426 for yearly wages, matching the same moments of the persistent component of their wage process.⁹ The wage process is approximated by a 12-state Markov chain, following the approach in Tauchen (1986). For the transitions between employment and unemployment we follow Shimer (2005), who estimates the finding rate and the separation rate from CPS data. At a quar-

⁹See Table IV in Floden and Lindé (2001). The relation between quarterly and annual parameters (denoted by the subscript a) is $\rho = \rho_a^{1/4}$ and

$$\sigma_\varepsilon^2 = \frac{8(1-\rho^2)}{2+3\rho+2\rho^2+\rho^3} \frac{\sigma_{\varepsilon,a}^2}{1-\rho_a^2}.$$

terly frequency, we then choose transition probabilities equal to 0.057 from employment to unemployment and equal to 0.882 from unemployment to employment. When first employed workers draw θ from its unconditional distribution. For the unemployment benefit v , we also follow Shimer (2005) and set it to 40% of average labor income.

Finally, we choose values for the bonds supply B and the borrowing limit ϕ to reflect U.S. households' balance sheets in 2006, before the onset of the financial crisis. Defining liquid assets broadly as the sum of all deposits plus securities held directly by households, the liquid assets to GDP ratio in 2006 was equal to 1.78.¹⁰ We choose B to match this ratio, computing liquid assets as the sum of households' positive bond holdings. Second, we match debt in our model to consumer credit, which was 18% of GDP in 2006.¹¹ We choose ϕ to match this ratio, computing debt as the sum of households' negative bond holdings. The value of ϕ that we obtain in this way is equal to about 1 year of the average income. At this stage, we are leaving aside two important elements of the households' balance sheet: housing wealth and mortgage debt. The model with durable goods in Section 6 will bring these elements back into the picture.

2.3 Steady state

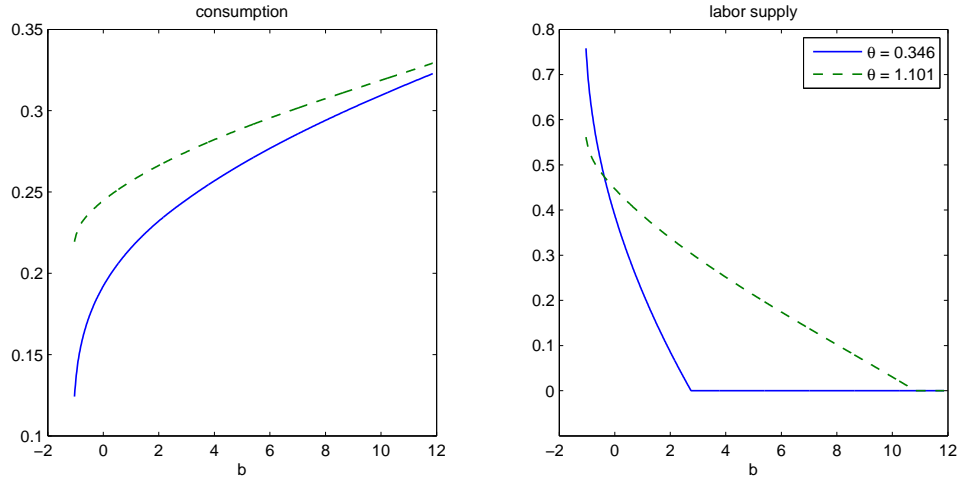
To conclude this section, we briefly describe the household policies in steady state. Figure 1 shows the optimal values of consumption and labor supply as a function of the initial level of bond holdings, for two productivity levels: $\theta = 0.346$, the lowest positive level in our grid (solid blue line), and $\theta = 1.101$, which is the average value (dashed green line).

Different responses at different levels of bond holdings are apparent. At high levels of b , consumer behavior is close to the permanent income hypothesis and the consumption function is almost linear in b . For lower levels of bond holdings, the consumption function is concave, as is common in precautionary savings models (Carroll and Kimball, 1996). The optimality condition for labor supply implies that the relation between bond holdings and labor supply mirrors that of consumption, capturing an income effect. In particular, at low levels of b a steeply increasing consumption function translates

¹⁰Federal Reserve Board Flow of Funds (Z.1) table B.100, sum of lines 9, 16, 19, 20, 21, 24, and 25.

¹¹Also in table B.100, line 34, which essentially corresponds to total household liabilities minus mortgage debt.

Figure 1: *Optimal Consumption and Labor Supply in Steady State*



Note: Consumption and bonds are in terms of steady state yearly aggregate output.

into a steeply decreasing labor supply function, making it convex. For most levels of b the substitution effect dominates the income effect and higher wages are associated to higher labor supply. For very low levels of b , however, the income effect dominates and low wage households supply more hours than high wage households.

3 Credit Crunch

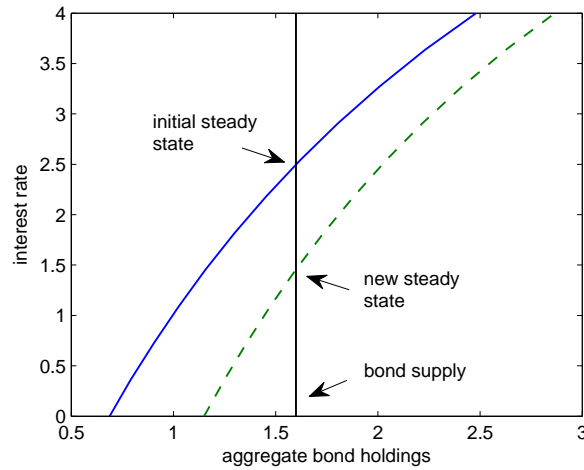
We now explore the response of our economy to a credit crunch. We consider an economy that at $t = 0$ is in steady state with the borrowing limit $\phi = 1.04$. We then look at the effects of an unexpected shock at $t = 1$ that gradually and permanently decreases the borrowing limit to $\phi' = 0.58$. The size of the shock is chosen so that the debt-to-GDP ratio drops by 10 percentage points in the new steady state.

Starting at $t = 1$, the borrowing limit ϕ_t follows the linear adjustment path

$$\phi_t = \max \{ \phi', \phi - \Delta t \},$$

and households perfectly anticipate this path. We choose Δ so that the adjustment lasts 6 quarters. Since all debt in the model has a one-quarter maturity, a sudden adjustment

Figure 2: *Bond Market Equilibrium in Steady State*

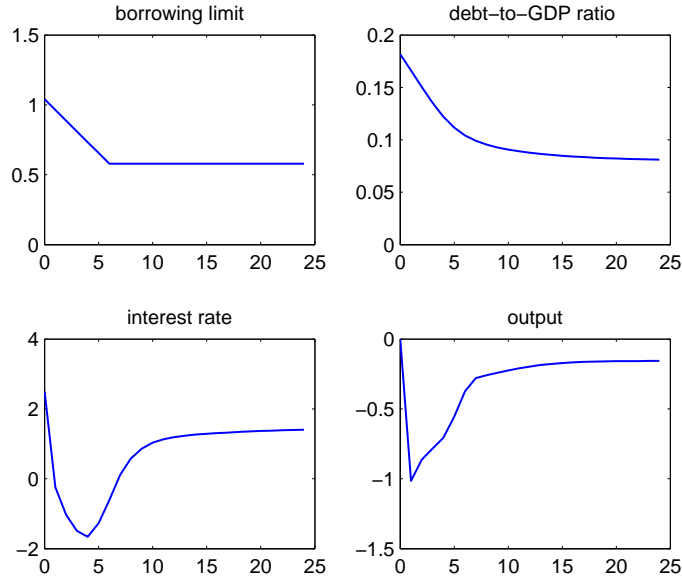


Note: Interest rate is in annual terms.

in the debt limit would require unrealistically large repayments by the most indebted households. An assumption of gradual adjustment of the debt limit is a simple way of capturing the fact that actual debt maturities are longer than a quarter, so that after a credit crunch households can gradually pay back their debt. An adjustment period of 6 quarters ensures that no household is forced into default. Default and bankruptcy are clearly an important element of the adjustment to a tighter credit regime, but we abstract from them.

Before looking at transitional dynamics, let us briefly compare the interest rate in the two steady states. In Figure 2 we plot the aggregate bond demand in the initial steady state (solid blue line) and in the new steady state (dashed green line). Two effects contribute to shifting the demand curve to the right in the new steady state. First there is a mechanical effect, as all households with debt larger than ϕ' need to reduce their debt. Second there is a precautionary effect, as households accumulate more wealth to stay away from the borrowing limit. As the supply of bonds is fixed at B , the shift in bond demand leads to a lower equilibrium interest rate.

Figure 3: *Interest Rate and Output Responses*



Note: Interest rate is in annual terms. Output is in percent deviation from initial steady state.

