Aligned Delegation

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Abstract

A principal delegates multiple decisions to an agent, who has private information relevant to each decision. The principal is uncertain about the agent’s preferences. I solve for the max-min optimal mechanisms – those which maximize the principal’s payoff against the worst possible agent preference type – in a variety of settings. These mechanisms are characterized by a property I call “aligned delegation.” In an aligned delegation mechanism all agent types play identically, as if they shared the principal’s preferences.

Max-min optimal mechanisms may take the simple forms of simultaneous ranking mechanisms, sequential quotas, or budgets. This work motivates the use of these common contracts.

1 Introduction

Consider a problem in which a principal (he) delegates a number of decisions to an agent (she). A school requires a teacher to assign grades to all of her students; a firm appoints a manager to determine investment levels in different projects; an organization asks a supervisor to evaluate her employees and make promotion and firing decisions, or give out bonuses. In each of these cases, the principal relies on the agent because she observes “states of the world” relevant to the principal’s preferences. The teacher knows how well the students have done in the class; the manager sees the quality of the investment opportunities; the supervisor has observed the performance of her employees. How should the principal choose a delegation rule that specifies which actions the agent may take?

If the principal and agent had identical preferences, there would be no reason to restrict the agent’s choices. But preferences may only be partially aligned. For instance, a teacher and school agree that better students should receive higher grades, all else equal. However, the teacher may be biased towards low grades relative to the school’s wishes, or high grades, or something more complicated – failing too many students while giving out too many A’s, say. Giving the agent more

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freedom lets her make better use of her private information, but it also gives more opportunities for biased agents to take advantage of the principal.

I assume that the principal knows the distribution of states, but has limited information about the agent’s biases – her utility function or “type” mapping a decision’s state and action into a payoff. He may only know that her utility satisfies a certain property, or that it is of a particular functional form. The principal seeks a robust mechanism which will work well for any possible agent type.

Formally, I model this robustness by searching for a max-min optimal mechanism. For any set of agent types and for any mechanism the principal suggests, we can find the minimum expected principal payoff over all agent types. A max-min optimal mechanism maximizes this worst-type payoff.¹

In a “simultaneous” problem the agent observes all of the underlying states before any actions are taken. A teacher sees all of her students’ test scores before assigning grades, for instance. Suppose there is a simultaneous problem in which the principal and agent both prefer higher actions in higher states, formalized as an increasing difference (complementarity) condition on utility functions. If the principal believes that the agent might have any increasing difference utility, then a standard ranking mechanism is max-min optimal. The agent is only asked to rank states from lowest to highest. The decision with the lowest state is assigned to some predetermined low action, the next higher state is assigned to some higher action, etc. This corresponds to a “strict grading curve” where the top 10% of students get an A and the next 15% get an A-; a bonus rule which gives $50,000 to the best-performing employee in a group and $40,000 to the next one; or a policy under which a manager invests in the top three projects in a given cycle.²

It is more difficult to solve for the max-min optimal mechanism in a “sequential” problem, where each action is chosen before the next state is observed – employees are evaluated one by one, and the

¹The set of possible agent utility functions may be many- or even infinite-dimensional, so it may be difficult for the principal to express a prior belief over the distribution of agent types. Even given a well-specified prior over agent types, the standard tools of Bayesian optimal mechanism design – see, for example, Pavan, Segal, and Toikka (2010) – assume that the agent has one-dimensional private information at every period. In my model the agent may have arbitrarily high dimensional private information on her utility function and may also have many-dimensional information on the observed states of the world, greatly complicating the Bayesian analysis. It is also standard to assume that the parties have quasilinear utility and that transfer payments can be used, which is not assumed here.

²Chakraborty and Harbaugh (2007) proposes ranking protocols for a similar environment in which the principal cannot commit to a mechanism – this paper is discussed further in the literature review. Campbell (1998) and Pesendorfer (2000) propose ranking mechanisms in a very different context, as a way for different bidders in auctions to credibly collude. In these three papers ranking is shown to achieve high payoffs when there are many decisions. Another instance of ranking contracts in economics is in tournament incentive structures. The literature on tournaments (see, e.g., Lazear and Rosen (1981)) focuses on the incentives which tournaments provide to those being evaluated – motivating employees to work hard, for instance. In the current paper I take the qualities of the evaluated to be exogenous, and look instead at the incentives of the evaluator – the supervisor who observes her employees and can fire or promote them.

Malcomson (1984) and Fuchs (2007) have previously pointed out that tournaments may have good incentive properties for evaluators who are themselves residual claimants on their employees’ compensation. The evaluator who commits to paying bonuses based on relative rankings cannot benefit ex post by misreporting performance, whereas payments based on nonverifiable output levels could be profitably manipulated.
supervisor must choose one employee’s bonus before observing the next employee’s performance. Under a stronger assumption than increasing differences, that the principal and agent utilities have a quadratic loss functional form, the max-min optimal mechanism is a sequential quota. In a sequential quota, the agent is given a list of $N$ actions to assign to the $N$ decisions. Every period, she observes the current state then chooses one of the actions from the list.

The quadratic loss utilities satisfy increasing differences, and also have the property that adding variance to a player’s belief about the state has no effect on his or her preferences over actions. This set of utility functions allows for any increasing mapping from states into preferred actions. Two special cases are the quadratic loss utilities in which the agent has a constant bias or a linear bias relative to the principal.

