

Liquidity and Trading Dynamics*

Veronica Guerrieri
University of Chicago and NBER

Guido Lorenzoni
MIT and NBER

May 2009

Abstract

In this paper, we build a model where the presence of liquidity constraints tends to magnify the economy's response to aggregate shocks. We consider a decentralized model of trade, where agents may use money or credit to buy goods. When agents do not have access to credit and the real value of money balances is low, agents are more likely to be liquidity constrained. This makes them more concerned about their short-term earning prospects when making their consumption decisions, and about their short-term spending opportunities when making their production decisions. This generates a coordination element in spending and production, which leads to greater aggregate volatility and greater comovement across producers.

Keywords: Liquidity, Money, Search, Aggregate Volatility, Amplification.

JEL codes: D83, E41, E44.

*Email addresses: veronica.guerrieri@chicagogsb.edu; glorenzo@mit.edu. We thank for helpful comments Larry Samuelson (the editor), three anonymous referees, Daron Acemoglu, Fernando Alvarez, Marios Angeletos, Boragan Aruoba, Gabriele Camera, Ricardo Cavalcanti, Chris Edmond, Ricardo Lagos, Robert Lucas, David Marshall, Robert Shimer, Nancy Stokey, Christopher Waller, Dimitri Vayanos, Iván Werning, Randall Wright, and seminar participants at MIT, the Cleveland Fed (Summer Workshop on Money, Banking, and Payments), University of Maryland, University of Notre Dame, Bank of Italy, AEA Meetings (Chicago), the Chicago Fed, Stanford University, the Philadelphia Fed Monetary Macro Workshop, UCLA, San Francisco Fed, St. Louis Fed, University of Chicago GSB, NYU, IMF, Bank of France/Toulouse Liquidity Conference (Paris), and the Minneapolis Fed/Bank of Canada Monetary Conference. Lorenzoni thanks the Federal Reserve Bank of Chicago for its generous hospitality during part of this research.

1 Introduction

During recessions, consumers, facing a higher risk of unemployment and temporary income losses, tend to hold on to their reserves of cash, bonds, and other liquid assets, as a form of self-insurance. This precautionary behavior can lead to reduced spending and magnify the initial decline in aggregate activity. In this paper, we explore formally this idea and show that this amplification mechanism depends crucially on the consumers' access to liquidity in a broad sense. The scarcer the access to liquidity, the more consumers are likely to be liquidity constrained in the near future. This means that they are more concerned about their short-term earnings prospects when making their spending decisions, and these decisions become more responsive to aggregate cyclical conditions. In other words, there is a stronger "coordination effect" which can magnify the effect of aggregate shocks on aggregate output and induce more comovement across different sectors of the economy.

We consider a decentralized model of production and exchange in the tradition of search models of money, where credit frictions arise from the limited ability to verify the agents' identity. There is a large number of households, each composed of a consumer and a producer. Consumers and producers from different households meet and trade in spatially separated markets, or islands. In each island, the gains from trade are determined by a local productivity shock. An exogenous aggregate shock determines the distribution of local shocks across islands. A good aggregate shock reduces the proportion of low productivity islands and increases that of high productivity islands, that is, it leads to a first-order stochastic shift in the distribution of local productivities. Due to limited credit access, households accumulate precautionary money balances to bridge the gap between current spending and current income. Money is supplied by the government and grows at a constant rate. In a stationary equilibrium, the rate of return on money is equal to the inverse of the money growth rate. A lower rate of return on money reduces the equilibrium real value of the money stock.

In the model, we distinguish different regimes along two dimensions: credit access and the rate of return on money. In regimes with less credit access and a lower rate of return on money, agents are more likely to face binding liquidity constraints. In such regimes, we show that there is a coordination element both in spending and in production decisions: agents are less willing to trade (buy and sell) when they expect others to trade less. This

leads both to greater aggregate volatility and to greater comovement among islands.

We first obtain analytical results in two polar cases which we call “unconstrained” and “fully constrained” regimes. An unconstrained regime arises when either households have unrestricted access to credit or the value of real money balances is sufficiently high. In this case, households are never liquidity constrained in equilibrium. Our first result is that, in an unconstrained regime, the quantity traded in each island is independent of what happens in other islands. The result follows from the fact that households are essentially fully insured against idiosyncratic shocks. This makes their expected marginal value of money constant, allowing the consumer and the producer from the same household to make their trading decisions independently. At the opposite end of the spectrum, a fully constrained regime arises when households have no credit access and the value of real money balances is sufficiently low, so that households expect to be liquidity constrained for all realizations of the idiosyncratic shocks. In this case, the decisions of the consumer and the producer are tightly linked. The consumer needs to forecast the producer’s earnings and the producer needs to forecast the consumer’s spending in order to evaluate the household’s marginal value of money.

Next, we look at the aggregate implications of these linkages. In all regimes, a bad aggregate shock has a negative compositional effect: as fewer islands have high productivity, aggregate output decreases. However, in an unconstrained regime there is no feedback from this aggregate fall in output to the level of trading in an island with a given local shock. In a fully constrained regime, instead, the linkage between trading decisions in different islands generates an additional effect on trading and output. A bad aggregate shock reduces the probability of high earnings for the producer, inducing the consumer to reduce spending. At the same time, the producer expects his partner to spend less, reducing his incentive to produce. These two effects imply that a lower level of aggregate activity induces lower levels of activity in each island, conditional on the local shock, leading to a magnified fall in aggregate activity. Numerical results show that our mechanism is also at work in intermediate regimes, where the liquidity constraint is occasionally binding, and that reduced credit access and a lower rate of return on money lead to higher volatility and comovement.

This paper is related to the literature on search models of decentralized trading, going back to Diamond (1982, 1984) and Kiyotaki and Wright (1989). In particular, Diamond

(1982) puts forth the idea that “the difficulty of coordination of trade” may contribute to cyclical volatility. The contribution of our paper is to show that the presence of this coordination effect depends crucially on credit market conditions and on the monetary regime. This allows us to identify a novel connection between financial development, liquidity supply, and aggregate dynamics. Our model allows for divisible money and uses the Lagos and Wright (2005) approach to make the model tractable. In Lagos and Wright (2005) agents alternate trading in a decentralized market to trading in a centralized competitive market. The combination of quasi-linear preferences and periodic access to a centralized market ensures that the distribution of money holdings is degenerate when agents enter the decentralized market. Here we use these same two ingredients, with a modified periodic structure. In our model, agents have access to a centralized market every three periods. The extra period of decentralized trading is necessary to make the precautionary motive matter for trading decisions in the decentralized market of the previous period. This is at the core of our amplification mechanism. A three-period structure is also used by Berentsen, Camera and Waller (2005) to study the short-run neutrality of money. They show that, away from the Friedman rule, random monetary injections can be non-neutral, since they have a differential effect on agents with heterogeneous money holdings. Although different in its objectives, their analysis also relies on the lack of consumption insurance. Our work is also related to a large number of papers who have explored the implications of different monetary regimes for risk sharing, in environments with idiosyncratic risk (e.g. Aiyagari and Williamson, 2000, Reed and Waller, 2006) and is related to Rocheteau and Wright (2005) for the use of competitive pricing *à la* Lucas and Prescott (1974) in a money search model.

More broadly, the paper is related to the literature exploring the relation between financial frictions and aggregate volatility, including Bernanke and Gertler (1989), Bencivenga and Smith (1991), Acemoglu and Zilibotti (1997), and Kiyotaki and Moore (1997). In particular, Kiyotaki and Moore (2001) also emphasize the effect of a limited supply of liquid assets (money) on aggregate dynamics. Their paper studies a different channel by which limited liquidity can affect the transmission of aggregate shocks, focusing on the effects on investment and capital accumulation.

Our paper is also related to the vast literature on the effect of liquidity constraints on consumption decisions. In particular, our argument relies on the idea that when liquidity

constraints are binding less often, consumption becomes less sensitive to short-term income expectations. Some evidence consistent with this idea is in Jappelli and Pagano (1989), who show that the excess sensitivity of consumption to current income is less pronounced in countries with more developed credit markets, and in Bacchetta and Gerlach (1997), who show that excess sensitivity has declined in the United States as a consequence of financial innovations.

The rest of the paper is organized as follows. In Section 2, we introduce a baseline model, with simple binary shocks and no credit access, and derive our main analytical results. In Section 3, we analyze an extended version of the model, generalizing the shock distribution and allowing for credit access. Section 5 presents a further extension with imperfect information and public signals. Section 6 concludes. The appendix contains all the proofs not in the text.

2 The Model

The economy is populated by a unit mass of infinitely-lived households, composed of two agents, a consumer and a producer. Time is discrete and each period agents produce and consume a single, perishable consumption good. The economy has a simple periodic structure: each time period t is divided into three subperiods, $s = 1, 2, 3$. We will call them “periods” whenever there is no risk of confusion.

In periods 1 and 2, the consumer and the producer from each household travel to spatially separated markets, or islands, where they interact with consumers and producers from other households. There is a continuum of islands and each island receives the same mass of consumers and producers in both periods 1 and 2. The assignment of agents to islands is random and satisfies a law of large numbers, so that each island receives a representative sample of consumers and producers. In each island there is a competitive goods market, as in Lucas and Prescott (1974). The consumer and the producer from the same household do not communicate while traveling in periods 1 and 2, but get together at the end of each period. In period 3, all consumers and producers trade in a single centralized market.¹

¹The use of a household made of two agents who are spatially separated during a trading period, goes back to Lucas (1990) and Fuerst (1992).

In period 1 of time t , a producer located in island k , has access to the linear technology

$$y_{t,1} = \theta_t^k n_t,$$

where $y_{t,1}$ is output, n_t is labor effort, and θ_t^k is the local level of productivity, which is random and can take two values: 0 and $\bar{\theta} > 0$. At time t , a fraction ζ_t of islands is randomly assigned the high productivity level $\bar{\theta}$, while a fraction $1 - \zeta_t$ is unproductive. The aggregate shock ζ_t is independently drawn and publicly revealed at the beginning of period 1, and takes two values, ζ_H and ζ_L , in $(0, 1)$, with probabilities α and $1 - \alpha$. The island-specific productivity θ_t^k is only observed by the consumers and producers located in island k . In Section 3, we will generalize the distributions of local and aggregate shocks.

In periods 2 and 3, each producer has a fixed endowment of consumption goods, $y_{t,2} = e_2$ and $y_{t,3} = e_3$. We assume that the value of e_3 is large, so as to ensure that equilibrium consumption in period 3 is strictly positive for all households.

The household's preferences are represented by the utility function

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t (u(c_{t,1}) - v(n_t) + U(c_{t,2}) + c_{t,3}) \right],$$

where $c_{t,s}$ is consumption in subperiod (t, s) and $\beta \in (0, 1)$ is the discount factor. Both u and U are increasing, strictly concave, with continuous first and second derivatives on $(0, \infty)$. The function u is bounded below, with $u(0) = 0$, has finite right-derivative at 0 and satisfies the Inada condition $\lim_{c \rightarrow \infty} u'(c) = 0$. The function U satisfies the Inada conditions $\lim_{c \rightarrow 0} U'(c) = \infty$ and $\lim_{c \rightarrow \infty} U'(c) = 0$. The function v represents the disutility of effort, is increasing and convex, has continuous first and second derivatives on $[0, \bar{n})$ and satisfies $v'(0) = 0$ and $\lim_{n \rightarrow \bar{n}} v'(n) = \infty$.

We assume that the consumers' identity cannot be verified in the islands they visit in periods 1 and 2, so credit contracts are not feasible. There is an exogenous supply of perfectly divisible notes issued by the government, money, and the only feasible trades in periods 1 and 2 are trades of money for goods. Each household is endowed with a stock of money M_0 at date 0. At the end of each subperiod 3, the government injects $(\gamma - 1) M_t$ units of money by a lump-sum transfer to each household (a lump-sum tax if $\gamma < 1$). Therefore, the stock of money M_t grows at the constant gross rate γ . In this paper we make no attempt to explain the government's choice of the monetary regime, but simply explore the effect of different regimes on equilibrium behavior.

Let us comment briefly on two of the assumptions made. First, the fact that in subperiod 3 consumers and producers trade in a centralized market and have linear utility is essential for tractability, as it allows us to derive an equilibrium with a degenerate distribution of money balances at the beginning of $(t, 1)$, as in Lagos and Wright (2005).² Second, we assume that the household is split in a consumer and a producer who make separate decisions in period 1, without observing the shock of the partner. This assumption allows us to capture in a compact way the effects of short-term income uncertainty on consumption and production decisions.

2.1 First-best

The first-best allocation provides a useful benchmark for the rest of the analysis. Consider a social planner with perfect information who can choose the consumption and labor effort of the households. Given that there is no capital, there is no real intertemporal link between times t and $t + 1$. Therefore, we can look at a three-period planner's problem.

Each household is characterized by a pair $(\theta, \tilde{\theta})$, where the first element represents the shock in the producer's island and the second the one in the consumer's island. An allocation is given by consumption functions $\{c_s(\theta, \tilde{\theta}, \zeta)\}_{s \in \{1, 2, 3\}}$ and an effort function $n(\theta, \tilde{\theta}, \zeta)$. The planner chooses an allocation that maximizes the ex ante utility of the representative household

$$\mathbb{E}[u(c_1(\theta, \tilde{\theta}, \zeta)) - v(n(\theta, \tilde{\theta}, \zeta)) + U(c_2(\theta, \tilde{\theta}, \zeta)) + c_3(\theta, \tilde{\theta}, \zeta)],$$

subject to the economy's resource constraints. In period 1, given the aggregate shock ζ , there is one resource constraint for each island θ^3

$$\mathbb{E}[c_1(\tilde{\theta}, \theta, \zeta) | \theta, \zeta] \leq \mathbb{E}[y_1(\theta, \tilde{\theta}, \zeta) | \theta, \zeta],$$

where $y_1(\theta, \tilde{\theta}, \zeta) = \theta n(\theta, \tilde{\theta}, \zeta)$. In period $s = 2, 3$, the resource constraint is

$$\mathbb{E}[c_s(\theta, \tilde{\theta}, \zeta) | \zeta] \leq e_s.$$

The resource constraints for periods 1 and 2 reflect the assumption that each island receives a representative sample of consumers and producers.

²See Shi (1997) for a different approach to obtain a degenerate distribution of money holdings.

³From now on, "island θ " is short for "an island with productivity shock θ ."