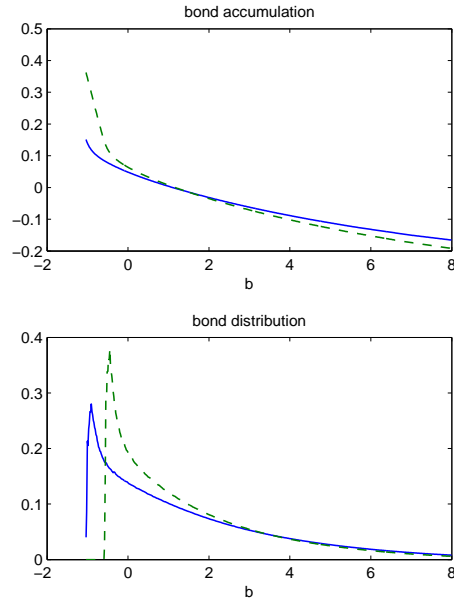
3.1 Transitional dynamics: interest rate

Figure 3 illustrates the economy’s response to the debt limit contraction. In the top left panel, we plot the exogenous adjustment path for ϕ_t . The remaining panels show the responses of the debt-to-GDP ratio (top right panel), of the interest rate (bottom left panel), and of output (bottom right panel).

The interest rate drops sharply after the shock, going negative for 6 quarters. The interest rate overshooting after a debt contraction is our first main result. From numerical experiments, this result seems a fairly general qualitative outcome of this class of models and not just the consequence of our choice of parameters. To provide some intuition, we now identify some properties of the household policy functions and of the steady state distributions that help explain the result.

Let us first look at the policy functions. The top panel of Figure 4 plots the optimal bond accumulation $b_{it+1} - b_{it}$ (averaged over θ) as a function of the initial bond holdings b_{it} , at the initial steady state (solid blue line) and at the new steady state (dashed green line). The function is very steep at low levels of bond holdings and flatter at higher

Figure 4: *Bond Accumulation and Distributions in the Two Steady States*



levels, reflecting the strong incentives to save for households at the left tail of the distribution, who want to move away from their borrowing limit. Notice that the convexity of the bond accumulation function follows from the budget constraint, the concavity of the consumption function and the convexity of the labor supply function (see Figure 1).

Consider next the stationary bond distributions. The bottom panel of Figure 4 shows the marginal density of bond holdings at the initial steady state (solid blue line) and at the new steady state (dashed green line). The two distributions have the same average, as the bond supply is the same in the two steady states, but the new distribution is more concentrated.¹² A comparison of the policies in the top panel helps to explain why. At low levels of bond holdings, the precautionary behavior induces agents in the new steady state to accumulate bonds faster. At high levels of bond holdings, the low equilibrium interest rate induces agents to decumulate bonds faster. This makes bond holdings mean-revert faster and makes the stationary distribution more concentrated.

We are now ready to put the pieces together. In equilibrium, aggregate net bond accumulation must be zero as the bond supply is fixed. In steady state, this means the

¹²Formally, the initial distribution is a mean-preserving spread of the new distribution. We checked this property numerically plotting the integral of the CDF of b for the two distributions.

integral of the solid (dashed) function in the top panel weighted by the solid (dashed) density in the bottom panel is equal to zero. Let us make a “disequilibrium” experiment and suppose the interest rate jumped to its new steady state value at $t = 1$ and stayed there from then on. Average bond accumulation could then be computed by integrating the dashed function in the top panel weighted by the solid density in the bottom panel. This gives a positive number, because the bond accumulation function is convex and the solid distribution is a mean-preserving spread of the dashed one. Therefore, at the conjectured interest rate path, there is excess demand of bonds and we need a lower interest in the initial periods to equilibrate the bonds market. Intuitively, the economy begins with too many households at low levels of debt, with a strong incentive to save, and this pushes down equilibrium interest rates. As the economy reaches its new steady state, the lower tail of the distribution converges towards higher levels of bond holdings, the saving pressure subsides, and the interest rate adjusts up.

3.2 Transitional dynamics: output

Next, we want to understand what happens to output. The bottom right panel of Figure 3 shows that output contracts by 1% on impact and then recovers, converging towards a level slightly lower than the initial steady state. We will consider below variations leading to larger output responses. But first let us understand the mechanism in our baseline exercise.

The output response depends on the combination of consumption and labor supply decisions. To understand the transitional dynamics of output, let us make the same disequilibrium experiment of the last subsection and suppose the interest rate jumped directly to its new steady state level at $t = 1$. Recall that the consumption and labor supply policies in Figure 1 are, respectively, a concave function and a convex function of bond holdings. Then, given that the initial bond distribution is a mean-preserving spread of the new steady state distribution, average consumption demand is lower than at the new steady state and average labor supply is higher. Therefore, at the conjectured interest path, there is excess supply in the goods market, which corresponds to the excess demand in the bond market discussed above.

The short-run drop in the interest rate equilibrates the goods market both by increas-

ing consumption and by lowering labor supply, via intertemporal substitution channels. The market clearing level of output can then, in general, be above or below its new steady state level, depending on whether the adjustment is more on the consumption side or on the labor supply side. Two sets of considerations determine which side of the goods market dominates the adjustment path: (i) how large are the shifts in consumption and labor supply for a given interest rate, and (ii) how interest rate elastic are consumption and labor supply. Our parameters imply that the fall in consumption demand is the dominating factor, and output falls below its new steady state value.

Our numerical results show that different workers respond in different ways to the forces just described. In equilibrium, labor supply increases for low-productivity workers at the bottom end of the bond distribution, who are closer to the borrowing limit and are least interest-sensitive. At the same time, labor supply drops for high-productivity workers with high bond holdings, who are farther from the limit and are more interest-sensitive. So behind the drop in output there is a misallocation effect and a drop in average labor productivity.¹³

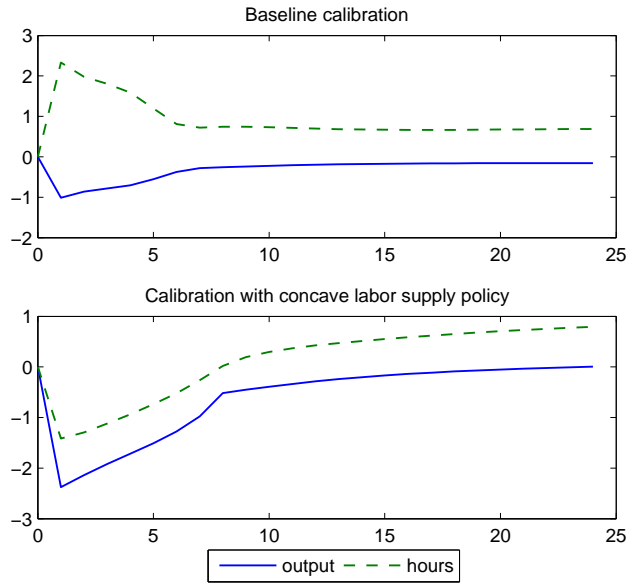
The top panel of Figure 5 shows the paths for output (solid blue line) and total hours (dashed green line) in our baseline calibration. Hours actually increase following a credit shock, reflecting the responses of the agents at the bottom of the bonds distribution. However, total output drops due to the misallocation effect.

The shape of the labor supply policies is crucial in determining the dynamics of hours. In particular, when the labor supply policy is less convex, the labor supply of poor households is less sensitive to a credit crunch. In the bottom panel of Figure 5, we plot output and hours for an alternative calibration in which the labor supply policy is less convex. To plot this figure, we reduce the upper bound on hours per week used to calibrate ψ .¹⁴ In this case, the labor supply policy function is actually concave in the relevant range. The result is that aggregate employment drops by more than one percentage point and the output drop is larger than 2%. The misallocation effect is still present, so average productivity decreases, although less than in the baseline calibration.

¹³This misallocation effect is closely related to the steady state labor misallocation analyzed in Heathcote, Storesletten, and Violante (2008).

¹⁴In particular, we assume that the marginal utility of leisure goes to infinity at 50 hours worked per week instead that at 98 hours. This implies that the ratio of average hours worked to the upper bound is 0.8 instead of 0.4 (see footnote 8).

Figure 5: *Output and Employment Responses*



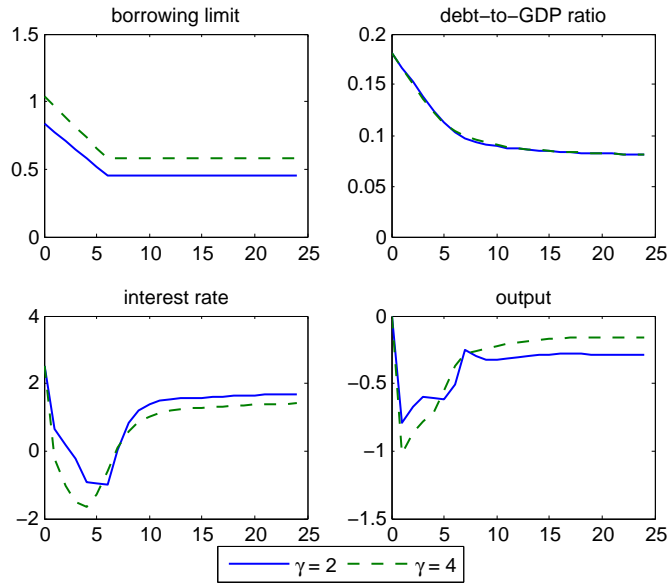
Note: Percent deviations from initial steady state.