The max-min optimality of ranking and quota mechanisms follow as corollaries of the results I develop in Section 3. In this general analysis, I allow not only for simultaneous or sequential problems but also for asymmetrically significant decisions – everyone may agree that decision $i$ is 37% more important than decision $j$. Perhaps this is a policy decision which affects 37% more people, or a decision about whether to invest in a 37% larger project. In a sequential environment, both parties might also discount the importance of later decisions.

On the way to finding max-min optimal mechanisms, I investigate mechanisms which have a property that I call aligned delegation. A mechanism satisfies aligned delegation with respect to a set of types if every agent type acts as though she were maximizing the principal’s payoff. Think of the ranking mechanism: any increasing-difference agent type submits honest rankings. Because all types would play identically, the principal can calculate his expected payoff without positing a prior distribution over the agent’s types. This makes it meaningful for the principal to search for the optimal (expected-payoff maximizing) aligned delegation mechanism.

Probability assignment mechanisms are generalizations of quotas which naturally arise in the search for optimal aligned delegation mechanisms. These simultaneous or sequential contracts let the agent take any actions she wants so long as each action is ultimately played an appropriate number of times. (The agent is allowed to assign a probability distribution over actions to a decision; the “number of times” accounts for stochastic actions and also for asymmetrically significant decisions). Proposition 1 shows that probability assignment mechanisms are optimal out of all aligned delegation mechanisms if two conditions hold. First, utilities are such that probability assignment mechanisms satisfy aligned delegation – I call this condition PA-alignment. Second, the agent has a rich set of possible utilities which contains certain extreme preferences.

In Proposition 2 I show that, under PA-alignment, these mechanisms provide high payoffs to the principal when there are many decisions. More precisely, if each decision’s underlying state is drawn iid from a known distribution and the number of decisions is large, then probability assignment

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3That is, a sequential quota is max-min optimal if the principal has some quadratic loss utility function and believes that the agent may have any quadratic loss utility.
mechanisms approximate the principal’s full information first-best payoffs. The main technical result of the paper is Theorem 1. Under the assumptions of PA-alignment and richness, the optimal probability assignment mechanism is max-min optimal over all mechanisms.

Section 4 shows that in a simultaneous problem, preferences are PA-aligned if the principal and agent both have increasing-difference utility. When the agent can have any increasing difference utility function, her utility set is rich. So by Theorem 1 a probability assignment mechanism is max-min optimal. When all significances are equal, the optimal probability assignment mechanism can be implemented as a standard ranking mechanism – see Observation 3. (Under asymmetric significances, probability assignment mechanisms unavoidably yield stochastic actions.)

Section 5 shows that in a sequential problem preferences are PA-aligned when both parties have quadratic loss utility. The agent’s utility set is rich if she can have any quadratic loss utility function, and so probability assignment mechanisms are max-min optimal. When all significances are equal, the optimal such mechanism can be implemented as a sequential quota (Lemma 12).

I study one last set of PA-aligned preferences in Section 6. For either a simultaneous or sequential problem, preferences are PA-aligned when the agent is known to be altruistic with private costs. Here the agent’s payoff from a decision is a weighted average of the principal’s payoff (the altruistic term) and some state-independent costs of actions. This is another rich class of utility functions for which probability assignment mechanisms are max-min optimal.

Section 7 shows that in the absence of richness, other aligned delegation mechanisms may strictly improve upon probability assignment. Suppose the principal and agent have quadratic loss constant bias utility, but the principal does not know the agent’s bias. The optimal aligned delegation and max-min optimal mechanism is a budget mechanism, for either a simultaneous or sequential problem. The agent is allowed to choose any actions whose sum, or weighted sum, equals some specified level. This corresponds to a grading curve which lets the professor assign any grades so long as the average GPA is 3.0, or a bonus pool where a supervisor can divide $150,000 among her employees as she sees fit.

Under quadratic loss utility with an unknown linear bias, the principal should instead use a two-moment mechanism. The agent is given freedom to choose any actions, subject to a predetermined sum of actions (budget) and sum-of-squared actions. In other words, the mechanism fixes both the mean and variance of actions.

1.1 Literature Review

A max-min optimality criterion, as opposed to a Bayesian one, is rare in the theory of contracting or mechanism design. One notable early exception is Hurwicz and Shapiro (1978), which shows that a

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4 Similar asymptotic first-best payoff results are found in Jackson and Sonnenschein (2007) and Chakraborty and Harbaugh (2007); see further discussion of these papers in the literature review, below.

5 Subsets such as the concave increasing difference functions are also rich.
50-50 sharing rule may be a max-min optimal sharecropping contract. More recently, Satterthwaite and Williams (2002) justifies double-auctions as worst-case asymptotic optimal in terms of efficiency loss. Other applications of max-min in economics include behavioral analyses of ambiguity aversion (see Gilboa and Schmeidler (1989) for an axiomatization) and macroeconomic work on robust control (see Sargent and Hansen (2007)). In computer science, algorithms are commonly evaluated by their max-min or worst-case performance; this approach has been applied to auction theory in work reviewed by Hartline and Karlin (2007).6

I borrow the basic set-up of my stage game from the literature on the delegation problem, introduced by Holmström (1977). An agent is privately informed about a state of the world which affects both her own preferences over actions and a principal’s. The principal “delegates” the decision by specifying a set of actions from which the agent may choose. Actions and states are elements of the real line and contracting is through restrictions on actions, without transfer payments conditional on actions or on outcomes.