An optimal allocation is easy to characterize. Due to the separability of the utility function, the optimal consumption and output levels in a given island are not affected by the fraction ζ of productive islands in the economy. Namely, $c_1(\tilde{\theta}, \theta, \zeta) = y_1(\theta, \tilde{\theta}, \zeta) = y_1^*(\theta)$ for all $\tilde{\theta}$ and ζ , where $y_1^*(0) = 0$ and $y_1^*(\bar{\theta})$ satisfies

$$\bar{\theta}u'(y_1^*(\bar{\theta})) = v'(y_1^*(\bar{\theta})/\bar{\theta}). \quad (1)$$

Moreover, at the optimum, $c_2(\theta, \tilde{\theta}, \zeta) = e_2$ for all $\theta, \tilde{\theta}$ and ζ , that is, households are fully insured against the shocks θ and $\tilde{\theta}$. Finally, given linearity, the consumption levels in period 3 are not pinned down, as consumers are ex ante indifferent among all profiles $c_3(\theta, \tilde{\theta}, \zeta)$ such that $\mathbb{E}[c_3(\theta, \tilde{\theta}, \zeta)] = e_3$.

2.2 Equilibrium

Let us normalize all nominal variables (prices and money holdings) dividing them by the aggregate money stock M_t . Then, we can focus on stationary monetary equilibria where money is valued and where normalized prices only depend on the current aggregate shock ζ_t . Therefore, we drop the time index t .

We begin by characterizing optimal individual behavior. Let $p_1(\theta, \zeta)$ denote the normalized price of goods in period 1 in island θ , and $p_2(\zeta)$ and $p_3(\zeta)$ denote the normalized prices in periods 2 and 3. Consider a household with an initial stock of money m (normalized), at the beginning of period 1 after the realization of ζ . The consumer travels to island $\tilde{\theta}$ and consumes $c_1(\tilde{\theta}, \zeta)$. Since money holdings are non-negative, the budget constraint and the liquidity constraint in period 1 are

$$\begin{aligned} m_1(\tilde{\theta}, \zeta) + p_1(\tilde{\theta}, \zeta)c_1(\tilde{\theta}, \zeta) &\leq m, \\ m_1(\tilde{\theta}, \zeta) &\geq 0, \end{aligned} \quad (2)$$

where $m_1(\tilde{\theta}, \zeta)$ denotes the consumer's normalized money holdings at the end of period 1. In the meantime, the producer, located in island θ , produces and sells $y_1(\theta, \zeta) = \theta n(\theta, \zeta)$. At the end of period 1, the consumer and the producer get together and pool their money holdings. Therefore, in period 2 the budget and liquidity constraints are

$$\begin{aligned} m_2(\theta, \tilde{\theta}, \zeta) + p_2(\zeta)c_2(\theta, \tilde{\theta}, \zeta) &\leq m_1(\tilde{\theta}, \zeta) + p_1(\theta, \zeta)\theta n(\theta, \zeta), \\ m_2(\theta, \tilde{\theta}, \zeta) &\geq 0, \end{aligned} \quad (3)$$

where consumption, $c_2(\theta, \tilde{\theta}, \zeta)$, and end-of-period normalized money holdings, $m_2(\theta, \tilde{\theta}, \zeta)$, are now contingent on both shocks θ and $\tilde{\theta}$. Finally, in period 3, the consumer and the producer are located in the same island and the revenues $p_3(\zeta) e_3$ are immediately available. Moreover, the household receives a lump-sum nominal transfer equal to $\gamma - 1$, in normalized terms. The constraints in period 3 are then

$$\begin{aligned} m_3(\theta, \tilde{\theta}, \zeta) + p_3(\zeta) c_3(\theta, \tilde{\theta}, \zeta) &\leq m_2(\theta, \tilde{\theta}, \zeta) + p_2(\zeta) e_2 + p_3(\zeta) e_3 + \gamma - 1, \\ m_3(\theta, \tilde{\theta}, \zeta) &\geq 0. \end{aligned} \quad (4)$$

A household with normalized money balances $m_3(\theta, \tilde{\theta}, \zeta)$ at the end of subperiod 3, will have normalized balances $\gamma^{-1} m_3(\theta, \tilde{\theta}, \zeta)$ at the beginning of the following subperiod 1, as the rate of return on normalized money holdings between $(t, 3)$ and $(t + 1, 1)$ is equal to the inverse of the growth rate of the money stock, γ . Let $V(m)$ denote the expected utility of a household with money balances m at the beginning of period 1, before the realization of the aggregate shock ζ . The household's problem is then characterized by the Bellman equation

$$\begin{aligned} V(m) = \max_{\{c_s\}, \{m_s\}, n} & \mathbb{E}[u(c_1(\tilde{\theta}, \zeta)) - v(n(\theta, \zeta)) + U(c_2(\theta, \tilde{\theta}, \zeta)) + c_3(\theta, \tilde{\theta}, \zeta) \\ & + \beta V(\gamma^{-1} m_3(\theta, \tilde{\theta}, \zeta))], \end{aligned} \quad (5)$$

subject to the budget and liquidity constraints specified above.⁴ The solution to this problem gives us the optimal household's choices as functions of the shocks and of the initial money balances m , which we denote by $c_1(\theta, \zeta; m)$, $c_2(\theta, \tilde{\theta}, \zeta; m)$, etc.

We are now in a position to define an equilibrium.⁵

Definition 1 *A stationary monetary equilibrium is given by prices $\{p_1(\theta, \zeta), p_2(\zeta), p_3(\zeta)\}$, a distribution of money holdings with c.d.f. $H(\cdot)$ and support \mathcal{M} , and an allocation $\{n(\theta, \zeta; m), c_1(\theta, \zeta; m), c_2(\theta, \tilde{\theta}, \zeta; m), c_3(\theta, \tilde{\theta}, \zeta; m), m_1(\theta, \zeta; m), m_2(\theta, \tilde{\theta}, \zeta; m), m_3(\theta, \tilde{\theta}, \zeta; m)\}$ such that:*

(i) *the allocation solves problem (5) for each $m \in \mathcal{M}$;*

⁴Standard dynamic programming techniques can be applied. To take care of the unboundedness of the per-period utility function, one can extend the argument in Lemma 7 of Lagos and Wright (2004).

⁵We focus on monetary equilibria, that is, equilibria where money has positive value. As usual in money search environments, non-monetary equilibria also exist.

(ii) goods markets clear

$$\int_{\mathcal{M}} \mathbb{E}[c_1(\theta, \zeta; m)|\theta, \zeta] dH(m) = \theta \int_{\mathcal{M}} \mathbb{E}[n(\theta, \zeta; m)|\theta, \zeta] dH(m) \text{ for all } \theta, \zeta,$$

$$\int_{\mathcal{M}} \mathbb{E}[c_s(\theta, \tilde{\theta}, \zeta; m)|\zeta] dH(m) = e_s \text{ for } s = 2, 3 \text{ and all } \zeta;$$

(iii) the distribution $H(\cdot)$ satisfies $\int_{\mathcal{M}} m dH(m) = 1$ and

$$H(m) = \Pr[(\theta, \tilde{\theta}, \tilde{m}) : \gamma^{-1} m_3(\theta, \tilde{\theta}, \zeta; \tilde{m}) \leq m | \zeta] \text{ for all } m \text{ and } \zeta.$$

Condition (iii) ensures that the distribution $H(\cdot)$ is stationary. As we will see below, we can focus on equilibria where the distribution of money balances is degenerate at $m = 1$. Therefore, from now on, we drop the argument m from the equilibrium allocations.

In order to characterize the equilibrium, it is useful to derive the household's first order conditions. From problem (5) we obtain three Euler equations, with respective complementary slackness conditions,

$$u'(c_1(\tilde{\theta}, \zeta)) \geq \frac{p_1(\tilde{\theta}, \zeta)}{p_2(\zeta)} \mathbb{E}[U'(c_2(\theta, \tilde{\theta}, \zeta)) | \tilde{\theta}, \zeta] \quad (m_1(\tilde{\theta}, \zeta) \geq 0) \quad \text{for all } \tilde{\theta}, \zeta, \quad (6)$$

$$U'(c_2(\theta, \tilde{\theta}, \zeta)) \geq \frac{p_2(\zeta)}{p_3(\zeta)} \quad (m_2(\theta, \tilde{\theta}, \zeta) \geq 0) \quad \text{for all } \theta, \tilde{\theta}, \zeta, \quad (7)$$

$$1 \geq p_3(\zeta) \beta \gamma^{-1} V'(\gamma^{-1} m_3(\theta, \tilde{\theta}, \zeta)) \quad (m_3(\theta, \tilde{\theta}, \zeta) \geq 0) \quad \text{for all } \theta, \tilde{\theta}, \zeta, \quad (8)$$

the optimality condition for labor supply

$$v'(n(\theta, \zeta)) = \theta \frac{p_1(\theta, \zeta)}{p_2(\zeta)} \mathbb{E}[U'(c_2(\theta, \tilde{\theta}, \zeta)) | \theta, \zeta] \quad \text{for all } \theta, \zeta, \quad (9)$$

and the envelope condition

$$V'(m) = \mathbb{E} \left[\frac{u'(c_1(\tilde{\theta}, \zeta))}{p_1(\tilde{\theta}, \zeta)} \right]. \quad (10)$$

Our assumptions allow us to simplify the equilibrium characterization as follows. Since $\theta = 0$ with probability $\zeta > 0$, the Inada condition for U implies that $m_1(\tilde{\theta}, \zeta)$ and $m_3(\theta, \tilde{\theta}, \zeta)$ are strictly positive for all $\theta, \tilde{\theta}$, and ζ . To insure against the risk of entering period 2 with zero money balances, households always keep positive balances at the end of periods 1 and 3. This implies that (6) and (8) always hold as equalities.

Condition (8), holding with equality, shows why we obtain equilibria with a degenerate distribution of money balances, as in Lagos and Wright (2005). Given that the normalized supply of money is equal to 1, a stationary equilibrium with a degenerate distribution $H(\cdot)$ must satisfy

$$\gamma^{-1}m_3(\theta, \tilde{\theta}, \zeta) = 1 \text{ for all } \theta, \tilde{\theta}, \zeta.$$

In equilibrium, all agents adjust their consumption in period 3, so as to reach the same level of m_3 , irrespective of their current shocks. The assumptions that utility is linear in period 3 and that e_3 is large enough imply that the marginal utility of consumption in period 3 is constant, ensuring that this behavior is optimal.⁶ Moreover, equation (8), as an equality, implies that in all stationary equilibria $p_3(\zeta)$ is independent of the aggregate shock ζ and equal to $\gamma/(\beta V'(1))$. From now on, we just denote it as p_3 .

This leaves us with condition (7). In general, this condition can be either binding or slack for different pairs $(\theta, \tilde{\theta})$, depending on the parameters of the model. However, we are able to give a full characterization of the equilibrium by looking at specific monetary regimes, namely, by making assumptions about γ . First, we look at equilibria where the liquidity constraint $m_2(\theta, \tilde{\theta}, \zeta) \geq 0$ is never binding. We will show that this case arises if and only if $\gamma = \beta$, that is, in a monetary regime that follows the Friedman rule. Second, we look at equilibria where the constraint $m_2(\theta, \tilde{\theta}, \zeta) \geq 0$ is binding for all pairs $(\theta, \tilde{\theta})$ and for all ζ . We will show that this case arises if and only if the rate of money growth is sufficiently high, that is, when $\gamma \geq \hat{\gamma}$ for a given cutoff $\hat{\gamma} > \beta$.

These two polar cases provide analytically tractable benchmarks which illustrate the mechanism at the core of our model. The numerical example in Section 3.3 considers the case of economies with $\gamma \in (\beta, \hat{\gamma})$, where the liquidity constraint in period 2 is binding for a subset of agents.

2.2.1 Unconstrained equilibrium

We begin by considering “unconstrained equilibria,” that is, stationary monetary equilibria where the liquidity constraint in period 2 is never binding. In this case, condition (7) always

⁶When $\gamma > \beta$, it can be shown that all stationary equilibria are characterized by a degenerate distribution of money holdings.

holds as an equality. Combining conditions (6)-(8), all as equalities, and (10) gives

$$\frac{u'(c_1(\tilde{\theta}, \zeta))}{p_1(\tilde{\theta}, \zeta)} = \beta\gamma^{-1}\mathbb{E}\left[\frac{u'(c_1(\tilde{\theta}', \zeta'))}{p_1(\tilde{\theta}', \zeta')}\right], \quad (11)$$

where $\tilde{\theta}'$ and ζ' represent variables in the next time period. Taking expectations with respect to $\tilde{\theta}$ and ζ on both sides shows that a necessary condition for an unconstrained equilibrium is $\gamma = \beta$. The following proposition shows that this condition is also sufficient. Moreover, under this monetary regime, the equilibrium achieves an efficient allocation.⁷

Proposition 1 *An unconstrained stationary monetary equilibrium exists if and only if $\gamma = \beta$ and achieves a first-best allocation.*

For our purposes, it is especially interesting to understand how the level of activity is determined in a productive island in period 1. Let $\bar{p}_1(\zeta)$ and $\bar{y}_1(\zeta)$ denote $p_1(\bar{\theta}, \zeta)$ and $y_1(\bar{\theta}, \zeta)$. Substituting (7) into (6) (both as equalities), we can rewrite the consumer's optimality condition in period 1 as

$$u'(\bar{y}_1(\zeta)) = \frac{\bar{p}_1(\zeta)}{p_3}. \quad (12)$$

Similarly, the producer's optimality condition (9) can be rewritten as

$$v'(\bar{y}_1(\zeta)/\bar{\theta}) = \frac{\bar{\theta}\bar{p}_1(\zeta)}{p_3}. \quad (13)$$

These two equations describe, respectively, the demand and the supply of consumption goods in island $\bar{\theta}$, as a function of the price $\bar{p}_1(\zeta)$. Jointly, they determine the equilibrium values of $\bar{p}_1(\zeta)$ and $\bar{y}_1(\zeta)$ for each ζ . These equations highlight that, in an unconstrained equilibrium, consumers and producers do not need to forecast the income/spending of their partners when making their optimal choices, given that their marginal value of money is constant and equal to $1/p_3$. This implies that trading decisions in a given island are independent of trading decisions in all other islands. We will see that this is no longer true when we move to a constrained equilibrium. Conditions (12) and (13) can be easily manipulated to obtain the planner's first order condition (1), showing that in an unconstrained equilibrium $\bar{y}_1(\zeta)$ is independent of ζ and equal to the first best.

⁷This result is closely related to the analysis in Section 4 of Rocheteau and Wright (2005).

2.2.2 Fully constrained equilibrium

We now turn to stationary monetary equilibria where the liquidity constraint is always binding in period 2, that is, $m_2(\theta, \tilde{\theta}, \zeta) = 0$ for all $\theta, \tilde{\theta}$ and ζ . We refer to them as “fully constrained equilibria.” We will show that such equilibria arise when the money growth rate γ is sufficiently high.

Again, our main objective is to characterize how output is determined in period 1. First, however, we need to derive the equilibrium value of $p_2(\zeta)$. At the beginning of each period the entire money supply is in the hands of the consumers. Since in a fully constrained equilibrium consumers spend all their money in period 2 and normalized money balances are equal to 1, market clearing gives us

$$p_2(\zeta)e_2 = 1, \tag{14}$$

which pins down $p_2(\zeta)$. To simplify notation, we normalize $e_2 = 1$, so as to have $p_2(\zeta) = 1$.