Our model is able to generate a recession even with perfectly competitive goods and labor markets. Clearly, adding frictions on the supply side of the model can help in getting a more realistic picture of the effects of a credit crunch on aggregate activity and possibly on unemployment. The introduction of nominal rigidities in the next section is a step in that direction.

To further investigate the consumption response, it is useful to experiment with different values of γ . Figure 6 shows the behavior of the interest rate and output with $\gamma = 2$ (solid blue lines) and, for comparison, in our baseline with $\gamma = 4$ (dashed green lines).¹⁵ Different effects are at work here. On the one hand, the effect of lower risk aversion is to make the precautionary effect weaker and thus the consumption policy less concave. On the other hand, lower risk aversion also implies that consumers tend to borrow more in the initial steady state, so the initial distribution is more spread out to the left. These two effects go in opposite directions, the first decreasing and the second increasing the initial shift in consumer demand. Finally, a reduction in γ also implies a lower elastic-

¹⁵All calibrated parameters are re-calibrated when we change γ .

Figure 6: *Changing the Coefficient of Relative Risk Aversion γ*



Note: Interest rate is in annual terms. Output is in percent deviation from initial steady state.

ity of intertemporal substitution, which implies that consumer demand is more interest rate elastic. The overall effect is that we obtain a slightly smaller drop in the interest rate and output. However, since opposing effects are present the relation between γ and the initial drops in the interest rate and output is in general non-monotone.

4 Nominal Rigidities

Under flexible prices, the real interest rate is free to adjust to its equilibrium path to equilibrate the demand and supply of bonds, or—equivalently—the demand and supply of goods. In this section we explore what happens in a variant of the model with nominal rigidities. In presence of nominal rigidities, the central bank can affect the path of the real interest rate by setting the nominal interest rate. However, the zero lower bound for the nominal interest rate, together with nominal rigidities, implies that the central bank may not be able to replicate the real interest rate path corresponding to the flexible price equilibrium. Therefore, a credit crisis which produces a large drop in real interest

rates under flexible prices can drive the economy into a liquidity trap and into a deeper recession under sticky prices.

In order to introduce nominal rigidities, we first enrich the model with monopolistic competition. The set up is identical to the baseline, except that output is now produced by a continuum of monopolistically competitive firms owned by the households in equal shares. Each firm produces a good $j \in [0, 1]$ and consumption is a Dixit-Stiglitz aggregate of these goods. Namely, consumption of household i is given by

$$c_{it} = \left(\int_0^1 c_{it}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where $c_{it}(j)$ is household i consumption of good j . We interpret the shock θ_{it} as a shock to the efficiency of household i labor. Each firm produces with a linear technology which produces one unit of good with one efficiency unit of labor, that is, θ_{it} goods are produced with one hour of work of household i . The labor market is perfectly competitive and w_t denotes the nominal wage rate per efficiency unit, so the hourly wage for household i is $\theta_{it}w_t$.

Firms are owned by households, so the budget constraint is

$$q_t b_{it+1} + p_t c_{it} = b_{it} + w_t \theta_{it} n_{it} - \tilde{\tau}_{it} + \pi_t,$$

where p_t is the appropriate price index and π_t denotes total nominal profits. Bond holdings, bond prices, and taxes are expressed in nominal terms. Solving the consumers' expenditure minimization problem, it follows that monopolist j faces the demand

$$y_{jt} = (p_{jt}/p_t)^{-\varepsilon} C_t,$$

where C_t is aggregate consumption in the economy.

Consider first the case of perfectly flexible prices. In this case, the equilibrium is very similar to that of the perfectly competitive economy of the previous section. The only differences are that households receive some profit income on top of labor income and that the real wage per efficiency unit is $w_t/p_t = (\varepsilon - 1)/\varepsilon$, which is smaller than 1 because of the firms' markup. Therefore, as long as markups are not too big, the response of the economy to the credit tightening is similar to that of the baseline model.

Turning to the case of sticky prices, assume that firms face menu costs to adjust nominal prices. In particular, we make the extreme assumption that menu costs are so large

that firms leave prices at their initial level, in the equilibrium we analyze. Therefore, we have $p_{jt} = p_t = p_0$ for all j and t .

How does an equilibrium with fixed prices work? Since the price level is constant at p_0 , the nominal interest rate is equal to the real interest rate. The central bank, by choosing a sequence of nominal interest rates, chooses a sequence of real rates r_t . Since nominal wages are flexible, we need to find a sequence of wage rates w_t that ensures labor market clearing. These wage rates are in general different from $(\varepsilon - 1) / \varepsilon$.¹⁶ The assumption of sticky prices and flexible wages simplifies the analysis as it keeps the households problem essentially identical to the baseline model. In particular, the optimality condition for labor supply is now

$$(w_t / p_t) \theta_{it} U_c(c_{it}, n_{it}) + U_n(c_{it}, n_{it}) = 0, \quad (4)$$

for each household with $\theta_{it} > 0$ and the Euler equation is unchanged. Clearly, this assumption also means that when output contracts below its flexible equilibrium path, the real wage needs to fall to be consistent with an aggregate reduction in labor supply. It would be interesting, but outside the scope of this paper, to extend the analysis to allow for frictional labor markets and wage rigidities.

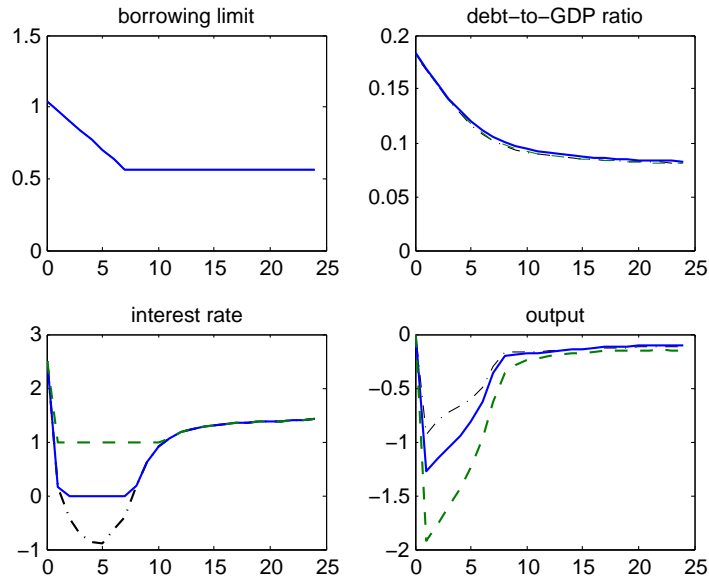
In the following exercises, we assume that the central bank chooses a path for the nominal interest rate r_t which converges to its flexible price steady state level. This implies that real wages converge to $(\varepsilon - 1) / \varepsilon$ so that that the firms' incentive to change nominal prices vanishes in the long run.¹⁷

In Figure 7 we consider three scenarios. The dash-dotted black line is the flexible price baseline. The solid blue line corresponds to a monetary policy in which the central bank tries to replicate the flexible-price interest rate path, with the only constraint that the interest rate cannot go negative. The dashed green line corresponds to a case in which the real interest rate is stuck at a higher value (1%). This can be interpreted either as a less aggressive monetary authority or as a situation in which the channel of transmission between the federal funds rate and longer term rates (relevant for consumers' saving and borrowing decisions) imposes a higher lower bound for the latter. The bot-

¹⁶The markup p_t / w_t is different from firms' desired markup $\varepsilon / (\varepsilon - 1)$, but firms cannot adjust prices due to the menu cost.

¹⁷All the parameters are calibrated as in the baseline and ε is chosen to yield profits to GDP equal to 5%.

Figure 7: Responses with Fixed Prices



Note: Interest rate is in annual terms. Output is in percent deviation from initial steady state.

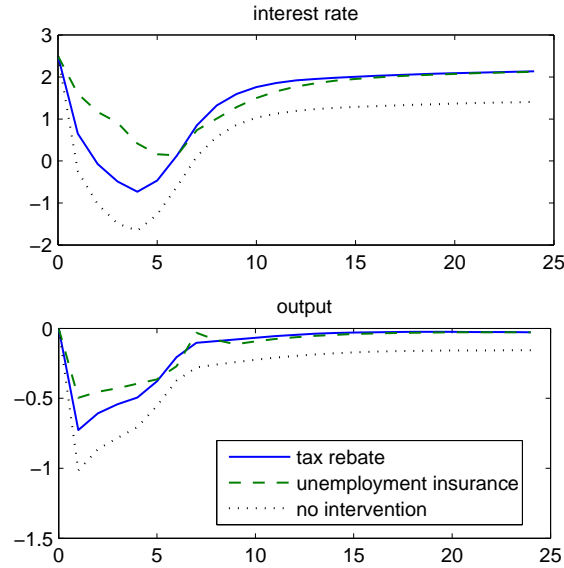
tom right panel of Figure 7 shows that the output responses are larger when the interest rate fails to adjust.