While I allow for multiple decisions (N ≥ 1) and I assume that the principal is uncertain over the agent’s utility function, the delegation literature tends to assume a single decision (N = 1) and a commonly known agent utility function. In this framework, Holmström (1977, 1984) proposed the use of interval delegation sets, in which the agent is allowed to choose any actions from a specified interval. Melumad and Shibano (1991) characterize the optimal delegation region when the state is uniform on [0,1] and when the players have quadratic loss utility with a bias that may be linear in the state. The optimal delegation set switches from an interval to a disconnected region when the agent is relatively insensitive to changes in the state. Martimort and Semenov (2006) and Alonso and Matouschek (2008) extend this analysis to allow for more general state distributions and a larger class of utility functions.7

Armstrong (1995) analyzes a one-decision delegation problem in which the principal may have a very general form of uncertainty over the agent’s preferences, but only considers the use of interval delegation sets which do not vary with the agent’s type. This paper characterizes the Bayesian optimal interval, and looks at comparative statics on the endpoints as the parameters of the problem vary.

Athey et al. (2005) and Amador et al. (2006) consider what are effectively sequential delegation

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6Sargent and Hansen (2007) considers worst cases over a small neighborhood of types, while the other papers – including the current one – look at worst cases over large or unbounded sets. In a monopolist’s pricing problem, Bergemann and Schlag (2008) follow an approach similar to Sargent and Hansen of looking only at a neighborhood of uncertainty.

7Other recent work on delegation problems – all with N = 1, and common knowledge of preferences – includes Krishna and Morgan (2008), which compares optimal delegation schemes under varying assumptions about the principal’s power to commit to mechanisms or to transfer schemes; Ambrus and Egorov (2009), which allows for contracts where the agent may be required to burn money; Kovac and Mylovanov (2009), which considers whether stochastic delegation mechanisms may dominate deterministic ones; Amador and Bagwell (2010), which looks at a bilateral trade negotiation in the framework of delegation; and Armstrong and Vickers (2010), in which the agent’s private information is on the set of available actions, rather than on a state variable which affects preferences over a fixed set of actions.
problems \((N \geq 1)\) in which the principal and agent share a commonly known stage utility, but the agent has time-inconsistent preferences. They find conditions under which interval delegation is optimal in each period – the agent is allowed to choose any actions below a cutoff level. In the case of Athey et al. (2005), past decisions do not directly affect current choice sets and the optimal cutoff is history independent.

Delegation is closely related to cheap talk, wherein the principal cannot commit to a mechanism. The agent reports some signal, and then the principal chooses his preferred action conditional on his posterior beliefs.\(^8\) While most of the work on cheap talk considers a single decision with common knowledge of agent’s utility, three papers which go beyond that in a sequential setting are Sobel (1985), Benabou and Laroque (1992), and Morris (2001). They suppose that the agent has one of two types – she is altruistic with some probability, and otherwise has some specified bias. Agents will then shade their reports in one period in order to earn the principal’s trust in the future. These papers differ from the current one in searching for equilibria of reputation games rather than solving for an optimal contract, and only allowing for two possible agent types. But they share an interest in studying how a principal who makes multiple decisions can extract information from an agent with a persistent, hidden bias.\(^9\)

Chakraborty and Harbaugh (2007) looks at a cheap talk problem over multiple simultaneous and equally significant decisions, where each player’s stage utility function has increasing differences. This is the environment for which I derive ranking mechanisms as optimal max-min mechanisms, and optimal in the class of aligned delegation. Chakraborty and Harbaugh show that when states are iid, the optimal ranking mechanisms can be implemented as an equilibrium of a cheap talk game – the principal does not need commitment power. As the number of iid decisions increases, this ranking outcome gives the principal asymptotically first-best payoffs. While they do not explicitly consider the possibility that an agent’s utility function may be unknown to the principal, their results do not depend on the principal’s knowledge of the agent’s preferences.

Jackson and Sonnenschein (2007) considers a mechanism design problem over a number of decisions, without transfers. When multiple players have iid private values over a large number of allocation decisions, approximately efficient outcomes are possible through forms of simultaneous or sequential quota mechanisms.\(^{10}\) (In the current paper players do not have private values, because utilities depend on a common state). Escobar and Toikka (2009) extends this analysis from iid decisions to Markovian ones. Cohn (2010) modifies the quota mechanism of Jackson and Sonnenschein to speed its convergence to efficiency in a bilateral trading environment.

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\(^8\)The seminal paper on cheap talk is Crawford and Sobel (1982), which constructs informative equilibria under the assumption of quadratic loss, constant bias utility (with known bias), and uniformly distributed states. There has been a huge amount of follow-up work, which I will not attempt to review here.

\(^9\)Alonso and Matouschek (2007) considers a sequential cheap talk problem, but the existence of future decisions only serves to endogenously support the principal’s commitment to a stationary decision rule.