Consider now a consumer and a producer in a productive island in period 1. Given that the consumer will be liquidity constrained in period 2, his consumption in that period will be fully determined by his money balances. In period 1, the consumer is spending $\bar{p}_1(\zeta)\bar{y}_1(\zeta)$ and expects his partner’s income to be $\bar{p}_1(\zeta)\bar{y}_1(\zeta)$ with probability ζ , and zero otherwise. Therefore, he expects total money balances at the beginning of period 2 to be 1 in the first case and $1 - \bar{p}_1(\zeta)\bar{y}_1(\zeta)$ in the second. Using $p_2(\zeta) = 1$, we can then rewrite the Euler equation (6) as

$$u'(\bar{y}_1(\zeta)) = \bar{p}_1(\zeta) [\zeta U'(1) + (1 - \zeta) U'(1 - \bar{p}_1(\zeta)\bar{y}_1(\zeta))]. \tag{15}$$

A symmetric argument on the producer’s side shows that the optimality condition (9) can be written as

$$v'(\bar{y}_1(\zeta)/\bar{\theta}) = \bar{\theta}\bar{p}_1(\zeta) [\zeta U'(1) + (1 - \zeta) U'(1 + \bar{p}_1(\zeta)\bar{y}_1(\zeta))]. \tag{16}$$

These two equations correspond to (12) and (13) in the unconstrained case and jointly determine $\bar{p}_1(\zeta)$ and $\bar{y}_1(\zeta)$ for each ζ . The crucial difference with the unconstrained case is that now ζ , the fraction of productive islands in the economy, enters the optimal decisions of consumers and producers in a given productive island, since it affects their expected income and consumption in the following period. We will see in a moment how this affects aggregate volatility and comovement.

Notice that (15) and (16) implicitly define a “demand curve” and a “supply curve,” $\bar{y}^D(\bar{p}_1, \zeta)$ and $\bar{y}^S(\bar{p}_1, \zeta)$.⁸ It is easy to show that, for any ζ , there exists a price where the two curves intersect. For comparative statics, it is useful to make the additional assumption

$$-(1 - \zeta_H)cU''(c)/U'(c) \leq 1 \text{ for all } c, \quad (\text{A1})$$

which ensures that the income effect on labor supply is not too strong and that the supply curve is positively sloped.

Lemma 1 *The function $\bar{y}^D(\bar{p}_1, \zeta)$ is decreasing in \bar{p}_1 . Under assumption A1, the function $\bar{y}^S(\bar{p}_1, \zeta)$ is increasing in \bar{p}_1 and, for given ζ , there is a unique pair $(\bar{p}_1(\zeta), \bar{y}_1(\zeta))$ which solves (15)-(16).*

To complete the equilibrium characterization, it remains to find p_3 and check that consumers are indeed constrained in period 2, that is, that (7) holds. In the next proposition, we show that this condition is satisfied as long as γ is above some cutoff $\hat{\gamma}$.

Proposition 2 *There is a $\hat{\gamma} > \beta$ such that a fully constrained stationary monetary equilibrium exists if and only if $\gamma \geq \hat{\gamma}$.⁹*

It is useful to clarify the role of the rate of return on money γ^{-1} in determining whether we are in a constrained or unconstrained equilibrium. Notice that in an unconstrained equilibrium the household’s normalized money balances at the beginning of period 1, which are equal to 1, must be sufficient to purchase both $\bar{p}_1(\zeta)\bar{y}_1(\zeta)$ and $p_2(\zeta)e_2$, in case the consumer is assigned to a productive island and the producer to an unproductive one. Therefore in an unconstrained equilibrium the following inequality holds for all ζ

$$\frac{1}{p_2(\zeta)} \geq e_2 + \frac{\bar{p}_1(\zeta)}{p_2(\zeta)}\bar{y}_1(\zeta).$$

On the other hand, (14) shows that $1/p_2(\zeta)$ is constant and equal to e_2 in a fully constrained equilibrium. That is, the real value of money balances in terms of period 2 consumption

⁸These are not standard partial-equilibrium demand and supply functions, as they represent the relation between the price \bar{p}_1 and the demand/supply of goods in a symmetric equilibrium where prices and quantities are identical in all productive islands.

⁹Under assumption A1, if $\gamma \geq \hat{\gamma}$ there is a unique fully constrained equilibrium. However, we cannot rule out, in general, the existence of other partially constrained equilibria.

is uniformly lower in a fully constrained equilibrium. This is due to the fact that the rate of return on money is low. This reduces the agents' willingness to hold money, reducing the equilibrium real value of money balances. Through this channel high money growth reduces the households' ability to self-insure.

2.3 Coordination, amplification and comovement

We now turn to the effects of the aggregate shock ζ on the equilibrium allocation in the various regimes considered. Aggregate output in period 1 is given by

$$Y_1(\zeta) = \zeta \bar{y}_1(\zeta). \quad (17)$$

Consider the proportional effect of a small change in ζ on aggregate output,

$$\frac{d \log Y_1(\zeta)}{d\zeta} = \frac{1}{\zeta} + \frac{d \ln \bar{y}_1(\zeta)}{d\zeta}. \quad (18)$$

When ζ decreases, there is a smaller fraction of productive islands, so aggregate output mechanically decreases in proportion to ζ . This “composition effect” corresponds to the first term in (18). The open question is whether a change in ζ also affects the endogenous level of activity in a productive island. This effect is captured by the second term in (18) and will be called “coordination effect.”

In an unconstrained equilibrium, we know that $\bar{y}_1(\zeta)$ is independent of ζ . Therefore, if money growth follows the Friedman rule and the rate of return of money is equal to β^{-1} , the coordination effect is absent. What happens in a fully constrained equilibrium, that is, when the rate of return on money is sufficiently low? Consider the demand and supply curves in a productive island, $\bar{y}^D(\bar{p}_1, \zeta)$ and $\bar{y}^S(\bar{p}_1, \zeta)$, derived above. Applying the implicit function theorem to (15) and (16) yields

$$\frac{\partial \bar{y}^D(\bar{p}_1(\zeta), \zeta)}{\partial \zeta} = p_1 \frac{U'(1) - U'(1 - \bar{p}_1(\zeta)\bar{y}_1(\zeta))}{u''(\bar{y}_1(\zeta)) + (\bar{p}_1(\zeta))^2 (1 - \zeta) U''(1 - \bar{p}_1(\zeta)\bar{y}_1(\zeta))} > 0,$$

and

$$\frac{\partial \bar{y}^S(\bar{p}_1(\zeta), \zeta)}{\partial \zeta} = p_1 \frac{U'(1) - U'(1 + \bar{p}_1(\zeta)\bar{y}_1(\zeta))}{v''(\bar{y}_1(\zeta)/\bar{\theta})/\bar{\theta} - \bar{\theta}(\bar{p}_1(\zeta))^2 (1 - \zeta) U''(1 + \bar{p}_1(\zeta)\bar{y}_1(\zeta))} > 0.$$

Both inequalities follow from the strict concavity of U . On the demand side, the intuition is the following. In period 1, a consumer in a productive island is concerned about receiving

a bad income shock. Given that he is liquidity constrained, this shock will directly lower his consumption from 1 to $1 - \bar{p}_1(\zeta)\bar{y}_1(\zeta)$. An increase in ζ lowers the probability of a bad shock, decreasing the expected marginal value of money and increasing the consumer's willingness to spend, for any given price. On the supply side, as ζ increases, a producer in a productive island expects his partner to spend $\bar{p}_1(\zeta)\bar{y}_1(\zeta)$ with higher probability. This generates a negative income effect which induces him to produce more, for any given price. These two effects shift both demand and supply to the right and, under assumption A1, lead to an increase in equilibrium output.¹⁰ This proves the following result.

Proposition 3 (*Coordination*) *Under assumption A1, in a fully constrained equilibrium, the output in the productive islands, $\bar{y}_1(\zeta)$, is increasing in ζ .*

This is the central result of our paper and shows that when liquidity constraints are binding there is a positive coordination effect, as consumers and producers try to keep their spending and income decisions aligned. Consumers spend more when they expect their partners to earn more, and producers work more when they expect their partners to spend more. This has two main consequences. First, the impact of an aggregate shock on the aggregate level of activity is magnified, leading to increased volatility. Second, there is a stronger degree of comovement across islands. Let us analyze these two implications formally.

Since $\bar{y}_1(\zeta)$ is independent of ζ in an unconstrained equilibrium and increasing in ζ in a fully constrained one, equation (18) implies immediately that $\partial \log Y_1(\zeta) / \partial \zeta$ is larger in a fully constrained equilibrium than in an unconstrained equilibrium. This leads to the following result.

Proposition 4 (*Amplification*) *Under assumption A1, $Var[\log Y_1(\zeta)]$ is larger in a fully constrained equilibrium than in an unconstrained equilibrium.*

¹⁰If is useful to mention what would happen in an environment where the producer and consumer from the same household can communicate (but not exchange money) in period 1. In that case, in a productive island there will be two types of consumers and producers, distinguished by the local shock of their partners. Consumers (producers) paired with a low productivity partner, will have lower demand (supply). So also in that case an increase in ζ would lead to an increase in activity in the productive island. However, that case is less tractable, due to the four types of agents involved, and it fails to capture the effect of uncertainty on the agents' decisions.

To measure comovement we look at the coefficient of correlation between local output in any given island and aggregate output. In an unconstrained equilibrium, there is some degree of correlation between the two, simply because an increase in ζ increases both aggregate output and the probability of a high productivity shock in any given island. However, in a fully constrained equilibrium the correlation tends to be stronger. Now, even conditionally on the island receiving the high productivity shock, an increase in ζ tends to increase both local and aggregate output, due to the coordination effect. This leads to the following result.¹¹

Proposition 5 (*Comovement*) *Under assumption A1, $Corr[y_1(\theta, \zeta), Y_1(\zeta)]$ is larger in a fully constrained equilibrium than in an unconstrained equilibrium.*

3 The Extended Model

In the baseline model, the rate of return on money is the only determinant of the households' access to liquidity. In this section, we allow a fraction of households to have access to credit each period. This introduces an additional dimension of liquidity, which makes the model better suited to interpret the effects of financial innovation and of financial crises, which can be described as changes in the fraction of households with credit access. We also generalize the distribution of local and aggregate shocks. Using this extended model, we first focus on the polar cases of unconstrained and fully constrained equilibria, to generalize the main result of the previous section, Proposition 3. Next, we use a numerical example to analyze the model's implications for amplification and comovement, both in the polar cases above and in intermediate cases.

The setup is as in Section 2 except for two differences. First, we allow for general distributions of aggregate and local shocks. The aggregate shock ζ_t has c.d.f. $G(\cdot)$ and support $[\zeta, \bar{\zeta}]$. Conditional on ζ_t , the local productivity shock θ_t^k has c.d.f. $F(\cdot|\zeta_t)$ with support $[0, \bar{\theta}]$. We assume that $F(\theta|\zeta)$ has an atom at 0, is continuous in θ on $(0, \bar{\theta}]$, and is continuous and decreasing in ζ , for each θ . The latter property implies that a larger ζ leads to a distribution of θ that first-order stochastically dominates a distribution associated

¹¹An alternative measure of comovement is the correlation between the level of activity in any given pair of islands, that is, $Corr[y_1(\theta, \zeta), y_1(\tilde{\theta}, \zeta)]$. In a setup with i.i.d. idiosyncratic shocks the two measures are interchangeable as it is possible to prove that $Corr[y(\theta, \zeta), y(\tilde{\theta}, \zeta)] = (Corr[y(\theta, \zeta), Y(\zeta)])^2$.

with a lower ζ . As before, a law of large numbers applies, so $F(\cdot|\zeta)$ also represents the distribution of productivity shocks across islands.

Second, we assume that at the beginning of each period t , a fraction ϕ of households is randomly selected and their identity can be fully verified during that period, so that they have full credit access. Each island is visited by a representative sample of households: a fraction ϕ of households with credit access, or credit households, and a fraction $1 - \phi$ of anonymous households who need money to trade goods, or money households.

We study stationary equilibria, defined along the lines of Definition 1 and focus on equilibria where the distribution of beginning-of-period money balances is degenerate at $m = 1$. In subperiod 3, households do not know if they will have access or not to credit in the next period and hence all choose the same money holdings. Without loss of generality, we can assume that all loans are repaid in subperiod 3. Moreover, we assume that IOUs issued by credit household can circulate and so are a perfect substitute for money. Therefore, we can represent the problem of credit households using the same budget constraints (2)-(4), simply omitting the non-negativity constraints for $m_1(\tilde{\theta}, \zeta)$ and $m_2(\theta, \tilde{\theta}, \zeta)$.

In a stationary equilibrium, the behavior of money households is characterized by the optimality conditions (6)-(9), as in Section 2.2. The assumption that $F(\cdot|\zeta)$ has an atom at 0, together with the Inada condition for U , ensures that (6) and (8) always hold as equalities, as in the binary case, while (7) can hold with equality or not depending on the shocks θ and $\tilde{\theta}$ and on the monetary regime. The behavior of credit households is described by the same optimality conditions, except for one difference: since they can hold negative nominal balances in subperiod 2, equation (7) always hold as an equality.

3.1 Unconstrained and fully constrained equilibria

First, let us look at unconstrained equilibria, which arise when either monetary policy follows the Friedman rule or when all households have access to credit, that is, when either $\gamma = \beta$ or $\phi = 1$.

Proposition 6 *In the extended model, an unconstrained stationary monetary equilibrium exists if and only if $\gamma = \beta$ and it achieves a first-best allocation. Output in period 1 is $y_1(\theta, \zeta) = y_1^*(\theta)$ where $y_1^*(\theta)$ satisfies $\theta u'(y_1^*(\theta)) = v'(y_1^*(\theta)/\theta)$ for all $\theta > 0$, is increasing in θ , and is independent of ζ .*

If $\phi = 1$ it is easy to show that there exists an unconstrained stationary equilibrium that also achieves a first-best allocation. The only difference is that since all agents have access to credit, money is not valued in equilibrium except if $\gamma = \beta$.

In an unconstrained equilibrium the real allocation is the same for all households, regardless of their access to credit. In particular, they consume and produce the first-best level of output in all islands. As in the binary model, the separability of the utility function implies that equilibrium output in island θ depends only on the local productivity and is not affected by the distribution of productivities in other islands.