5 Fiscal policy

We now explore the role of fiscal policy in mitigating the recession. In particular, we focus on simple policies in which the government changes the supply of government bonds. Increasing the supply of bonds can be beneficial for two reasons. First, there is a direct increase in the supply of liquid assets that reduces the downward pressure on the real interest rate. Second, as the government increases bond supply, the associated deficit can be used to reduce taxes or increase transfers in the short run. In our economy, this has a positive effect on spending given that Ricardian equivalence does not hold.

Since we assume lump sum taxation, an equivalence result holds between government supplied and privately supplied liquidity. Namely, an increase in the supply of government bonds B_t can exactly offset a change in the borrowing limit ϕ . In particular,

Figure 8: *Fiscal Policy*



Note: Interest rate is in annual terms. Output is in percent deviation from initial steady state.

the only thing that matters for the equilibrium is the sum $B_t + \phi$. This is a common result in this class of models, and it implies that in principle the government could completely neutralize the effect of a credit shock, by a sufficiently large increase in the supply of government bonds. However, for the sake of realism, here we look at the effects of policies that only partially offset the long run change in ϕ , possibly because of unmodelled concerns with the distortionary effects of higher taxation in the long run.¹⁸

Consider, in particular, a policy of increasing gradually the supply of real bonds to a level B' that is 20% higher in the new steady state. Namely, assume that B_t follows the path

$$B_t = \rho_b^t B + (1 - \rho_b^t) B',$$

with $\rho_b = 0.95$. We then consider two different ways of spending the deficit associated to this increase in bond supply. First, we look at a policy where taxes adjust to balance the government budget in every period. Second, we look at a policy where the gov-

¹⁸Aiyagari and McGrattan (1998) study the trade off between distortionary taxation and the self-insurance benefits of government bonds in steady state.

ernment deficit is used to finance a temporary increase in the unemployment benefit. In particular, we let the unemployment benefit to be 50% higher for the first two years after the shock. Figure 8 shows what happens to the interest rate and output under these two policies. The solid blue lines represent the policy in which the increase in B_t finances a temporary reduction in the tax τ_t ; the dashed green lines represent the policy in which the deficit goes partly to finance an increase in the unemployment benefit v_t .

The figure shows that increasing the supply of government bonds dampens the responses of both interest rates and output. Moreover, increasing the unemployment benefit in the short run has larger effects than reducing the lump-sum tax, because it is a policy targeted towards households that are more likely to be credit constrained.

6 Durable Goods

In this section, we extend the baseline model to include durable goods. A large part of household borrowing is associated to durable purchases and takes the form of secured debt, in which durables serve as collateral. Therefore, a model with durables is more realistic in capturing both the motive for borrowing and the nature of the credit limit.

A model with durables enriches the household portfolio decision.¹⁹ As durables offer an alternative store of value, when the precautionary demand for assets increases, it can be directed not only towards bonds but also towards durables. This can potentially lead to an increase in durable accumulation as a result of an increase in precautionary savings. An opposing force is at work on the borrowers' side: reduced credit access implies that borrowers need to sell durables in order to reduce their debt. This leads to durable goods decumulation. Whether the force on the savers' side or on the borrowers' side dominates, depends on the model parameters and on the nature of the shock hitting the economy, as we will see shortly.

Households' portfolio decisions are also affected by the fact that durables are a less liquid form of savings than bonds. To capture the illiquidity of durable goods, we assume that households face a discount when re-selling durables. When households build up precautionary reserves following a credit shock, they tend to prefer more liquid as-

¹⁹Krueger and Fernandez-Villaverde (2011) emphasize the importance of durable wealth to understand households' life cycle consumption and portfolio allocation.

sets, favoring bonds over durable goods. This reduces the increase in durable demand by savers and tends to generate an overall reduction in durable purchases. The interesting finding here is that in a model with liquid and illiquid assets a credit shock can lead, at the same time, to an increase in demand for the liquid asset and to a reduction in demand for the illiquid asset. This captures a form of “flight to liquidity” on the household side.

Households preferences are now represented by the utility function

$$E \left[\sum_{t=0}^{\infty} \beta^t U(c_{it}, k_{it}, n_{it}) \right],$$

where c_{it} is non-durable consumption, the service flow from durables is proportional to the stock of durables k_{it} , and n_{it} is labor effort.

Each period durables depreciate at the rate δ and the household chooses whether to increase or decrease its durable stock. A household that wants to increase its durable stock to $k_{it+1} > k_{it}$ needs to spend $k_{it+1} - k_{it}$ plus δk_{it} to cover the depreciation of the existing stock. A household that wants instead to reduce its durable stock to $k_{it+1} < k_{it}$ faces additional reselling costs proportional to the capital sold, $\zeta (k_{it} - k_{it+1})$. The parameter $\zeta > 0$ determines the illiquidity of durable goods. These assumptions are summarized in the adjustment cost function

$$g(k_{it+1}, k_{it}) = \begin{cases} k_{it+1} - k_{it} + \delta k_{it} & \text{if } k_{it+1} \geq k_{it} \\ (1 - \zeta)(k_{it+1} - k_{it}) + \delta k_{it} & \text{if } k_{it+1} < k_{it} \end{cases}.$$

We assume that $1 - \zeta > \delta$, so the household can always liquidate part of its durable stock to cover for depreciation.²⁰

We also extend the model to introduce a spread between borrowing and lending interest rates. Specifically, we assume that if a household is a net seller of bonds, i.e., if $b_{it+1} < 0$, the household needs to buy intermediation services from a competitive banking sector. A banking firm incurs a proportional intermediation cost of χ per dollar of bonds issued, which captures monitoring and collection costs. This implies that the

²⁰An alternative approach to modeling transaction costs—made in Grossman and Laroque (1990) and Gruber and Martin (2003)—is to assume that when agents choose $k_{t+1} \neq k_t$ they have to sell k_t at price $(1 - \zeta)k_t$ and buy k_{t+1} at full price. So $g(k_{t+1}, k_t) = k_{t+1} - (1 - \zeta)k_t + \delta k_t$ if $k_{t+1} \neq k_t$ and $g(k_{t+1}, k_t) = \delta k_t$ otherwise. A large literature takes explicitly into account the lumpiness of durable purchases associated to various fixed costs of adjustment (e.g., Caballero, 1990, and more recently Leahy and Zeira, 2005). An advantage of our approach is that it keeps the household’s problem concave.

household receives a net price $\hat{q}_t = (1 - \chi) q_t$ per bond issued and banks make zero profits. Letting b_{it}^+ denote positive bond holdings and b_{it}^- denote negative holdings, the household's budget constraint is then

$$q_t b_{it+1}^+ + \hat{q}_t b_{it+1}^- + g(k_{it+1}, k_{it}) + c_{it} + \tilde{\tau}_{it} \leq b_{it} + y_{it},$$

where the tax $\tilde{\tau}_{it}$ depends on the household's productivity θ_{it} as in the baseline model. The spread between borrowing and lending rates is equal to

$$\frac{1}{\hat{q}_t} - \frac{1}{q_t} = \frac{1}{q_t} \frac{\chi}{1 - \chi},$$

and is approximately equal to χ for low values of χ and r_t .

The production side of the model is as in the benchmark model, with a linear production function $y_{it} = \theta_{it} n_{it}$ and an exogenous Markov process for θ_{it} . For simplicity, durable and non-durable goods are produced with the same technology, so the relative price of durables is 1.²¹

The household's borrowing constraint is

$$b_{it+1} \geq -\phi_k k_{it+1}. \quad (5)$$

The household debt is collateralized by its durable holdings. The parameter ϕ_k denotes the fraction of the value of the durable that can be used as collateral, that is, the maximum loan-to-value ratio.

The government budget constraint is unchanged:

$$B_t + v_t u = q_t B_{t+1} + \tau_t.$$

As in the baseline, we fix the supply of government bonds and the unemployment benefit at the levels B and v , and let the tax τ_t adjust to satisfy budget balance.

6.1 Equilibrium and calibration

The main difference with the baseline model is that durable goods are now an additional state variable. Optimal household policies are now functions of the three-dimensional

²¹Different technological assumptions would introduce endogenous price dynamics for durables, possibly adding an amplification channel à la Kiyotaki and Moore (1997).

state (b, k, θ) : the initial stock of bonds, the initial stock of durables, and current productivity. These three states fully determine the household's choice of non-durable and durable purchases, labor supply and the optimal level of borrowing or lending.