\(^{10}\)They use the term “budget mechanisms,” which I reserve for mechanisms which fix a single budget over the sum of all actions.
I find an approximate first-best payoff result similar to that in Chakraborty and Harbaugh (2007) and Jackson and Sonnenschein (2007), extending to the case of asymmetric significances. But the main contribution of this paper is the derivation of ranking and quota contracts as max-min optimal, rather than merely approximately efficient.

Another paper which derives quota contracts as optimal, in a different sense, is Frankel (2010a). The paper finds conditions under which sequential “discounted quota” contracts are Bayesian optimal for any generic beliefs about the agent’s utility, and are optimal contracts even if transfer payments can be used. (They also achieve asymptotically first-best payoffs when there are many iid states). This contrasts with the current paper, in which quota-like probability assignment mechanisms are found to be max-min optimal but not Bayesian optimal, and transfers are exogenously forbidden. The models in the two papers are similar, the key difference being that Frankel (2010a) assumes that agent payoffs do not depend on the states of the world.

2 The Model

2.1 The Decision Problem

A decision problem is comprised of $N < \infty$ decisions, indexed by $i = 1, 2, ..., N$. For each decision $i$, a state $\theta_i \in \Theta \subseteq \mathbb{R}$ is realized and an action $a_i \in \mathcal{A} \subseteq \mathbb{R}$ is taken. There are two players, a principal and an agent. Each player’s payoff depends jointly on both actions and states, but only the agent observes the states of the world. The principal gets a stage utility $U_P(a_i | \theta_i)$ and the agent gets $U_A(a_i | \theta_i)$ from decision $i$.

I take both $\mathcal{A}$ and $\Theta$ to be compact, i.e., closed and bounded subsets of the real line. Let $\underline{a}$ and $\bar{a}$ be the minimum and maximum actions in $\mathcal{A}$, and let $\underline{\theta}$ and $\bar{\theta}$ be the minimum and maximum states in $\Theta$. I use $a \in \mathcal{A}^N$ and $\theta \in \Theta^N$ to refer to the vectors of all actions and all states.

The utility functions $U_A$ and $U_P$ are elements of $\mathcal{U}$, the set of continuous functions which map from $\mathcal{A} \times \Theta$ into $\mathbb{R}$.

The lifetime payoff of each player is a weighted sum of the stage utilities:

Principal: $\sum_{i=1}^{N} \gamma_i U_P(a_i | \theta_i)$

Agent: $\sum_{i=1}^{N} \gamma_i U_A(a_i | \theta_i)$

The principal and agent agree about the weighting factor $\gamma_i$ of decision $i$, which I call its “significance.” I assume that $N$ and all $\gamma_i$ values are common knowledge at the beginning of the game.

Let $\Gamma = \sum_{i=1}^{N} \gamma_i$ be the sum of all significances. Let $\Gamma_i = \sum_{j=i}^{N} \gamma_j$ be the sum of the significances of decisions from $i$ onward, with $\Gamma_1 = \Gamma$ and $\Gamma_{N+1} = 0$. 

When all significances are equal we can normalize \( \gamma_i \) to 1 for all \( i \), so that lifetime payoffs are unweighted sums of stage payoffs. If significances are not all equal, I say that decisions are asymmetrically significant.

One interpretation of asymmetric significances is that these weights are proportional to the amount of money at stake, or to the number of people affected by a decision. Another interpretation may be that decisions are implemented over time and decisions in the future have discounted payoffs. A standard model with discounting would have \( \gamma_i = \beta^i - 1 \) for some \( \beta \in (0, 1) \).

The principal knows his own utility function \( U_P \in U \) at the beginning of the game, but is uncertain about the agent’s utility function. The principal only knows a set \( U_A \subseteq U \) from which \( U_A \) is drawn. In my analysis the principal need not have a Bayesian prior belief about the agent’s utility function over \( U_A \).

By compactness, the principal always has at least one preferred action at any state \( \theta \). Let \( a^*_P(\theta) \) represent the principal’s preferred action. (If multiple actions are optimal, pick one arbitrarily).

It is useful to define a notion of equivalence for utility functions:

**Definition (Equivalence of Utilities).** Say two stage utility functions \( U \) and \( \tilde{U} \) in \( U \) are equivalent if there exist a positive constant \( \lambda \in \mathbb{R}_{++} \) and a function \( b : \Theta \to \mathbb{R} \) such that \( U(a|\theta) = \lambda \tilde{U}(a|\theta) + b(\theta) \) for all \( a \) and \( \theta \).

Equivalent utility functions imply identical preferences over actions.\(^{11}\) In later sections when I discuss any functional forms of principal or agent utilities, the conclusions should always be understood to generalize to the full equivalence classes.

Finally, I assume that the principal and agent share a prior belief at the beginning of time about the joint distribution of \( \theta = (\theta_1, ..., \theta_N) \).\(^{12}\) Certain results regarding asymptotic payoffs will restrict to the case where \( \theta_i \)'s are iid from a known distribution.

As mentioned above, only the agent observes the states of the world which affect both his and the principal’s preferences over actions. The principal doesn’t observe states directly, but tries to elicit this information from the agent through a mechanism.

### 2.2 Timing and Mechanisms

The principal designs a mechanism through which the agent can send messages, where each message leads to specified actions or (for stochastic mechanisms) distributions over actions. Then the agent observes the states and sends her messages. I assume that the agent must participate in any mechanism which the principal designs.