Second, consider fully constrained equilibria where money households are always constrained in period 2 and there are no credit households, that is, when $\gamma \geq \hat{\gamma}$ and $\phi = 0$. Following similar steps to Section 2.2, we can show that $p_2(\zeta)$ is constant and equal to 1 (under the normalization $e_2 = 1$). Consider a consumer and a producer in island θ . The consumer's Euler equation and the producer's optimality condition can be rewritten as

$$u'(y_1(\theta, \zeta)) = p_1(\theta, \zeta) \int_0^{\bar{\theta}} U'(c_2(\tilde{\theta}, \theta, \zeta)) dF(\tilde{\theta}|\zeta), \quad (19)$$

$$v'(y_1(\theta, \zeta)/\theta) = \theta p_1(\theta, \zeta) \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta}, \zeta)) dF(\tilde{\theta}|\zeta), \quad (20)$$

where, from the consumer's budget constraints in periods 1 and 2,

$$c_2(\theta, \tilde{\theta}, \zeta) = 1 - p_1(\tilde{\theta}, \zeta)y_1(\tilde{\theta}, \zeta) + p_1(\theta, \zeta)y_1(\theta, \zeta). \quad (21)$$

Equations (19) and (20) are analogous to (15) and (16) and represent the demand and supply in island θ , taking as given prices and quantities in other islands. They define two functional equations in $p_1(\cdot, \zeta)$ and $y_1(\cdot, \zeta)$. In the Appendix, we show that this pair of functional equations has a unique solution. To do so, we define nominal income $x(\theta, \zeta) \equiv p_1(\theta, \zeta)y_1(\theta, \zeta)$ and solve a fixed point problem for the function $x(\cdot, \zeta)$.

To solve our fixed point problem, we use a contraction mapping argument, making the following assumption:

$$-\frac{cu''(c)}{u'(c)} \in [\underline{\rho}, 1) \text{ for all } c, \quad (\text{A2})$$

for some $\underline{\rho} > 0$. The upper bound on $-cu''(c)/u'(c)$ is needed to ensure that the demand elasticity in a given island θ is high enough. This guarantees that in islands where productivity is higher prices do not fall too much, so that nominal income is increasing in θ .

That is, producers in more productive islands receive higher earnings. This property is both economically appealing and useful on technical grounds, as it allows us to prove the monotonicity of the mapping used in our fixed point argument.¹² The lower bound $\underline{\rho}$ is used to prove the discounting property of the same mapping.¹³

As in the binary model, we can then characterize a fully constrained equilibrium and find a cutoff $\hat{\gamma}$ such that such an equilibrium exists whenever $\gamma \geq \hat{\gamma}$.

Proposition 7 *In the extended model, under assumption A2, there is a cutoff $\hat{\gamma} > \beta$ such that a fully constrained stationary monetary equilibrium exists if and only if $\phi = 0$ and $\gamma \geq \hat{\gamma}$. In equilibrium, both output $y_1(\theta, \zeta)$ and nominal income $p_1(\theta, \zeta)y_1(\theta, \zeta)$ are monotone increasing in θ .*

We could prove existence under weaker conditions, using a different fixed point argument. However, the contraction mapping approach helps us derive the coordination result in Proposition 8 below.

3.2 Aggregate implications

We now turn to the analysis of the impact of the aggregate shock ζ on the equilibrium allocation. Aggregate output in period 1 is given by

$$Y_1(\zeta) \equiv \int_0^{\bar{\theta}} y_1(\theta, \zeta) dF(\theta|\zeta), \quad (22)$$

where $y_1(\theta, \zeta) \equiv \phi y_1^C(\theta, \zeta) + (1 - \phi) y_1^M(\theta, \zeta)$, given that in each island θ there is a fraction ϕ of credit households (C) and a fraction $1 - \phi$ of money households (M). The proportional response of output to a small change in ζ , can be decomposed as in the binary case,

$$\frac{d \ln Y_1}{d \zeta} = \frac{1}{Y_1} \int_0^{\bar{\theta}} y_1(\theta, \zeta) \frac{\partial f(\theta|\zeta)}{\partial \zeta} d\theta + \frac{1}{Y_1} \int_0^{\bar{\theta}} \frac{\partial y_1(\theta, \zeta)}{\partial \zeta} dF(\theta|\zeta). \quad (23)$$

¹²It is useful to mention alternative specifications which can deliver the same result (nominal income increasing in θ) without imposing restrictions on risk aversion in period 1. One possibility is to introduce local shocks as preference shocks. For example, we could assume that the production function is the same in all islands while the utility function takes the form $\theta u(c_1)$ where θ is the local shock. In this case, it is straightforward to show that both $p_1(\theta, \zeta)$ and $y_1(\theta, \zeta)$ are increasing in θ , irrespective of the curvature of u . This immediately implies that nominal income is increasing in θ . Another possibility is to use more general preferences, which allow to distinguish risk aversion from the elasticity of intertemporal substitution. For example, using a version of Epstein and Zin (1989) preferences, it is possible to show that our results only depends on the elasticity of substitution between c_1 and c_2 and not on risk aversion.

¹³This assumption is minimally restrictive, as $\underline{\rho}$ is only required to be non-zero.

The first term is the mechanical composition effect of having a larger fraction of more productive islands. This effect is positive both in an unconstrained and in a fully constrained equilibrium. This follows from the fact that an increase in ζ leads to a first-order stochastic shift in the distribution of θ and that $y_1(\theta, \zeta)$ is increasing in θ in both regimes, as shown in Propositions 6 and 7.

The second term in (23) is our coordination effect. As in the binary case, this effect is zero in an unconstrained equilibrium, since, by Proposition 6, output in any island θ is independent of the economy-wide distribution of productivity. Turning to a fully constrained equilibrium, we can generalize Proposition 3 and show that $y_1(\theta, \zeta)$ is increasing in ζ , for any realization of the local productivity shock θ . For this result, we make a stronger assumption than the one used in the binary model, that is, we assume that U has a coefficient of relative risk aversion smaller than one

$$-\frac{cU''(c)}{U'(c)} \leq 1 \text{ for all } c. \quad (\text{A1}')$$

This condition is sufficient to prove that the labor supply in each island is positively sloped. Assumption A1' is stronger than needed, as numerical examples show that the labor supply is positively sloped also for parametrizations with a coefficient of relative risk aversion greater than 1. In fact, the coordination effect can be more powerful when agents are more risk averse.

Proposition 8 (*Coordination*) *Consider the extended model. Under assumptions A1' and A2, in a fully constrained equilibrium, for each $\theta > 0$, the output $y_1(\theta, \zeta)$ is increasing in ζ .*

To understand the mechanism behind this result, consider the following partial equilibrium exercise. Focus on island θ , taking as given $p_1(\tilde{\theta}, \zeta)$ and $y_1(\tilde{\theta}, \zeta)$ for all $\tilde{\theta} \neq \theta$. Consider the demand and supply equations (19) and (20). Since, from Proposition 7, $p_1(\tilde{\theta}, \zeta)y_1(\tilde{\theta}, \zeta)$ is increasing in $\tilde{\theta}$, it follows that $U'(c_2(\tilde{\theta}, \theta, \zeta))$ is decreasing in $\tilde{\theta}$ and $U'(c_2(\theta, \tilde{\theta}, \zeta))$ is increasing in $\tilde{\theta}$. Hence, when ζ increases the integral on the right-hand side of (19) decreases, while the integral on the right-hand side of (20) increases.¹⁴ The intuition is similar to the one for the binary model. When a liquidity constrained consumer expects higher income,

¹⁴This follows immediately from the fact that an increase in ζ leads to a shift of the distribution of θ in the sense of first-order stochastic dominance.

his marginal value of money decreases and he increases consumption for any $p_1(\theta, \zeta)$. When a producer expects higher spending by his partner, he faces a negative income effect and produces more for any $p_1(\theta, \zeta)$. The first effect shifts the demand curve to the right, the second shifts the supply curve also to the right. Therefore, equilibrium output increases in island θ .

On top of this partial equilibrium mechanism, there is a general equilibrium feed-back due to the endogenous response of prices and quantities in islands $\tilde{\theta} \neq \theta$. This magnifies the initial effect. As nominal output in all other islands increases, there is a further increase in the marginal value of money for the consumers and a further decrease for the producers, leading to an additional increase in output.

Summing up, the coordination effect identified in Proposition 8 is driven by the agents' expectations regarding nominal income in other islands. This effect tends to magnify the output response to aggregate shocks in a fully constrained economy and to generate more comovement across islands.

Going back to equation (23), we have established that the coordination effect is zero in the unconstrained case and positive in the fully constrained one. However, this is not sufficient to establish that output volatility is greater in the constrained economy, since we have not compared the relative magnitude of the compositional effect, which is positive in both cases. In the binary model, the comparison was unambiguous, given that this effect was identical in the two regimes. However, with general shock distributions, it is difficult to compare the relative size of this effect in the two regimes and obtain the analogues of Propositions 4 and 5. Therefore, to gauge the implications of our coordination effect for volatility and comovement, we turn to a numerical example.

3.3 Numerical example

We use a numerical example both to analyze the aggregate implications of our coordination effect in the two polar cases analyzed above and to study intermediate cases in which the fraction of households with credit access ϕ is in $(0, 1)$ and the rate of return on money γ is in the intermediate range $(\beta, \hat{\gamma})$.

We choose isoelastic instantaneous utility functions: $u(c) = c^{1-\rho_1}/(1-\rho_1)$, $U(c) = c^{1-\rho_2}/(1-\rho_2)$, and $v(n) = n^{1+\eta}/(1+\eta)$. There are two aggregate states, ζ_L and ζ_H , with

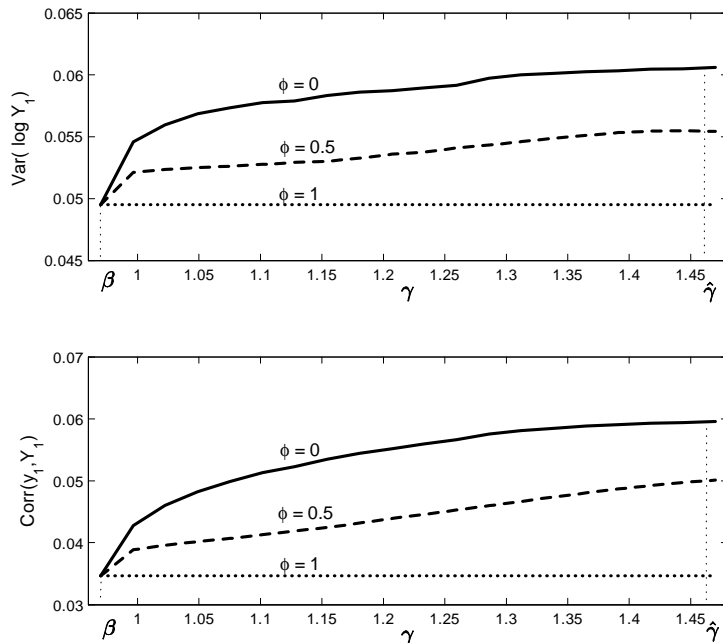


Figure 1: Volatility and comovement

probabilities α and $1 - \alpha$. Conditional on the aggregate state, the shock θ is log-normally distributed with mean μ_H in state ζ_H , μ_L in state ζ_L , and variance σ^2 .¹⁵ We interpret each sequence of three subperiods as a year and set the discount factor β to 0.97. We normalize $e_2 = 1$ and set $\rho_1 = 0.5$, $\rho_2 = 1$, $\eta = 0.3$, $e_3 = 3$.¹⁶ For the shock distribution we choose $\alpha = 0.2$, $\mu_L = 0.5$, $\mu_H = 0.56$, $\sigma^2 = 0.19$.¹⁷ The aggregate shock $\mu_H - \mu_L$ is chosen to deliver a standard deviation of $\log Y_1$ equal to 0.05 in the first-best allocation.

In Figure 1 we look at the effects of different liquidity regimes on volatility and comovement, plotting the standard deviation of $\log Y_1$ (top panel) and the correlation coefficient of y_1 and Y_1 (bottom panel), as functions of γ for different levels of ϕ . With $\phi = 1$, all consumers have perfect access to credit and the equilibrium achieves the first-best alloca-

¹⁵Even though this distribution does not have an atom at 0, in our example consumers never exhaust their money balances in period 1.

¹⁶The parameters ρ_1 , η , and e_3 are chosen to roughly match the empirical relation between money velocity and the nominal interest rate in the post-war US data (following Lucas, 2000, Lagos and Wright, 2005, and Craig and Rocheteau, 2007).

¹⁷The variance of the idiosyncratic shock σ^2 yields a standard deviation of income volatility at the household level equal to 0.2, consistent with estimates in Hubbard, Skinner and Zeldes (1994).

tion, so both volatility and comovement are independent of γ . With $\phi = 0$, volatility and comovement increase with γ until the economy reaches the fully constrained equilibrium for $\gamma \geq \hat{\gamma}$. In the intermediate case with $\phi = 0.5$, volatility and comovement are both increasing in γ , and, for each $\gamma > \beta$, achieve intermediate levels relative to the two extreme cases.

The figure shows that as the rate of return on money increases or the fraction of households with access to credit decreases, both volatility and comovement increase. In particular, a fully constrained economy is significantly more volatile than the unconstrained one, but the relation between γ and volatility is concave and relatively large effects already appear when γ is not far from the Friedman rule, e.g. at $\gamma = 1$. Experimenting with the parameters, shows that increasing the elasticity of labor supply, by lowering η , tends to lead to larger effects. Increasing the second period risk aversion ρ_2 can increase or decrease volatility, depending on the initial parameters. A higher ρ_2 increases the precautionary motive, making households more responsive to a negative shock. However, there is a countervailing force, as a higher ρ_2 also increases the equilibrium value of real balances.

4 News Shocks

Consider now the general model of Section 3, with the only difference that the aggregate shock ζ is not observed by the households in period 1. Instead, they all observe a public signal $\xi \in [\underline{\xi}, \bar{\xi}]$, which is drawn at the beginning of each period, together with the aggregate shock ζ , from a continuous distribution with joint density function $g(\zeta, \xi)$.

Take an agent located in an island with productivity θ , his posterior density regarding ζ can be derived using Bayes' rule:

$$g(\zeta|\xi, \theta) = \frac{f(\theta|\zeta)g(\zeta, \xi)}{\int_{\underline{\zeta}}^{\bar{\zeta}} f(\theta|\tilde{\zeta})g(\tilde{\zeta}, \xi)d\tilde{\zeta}}.$$

The distribution $g(\zeta|\xi, \theta)$ is then used to derive the agent's posterior beliefs regarding $\tilde{\theta}$ in the island where his partner is located

$$F(\tilde{\theta}|\xi, \theta) = \int_{\underline{\zeta}}^{\bar{\zeta}} F(\tilde{\theta}|\zeta)g(\zeta|\xi, \theta)d\zeta.$$

We will make the assumption that $F(\tilde{\theta}|\xi, \theta)$ is non-increasing in ξ , for any pair $(\theta, \tilde{\theta})$. This means, that conditional on θ , the signal ξ is “good news” for $\tilde{\theta}$, in the sense of Milgrom

(1981). We also make the natural assumption that $F(\tilde{\theta}|\xi, \theta)$ is non-increasing in θ . In period 2, the actual shock ζ is publicly revealed.