Let $\Psi_t(b, k, \theta)$ denote the joint distribution of b , k and θ in the population. Combining the household's optimal transition for bond holdings and durable goods with the exogenous Markov process for productivity, we obtain the transition probability of the individual state, and, aggregating, a transition for the distribution Ψ_t . The definition of equilibrium is then the natural generalization of definition 1, where the bonds market clearing condition is now

$$\int b d\Psi_t(b, k, \theta) = B.$$

To calibrate the model we adopt the utility function:

$$U(c, k, n) = \frac{(c^\alpha k^{1-\alpha})^{1-\gamma}}{1-\gamma} + \psi \frac{(1-n)^{1-\eta}}{1-\eta}.$$

We choose a simple Cobb-Douglas specification to aggregate durable and non-durable consumption.²² Therefore, α is the ratio of non-durable consumption to total consumption. To compute this ratio we compute durables as the sum of durable consumption and consumption of housing services from NIPA. We take all other consumption (non-durable goods and non-housing services) as nondurables. The average value for 2000-2010 gives us $\alpha = 0.7$. As in our baseline exercise, we choose β to get a 2.5% yearly interest rate, set the coefficient of risk aversion $\gamma = 4$, choose η to obtain an average Frisch elasticity of 1, and choose ψ so that average hours worked are 40% of the time endowment. Also for the wage process, the transitions between employment and unemployment, and the unemployment benefit we follow our baseline calibration.²³

For the accumulation of durable goods, we need to choose δ and ζ . We set $\delta = 1.29\%$ to match the depreciation rate from NIPA Fixed Assets Tables. The parameter ζ represents the cost of selling durable goods and captures their illiquidity. We set it to 15%.

Finally we need to choose ϕ_k , the intermediation cost χ , and the bond supply B . We set ϕ_k to 0.8, which is in the range of loan-to-value ratios in mortgages and durable

²²Ogaki and Reinhart (1998) offer evidence in favor of an elasticity of substitution between durables and non-durables close to 1.

²³However, we now approximate the wage with a 5-state Markov chain, for computational reasons.

Table 2: *Parameter Values: Durable Model*

Parameter	Explanation	Value	Target/Source
β	Discount factor	0.9713	Interest rate $r = 2.5\%$
γ	Coefficient of relative risk aversion	4	
η	Curvature of utility from leisure	1.50	Average Frisch elasticity = 1
ψ	Coefficient on leisure in utility	2.54	Average hours worked = 0.4
α	Coefficient on non-durables	0.7	Ratio of non-durable and non-housing services to total personal consumption expenditures in NIPA (2000-10 average)
δ	Durables depreciation rate	0.0129	BEA Fixed Asset Tables ratio of depreciation to net stock, (2000-8 average, Hall, 2011b)
ζ	Proportional loss on durable sales	0.15	
χ	Intermediation cost	0.01	
ρ	Persistence of productivity shock	0.967	Persistence of wage process in Floden and Lindé (2001)
σ_ε	Variance of productivity shock	0.017	Variance of wage process in Floden and Lindé (2001)
$\pi_{e,u}$	Transition to unemployment	0.057	Shimer (2005)
$\pi_{u,e}$	Transition to employment	0.882	Shimer (2005)
v	Unemployment benefit	0.160	40% of average labor income (Shimer, 2005)
ϕ_k	Max loan-to-value ratio	0.8	
B	Bond supply	1.60	Liquid-assets-to-GDP ratio of 1.78

Note: The quantities v and B are expressed in terms of yearly aggregate output. See the text for details on the targets.

loans.²⁴ The parameter χ is set at 1%, so our exercise starts from a fairly low initial spread. The supply of government bonds B is chosen as in the baseline, to match the ratio of liquid assets to GDP equal to 1.78. The parameters used are summarized in Table 2.

6.2 Characterization and steady state

The new ingredient, relative to the baseline problem, is that households face a portfolio choice. Each period, households choose their labor effort n_{it} and non-durable consumption c_{it} as in the baseline model. These choices determine their saving, gross of durable purchases, $y_{it} - c_{it} - \tilde{\tau}_{it}$ (from now on, “gross saving”). But then they also need to choose how to allocate this saving between durable purchases and bond accumulation. The optimality conditions characterizing household behavior are derived in the online appendix. Here we just provide some intuition for the optimal portfolio dynamics.

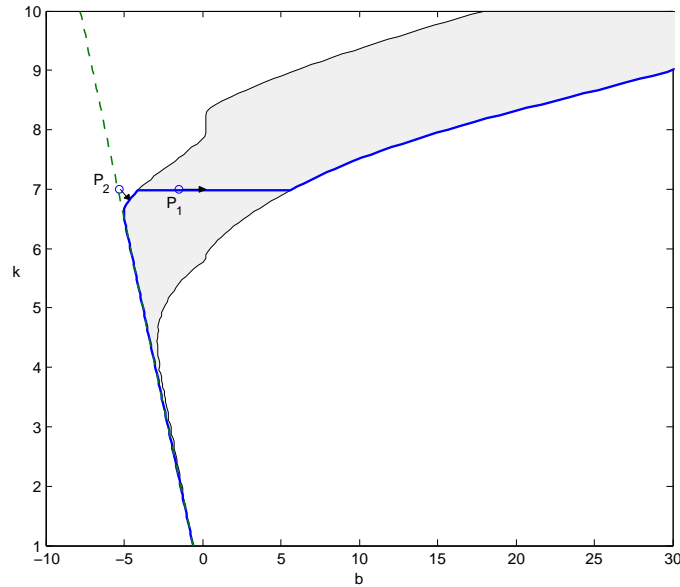
The kinked adjustment cost for durables implies that the optimal portfolio is characterized by two adjustment bands. In particular, for a given productivity θ_{it} , the optimal portfolio (b_{it+1}, k_{it+1}) always lies in a region like the grey region in Figure 9 (which corresponds to $\theta_{it} = 1$). The dashed green line corresponds to the borrowing limit, which is proportional to durable holdings. Given an initial capital stock, say $k_{it} = 7$, the locus of possible optimal portfolios is given by the solid blue line.²⁵ Our numerical analysis shows that a household starting at portfolio P_1 ($b_{it} = -1.5, k_{it} = 7$) will choose positive gross saving, keep its durable holdings unchanged, and allocate its gross saving to debt repayment, moving along the arrow originating at P_1 . A more indebted household, starting at portfolio P_2 ($b_{it} = -5.3, k_{it} = 7$) will also choose positive gross saving, but it will also sell some durables and use its gross saving plus the receipts from the durables sale to repay its debt, moving along the arrow originating at P_2 so as to reach the blue line. The common element is that all households starting at k_{it} , irrespective of their initial bond holdings, will choose an end-of-period portfolio on the blue line.²⁶

²⁴The debt-to-GDP ratio we obtain is 54%. We do not try to choose ϕ_k to match observed debt-to-GDP ratios in the household sector (which prior to the crisis went above 100%) because, given our other parameters, our model cannot deliver debt-to-GDP ratios above 70%.

²⁵See the appendix for a formal definition of this locus.

²⁶Notice that both boundaries of the adjustment region have a vertical segment at $b = 0$. This vertical

Figure 9: *Portfolio Choice in the Model with Durables*



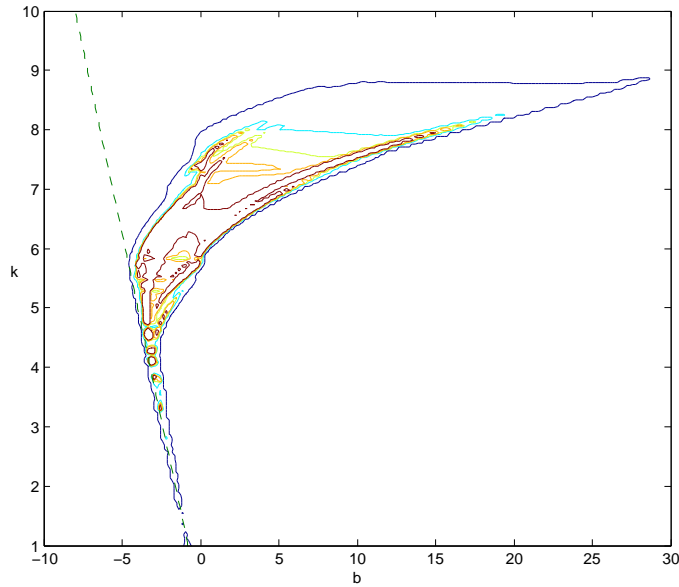
The shape of the adjustment region in Figure 9 is similar for all values of θ . Therefore, the support of the steady state distribution of bonds and durable holdings takes a similar shape, as can be seen in Figure 10. In this figure, we plot the contours of the steady state distribution. The dashed green line represents again the borrowing limit. At low levels of total wealth (bonds plus durables) we find households who hold small durable stocks and small amounts of debt. If a household receives a positive productivity shock, it responds by accumulating durables and taking on more debt, given that the shock is expected to persist. If the household stays at high productivity, it eventually starts to pay off its debt and then goes on to accumulate positive bond holdings. If instead the household is hit by a negative shock, in a first phase it will adjust only by selling bonds and, in a second phase, it will adjust also by selling durables.²⁷

The portfolio dynamics just described help to account for the fact that the distribution tends to be concentrated at the boundaries of the adjustment regions, given that if

segment is due to the spread between borrowing and lending rates.