I consider two possible timing environments, simultaneous and sequential. In the simultaneous problem, the agent observes all states before any actions are determined. In the sequential problem,

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\(^{11}\)The additive term \( b(\theta) \) is exogenous to the chosen actions, while the constant \( \lambda \) uniformly rescales the action-dependent component of utility.

\(^{12}\)See Section 8.3 for a discussion of this assumption.
the agent observes states one at a time; one decision’s action must be assigned before the next state is observed.

The only inputs into the mechanism are the agent’s reports – any outside information regarding the values of current or past states is noncontractible. In particular, the principal’s utility realizations play no role in the contract. This can be thought of as a restriction on the information available to the principal, or as a restriction on the set of mechanisms considered.

The only outputs of the mechanisms are the actions taken. For instance, there are no transfer payments.

After a mechanism is proposed and the agent’s utility is drawn, the agent is asked to submit an initial message. For a simultaneous problem, the agent then observes all states and submits an interim message. The mechanism determines the joint distribution over all actions as a function of both messages. For a sequential problem, the agent observes states one at a time, and reports interim messages after each observation. The mechanism determines the distribution of the current action as a function of the current message combined with the history of past messages and actions.

The mechanisms, formalized below, do not include all possible indirect mechanisms. However, a revelation principle applies (see, e.g., Myerson (1986)). Any equilibrium of any indirect mechanism – even one of a more general form than those considered – could be expressed as an equivalent truthful equilibrium of an incentive compatible direct mechanism. The mechanism forms I consider are without loss of generality in the sense that they include all direct mechanisms.

**Simultaneous Mechanisms**

In a simultaneous environment, a mechanism $D = (\mathcal{M}_0, \mathcal{M}, M)$ is

- an initial message space $\mathcal{M}_0$ and an interim message space $\mathcal{M}$
- a map $M$ from pairs of messages (in $\mathcal{M}_0 \times \mathcal{M}$) into joint distributions over actions (in $\Delta(\mathcal{A})^N$, where $\Delta(\cdot)$ represents the set of Borel measurable distributions).\(^{13}\)

The mechanism induces the following (single-player) game for the agent:

1. The agent observes $U_A \in \mathcal{U}_A$.
2. The agent sends initial report $m_0 \in \mathcal{M}_0$.
3. The agent observes $\theta \in \Theta^N$.\(^{14}\)
4. The agent sends interim report $m \in \mathcal{M}$.
5. $a$ is determined by a draw from the distribution $M(m_0, m)$.

A direct mechanism would have $\mathcal{M}_0 = \mathcal{U}_A$ and $\mathcal{M} = \Theta^N$.

\(^{13}\)It is sufficient to consider $\Delta(\mathcal{A})^N$ rather than the larger set $\Delta(\mathcal{A}^N)$ because lifetime utilities are additive across stages. Given any states, either player’s payoff from any joint distribution over actions is completely determined by the marginal distributions. So I only work with marginal distributions.

\(^{14}\)We could modify the timing and have the agent observe $U_A$ and $\theta$ all at once, prior to any reports. In the sequential game, the corresponding modification would allow the agent to observe $U_A$ and $\theta_1$ prior to any reports. Neither of these changes would affect any results of the paper.
An agent’s pure reporting strategy $\sigma$ is an initial message $m_0 \in M_0$ and a function mapping vectors of states into interim messages $m$. Let $\Sigma^D$ be the set of possible reporting strategies. As a matter of (nonstandard) notation, $U_A$ is observed and then a reporting strategy is chosen; the reporting strategy does not itself take $U_A$ as an argument.  

### Sequential Mechanisms

In a sequential environment, a mechanism $D = (M_0, M_1, ..., M_N, M_1, ..., M_N)$ is

- an initial message space $M_0$, and a set of $N$ interim message spaces $M_1, ..., M_N$
- a set of $N$ functions $M_1, ..., M_N$ which map “public histories” (the list of past messages and actions, in the set $M_0 \times A \times \cdots \times M_{i-1} \times A$ in period $i$) and current messages ($M_i$) into distributions over actions ($\Delta(A)$).

This mechanism induces the following game:

1. The agent observes $U_A \in U_A$.
2. The agent reports $m_0 \in M_0$.
3. In each period $i \geq 1$,
   - (a) The agent observes $\theta_i \in \Theta$.
   - (b) The agent reports $m_i \in M_i$.
   - (c) $a_i$ is determined by a draw from $M_i(m_0, m_1, a_1, ..., m_{i-1}, a_{i-1}, m_i)$.

A direct mechanism would have $M_0 = U_A$, and $M_i = \Theta$ for $i \geq 1$.

Message spaces $M_i$ are defined to be history-independent, but we can restrict messages $m_i$ to be in some history-dependent “feasible” subset of $M_i$. Any infeasible message would be mapped into an arbitrary feasible one.

An agent’s reporting strategy $\sigma$ is a message $m_0 \in M_0$ and a set of $N$ functions mapping public histories, privately observed past states, and current states into an interim message. Let $\Sigma^D$ be the set of possible reporting strategies. (Whenever possible I use the same notation across simultaneous and sequential environments, with the meaning determined by context.)