In this environment, we study a stationary equilibrium, along the lines of the one described in Section 3. Prices and allocations now depend on the local shocks and on the aggregate shocks ξ and ζ . In particular, prices and quantities in period 1 depend only on θ and ξ , given that ζ is not in the information set of the households in that period. Aggregate output in period 1 becomes

$$Y_1(\zeta, \xi) \equiv \int_0^{\bar{\theta}} y_1(\theta, \xi) dF(\theta|\zeta). \quad (24)$$

We can now look separately at the output response to the productivity shock ζ and to the news shock ξ . In particular, next proposition shows that the output response to ζ is positive both in an unconstrained and in a fully constrained equilibrium, while the output response to the signal ξ is positive only in the fully constrained case.

Proposition 9 *Consider an economy with imperfect information regarding the aggregate shock. Under assumptions A1' and A2, in an unconstrained equilibrium $\partial Y_1(\zeta, \xi)/\partial \zeta > 0$ and $\partial Y_1(\zeta, \xi)/\partial \xi = 0$. In a fully constrained equilibrium $\partial Y_1(\zeta, \xi)/\partial \zeta > 0$ and $\partial Y_1(\zeta, \xi)/\partial \xi > 0$.*

This result is not surprising, in light of the analysis in the previous sections. Compare the expression for aggregate output under imperfect information (24) with the correspondent expression in the case of full information (22). By definition, the productivity shock ζ affects the distribution of local shocks $F(\cdot|\zeta)$ in both cases. However, the trading decisions of anonymous households in island θ are affected only by the agents' expectations about that distribution, which, in the case of imperfect information, are driven by the signal ξ . It follows that the effect of ζ is analogous to the mechanical composition effect in the model with full information on ζ , while the effect of ξ is analogous to the coordination effect. The advantage of an environment with imperfect information, is that these two effects can be disentangled. In an unconstrained economy, as we know from Proposition 6, output in island θ is independent of the economy-wide distribution of productivity and thus does not respond to ξ . The result that the output response to ξ is positive in a fully constrained economy is a natural extension of Proposition 8. In island θ , trading is lower the more

pessimistic agents are about trading in all other islands. The only difference is how expectations are formed. The perceived distribution of productivities for an agent in island θ depends now on the signal ξ , instead that on the actual ζ . A negative signal ξ makes both consumers and producers in island θ more pessimistic about trading in other islands, even if the underlying ζ is unchanged. This highlights that expectations are at the core of our amplification result.

5 Concluding Remarks

In this paper, we have analyzed how different liquidity regimes affect the response of an economy to aggregate shocks. A liquidity regime depends both on the households' access to credit and on the value of their money holdings. We show that in regimes where liquidity constraints are binding more often, there is a coordination motive in the agents' trading decisions. This generates both an amplified response to aggregate shocks and a larger degree of comovement.

Our mechanism is driven by the combination of risk aversion, idiosyncratic uncertainty, and decentralized trade. All three ingredients are necessary for the mechanism to operate. Risk aversion and idiosyncratic risk give rise to an insurance problem. Decentralized trade implies that agents with no access to credit can only self-insure using their money holdings.¹⁸ A nice feature of our setup is that simply by changing the credit and monetary regimes, we move from an environment in which idiosyncratic risk is perfectly insurable (unconstrained equilibrium) to an environment in which idiosyncratic risk is completely uninsurable (fully constrained equilibrium). In this sense, the mechanism identified in this paper speaks more broadly about the effect of uninsurable idiosyncratic risk on aggregate behavior.

For analytical tractability, we have developed our argument in a periodic framework *à la* Lagos and Wright (2005). This framework is clearly special in many respects, and, in particular, displays no endogenous source of persistence. It would be interesting to investigate, numerically, the quantitative implications of our mechanism in a version of the model that allows for richer dynamics of individual asset positions.¹⁹ A similar extension would also help to clarify the relation between our results and the literature on the aggregate

¹⁸Reed and Waller (2006) also point out the risk sharing implications of different monetary regimes in a model *à la* Lagos and Wright (2005).

¹⁹See, for example, the computational approach in Molico (2006).

implications of imperfect risk sharing, such as Krusell and Smith (1989).²⁰

The current crisis in the U.S. is a good example of how anticipated changes in access to liquidity can have a substantial impact on aggregate activity. Our model provides a possible interpretation of these effects: a reduction in credit access induces consumers to be more cautious in their spending decisions and more concerned about their income prospects in the near future, making the recession worse. Our model can also be applied to interpret the effects of more gradual regime changes: for example, many have argued that a gradual increase in credit access for households and firms have contributed to a reduction in aggregate volatility in the U.S. after the mid 80's. The model's predictions are qualitatively consistent with this story and also emphasizes that the high inflation of the 70's, by reducing the value of the real balances in the hands of consumers, may have further increased volatility in the pre-1982 period.

Appendix

Proof of Proposition 1

In the main text we show that $\gamma = \beta$ is a necessary condition for an unconstrained equilibrium and that an unconstrained equilibrium achieves first-best efficiency in period 1. In period 2, if the liquidity constraint is slack, all households' consume the same amount, as $U'(c_2(\theta, \tilde{\theta}, \zeta)) = p_2(\zeta)/p_3$ for all θ and $\tilde{\theta}$. By market clearing $c_2(\theta, \tilde{\theta}, \zeta)$ must then be equal to e_2 . Since any stationary allocation $c_3(\theta, \tilde{\theta}, \zeta)$ is consistent with first-best efficiency, this completes the proof of efficiency. It remains to prove that $\gamma = \beta$ is sufficient for an unconstrained equilibrium to exist. To do so, we construct such an equilibrium. Let the prices be

$$\begin{aligned} p_1(\theta) &= p_3 u'(y_1^*(\theta)) \text{ for all } \theta, \\ p_2 &= p_3 U'(e_2), \end{aligned}$$

and let p_3 take any value in $(0, \hat{p}_3]$, where $\hat{p}_3 \equiv [u'(y_1^*(\bar{\theta}))y_1^*(\bar{\theta}) + U'(e_2)e_2]^{-1}$. From the argument above, the consumption levels in periods 1 and 2 must be at their first-best level.

²⁰In Krusell and Smith (1989) the entire capital stock of the economy is a liquid asset and the presence of uninsurable idiosyncratic risk has minor effects on aggregate dynamics. To explore our mechanism, it would be interesting to assume that capital is at least partially illiquid.

Substituting in the budget constraints the prices above and the first-best consumption levels in periods 1 and 2, we obtain

$$c_3(\theta, \tilde{\theta}, \zeta) = e_3 - u'(y_1^*(\tilde{\theta}))y_1^*(\tilde{\theta}) + u'(y_1^*(\theta))y_1^*(\theta).$$

The assumption that e_3 is large ensures that $c_3(\theta, \tilde{\theta}) > 0$ for all θ and $\tilde{\theta}$. Moreover, it is easy to show that money holdings are non-negative, thanks to the assumption $p_3 \leq \hat{p}_3$. It is also easy to check that the allocation is individually optimal and satisfies market clearing, completing the proof.

Proof of Lemma 1

Applying the implicit function theorem, to (15) and (16) we obtain

$$\frac{\partial \bar{y}^D(\bar{p}_1, \zeta)}{\partial \bar{p}_1} = \frac{\zeta U'(e_2) + (1 - \zeta) U'(e_2 - \bar{p}_1 \bar{y}_1) - \bar{p}_1 \bar{y}_1 (1 - \zeta) U''(e_2 - \bar{p}_1 \bar{y}_1)}{u''(\bar{y}_1) + \bar{p}_1^2 (1 - \zeta) U''(e_2 - \bar{p}_1 \bar{y}_1)}, \quad (25)$$

$$\frac{\partial \bar{y}^S(\bar{p}_1, \zeta)}{\partial \bar{p}_1} = \frac{\bar{\theta} \zeta U'(e_2) + (1 - \zeta) U'(e_2 + \bar{p}_1 \bar{y}_1) + \bar{p}_1 \bar{y}_1 (1 - \zeta) U''(e_2 + \bar{p}_1 \bar{y}_1)}{v''(\bar{y}_1/\bar{\theta})/\bar{\theta} - \bar{p}_1^2 (1 - \zeta) U''(e_2 + \bar{p}_1 \bar{y}_1)}. \quad (26)$$

The concavity of u and U imply that the numerator of (25) is positive and the numerator is negative, proving that $\partial \bar{y}^D(\bar{p}_1, \zeta)/\partial p_1 < 0$. The concavity of U and the convexity of v show that the denominator of (26) is positive. It remains to show that the numerator is also positive. The following chain of inequalities is sufficient for that:

$$\begin{aligned} \zeta U'(e_2) + (1 - \zeta) U'(e_2 + \bar{p}_1 \bar{y}_1) + (1 - \zeta) \bar{p}_1 \bar{y}_1 U''(e_2 + \bar{p}_1 \bar{y}_1) &> \\ U'(e_2 + \bar{p}_1 \bar{y}_1) + (1 - \zeta) (e_2 + \bar{p}_1 \bar{y}_1) U''(e_2 + \bar{p}_1 \bar{y}_1) &\geq 0. \end{aligned}$$

The first inequality follows because the concavity of U implies $U'(e_2) > U'(e_2 + \bar{p}_1 \bar{y}_1)$ and $e_2 U''(e_2 + \bar{p}_1 \bar{y}_1) < 0$. The second follows from assumption A1, completing the proof that $\partial \bar{y}^S(\bar{p}_1, \zeta)/\partial p_1 > 0$. Existence can be shown using similar arguments as in the proof of Lemma 2 below. Uniqueness follows immediately.

Proof of Proposition 2

First, we complete the characterization of a fully constrained equilibrium, presenting the steps omitted in the text. Then, we will define $\hat{\gamma}$ and prove that such an equilibrium exists iff $\gamma \geq \hat{\gamma}$. Suppose for the moment that (15) and (16) have a unique solution, $p_1(\bar{\theta}, \zeta)$ and

$y_1(\bar{\theta}, \zeta)$. In unproductive islands, output and nominal output are zero, $y_1(0, \zeta) = 0$ and $p_1(0, \zeta) y_1(0, \zeta) = 0$. From the consumer's budget constraint in period 2, we obtain

$$c_2(\theta, \tilde{\theta}, \zeta) = e_2 - p_1(\tilde{\theta}, \zeta) y_1(\tilde{\theta}, \zeta) + p_1(\theta, \zeta) y_1(\theta, \zeta).$$

The price level in unproductive islands is obtained from the Euler equation (6),

$$p_1(0, \zeta) = u'(0) \left(\mathbb{E}[U'(c_2(0, \tilde{\theta}, \zeta)) | \tilde{\theta}, \zeta] \right)^{-1}.$$

From the consumer's budget constraint in period 3 we obtain $c_3 = e_3$. Combining the Euler equations (6) and (8) and the envelope condition (10), p_3 is uniquely pinned down by

$$\frac{1}{p_3} = \beta \gamma^{-1} \mathbb{E}[U'(c_2(\theta, \tilde{\theta}, \zeta))]. \quad (27)$$

The only optimality condition that remains to be checked is the Euler equation in period 2, (7). Notice that given our construction of $c_2(\theta, \tilde{\theta}, \zeta)$ and the concavity of U , $U'(c_2(\theta, \tilde{\theta}, \zeta)) \geq \min_{\zeta} U'(c_2(\bar{\theta}, \underline{\theta}, \zeta))$ for all $\theta, \tilde{\theta}, \zeta$. It follows that a necessary and sufficient condition for (7) to hold for all $\theta, \tilde{\theta}, \zeta$ is

$$\min_{\zeta} U'(c_2(\bar{\theta}, \underline{\theta}, \zeta)) \geq \frac{1}{p_3}. \quad (28)$$

We now define the cutoff

$$\hat{\gamma} \equiv \beta \frac{\mathbb{E}[U'(c_2(\theta, \tilde{\theta}, \zeta))]}{\min_{\zeta} U'(c_2(\bar{\theta}, \underline{\theta}, \zeta))}$$

and prove the statement of the proposition. Using (27) to substitute for p_3 , condition (28) is equivalent to $\gamma \geq \hat{\gamma}$. Therefore, if an unconstrained equilibrium exists, (28) implies $\gamma \geq \hat{\gamma}$, proving necessity. If $\gamma \geq \hat{\gamma}$, the previous steps show how to construct a fully constrained equilibrium, proving sufficiency. In the case where (15) and (16) have multiple solutions, one can follow the steps above and find a value of $\hat{\gamma}$ for each solution. The smallest of these values gives us the desired cutoff. Under assumption A1, Lemma 1 ensures that (15) and (16) have a unique solution and, from the characterization above, the fully constrained equilibrium is unique.

Proof of Proposition 4

The argument in the text shows that $d \log Y_1(\zeta) / d\zeta$ is larger in a fully constrained equilibrium, for all $\zeta \in [\zeta_L, \zeta_H]$, which implies that $\log Y_1(\zeta_H) - \log Y_1(\zeta_L)$ is larger as well. This proves our statement, since $Var[\log Y_1(\zeta)] = \alpha(1 - \alpha)[\log Y_1(\zeta_H) - \log Y_1(\zeta_L)]^2$.

Proof of Proposition 5

Let $\mu_y = \mathbb{E}[y_1(\theta, \zeta)]$. Since $Y_1(\zeta) = \mathbb{E}[y_1(\theta, \zeta)|\zeta]$, we have

$$Cov[y_1(\theta, \zeta), Y_1(\zeta)] = \mathbb{E}[\mathbb{E}[(y_1(\theta, \zeta) - \mu_y)(Y_1(\zeta) - \mu_y)|\zeta]] = Var[Y_1(\zeta)],$$

and hence $Corr[y_1(\theta, \zeta), Y_1(\zeta)] = (Var[Y_1(\zeta)]/Var[y_1(\theta, \zeta)])^{1/2}$. Using the decomposition $Var[y_1(\theta, \zeta)] = Var[Y_1(\zeta)] + \mathbb{E}[Var[y_1(\theta, \zeta)|\zeta]]$, rewrite this correlation as

$$\begin{aligned} Corr[y_1(\theta, \zeta), Y_1(\zeta)] &= \left(1 + \frac{\mathbb{E}[Var[y_1(\theta, \zeta)|\zeta]]}{Var[Y_1(\zeta)]}\right)^{-1/2} = \\ &= \left(1 + \frac{(1-\alpha)\zeta_L(1-\zeta_L)(\bar{y}_1(\zeta_L))^2 + \alpha\zeta_H(1-\zeta_H)(\bar{y}_1(\zeta_H))^2}{\alpha(1-\alpha)(\zeta_H\bar{y}_1(\zeta_H) - \zeta_L\bar{y}_1(\zeta_L))^2}\right)^{-1/2}. \end{aligned}$$

Therefore, the correlation is lower in the unconstrained economy if and only if $f(\nu^U) < f(\nu^C)$, where ν^U and ν^C denote, respectively, the ratio $\bar{y}_1(\zeta_H)/\bar{y}_1(\zeta_L)$ in the unconstrained and in the fully constrained regimes and the function $f(\nu)$ is defined as

$$f(\nu) \equiv \frac{\alpha(1-\alpha)(\zeta_H\nu - \zeta_L)^2}{(1-\alpha)\zeta_L(1-\zeta_L) + \alpha\zeta_H(1-\zeta_H)\nu^2}.$$

Notice that $f(\nu)$ is continuous and differentiable, $\nu^U = 1$, from Proposition 1, and $\nu^C > 1$, from Proposition 3. Therefore, to prove our statement it is sufficient to show that $f'(\nu) > 0$ for $\nu \geq 1$. Differentiating $f(\nu)$ shows that $f'(\nu)$ has the same sign as

$$\zeta_H(\zeta_H\nu - \zeta_L)((1-\alpha)\zeta_L(1-\zeta_L) + \alpha\zeta_H(1-\zeta_H)\nu^2) - \alpha\zeta_H(1-\zeta_H)(\zeta_H\nu - \zeta_L)^2\nu.$$

Since $\zeta_H > \zeta_L$, if $\nu \geq 1$ then $\zeta_H\nu - \zeta_L > 0$. Some algebra shows that the expression above has the same sign as $(1-\alpha)(1-\zeta_L) + \alpha(1-\zeta_H)\nu$ and is always positive, completing the proof.