²⁷Depending on the shock and the initial bond holdings, the first phase can be absent and the household can start selling durables right away.

Figure 10: *Steady State Distribution of Bonds and Durables*



Note: The contour lines correspond to the 0.1%, 10%, 20%, 30%, and 50% percentiles.

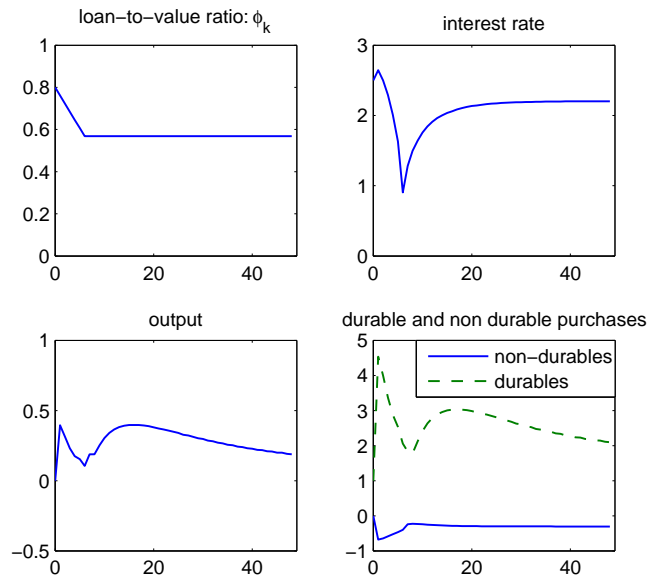
households are on the boundary and are not hit by a shock, they remain on the boundary. Moreover, there is a mass of households at, or near, the borrowing limit. Unlike in the baseline model, they are not only the households with the lowest total wealth (bonds plus durable holdings), but also middle-wealth households with levered holdings of durable goods.

6.3 Credit crunch

For the model with durables, we consider three different credit-tightening exercises, by looking at the effects of a permanent reduction in the borrowing limit ϕ_k , and a permanent and a temporary increase in the spread χ . As in our baseline exercise, all these shocks are unexpected and hit the economy in steady state. All three exercises feature overshooting in the interest rate as in the baseline, although with different strengths. On the output side, however, the implications of the three shocks are quite different.

Our first exercise is a permanent contraction in the borrowing limit ϕ_k from 0.8 to 0.56, which yields a 10 percentage points reduction in the household debt-to-GDP ratio

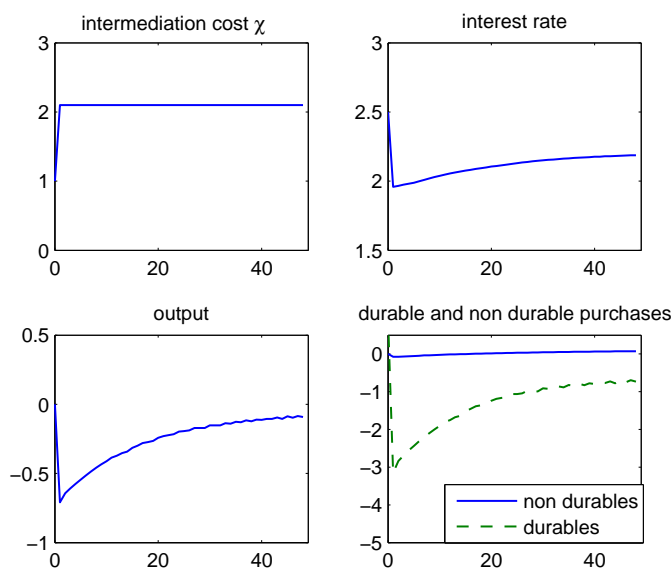
Figure 11: Responses to a Shock to the Borrowing Limit ϕ_k



Note: Interest rate is in annual terms. Quantities are in percent deviation from initial steady state.

from 54% to 44%. The contraction in ϕ_k is gradual and follows a linear path that lasts 6 quarters. Figure 11 shows that the contraction in the interest rate is less strong than in our baseline exercise and that output actually increases by a 0.4%. The reason behind these results is that durable purchases are a much more interest elastic component of consumer spending. So a smaller interest rate reduction is needed to equilibrate the goods market (and hence the bonds market) and total spending is actually higher at the new equilibrium. This is confirmed by the bottom right panel of Figure 11, which shows that there is a contraction in non-durable spending, similar in size to the contraction obtained in our baseline, but this contraction is more than compensated by a 4% increase in durable spending. Numerical experiments show that this increase in durable spending is due to the interest rate adjustment: a simple disequilibrium exercise shows that durable spending would drop by about 18% if the interest rate adjusted immediately to its new long run level, which is 2.2%. A short lived drop in the interest rate to 0.9% is sufficient to turn a 18% contraction in durable spending into a 4% increase. This may

Figure 12: Responses to a Permanent Shock to the Intermediation Cost χ

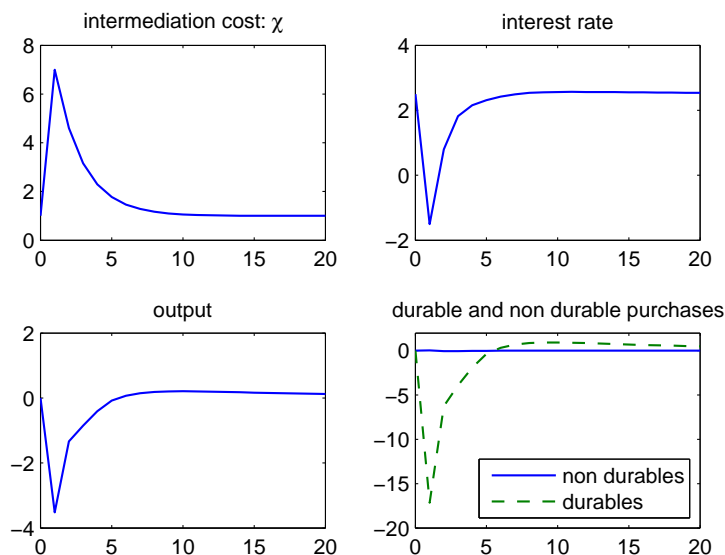


Note: Interest rate is in annual terms. Quantities are in percent deviation from initial steady state.

seem an unrealistically large interest elasticity of durable spending which indicates that in our model bonds and durables remain very good substitutes, despite the illiquidity cost. This points in the direction of extending the model using alternative specifications of the durables adjustment cost or accounting explicitly for the price risk associated with durable purchases (especially of housing), to reduce the substitutability between the two assets. We leave these developments to future work.

Our second experiment is to look at the effects of a spread shock: a permanent increase of the intermediation cost χ from 1% to 2.21%. As in previous experiments, the size of the shock is chosen to obtain a 10 percentage point long run reduction in the debt-to-GDP ratio. The results are presented in Figure 12. The effects of this shock are much more gradual, as there is no forced deleveraging and borrowers are allowed to adjust their borrowing positions over time. However, the shock is more pervasive, as it affects all borrowers and not just those near the borrowing limit. Therefore, the effect is a drop in output, with a contraction in durable purchases of about 3% and an almost

Figure 13: Responses to a Temporary Shock to the Intermediation Cost χ



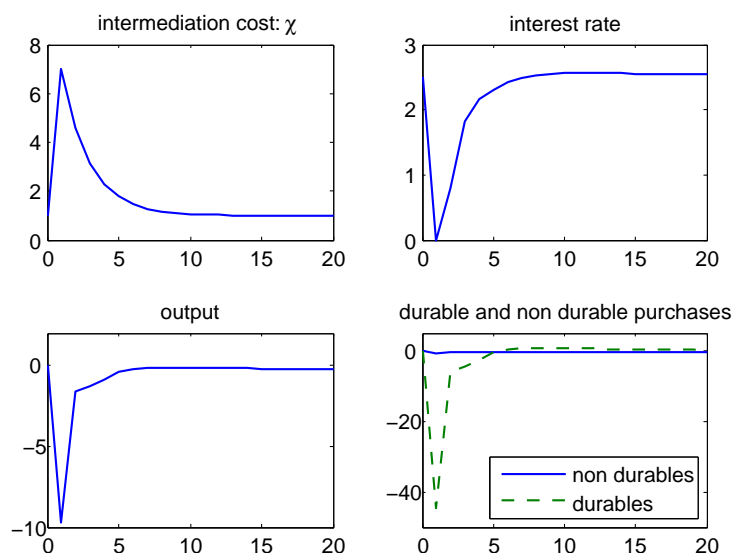
Note: Interest rate is in annual terms. Quantities are in percent deviation from initial steady state.

negligible drop in non-durables. The smoother response of non-durables is due to the fact that the shock is less concentrated on the agents near the borrowing limit, who have a higher marginal propensity to consume out of liquid wealth.