Given any mechanism $D$, an agent observes $U_A$ then chooses a sequentially rational reporting strategy $\sigma$ to maximize her expected lifetime utility going forward from each information node. Let $\Sigma^{*D}(U_A) \subseteq \Sigma^D$ be the set of sequentially rational reporting strategies for an agent with utility $U_A \in U$.

If a principal proposes mechanism $D$, and if strategy $\sigma$ is chosen, then a principal or agent with stage utility $U(a|\theta)$ gets a lifetime expected payoff of

$$E_{a,\theta} \left[ \sum_i \gamma_i U(a_i|\theta_i) \mid D, \sigma \right]$$

15 This terminology allows me to talk about different types’ playing the same strategy. In more standard notation, a “strategy” would be a map from $U_A$ into $\Sigma^D$. 

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The notation $E_{a, \theta}$ signifies that expectation is taken with respect to the exogenous random variables $\theta$ as well as the actions $a$, which – depending on $D$, $\theta$, and $\sigma$ – may be stochastic. The principal does not need to take expectation with respect to the agent’s type, since the type only affects actions through the choice of $\sigma$.

3 General Results

3.1 Richness and Measure

This section introduces the two concepts of the “richness” of an agent’s utility set, and the “induced measure” over actions. An agent has a rich set of utilities if she may have certain sequences of extreme preferences which do not strongly depend on the states of the world. The induced measure counts the number of times that each action is taken, given an agent’s strategy in a particular mechanism and the realized states. Lemma 1 puts these two definitions together: when the agent’s utility set is rich, there is some limiting type for which the induced measure is constant across states. This result forms the basis for the proofs of Proposition 1 and Theorem 1.

The reader is encouraged to skip over the definition of richness, and return to this as a reference when particular rich sets are discussed in later sections.

Definition (Richness). Say that a set of agent utilities $\mathcal{U}_A \subseteq \mathcal{U}$ is rich if there exists an increasing sequence of natural numbers $\langle n^{(j)} \rangle_{j=1}^{\infty}$ and a function $\psi : \mathcal{A} \times \Theta \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ – written as $\psi(a|\theta; \lambda, n)$ – such that:

- For each $n^{(j)}$, for each $\lambda > 0$, there is a value $\tilde{\lambda} \in \mathbb{R}$ with $|\lambda| > \tilde{\lambda}$ such that a function of $a$ and $\theta$ equivalent to either $\psi(a|\theta; \lambda, n^{(j)}) - (a - \lambda)^{2n^{(j)}}$ or $\psi(a|\theta; \lambda, n^{(j)}) + (a - \lambda)^{2n^{(j)}}$ is contained in $\mathcal{U}_A$.
- There exists $C > 0$ and $\phi \in \mathcal{U}$ such that $|\psi(a|\theta; \lambda, n)| \leq C \lambda^n \phi(a|\theta)$ for each $a, \theta, \lambda, n$.

This is a way of formalizing the notion that the agent may have extreme preferences which in the limit only care about taking high average actions ($\lambda \to \infty$) or low ones ($\lambda \to -\infty$). Taking $n$ large makes the preferences arbitrarily convex or concave in actions, which implies extreme preferences over higher moments of the actions.

I now introduce the measure over actions induced by an agent’s strategy, under some mechanism and some vector of realized states. A strategy and a vector of states imply a joint distribution over the actions that will be taken. We define the induced measure over $\mathcal{A}$ by “adding up” the marginal probability distributions for each action, weighted by the significances.

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16The definition provides a sufficient condition on utility sets to imply Lemma 1, below. It is by no means a necessary condition. One could find alternate definitions of “richness” which could take the place of this in Lemma 1 (and therefore Proposition 1 and Theorem 1), and which neither imply nor are implied by this definition. Indeed, I have not defined richness in this manner out of any consideration that the required sequences are economically meaningful. Rather, sequences of this form are contained in sets which I do consider economically meaningful, discussed in Sections 4 through 6.
As a matter of notation, given some measure $\mu$ over the set $\mathcal{A}$, I write $\mu(B)$ as the measure of a (Borel-measurable) set $B \subseteq \mathcal{A}$.

**Definition (Induced Measure).** Let the measure $\mu^D_{\sigma, \theta}$ induced by a mechanism $D$, a strategy $\sigma$, and a state realization $\theta$ be the measure over $\mathcal{A}$ defined by

$$
\mu^D_{\sigma, \theta}(B) = \sum_i \gamma_i \text{Prob}[a_i \in B|\sigma, \theta] \quad \text{for any } B \subseteq \mathcal{A}.
$$

Say that a measure is *proper* if it places a mass of $\Gamma$ on the full set of actions $\mathcal{A}$. Any induced measure $\mu^D_{\sigma, \theta}$ is proper.

Because I am conditioning on $\theta$ and because $\sigma$ is a pure strategy, the only randomness comes from the mechanism itself. If $D$ is a deterministic mechanism, then each probability in the construction of the measure $\mu^D_{\sigma, \theta}$ is either 0 or 1.

**Lemma 1.** Take any mechanism $D$ and any rich set of agent utilities $\mathcal{U}_A$. There exists some proper measure $\mu^D_{\infty}$ and some sequence of types $\langle U^j_A \rangle_{j=1}^\infty$ in $\mathcal{U}_A$ such that for all $\theta \in \Theta^N$ and for all sequences of strategies $\langle \sigma^j \rangle_{j=1}^\infty$ with $\sigma^j \in \Sigma^D(U^j_A)$,

$$
\mu^D_{\sigma^j, \theta} \text{ weakly converges to } \mu^D_{\infty} \text{ as } j \to \infty
$$

This holds for a simultaneous or sequential problem.