Proof of Proposition 6

It is easy to generalize the first-best allocation described in Section 2.1 for the binary model. Solving the planner problem for the extended model, the optimal output level in period 1, in island θ , is equal to the $y_1^*(\theta)$ that satisfies

$$\theta u'(y_1^*(\theta)) = v'(y_1^*(\theta)/\theta). \quad (29)$$

Optimal consumption in period 2 is $c_2^C(\theta, \tilde{\theta}, \zeta) = c_2^M(\theta, \tilde{\theta}, \zeta) = e_2$ for all $\theta, \tilde{\theta}$ and ζ , where C and M denote, respectively, credit and money households.

Next, we prove that any unconstrained equilibrium achieves a first-best allocation. Since (7) holds as an equality for all $\theta, \tilde{\theta}$ and ζ , both for credit and money households, it follows that $c_2^i(\theta, \tilde{\theta}, \zeta)$ is equal to a constant e_2 for all $\theta, \tilde{\theta}, \zeta$, and $i = C, M$. Then, market clearing requires $c_2 = e_2$. Substituting in (6) (for a consumer in island θ) and (9) (for a producer in island θ), and given that (6) holds as an equality, we obtain $c_1^C(\theta, \zeta) = c_1^M(\theta, \zeta) = c_1(\theta)$ and $n^C(\theta, \zeta) = n^M(\theta, \zeta) = n(\theta)$ for all θ and ζ , where

$$u'(c_1(\theta)) = \frac{p_1(\theta)}{p_2} U'(e_2) \text{ and } v'(n(\theta)) = \theta \frac{p_1(\theta)}{p_2} U'(e_2).$$

These two conditions, and market clearing in island θ , imply that $y_1^C(\theta, \zeta) = y_1^M(\theta, \zeta) = y_1^*(\theta)$ as defined by the planner optimality condition (29). Therefore, consumption levels in periods 1 and 2 achieve the first best. Since any consumption allocation in period 3 is consistent with first-best efficiency, this completes the argument.

The proof that $\gamma = \beta$ is necessary for an unconstrained equilibrium to exist is the same as in the binary model. To prove sufficiency, when $\gamma = \beta$ we can construct an unconstrained equilibrium with prices

$$\begin{aligned} p_1(\theta) &= p_3 u'(y_1^*(\theta)) \text{ for all } \theta, \\ p_2 &= p_3 U'(e_2), \end{aligned}$$

for some $p_3 \in (0, \hat{p}_3]$, where $\hat{p}_3 \equiv [u'(y_1^*(\bar{\theta}))y_1^*(\bar{\theta}) + U'(e_2)e_2]^{-1}$. From the argument above, consumption levels in periods 1 and 2 are at their first-best level. Substituting in the budget constraints the prices above and the first-best consumption levels in periods 1 and 2, we obtain

$$c_3^C(\theta, \tilde{\theta}, \zeta) = c_3^M(\theta, \tilde{\theta}, \zeta) = e_3 - u'(y_1^*(\tilde{\theta}))y_1^*(\tilde{\theta}) + u'(y_1^*(\theta))y_1^*(\theta).$$

Moreover, choosing any $p_3 \leq \hat{p}_3$ ensures that money holdings are non-negative. It is straightforward to check that this allocation satisfies market clearing and that it is individually optimal, completing the proof.

Finally, it is easy to show that $y_1^*(\theta)$ is increasing, by applying the implicit function theorem to the planner's optimality condition (29).

Preliminary results for Proposition 7

In order to prove Proposition 7, it is useful to prove some preliminary lemmas, which will be used to show that the system of functional equations (19)-(20) has a unique solution $(p_1(\cdot, \zeta), y_1(\cdot, \zeta))$, for a given ζ . These results will also be useful to prove Proposition 8.

Let us define a fixed point problem for the function $x(\cdot, \zeta)$. Recall from the text that $x(\theta, \zeta) \equiv p_1(\theta, \zeta)y_1(\theta, \zeta)$. To save on notation, in the lemmas we fix ζ and refer to $p_1(\cdot, \zeta)$, $y_1(\cdot, \zeta)$, $x(\cdot, \zeta)$, and $F(\cdot, \zeta)$, as $p(\cdot)$, $y(\cdot)$, $x(\cdot)$, and $F(\cdot)$. Notice that, in an island where $\theta = 0$, $x(0) = 0$. Moreover, non-negativity of consumption in period 2 requires that $x(\theta) \leq 1$ for all θ . Therefore, we restrict attention to the set of measurable, bounded functions $x : [0, \bar{\theta}] \rightarrow [0, 1]$ that satisfy $x(0) = 0$. We use X to denote this set.

Lemma 2 *Given $\theta > 0$ and a function $x \in X$, there exists a unique pair (p, y) which solves the system of equations*

$$u'(y) - p \int_0^{\bar{\theta}} U'(1 - py + x(\tilde{\theta})) dF(\tilde{\theta}) = 0, \quad (30)$$

$$v'(y/\theta) - \theta p \int_0^{\bar{\theta}} U'(1 - x(\tilde{\theta}) + py) dF(\tilde{\theta}) = 0. \quad (31)$$

The pair (p, y) satisfies $py \in [0, 1]$.

Proof. We proceed in two steps, first we prove existence, then uniqueness.

Step 1. Existence. For a given $p \in (0, \infty)$, it is easy to show that there is a unique y which solves (30) and a unique y which solves (31), which we denote, respectively, by $y^D(p)$ and $y^S(p)$. Finding a solution to (30)-(31), is equivalent to finding a p that solves

$$y^D(p) - y^S(p) = 0. \quad (32)$$

It is straightforward to prove that $y^D(p)$ and $y^S(p)$ are continuous on $(0, \infty)$. We now prove that they satisfy four properties: (a) $py^D(p) < 1$ for all $p \in (0, \infty)$, (b) $y^S(p) < \theta \bar{n}$ for all $p \in (0, \infty)$, (c) $\limsup_{p \rightarrow 0} y^D(p) = \infty$, and (d) $\limsup_{p \rightarrow \infty} py^S(p) = \infty$. Notice that $x(0) = 0$ with positive probability, so the Inada condition for U can be used to prove property (a). Similarly, to prove property (b), we can use the assumption $\lim_{n \rightarrow \bar{n}} v'(n) = \infty$. To prove (c) notice that (a) implies $\limsup_{p \rightarrow 0} py^D(p) \leq 1$. If $\limsup_{p \rightarrow 0} py^D(p) = 1$,

then, we immediately have $\limsup_{p \rightarrow 0} y^D(p) = \infty$. If, instead, $\limsup_{p \rightarrow 0} py^D(p) < 1$, then there exists a $K \in (0, 1)$ and an $\epsilon > 0$ such that $py^D(p) < K$ for all $p \in (0, \epsilon)$. Since U' is decreasing, this implies that $U'(1 - py^D(p) + x(\tilde{\theta}))$ is bounded above by $U'(1 - K) < \infty$ for all $p \in (0, \epsilon)$, which implies

$$\lim_{p \rightarrow 0} p \int_0^{\bar{\theta}} U'(1 - py^D(p) + x(\tilde{\theta})) dF(\tilde{\theta}) = 0.$$

Using (30), this requires $\lim_{p \rightarrow 0} u'(y^D(p)) = 0$ and, hence, $\lim_{p \rightarrow 0} y^D(p) = \infty$. To prove property (d), suppose, by contradiction, that there exist a $K > 0$ and a $P > 0$, such that $py^S(p) \leq K$ for all $p \geq P$. Then $U'(1 - x(\tilde{\theta}) + py^S(p))$ is bounded below by $U'(1 + K) > 0$ for all $p \in (P, \infty)$, which implies

$$\lim_{p \rightarrow \infty} p \int_0^{\bar{\theta}} U'(1 - x(\tilde{\theta}) + py^S(p)) dF(\tilde{\theta}) = \infty. \quad (33)$$

Moreover, since $0 \leq py^S(p) \leq K$ for all $p \geq P$, it follows that $\lim_{p \rightarrow \infty} y^S(p) = 0$ and thus

$$\lim_{p \rightarrow \infty} v'(y^S(p)/\theta) < \infty. \quad (34)$$

Using equation (31), conditions (33) and (34) lead to a contradiction, completing the proof of (d). Properties (a) and (d) immediately imply $\limsup_{p \rightarrow \infty} (py^S(p) - py^D(p)) = \infty$, while (b) and (c) imply $\limsup_{p \rightarrow 0} (y^D(p) - y^S(p)) = \infty$. It follows that there exists a pair (p', p'') , with $p' < p''$, such that $y^D(p') - y^S(p') > 0$ and $y^D(p'') - y^S(p'') < 0$. By the intermediate value theorem there exists a p which solves (32). Property (a) immediately implies that $py \in [0, 1]$, where $y = y^D(p) = y^S(p)$.

Step 2. Uniqueness. Let \hat{p} be a zero of (32), and $\hat{y} = y^D(\hat{p}) = y^S(\hat{p})$. To show uniqueness, it is sufficient to show that $dy^D(p)/dp - dy^S(p)/dp < 0$ at $p = \hat{p}$. Applying the implicit function theorem gives

$$\left[\frac{dy^D(p)}{dp} \right]_{p=\hat{p}} = \frac{\int_0^{\bar{\theta}} U'(\tilde{c}_2^D) dF(\tilde{\theta}) - \hat{p}\hat{y} \int_0^{\bar{\theta}} U''(\tilde{c}_2^D) dF(\tilde{\theta})}{u''(\hat{y}) + \hat{p}^2 \int_0^{\bar{\theta}} U''(\tilde{c}_2^D) dF(\tilde{\theta})},$$

where $\tilde{c}_2^D = 1 - \hat{p}\hat{y} + x(\tilde{\theta})$ and

$$\left[\frac{dy^S(p)}{dp} \right]_{p=\hat{p}} = \frac{\int_0^{\bar{\theta}} U'(\tilde{c}_2^S) dF(\tilde{\theta}) + \hat{p}\hat{y} \int_0^{\bar{\theta}} U''(\tilde{c}_2^S) dF(\tilde{\theta})}{v''(\hat{y}/\theta)/\theta^2 - \hat{p}^2 \int_0^{\bar{\theta}} U''(\tilde{c}_2^S) dF(\tilde{\theta})}.$$

where $\tilde{c}_2^S = 1 - x(\tilde{\theta}) + \hat{p}\hat{y}$. Using (30)-(31), the required inequality can then be rewritten as

$$\begin{aligned} \frac{v''(\hat{y}/\theta)}{\theta^2} \left(\frac{u'(\hat{y})}{\hat{p}} - \hat{p}\hat{y} \int_0^{\bar{\theta}} U''(\tilde{c}_2^D) dF(\tilde{\theta}) \right) - \frac{v'(\hat{y}/\theta)}{\theta\hat{p}} \left(u''(\hat{y}) + \hat{p}^2 \int_0^{\bar{\theta}} U''(\tilde{c}_2^D) dF(\tilde{\theta}) \right) \\ + \hat{p} \int_0^{\bar{\theta}} U''(\tilde{c}_2^S) dF(\tilde{\theta}) (u'(\hat{y}) + \hat{y}u''(\hat{y})) > 0. \end{aligned}$$

The first two terms on the left-hand side are positive. Assumption A2 implies that also the last term is positive, completing the argument. ■

Lemma 3 *Given a function $x \in X$, for any $\theta > 0$ let $(p(\theta), y(\theta))$ be the unique pair solving the system (30)-(31) and define $z(\theta) \equiv p(\theta)y(\theta)$. The function $z(\theta)$ is monotone increasing.*

Proof. Define the two functions

$$\begin{aligned} h_1(z, y; \theta) &\equiv u'(y)y - z \int_0^{\bar{\theta}} U'(1 - z + x(\tilde{\theta})) dF(\tilde{\theta}), \\ h_2(z, y; \theta) &\equiv v'(y/\theta)y/\theta - z \int_0^{\bar{\theta}} U'(1 - x(\tilde{\theta}) + z) dF(\tilde{\theta}), \end{aligned}$$

which correspond to the left-hand sides of (30) and (31) multiplied, respectively, by y and y/θ . Lemma 2 ensures that for each $\theta > 0$ there is a unique positive pair $(z(\theta), y(\theta))$ which satisfies

$$h_1(z(\theta), y(\theta); \theta) = 0 \text{ and } h_2(z(\theta), y(\theta); \theta) = 0.$$

Applying the implicit function theorem, gives

$$z'(\theta) = \frac{\frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial \theta} - \frac{\partial h_2}{\partial y} \frac{\partial h_1}{\partial \theta}}{\frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial y}}. \quad (35)$$

To prove the lemma it is sufficient to show that $z'(\theta) > 0$ for all $\theta \in (0, \bar{\theta}]$. Using z and y as shorthand for $z(\theta)$ and $y(\theta)$, the numerator on the right-hand side of (35) can be written as

$$-\frac{y}{\theta^2} [v'(y/\theta) + v''(y/\theta)y/\theta] [u'(y) + u''(y)y],$$

and the denominator can be written, after some algebra, as

$$\begin{aligned} [v'(y/\theta) + v''(y/\theta)y/\theta] \frac{z}{\theta} \int_0^{\bar{\theta}} U''(1 - z + x(\tilde{\theta})) dF(\tilde{\theta}) + \\ + [u'(y) + u''(y)y] z \int_0^{\bar{\theta}} U''(1 - x(\tilde{\theta}) + z) dF(\tilde{\theta}) + \frac{y^2}{z\theta^2} [u''(y)v'(y/\theta)\theta - u'(y)v''(y/\theta)]. \end{aligned} \quad (36)$$

Assumption A2 ensures that both numerator and denominator are negative, completing the proof. ■

We can now define a map T from the space X into itself.