Given the gradual nature of the spread shock just analyzed, it is useful to also consider a larger, but temporary spread shock. Therefore, we now look at the effects of a shock that increases the intermediation cost by 6 percentage points at an annual rate. We assume the shock decays geometrically with a rate of decay of 0.6.²⁸ This implies that the rate on a 1 year loan goes up by about 3.9% in the first quarter after the shock. Hall (2011a) uses the same shock (in the context of a different model) and argues that it is a reasonable representation of the credit shock experienced in the U.S. 2008-2009 recession. The responses to this shock are in Figure 13. The shock has a much larger, but short lived effects on quantities, with a 3.5% output drop. As in the case of a persistent shock, the adjustment is now all in durables (−17.2%), while non-durables are essen-

²⁸That is, we have $\chi_t = 0.0025 + 0.015 \cdot 0.6^{-(t-1)}$, for $t = 1, 2, \dots$

Figure 14: Responses to a Temporary Shock to the Intermediation Cost χ , with Fixed Prices



Note: Interest rate is in annual terms. Quantities are in percent deviation from initial steady state.

tially unchanged. This shock is sufficiently large to drive the interest rate into negative values. Therefore, we can ask what happens with fixed prices and a central bank that tries to replicate the flexible price interest rate path subject to the zero lower bound. As in Section 4, we extend the model by introducing monopolistic competition and large menu costs. The results for this case are in Figure 14. Now there is a very large contraction in durable purchases (-44.5%), which leads to a 9.7% output contraction. Once more, we see the effects of the very large interest elasticity of durable purchases. Sticky prices cause the real interest rate to be off by about 1.5% , relative to the flexible price case, and this is sufficient to reduce durable purchases by an additional 27% , causing a much deeper recession. As argued above, it would be interesting to develop models where durable purchases are less interest elastic.

It is useful to remark two differences between our baseline model and our model with durables. First, the two calibrations lead to very different values for household total net worth. In the baseline, households only hold liquid wealth, and net worth

over GDP is 1.60.²⁹ In the model with durables, households also hold durable wealth, and net worth over GDP is 5.27.³⁰ However, net worth to GDP is not the only variable determining how important are liquidity constraints, given the different liquidity of the two assets. This point is related to Kaplan and Violante (2011), who also emphasize that adjustment costs imply that “rich” households’ consumption behavior can still be far from permanent income predictions.

Second, as argued above, in the baseline model the most indebted households are the households with lowest total wealth, while here they are intermediate-wealth households, with large levered positions in durables. These households can still be induced to adjust nondurable consumption if they are close to their constraint, as seen from the nondurable response in our first exercise (Figure 11). However, when we look at a spread shock that hits all indebted households equally, the typical household hit by the shock now prefers to smooth nondurable consumption and adjust to the shock by selling durables. This helps to explain why the nondurable response is muted in our second and third exercises (Figures 12 and 13) while there is a larger adjustment in durables. In future work, it will be interesting to explore further different combinations of shocks to loan-to-value ratios and to spreads, to understand how they affect differentially households with different initial portfolios, and to compare these results with empirical evidence on the disaggregated response of consumption.

7 Concluding Remarks

We have proposed a model with uninsurable idiosyncratic risk to show how a credit crunch can generate a recession with low interest rates, due to a combination of debt repayments and an increase in precautionary savings. This helps to explain why recessions driven by financial market trouble are more likely to drive the economy into a liquidity trap.

A simplifying assumption in our model is that the unemployment risk is exogenous and not affected by the credit crunch. It would be interesting to develop a version of the model with an explicit treatment of labor market frictions, in which the labor market

²⁹Liquid wealth to GDP is 1.78 and debt to GDP is 0.18.

³⁰Liquid wealth to GDP is 1.78, durable wealth over GDP is 4.03, and debt to GDP is 0.54.

response to a drop in consumer demand leads to an endogenous increase in unemployment.³¹

Finally, a missing element in the analysis is capital. Adding capital to the model requires a theory of why claims to physical capital cannot be costlessly transformed into perfectly liquid assets like the bonds of our model. A way to move in this direction would be to combine our analysis of the household sector with financial frictions on the firms' side or a richer model of intermediation.

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³¹Krusell, Mukoyama, and Sahin (2010) introduce search frictions in an Aiyagari (1994) incomplete markets environment. Hall (2011c) discusses ways of adding search frictions to monetary environments with a binding zero lower bound on the nominal interest rate.

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Appendix [For Online Publication]

A.1 Baseline model

Here we describe the algorithm used to compute steady states and transitional dynamics of the baseline model. The MATLAB codes are available on our web pages.

Let us begin from the steady state computations. First, we describe how optimal policies and the bond distribution are computed for a given steady state interest rate r . To compute the policy functions $C(b, \theta)$ and $N(b, \theta)$, we iterate on the Euler equation and the optimality condition for labor supply on a discrete grid for the state variable b . To iterate on the policy functions, we use the endogenous gridpoints method of Carroll (2006). To compute the invariant distribution $\Psi(b, \theta)$ we derive the inverse of the bond accumulation policy, denoted by $g(b, \theta)$, from the policy functions, and update the conditional bond distribution using the formula $\Psi_{(k)}(b|\theta) = \sum_{\tilde{\theta}} \Psi_{(k-1)}(g(b, \tilde{\theta})|\tilde{\theta})P(\tilde{\theta}|\theta)$ for all $b \geq -\phi$, where k stands for the k -th iteration and $P(\tilde{\theta}|\theta)$ is the probability of $\theta_{t-1} = \tilde{\theta}$ conditional on $\theta_t = \theta$. Due to the borrowing constraint, the bond accumulation policy is not invertible at $b = -\phi$, but the formula above still holds defining $g(-\phi, \theta)$ as the largest b such that $b' = -\phi$ is optimal. Finally, we search for the interest rate r that clears the bond market.

To compute transitional dynamics, we get the initial bond distribution $\Psi_0(b, \theta)$ from the initial steady state. We then compute the final steady at $\phi = \phi'$. We choose T large enough that the economy is approximately at the new steady state at $t = T$ (we use $T = 200$ in the simulations reported). Next, we guess a path of interest rates $\{r_t\}$ with $r_T = r'$. We take the consumption policy to be at the final steady state level at $t = T$, setting $C_T(b, \theta) = C'(b, \theta)$, and we compute the sequence of policies $\{C_t(b, \theta), N_t(b, \theta)\}$ using the Euler equation and the optimality condition for labor supply, going backward from $t = T - 1$ to $t = 0$ (using the endogenous gridpoints method). Next, we compute the sequence of distributions $\Psi_t(b, \theta)$ going forward from $t = 0$ to $t = T$, starting at $\Psi_0(b, \theta)$, using the optimal policies $\{C_t(b, \theta), N_t(b, \theta)\}$ to derive the bond accumulation policy (using the same updating formula as in the steady state). We then compute the aggregate bond demand B_t for $t = 0, \dots, T$ and update the interest rate path using the simple linear updating rule $r_t^{(k)} = r_t^{(k-1)} - \epsilon(B_t^{(k)} - \bar{B})$. Choosing the parameter $\epsilon > 0$ small enough the algorithm converges to bond market clearing for all $t = 0, \dots, T$.

A.2 Model with durables

A.2.1 Derivations

Here we derive optimality conditions for the model with durables. We focus on the steady state for ease of notation, but analogous derivations apply to the transitional dynamics (adding time subscripts). The Bellman equation is

$$V(b, k, \theta) = \max_{c, n, k', b'} U(c, k, n) + \beta E [V(b', k', \theta') | \theta]$$

$$s.t. \quad b + \theta n - \tau(\theta) \geq qb'_+ + \hat{q}b'_- + g(k', k) + c,$$

$$b' + \phi_k k' \geq 0.$$

The first order conditions for this problem are as follows. For c and n :

$$U_c(c, k, n) = \lambda,$$

$$-U_n(c, k, n) = \theta\lambda \text{ and } n > 0 \text{ or } -U_n(c, k, n) \geq \theta\lambda \text{ and } n = 0.$$

For b' and k' :

$$\beta E [V_b(b', k', \theta') | \theta] + \mu = q\lambda \text{ if } b' > 0,$$

$$\beta E [V_b(b', k', \theta') | \theta] + \mu = \hat{q}\lambda \text{ if } b' < 0,$$

$$q\lambda \geq \beta E [V_b(b', k', \theta') | \theta] + \mu \geq \hat{q}\lambda \text{ if } b' = 0$$

$$\beta E [V_k(b', k', \theta') | \theta] + \mu\phi_k = \lambda \text{ if } k' > k,$$

$$\beta E [V_k(b', k', \theta') | \theta] + \mu\phi_k = \lambda(1 - \zeta) \text{ if } k' < k,$$

$$\lambda \geq \beta E [V_k(b', k', \theta') | \theta] + \mu\phi_k \geq \lambda(1 - \zeta) \text{ if } k' = k.$$

The complementary slackness condition for the borrowing constraint requires that $b' + \phi_k k' = 0$ if $\mu > 0$ and $\mu = 0$ if $b' + \phi_k k' > 0$. The locus of optimal portfolios depicted in Figure 9 corresponds to the set of pairs (b', k') that satisfy the optimality conditions for b' and k' and the complementary slackness condition for μ , for some positive λ .