The cumulative mass function of $\mu$ at action $a$, written as $\mu(({-}\infty, a])$, is the measure placed by $\mu$ on the set of actions less than or equal to $a$.\footnote{Because $\mu$ is defined as a measure on $A \subseteq \mathbb{R}$, $({-}\infty, a]$ should more properly be written as $({-}\infty, a] \cap A$.} A sequence of measures $\langle \mu^j \rangle_{j=1}^\infty$ is said to weakly converge to a limiting measure $\mu$ if, at all continuity points $a$ of the cumulative mass function of $\mu$, it holds that $\mu^j(({-}\infty, a]) \xrightarrow{i \to \infty} \mu(({-}\infty, a])$.

**Proof.** Proof outline – for formalization, see Appendix A.

Rich utility sets include utility functions which have a term of the form $\pm(a - \lambda)^{2n}$, for $n$ large and $|\lambda|$ large. As we increase $|\lambda|$, the agent’s preference becomes arbitrarily close to one which lexicographically either minimizes or maximizes the first moment $\sum_i a_i$, then the second moment $\sum_i (a_i)^2$, and so forth through the $n - 1^{st}$ moment. Only after these moments are fixed does the agent consider the state of the world $\theta$. So an agent with $n$ and $|\lambda|$ going off to infinity plays a strategy in which all moments are independent of the realized states. Any measures with identical moments must be equal.

**3.2 Aligned Delegation**

I now introduce the property of “aligned delegation,” which will play a central role in this paper. A mechanism satisfies aligned delegation if it leads all agent types to act exactly in the principal’s
Lemma 1 shows that if the agent has a rich set of types, then in any mechanism some limiting type plays so that the induced measure – the number of times an action is taken – is independent of the realized states. Lemma 2, below, says that in an aligned delegation mechanism this holds for all types rather than a single limiting one.

**Definition** (Aligned Delegation). Fix a principal’s utility $U_P$. A mechanism $D$ is **aligned delegation** over $U_A$ if there exists some $\sigma^*$ which is optimal (i.e., sequentially rational) for every type $U_A \in U_A$ and (if $U_P$ is not already contained in $U_A$) would also be optimal for an agent of type $U_A = U_P$:

$$\exists \sigma^* \text{ s.t. } \sigma^* \in \Sigma^*D(U_A) \text{ for all } U_A \in U_A \cup \{U_P\}$$

I call $\sigma^*$ an **aligned strategy**.

If every strategy which is optimal for some type $U_A \in U_A$ is optimal for every other type in $U_A \cup \{U_P\}$, then I say that the mechanism is **strict** aligned delegation.

Any time I talk about “the payoff” to the principal of an aligned delegation mechanism, it should be understood to mean the payoff under an aligned strategy:

$$\mathbb{E}_{\theta,a} \left[ \sum_i \gamma_i U_P(a_i|\theta_i)|\sigma^* \right]$$

If multiple aligned strategies $\sigma^*$ exist, they will all be payoff equivalent. The payoff can be evaluated without specifying a prior belief over agent types in $U_A$, since all types play identically.

**Observation 1** (Aligned Delegation for Direct Mechanisms). In a direct mechanism, the agent reports her utility function and then reports the states that she observes. Loosely speaking, an incentive compatible direct mechanism is aligned delegation if and only if the actions taken are independent of the reported utility function.

This is precise when $U_P \in U_A$; otherwise, it holds for an augmented direct mechanism which would also be incentive compatible for an agent of type $U_A = U_P$.

**Lemma 2.** Fix some principal utility function and some rich set of agent utilities. In any aligned delegation mechanism $D$, the induced measure $\mu^D_{\sigma^*,\theta}$ is constant across all states $\theta$ and all aligned strategies $\sigma^*$. I call this induced measure $\mu^D$.

This result holds for a simultaneous or sequential problem.

**Proof.** Find some $\mu^D_{\infty}$ implied by Lemma 1. There is some sequence of types for which $\mu^D_{\sigma^*,\theta}$ approaches $\mu^D_{\infty}$, under any states $\theta$ and any optimal strategies $\langle \sigma^j \rangle$. Take some aligned strategy $\sigma^*$, and let $\sigma^j = \sigma^*$ for all $j$. Then $\mu^D_{\sigma^*,\theta}$ is constant in $j$, and so must be everywhere equal to $\mu^D_{\infty}$. Let $\mu^D = \mu^D_{\infty}$. \[\square\]
3.3 Probability Assignment Mechanisms

In this section I apply Lemma 2 to show that if “probability assignment” mechanisms satisfy aligned delegation over a rich utility set, then these mechanisms must be optimal in the class of all aligned delegation mechanisms.

A probability assignment mechanism specifies some proper measure \( \mu \), then asks the agent to declare probability distributions from which each action is to be drawn. Any distributions are allowed so long as the induced measure over all actions – the significance-weighted sum of the distributions – is \( \mu \). In a simultaneous problem she reports all the distributions at once, while in a sequential problem she reports these one by one.