Definition 2 *Given a function $x \in X$, for any $\theta > 0$ let $(p(\theta), y(\theta))$ be the unique pair solving the system (30)-(31). Define a map $T : X \rightarrow X$ as follows. Set $(Tx)(\theta) = p(\theta)y(\theta)$ if $\theta > 0$ and $(Tx)(\theta) = 0$ if $\theta = 0$.*

The following lemmas prove monotonicity and discounting for the map T . These properties will be used to find a fixed point of T . In turns, this fixed point will be used to construct the equilibrium in Proposition 7.

Lemma 4 *Take any $x^0, x^1 \in X$, with $x^1(\theta) \geq x^0(\theta)$ for all θ . Then $(Tx^1)(\theta) \geq (Tx^0)(\theta)$ for all θ .*

Proof. For each $\tilde{\theta} \in [0, \bar{\theta}]$ and any scalar $\lambda \in [0, 1]$, with a slight abuse of notation, we define $x(\tilde{\theta}, \lambda) \equiv x^0(\tilde{\theta}) + \lambda\Delta(\tilde{\theta})$, where $\Delta(\tilde{\theta}) \equiv x^1(\tilde{\theta}) - x^0(\tilde{\theta}) \geq 0$. Notice that $x(\tilde{\theta}, 0) = x^0(\tilde{\theta})$ and $x(\tilde{\theta}, 1) = x^1(\tilde{\theta})$. Fix a value for θ and define the two functions

$$\begin{aligned} h_1(z, y; \lambda) &\equiv yu'(y) - z \int_0^{\bar{\theta}} U'(1 - z + x(\tilde{\theta}, \lambda))dF(\tilde{\theta}), \\ h_2(z, y; \lambda) &\equiv v'(y/\theta)y/\theta - z \int_0^{\bar{\theta}} U'(1 - x(\tilde{\theta}, \lambda) + z)dF(\tilde{\theta}). \end{aligned}$$

Applying Lemma 2, for each $\lambda \in [0, 1]$ we can find a unique positive pair $(z(\lambda), y(\lambda))$ that satisfies

$$h_1(z(\lambda), y(\lambda); \lambda) = 0 \text{ and } h_2(z(\lambda), y(\lambda); \lambda) = 0.$$

We are abusing notation in the definition of $h_1(\cdot, \cdot; \lambda)$, $h_2(\cdot, \cdot; \lambda)$, $z(\lambda)$, and $y(\lambda)$, given that the same symbols were used above to define functions of θ . Here we keep θ constant throughout the proof, so no confusion should arise. Notice that, by construction, $(Tx^0)(\theta) = z(0)$ and $(Tx^1)(\theta) = z(1)$. Therefore, to prove our statement it is sufficient to show that $z'(\lambda) \geq 0$ for all $\lambda \in [0, 1]$.

Applying the implicit function theorem yields

$$z'(\lambda) = \frac{\frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial \lambda} - \frac{\partial h_2}{\partial y} \frac{\partial h_1}{\partial \lambda}}{\frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial y}}. \quad (37)$$

Using z and y as shorthand for $z(\lambda)$ and $y(\lambda)$, the numerator on the right-hand side of (37) can be written as

$$\begin{aligned} & [u'(y) + u''(y)y] z \int_0^{\bar{\theta}} U''(1 - x(\tilde{\theta}, \lambda) + z) \Delta(\tilde{\theta}) dF(\tilde{\theta}) + \\ & + \frac{z}{\theta} [v'(y/\theta) + v''(y/\theta)y/\theta] \int_0^{\bar{\theta}} U''(1 - z + x(\tilde{\theta}, \lambda)) \Delta(\tilde{\theta}) dF(\tilde{\theta}). \end{aligned}$$

The denominator takes a form analogous to (36). Again, assumption A2 ensures that both the numerator and the denominator are negative, completing the argument. ■

Before proving the discounting property, it is convenient to restrict the space X to the space \tilde{X} of functions bounded in $[0, \bar{z}]$ for an appropriate $\bar{z} < 1$. The following lemma shows that the map T maps \tilde{X} into itself, and that any fixed point of T in X must lie in \tilde{X} .

Lemma 5 *There exists a $\bar{z} < 1$, such that if $x \in X$ then $(Tx)(\theta) \leq \bar{z}$ for all θ .*

Proof. Set $\bar{x}(0) = 0$ and $\bar{x}(\theta) = 1$ for all $\theta > 0$. Setting $x(\cdot) = \bar{x}(\cdot)$ and $\theta = \bar{\theta}$, equations (30)-(31) take the form

$$\begin{aligned} u'(y) &= p[F(0)U'(1 - py) + (1 - F(0))U'(2 - py)], \\ v'(y/\bar{\theta}) &= \bar{\theta}p[F(0)U'(1 + py) + (1 - F(0))U'(py)]. \end{aligned}$$

Let (\hat{p}, \hat{y}) denote the pair solving these equations, and let $\bar{z} \equiv \hat{p}\hat{y}$. Since $F(0) > 0$ and U satisfies the Inada condition, $\lim_{c \rightarrow 0} U'(c) = \infty$, inspecting the first equation shows that $\bar{z} < 1$. Now take any $x \in X$. Since $x(\theta) \leq \bar{x}(\theta)$ for all θ , Lemma 4 implies that $(Tx)(\theta) \leq (T\bar{x})(\theta)$. Moreover, Lemma 3 implies that $(T\bar{x})(\theta) \leq (T\bar{x})(\bar{\theta}) = \bar{z}$. Combining these inequalities we obtain $(Tx)(\theta) \leq \bar{z}$. ■

Lemma 6 *There exists a $\delta \in (0, 1)$ such that the map T satisfies the discounting property: for any $x^0, x^1 \in \tilde{X}$ such that $x^1(\theta) = x^0(\theta) + a$ for some $a > 0$, the follow inequality holds*

$$|(Tx^1)(\theta) - (Tx^0)(\theta)| \leq \delta a \text{ for all } \theta \in [0, \bar{\theta}].$$

Proof. Proceeding as in the proof of Lemma 4, define $x(\tilde{\theta}, \lambda) \equiv x^0(\tilde{\theta}) + \lambda\Delta(\tilde{\theta})$, where now $\Delta(\tilde{\theta}) = a$ for all $\tilde{\theta}$. After some algebra, we obtain

$$z'(\lambda) = \frac{\left(1 + \frac{yu''(y)}{u'(y)}\right) A + \left(1 + \frac{nv''(n)}{v'(n)}\right) B}{\left(1 + \frac{yu''(y)}{u'(y)}\right) A + \left(1 + \frac{nv''(n)}{v'(n)}\right) B + \frac{nv''(n)}{v'(n)} - \frac{yu''(y)}{u'(y)}} a, \quad (38)$$

where y and n are shorthand for $y(\lambda)$ and $y(\lambda)/\theta$ and

$$A = -\frac{z(\lambda) \int_0^{\bar{\theta}} U'' \left(1 - x(\tilde{\theta}, \lambda) + z(\lambda)\right) dF(\tilde{\theta})}{\int_0^{\bar{\theta}} U' \left(1 - x(\tilde{\theta}, \lambda) + z(\lambda)\right) dF(\tilde{\theta})},$$

$$B = -\frac{z(\lambda) \int_0^{\bar{\theta}} U'' \left(1 - z(\lambda) + x(\tilde{\theta}, \lambda)\right) dF(\tilde{\theta})}{\int_0^{\bar{\theta}} U' \left(1 - z(\lambda) + x(\tilde{\theta}, \lambda)\right) dF(\tilde{\theta})}.$$

Now, given that $z(\lambda)$ and $x(\tilde{\theta}, \lambda)$ are both in $[0, \bar{z}]$ and $\bar{z} < e_2$, and given that U has continuous first and second derivatives on $(0, \infty)$, it follows that both A and B are bounded above. We can then find a uniform upper bound on both A and B , independent of λ and of the functions x^0 and x^1 chosen. Let C be this upper bound. Given that $u''(y) \leq 0$, then

$$\left(1 + \frac{yu''(y)}{u'(y)}\right) A + \left(1 + \frac{nv''(n)}{v'(n)}\right) B \leq \left(2 + \frac{nv''(n)}{v'(n)}\right) C.$$

Therefore, (38) implies

$$z'(\lambda) \leq \left(1 + \frac{nv''(n)/v'(n) - yu''(y)/u'(y)}{(2 + nv''(n)/v'(n))C}\right)^{-1} a.$$

Recall that $\underline{\rho} > 0$ is a lower bound for $-yu''(y)/u'(y)$. Then

$$\frac{nv''(n)/v'(n) - yu''(y)/u'(y)}{(2 + nv''(n)/v'(n))C} \geq \frac{-yu''(y)/u'(y)}{2C} \geq \frac{\underline{\rho}}{2C}.$$

Setting $\delta \equiv 1/[1 + \underline{\rho}/(2C)] < 1$, it follows that $z'(\lambda) \leq \delta a$ for all $\lambda \in [0, 1]$. Integrating both sides of the last inequality over $[0, 1]$, gives $z(1) - z(0) \leq \delta a$. By construction $(Tx^1)(\theta) = z(1)$ and $(Tx^0)(\theta) = z(0)$, completing the proof. ■

Proof of Proposition 7

We first uniquely characterize prices and allocations in a fully constrained equilibrium. Next, we will use this characterization to prove our claim. The argument in the text and the

preliminary results above show that if there exists an equilibrium with $m_2(\theta, \tilde{\theta}, \zeta) = 0$ for all θ and $\tilde{\theta}$, then $p_1(\theta, \zeta)$ and $y_1(\theta, \zeta)$ must solve the functional equations (19)-(20) for any given ζ . To find the equilibrium pair $(p_1(\theta, \zeta), y_1(\theta, \zeta))$ we first find a fixed point of the map T defined above (Definition 2). Lemmas 4 and 6 show that T is a map from a space of bounded functions into itself and satisfies the assumptions of Blackwell's theorem. Therefore, a fixed point exists and is unique. Let x denote the fixed point, then Lemma 2 shows that we can find two functions $p_1(\theta, \zeta)$ and $y_1(\theta, \zeta)$ for a given ζ that satisfy (30)-(31). Since $x(\theta, \zeta)$ is a fixed point of T we have $x(\theta, \zeta) = p_1(\theta, \zeta)y_1(\theta, \zeta)$, and substituting in (30)-(31) shows that (19)-(20) are satisfied. Therefore, in a fully constrained equilibrium $p_1(\theta, \zeta)$ and $y_1(\theta, \zeta)$ are uniquely determined, and so is labor supply $n(\theta, \zeta) = y_1(\theta, \zeta)/\theta$. Moreover, from the budget constraint and the market clearing condition in period 2, consumption in period 2 is uniquely determined by (21). The price p_2 is equal to 1, given the normalization in the text. From the consumer's budget constraint in period 3 we obtain $c_3 = e_3$. Combining the Euler equations (6) and (8) and the envelope condition (10), p_3 is uniquely pinned down by

$$\frac{1}{p_3} = \beta\gamma^{-1}\mathbb{E}[U'(c_2(\theta, \tilde{\theta}, \zeta))]. \quad (39)$$

Moreover, equilibrium money holdings are $m_1(\theta, \zeta) = 1 - p_1(\theta, \zeta)y_1(\theta, \zeta)$, $m_2(\theta, \tilde{\theta}, \zeta) = 0$, and $m_3(\theta, \tilde{\theta}, \zeta) = \gamma$. Define the cutoff

$$\hat{\gamma} \equiv \beta \frac{\mathbb{E}[U'(c_2(\theta, \tilde{\theta}, \zeta))]}{\min_{\zeta}\{U'(c_2(\bar{\theta}, \underline{\theta}, \zeta))\}}.$$

The only optimality condition that remains to be checked is the Euler equation in period 2, that is, equation (7). Given the definition of $c_2(\theta, \tilde{\theta}, \zeta)$, Lemma 3 implies that it is an increasing function of θ and a decreasing function of $\tilde{\theta}$. It follows that a necessary and sufficient condition for (7) to hold for all $\theta, \tilde{\theta}$ and ζ is

$$\min_{\zeta}\{U'(c_2(\bar{\theta}, \underline{\theta}, \zeta))\} \geq \frac{1}{p_3}. \quad (40)$$

Substituting the expression (39) for $1/p_3$, this condition is equivalent to $\gamma \geq \hat{\gamma}$. Therefore, if a fully constrained equilibrium exists, $c_2(\theta, \tilde{\theta}, \zeta)$ is uniquely determined and condition (40) implies that $\gamma \geq \hat{\gamma}$, proving necessity. Moreover, if $\gamma \geq \hat{\gamma}$, the previous steps show how to construct a fully constrained equilibrium, proving sufficiency.

Finally, the proof that nominal income $p_1(\theta, \zeta) y_1(\theta, \zeta)$ is monotone increasing in θ , for a given ζ , follows immediately from Lemma 3. To prove that also output $y_1(\theta, \zeta)$ is monotone increasing in θ , let us use the same functions $h_1(z, y; \theta)$ and $h_2(z, y; \theta)$ and the same notation as in the proof of Lemma 3. For a given ζ , apply the implicit function theorem to get

$$y'(\theta) = \frac{\frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial \theta} - \frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial \theta}}{\frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial y}}. \quad (41)$$

Then it is sufficient to show that $y'(\theta) > 0$ for all $\theta \in (0, \bar{\theta}]$. Using z and y as shorthand for $z(\theta)$ and $y(\theta)$, the numerator on the right-hand side of (41) can be written as

$$\frac{y}{\theta^2} [v'(y/\theta) + v''(y/\theta) y/\theta] \left[z \int_0^{\bar{\theta}} U''(1 - z + x(\tilde{\theta})) dF(\tilde{\theta}) - \int_0^{\bar{\theta}} U'(1 - z + x(\tilde{\theta})) dF(\tilde{\theta}) \right],$$

and is negative. Finally, the denominator is equal to (36) and is negative thanks to assumption A2, as we have argued in the proof of Lemma 3. This completes the argument.

Proof of Proposition 8

The proof proceeds in three steps. The first two steps prove that, for each θ , the nominal income in island θ , $x(\theta, \zeta)$, is increasing with the aggregate shock ζ . Using this result, the third step shows that $y_1(\theta, \zeta)$ is increasing in ζ . Consider two values ζ^I and ζ^{II} , with $\zeta^{II} > \zeta^I$. Denote, respectively, by T_I and T_{II} the maps defined in Definition 2 under the distributions $F(\theta|\zeta_I)$ and $F(\theta|\zeta_{II})$. Let x^I and x^{II} be the fixed points of T_I and T_{II} , that is, $x^I(\theta) \equiv x(\theta, \zeta^I)$ and $x^{II}(\theta) \equiv x(\theta, \zeta^{II})$ for any θ . Again, to save on notation, we drop the period index for y_1 .