The envelope conditions are as follows. For b ,

$$V_b(b, k, \theta) = \lambda;$$

for k ,

$$V_k(b, k, \theta) = U_k(c, k, n) + \lambda(1 - \delta) \text{ if } k' > k$$

$$V_k(b, k, \theta) = U_k(c, k, n) + \lambda(1 - \delta - \zeta) \text{ if } k' < k$$

and

$$V_k(b, k, \theta) = U_k(c, k, n) - \delta\lambda + \beta E [V_k(b', k', \theta') | \theta] + \mu\phi_k \text{ if } k' = k.$$

Using the first order conditions, the envelope condition for k can be written compactly as

$$V_k(b, k, \theta) = U_k(c, k, n) - \delta\lambda + \beta E [V_k(b', k', \theta') | \theta] + \mu\phi_k.$$

A.2.2 Computation

Here we describe the algorithm used to compute the model with durables. The MATLAB codes are available on our web pages.

The computation of the model with durables also exploits the endogenous gridpoints method. However, adapting this method to the case of two endogenous state variables requires some extra steps, which are described here. Our approach is similar to Hintermeier and Koeniger (2010), in that we first find the subset of potentially optimal portfolios in the space (b', k') and then take the backward step typical of the endogenous gridpoints method only starting from pairs (b', k') in this subspace. However, unlike Hintermeier and Koeniger (2010), our approach focuses on computing the partial derivatives of the value function instead that on the policy functions.

Define

$$V_b(b', k', \theta) \equiv E [V_b(b', k', \theta') | \theta], \quad (6)$$

$$V_k(b', k', \theta) \equiv E [V_k(b', k', \theta') | \theta]. \quad (7)$$

Our objective is to approximate the functions V_b and V_k with piecewise linear functions on the discrete grids $\{b^0, \dots, b^n\}$ and $\{k^0, \dots, k^m\}$. We start with an initial guess for V_b and V_k and proceed as follows.

1. *Find the set of pairs (b', k') that are optimal for some state (b, k, θ) .* To do so, let k and k' vary (independently) on the grid $\{k^0, \dots, k^m\}$ and let θ vary on $\{\theta^0, \dots, \theta^S\}$. For each tripe (k, k', θ) , three cases are possible: $k' > k$, $k' < k$ and $k' = k$. In each case, we want to find the value(s) of b' consistent with optimality.
 - (a) If $k' > k$, choose the value of b' that satisfies one of the following optimality

conditions:

$$\begin{aligned}
& qV_k(b', k', \theta) = V_b(b', k', \theta) \text{ and } b' > 0, \\
& \text{or } qV_k(b', k', \theta) \geq V_b(b', k', \theta) \geq \hat{q}V_k(b', k', \theta) \text{ and } b' = 0, \\
& \text{or } \hat{q}V_k(b', k', \theta) = V_b(b', k', \theta) \text{ and } -\phi_k k' < b' < 0, \\
& \text{or } \hat{q}V_k(b', k', \theta) \geq V_b(b', k', \theta) \text{ and } b' = -\phi_k k'.
\end{aligned}$$

(b) If $k' < k$, choose the value of b' that satisfies one of the following optimality conditions:

$$\begin{aligned}
& qV_k(b', k', \theta) = (1 - \zeta) V_b(b', k', \theta) \text{ and } b' > 0, \\
& \text{or } qV_k(b', k', \theta) \geq (1 - \zeta) V_b(b', k', \theta) \geq \hat{q}V_k(b', k', \theta) \text{ and } b' = 0, \\
& \text{or } \hat{q}V_k(b', k', \theta) = (1 - \zeta) V_b(b', k', \theta) \text{ and } -\phi_k k' < b' < 0, \\
& \text{or } \hat{q}V_k(b', k', \theta) \geq (1 - \zeta) V_b(b', k', \theta) \text{ and } b' = -\phi_k k'.
\end{aligned}$$

(c) If $k' = k$, there is an interval of values of b' consistent with optimality, which we denote $[b'_L, b'_U]$. b'_L is the value that solves the conditions in (1.a) and b'_U is the value that solves the conditions in (1.b). Clearly, in some cases $b'_L = b'_U$ and the interval is degenerate.

2. *Derive the associated values of the Lagrange multipliers.* For each tripe (k, k', θ) , given the value(s) of b' found in step 1, we derive values for the Lagrange multipliers λ and μ . Again, there are three cases.

(a) $k' > k$. If the associated b' is equal to $-\phi_k k'$ find μ that solves

$$\beta V_b(b', k', \theta) + \mu = \beta \hat{q}V_k(b', k', \theta) + \phi_k \hat{q}\mu,$$

otherwise set $\mu = 0$. Then set $\lambda = \beta V_k(b', k', \theta) + \phi_k \mu$.

(b) $k' < k$. If the associated b' is equal to $-\phi_k k'$ find μ that solves

$$\beta (1 - \zeta) V_b(b', k', \theta) + (1 - \zeta) \mu = \beta \hat{q}V_k(b', k', \theta) + \phi_k \hat{q}\mu,$$

otherwise set $\mu = 0$. Then set $\lambda = \beta V_k(b', k', \theta) + \phi_k \mu$.

(c) $k' = k$. Now there are in general different triples (b', μ, λ) consistent with optimality. We derive them as follows, depending on the values b'_L and b'_U derived in (1.c).

i. If $b'_L = b'_U = -\phi_k k'$, form a sequence of triples (b', μ, λ) with $b' = -\phi_k k'$, μ taking values in the interval

$$\left[\beta \frac{\hat{q}V_k(b', k', \theta) - V_b(b', k', \theta)}{1 - \phi_k \hat{q}}, \beta \frac{\hat{q}V_k(b', k', \theta) - (1 - \zeta)V_b(b', k', \theta)}{1 - \zeta - \phi_k \hat{q}} \right],$$

and $\lambda = (\beta V_b(b', k', \theta) + \mu) / \hat{q}$.

ii. If $b'_L = -\phi_k k' < b'_U$, form a sequence of triples (b', μ, λ) as follows: first, a sequence of triples with $b' = -\phi_k k'$, μ taking values in the interval

$$\left[0, \beta \frac{\hat{q}V_k(b', k', \theta) - (1 - \zeta)V_b(b', k', \theta)}{1 - \zeta - \phi_k \hat{q}} \right],$$

and $\lambda = (\beta V_b(b', k', \theta) + \mu) / \hat{q}$; next, a sequence with b' taking values in the interval $(b'_L, b'_U]$, $\mu = 0$, and $\lambda = \beta V_b(b', k', \theta) / \hat{q}$ if $b' < 0$ and $\lambda = \beta V_b(b', k', \theta) / q$ if $b' > 0$.

iii. If $-\phi_k k' < b'_L < b'_U$, form a sequence of triples (b', μ, λ) with b' taking values in $[b'_L, b'_U]$, $\mu = 0$, and $\lambda = \beta V_b(b', k', \theta) / \hat{q}$ if $b' < 0$ and $\lambda = \beta V_b(b', k', \theta) / q$ if $b' > 0$.

iv. Finally, if $b'_U \geq 0 \geq b'_L$, add to the sequences of triples (b', μ, λ) in (i)-(iii) a sequence with $b' = 0$, $\mu = 0$, and λ taking values in the interval $[\beta \max \{V_b/q, V_k\}, \beta \min \{V_b/\hat{q}, V_k/(1 - \zeta)\}]$.

3. Derive the associated values of the control variables and of the initial state b . For each combination $(k, k', \theta, b', \mu, \lambda)$ derived in 1 and 2, compute c, n, b that solve

$$\begin{aligned} U_c(c, k, n) &= \lambda, \\ -U_n(c, k, n) &= \theta \lambda, \end{aligned}$$

and

$$b = qb^{'+} + \hat{q}b'^{-} + g(k', k) + c - \theta n + \tilde{\tau}(\theta).$$

4. Update V_b and V_k . For each combination $(k, k', \theta, b', \mu, \lambda, c, n, b)$ derived in 1-3, use the envelope conditions

$$\begin{aligned} V_b(b, k, \theta) &= \lambda, \\ V_k(b, k, \theta) &= U_k(c, k, n) - \delta\lambda + \beta V_k(b', k', \theta) + \phi_k \mu, \end{aligned}$$

conditions (6)-(7) and the Markov process for θ to compute new values of V_b and V_k . The values of k are on the grid $\{k^0, \dots, k^m\}$ by construction, but the values of b are not in $\{b^0, \dots, b^n\}$, so in this step we use a linear interpolation to compute the values on the grid $\{b^0, \dots, b^n\}$.

Steps 1 to 4 are repeated until convergence of the functions V_b and V_k .

The computation of the optimal policy for the transitional dynamics follow the same approach, except that the functions V_b and V_k have a time index.

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