Step 1. Let the function x^0 be defined as $x^0 = T_{II}x^I$. In this step, we want to prove that $x^0(\theta) > x^I(\theta)$ for all $\theta > 0$. We will prove it pointwise for each θ . Fix $\theta > 0$ and define the functions

$$\begin{aligned} h_1(z, y; \zeta) &\equiv yu'(y) - z \int_0^{\bar{\theta}} U'(1 - z + x^I(\tilde{\theta})) dF(\tilde{\theta}|\zeta), \\ h_2(z, y; \zeta) &\equiv v'(y/\theta) y/\theta - z \int_0^{\bar{\theta}} U'(1 - x^I(\tilde{\theta}) + z) dF(\tilde{\theta}|\zeta), \end{aligned}$$

for $\zeta \in [\zeta^I, \zeta^{II}]$. Lemma 2 implies that we can find a unique pair $(z(\zeta), y(\zeta))$ that satisfies

$$h_1(z(\zeta), y(\zeta); \zeta) = 0 \text{ and } h_2(z(\zeta), y(\zeta); \zeta) = 0.$$

Once more, we are abusing notation in the definition of $h_1(\cdot, \cdot; \zeta)$, $h_2(\cdot, \cdot; \zeta)$, $z(\zeta)$, and $y(\zeta)$. However, as θ is kept constant, there is no room for confusion. Notice that $z(\zeta^I) = x^I(\theta)$, since x^I is a fixed point of T_I , and $z(\zeta^{II}) = x^0(\theta)$, by construction. Therefore, to prove our statement we need to show that $z(\zeta^{II}) > z(\zeta^I)$. It is sufficient to show that $z'(\zeta) > 0$ for all $\zeta \in [\zeta^I, \zeta^{II}]$. Applying the implicit function theorem gives

$$z'(\zeta) = \frac{\frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial \zeta} - \frac{\partial h_2}{\partial y} \frac{\partial h_1}{\partial \zeta}}{\frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial y}}. \quad (42)$$

Notice that $x^I(\tilde{\theta})$ is monotone increasing in $\tilde{\theta}$, by Lemma 3, and U is strictly concave. Therefore, $U'(1 - z + x^I(\tilde{\theta}))$ is decreasing in $\tilde{\theta}$ and $U'(1 - x^I(\tilde{\theta}) + z)$ is increasing in $\tilde{\theta}$. By the properties of first-order stochastic dominance, $\int_0^{\bar{\theta}} U'(1 - z + x^I(\tilde{\theta})) dF(\tilde{\theta}|\zeta)$ is decreasing in ζ and $\int_0^{\bar{\theta}} U'(1 - x^I(\tilde{\theta}) + z) dF(\tilde{\theta}|\zeta)$ is increasing in ζ . This implies that $\partial h_1 / \partial \zeta > 0$ and $\partial h_2 / \partial \zeta < 0$. Using y as shorthand for $y(\zeta)$, the numerator on the right-hand side of (42) is, with the usual notation,

$$[u'(y) + yu''(y)] \frac{\partial h_2}{\partial \zeta} - \frac{1}{\theta} [v'(y/\theta) + v''(y/\theta)y/\theta] \frac{\partial h_1}{\partial \zeta}.$$

The denominator is the analogue of (36). Once more, assumption A2 ensures that both numerator and denominator are negative, completing the argument.

Step 2. Define the sequence of functions (x^0, x^1, \dots) in X , using the recursion $x^{j+1} = T_{II}x^j$. Since, by step 1, $x^0 \geq x^I$ (where by $x^0 \geq x^I$ we mean $x^0(\theta) \geq x^I(\theta)$ for all $\theta > 0$) and, by Lemma 4, T_{II} is a monotone operator, it follows that this sequence is monotone, with $x^{j+1} \geq x^j$. Moreover, T_{II} is a contraction by Lemmas 4 and 6, so this sequence has a limit point, which coincides with the fixed point x^{II} . This implies that $x^{II} \geq x^0$ and, together with the result in step 1, shows that $x^{II} > x^I$, as we wanted to prove.

Step 3. Fix $\theta > 0$ and, with the usual abuse of notation, define the functions

$$\begin{aligned} h_1(z, y; \zeta) &\equiv yu'(y) - z \int_0^{\bar{\theta}} U'(1 - z + x(\tilde{\theta}, \zeta)) dF(\tilde{\theta}|\zeta), \\ h_2(z, y; \zeta) &\equiv v'(y/\theta)y/\theta - z \int_0^{\bar{\theta}} U'(1 - x(\tilde{\theta}, \zeta) + z) dF(\tilde{\theta}|\zeta). \end{aligned}$$

Notice the difference with the definitions of h_1 and h_2 in step 1, now $x(\tilde{\theta}, \zeta)$ replaces $x^I(\tilde{\theta})$. The functions $z(\zeta)$ and $y(\zeta)$ are defined in the usual way. Applying the implicit function

theorem, we get

$$y'(\zeta) = \frac{\frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial \zeta} - \frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial \zeta}}{\frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial y}}.$$

To evaluate the numerator, notice that

$$\begin{aligned} \frac{\partial h_1}{\partial z} &= - \int_0^{\bar{\theta}} U'(1 - z + x(\tilde{\theta}, \zeta)) dF(\tilde{\theta}|\zeta) + z \int_0^{\bar{\theta}} U''(1 - z + x(\tilde{\theta}, \zeta)) dF(\tilde{\theta}|\zeta) < 0, \\ \frac{\partial h_2}{\partial z} &= - \int_0^{\bar{\theta}} U'(1 - x(\tilde{\theta}, \zeta) + z) dF(\tilde{\theta}|\zeta) - z \int_0^{\bar{\theta}} U''(1 - x(\tilde{\theta}, \zeta) + z) dF(\tilde{\theta}|\zeta) \leq \\ &\leq - \int_0^{\bar{\theta}} \left[U'(1 - x(\tilde{\theta}, \zeta) + z) + (1 - x(\tilde{\theta}, \zeta) + z) U''(1 - x(\tilde{\theta}, \zeta) + z) \right] dF(\tilde{\theta}|\zeta) \leq 0, \end{aligned}$$

where the last inequality follows from assumption A1' (this is the only place where this assumption is used). Furthermore, notice that

$$\frac{\partial h_1}{\partial \zeta} = -z \int_0^{\bar{\theta}} U''(1 - z + x(\tilde{\theta}, \zeta)) \frac{\partial x(\tilde{\theta}, \zeta)}{\partial \zeta} dF(\tilde{\theta}|\zeta) - z \int_0^{\bar{\theta}} U'(1 - z + x(\tilde{\theta}, \zeta)) \frac{\partial f(\tilde{\theta}|\zeta)}{\partial \zeta} d\tilde{\theta} > 0$$

where the first element is positive from steps 1 and 2, and the second element is positive because ζ leads to a first order stochastic increase in $\tilde{\theta}$ and $U'(1 - z + x(\tilde{\theta}, \zeta))$ is decreasing in $\tilde{\theta}$. A similar reasoning shows that

$$\frac{\partial h_2}{\partial \zeta} = z \int_0^{\bar{\theta}} U''(1 - x(\tilde{\theta}, \zeta) + z) \frac{\partial x(\tilde{\theta}, \zeta)}{\partial \zeta} dF(\tilde{\theta}|\zeta) + z \int_0^{\bar{\theta}} U'(1 - x(\tilde{\theta}, \zeta) + z) \frac{\partial f(\tilde{\theta}|\zeta)}{\partial \zeta} d\tilde{\theta} < 0.$$

Putting together the four inequalities just derived shows that the numerator is negative. The denominator takes the usual form, analogous to (36), and is negative. This completes the proof.

Proof of Proposition 9

From expression (24) it follows that

$$\begin{aligned} \frac{\partial Y_1(\zeta, \xi)}{\partial \zeta} &= \int_0^{\bar{\theta}} y_1(\theta, \xi) \frac{\partial f(\theta|\zeta)}{\partial \zeta} d\theta, \\ \frac{\partial Y_1(\zeta, \xi)}{\partial \xi} &= \int_0^{\bar{\theta}} \frac{\partial y_1(\theta, \xi)}{\partial \xi} dF(\theta|\zeta). \end{aligned}$$

In the case of an unconstrained equilibrium, the analogue of Proposition 6 can be easily derived, showing that $\partial y_1(\theta, \xi)/\partial \xi = 0$ and $\partial y_1(\theta, \xi)/\partial \theta > 0$. These properties imply that

$\partial Y_1(\zeta, \xi)/\partial \zeta > 0$ and $\partial Y_1(\zeta, \xi)/\partial \xi = 0$. Next, consider a fully constrained equilibrium, where $\phi = 0$ and $\gamma \geq \hat{\gamma}$. For each value of ξ , the functions $p_1(\theta, \xi)$ and $y_1(\theta, \xi)$ can be derived solving the following system of functional equations, analogous to (19)-(20):

$$\begin{aligned} u'(y_1(\theta, \xi)) &= p_1(\theta, \xi) \int_0^{\bar{\theta}} U'(c_2(\tilde{\theta}, \theta, \xi)) dF(\tilde{\theta}|\xi, \theta), \\ v'(y_1(\theta, \xi)/\theta) &= \theta p_1(\theta, \xi) \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta}, \xi)) dF(\tilde{\theta}|\xi, \theta), \end{aligned}$$

where $c_2(\tilde{\theta}, \theta, \xi) = 1 - p_1(\theta, \xi)y_1(\theta, \xi) + p_1(\tilde{\theta}, \xi)y_1(\tilde{\theta}, \xi)$. The only formal difference between these and (19)-(20) is that the distribution $F(\tilde{\theta}|\xi, \theta)$ depends also on θ . However, this does not affect any of the steps of Proposition 7 (there is only a minor difference in the proof of the analogue of Lemma 3, the details are available on request). Therefore, this system has a unique solution for each ξ . Next, following the steps of Propositions 7 and 8, we can show that $y_1(\theta, \xi)$ is increasing in θ and ξ . This implies that $\partial Y_1(\zeta, \xi)/\partial \zeta > 0$ and $\partial Y_1(\zeta, \xi)/\partial \xi > 0$.

References

- [1] Acemoglu, Daron and Fabrizio Zilibotti, “Was Prometheus Unbounded by Chance? Risk, Diversification and Growth,” *Journal of Political Economy*, 1997, 105, 709-751.
- [2] Aiyagari Rao and Stephen Williamson, “Money and Dynamic Credit Arrangements with Private Information,” *Journal of Economic Theory*, 2000, 91(2), 248-279.
- [3] Bacchetta, P. and S. Gerlach, “Consumption and Credit Constraints: International Evidence,” *Journal of Monetary Economics*, 1997, 40 (2), 207-238.
- [4] Bencivenga, Valerie and Bruce Smith, “Financial Intermediation and Endogenous Growth,” *Review of Economic Studies*, 1991, 58, 195-209.
- [5] Berentsen, Aleksander, Camera, Gabriele and Christopher Waller, “The Distribution Of Money Balances And The Nonneutrality Of Money,” *International Economic Review*, 2005, 46, 465-486.

- [6] Bernanke, Ben and Mark Gertler, “Agency Costs, Net Worth, and Business Fluctuations,” *American Economic Review*, 1989, 79, 14-31.
- [7] Craig, Ben and Guillaume Rocheteau, “Inflation and Welfare: a Search Approach,” mimeo, 2007.
- [8] Diamond, Peter, “Aggregate Demand Management in Search Equilibrium,” *Journal of Political Economy*, 1982, 90, 881-94.
- [9] Diamond, Peter, “Money in Search Equilibrium,” *Econometrica*, 1984, 52, 1-20.
- [10] Epstein, Larry G. and Stanley E. Zin, “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework”, *Econometrica*, 1989, 57, 937-969
- [11] Fuerst, Timothy, “Liquidity, Loanable Funds, and Real Activity,” *Journal of Monetary Economics*, 1992, 29, 3-24.
- [12] Hubbard, R., J. Skinner, and S. Zeldes, “The importance of precautionary motives in explaining individual and aggregate savings,” *Carnegie Rochester Conference Series on Public Policy*, 1994, 40, 59–126.
- [13] Jappelli, Tullio and Marco Pagano, “Consumption and Capital Market Imperfections: an International Comparison”, *American Economic Review*, 1989, 79, 1088-1105.
- [14] Kiyotaki, Nobuhiro and John H. Moore, “Credit Cycles,” *Journal of Political Economy*, 1997, 105, 211-248.
- [15] Kiyotaki, Nobuhiro and John H. Moore, “Liquidity, Business Cycles and Monetary Policy,” 2001, mimeo, Princeton University.
- [16] Kiyotaki, Nobuhiro and Randall Wright, “On Money as a Medium of Exchange,” *Journal of Political Economy*, 1989, 97, 927-954.
- [17] Krusell, Per and Anthony A. Smith, Jr., “Income and Wealth Heterogeneity in the Macroeconomy,” *Journal of Political Economy*, 1998, 106, 867-896.

- [18] Lagos, Ricardo and Randall Wright, "A Unified Framework of Money Theory and Policy Analysis," Staff Report no. 346, September 2004, Res. Dept., Fed. Reserve Bank Minneapolis.
- [19] Lagos, Ricardo and Randall Wright, "A Unified Framework of Money Theory and Policy Analysis," *Journal of Political Economy*, 2005, 113, 463-484.
- [20] Lucas, Robert, "Liquidity and Interest Rates," *Journal of Economic Theory*, 1990, 50, 237-264.
- [21] Lucas, Robert, "Inflation and Welfare," *Econometrica*, 2000, 68, 247-274.
- [22] Lucas, Robert and Edward Prescott, "Equilibrium Search and Unemployment," *Journal of Political Economy*, 1974, 7, 188-209.
- [23] Milgrom, Paul, "Good News and Bad News: Representation Theorems and Applications," *Bell Journal of Economics*, 1981, 12, 380-91.
- [24] Molico, Miguel, "The Distribution Of Money And Prices In Search Equilibrium," *International Economic Review*, 2006, 47(3), 701-722.
- [25] Reed, Robert and Christopher Waller, "Money and Risk Sharing," *Journal of Money, Credit, and Banking*, 2006, 38(6) 1599-1618.
- [26] Rocheteau, Guillaume and Randall Wright, "Money in Competitive Equilibrium, in Search Equilibrium, and in Competitive Search Equilibrium," *Econometrica*, 2005, 73, 175-202.
- [27] Shi, Shouyong, "A Divisible Search Model of Fiat Money," *Econometrica*, 1997, 65, 75-102.