Before formally defining the probability assignment mechanisms, I introduce notation for multiplying measures by scalars, and for adding or subtracting measures.

For a measure \( \mu \) over \( A \) and a scalar \( \zeta > 0 \), let \( \zeta \mu \) be the measure such that \( (\zeta \mu)(B) = \zeta \cdot (\mu(B)) \) for any measurable set \( B \subseteq A \). For measures \( \mu^1 \) and \( \mu^2 \), let \( (\mu^1 + \mu^2)(B) = \mu^1(B) + \mu^2(B) \).

Define a partial order on measures such that \( \mu^1 \leq \mu^2 \) if and only if \( \mu^1(B) \leq \mu^2(B) \) for all measurable sets \( B \subseteq A \). For measures \( \mu^1 \leq \mu^2 \), let \( (\mu^2 - \mu^1)(B) = \mu^2(B) - \mu^1(B) \).

**Definition** (Simultaneous Probability Assignment Mechanism). In a simultaneous environment, a probability assignment mechanism \( PA^{sim}(\mu) \) is characterized by a proper measure \( \mu \) on the set of actions.

There is no time 0 message. The interim message space \( M \) is the subset of \( \Delta(A)^N \) (vectors of \( N \) distributions of actions) for which \( (m_1, ..., m_n) \in \Delta(A)^N \) satisfies \( \sum_i \gamma_i m_i = \mu \).

Given message \( (m_1, ..., m_n) \), action \( a_i \) is drawn according to distribution \( m_i \).

**Definition** (Sequential Probability Assignment Mechanism). In a sequential environment, a probability assignment mechanism \( PA^{seq}(\mu) \) is characterized by a proper measure \( \mu \) on the set of actions.

There is no time 0 message. The message space \( M_i \) is \( \Delta(A) \), the set of distributions over actions, and a message \( m_i \) is feasible if \( \gamma_i m_i \leq \mu - \sum_{j<i} \gamma_j m_j \). Call \( \mu_i = \mu - \sum_{j<i} \gamma_j m_j \) the “remaining measure” at period \( i \); it places total mass \( \Gamma_i \) on the set of actions.

After feasible message \( m_i \in \Delta(A) \) is chosen, action \( a_i \) is drawn according to the distribution \( m_i \).

**Lemma 3.** For any probability assignment mechanism (simultaneous or sequential) and for any agent utility function, there exists some optimal strategy for the agent.

**Proof.** See Appendix A.

I say that the principal’s and agent’s utilities are PA-aligned if they guarantee that any probability assignment mechanism is aligned delegation:
Definition (PA-alignment). Fix a set of actions $A$ and a set of states $\Theta$, and a timing for the game – simultaneous or sequential. Say that preferences are [strictly] PA-aligned if the principal’s utility $U_P \in U$ and the set of agent utilities $U_A$ are such that for any parameters of the problem (any $N$, any $\langle \gamma_i \rangle_{i=1}^N$, any joint distribution over states $\theta$) and for any proper measure $\mu$, the probability assignment mechanism characterized by $\mu$ is [strict] aligned delegation.

I will sometimes treat PA-alignment as a property of the agent’s utility set, for a fixed principal utility. It is trivially the case that the agent’s utility is PA-aligned if $U_A = \{U_P\}$. In Sections 4 through 6 I give more interesting families of $U_P$ and $U_A$ which imply PA-alignment. In general, it is more difficult to satisfy PA-alignment in a sequential problem than in a simultaneous one.\footnote{That is, if utilities are PA-aligned in a sequential problem, then those same utilities must also be PA-aligned in a simultaneous problem. The reverse does not hold. This is because in a simultaneous problem, the agent observes the full vector of states $\theta$ before making any interim reports – distributions over states are irrelevant. In a sequential problem, the agent’s choices must agree with the principal’s at every period for all possible future beliefs. One possible belief is that $\theta$ will be realized with probability one, in which case the agent will play as if she is in a simultaneous problem in which $\theta$ is realized.}

Lemma 4. When utilities are PA-aligned, an optimal measure exists. That is, in a simultaneous problem, there is some proper measure $\mu^{*\text{sim}}$ which maximizes the principal’s expected payoff from $\text{PA}^{\text{sim}}(\mu)$ over measures $\mu$. In a sequential problem, there is some $\mu^{*\text{seq}}$ which maximizes the principal’s expected payoff from $\text{PA}^{\text{seq}}(\mu)$.

Proof. See Appendix A.

Call the probability assignment mechanism characterized by the optimal measure the optimal probability assignment mechanism. The optimal measure depends on the principal’s utility function $U_P$ and on the distribution of states, but not on the agent’s utility set $U_A$ (as long as utilities are PA-aligned).

When the agent has a rich set of types, Lemma 2 states that any aligned delegation mechanism fixes some predetermined measure across all equilibrium reports. Probability assignment mechanisms give the most possible freedom to the agent, subject to fixing a measure. And if it so happens that probability assignment mechanisms satisfy aligned delegation, then the agent uses this freedom “for good” – her strategy maximizes the principal’s payoff across all mechanisms fixing the given measure. So any aligned delegation mechanism is weakly worse than some probability assignment mechanism, which is in turn worse than the optimal probability assignment mechanism.

Proposition 1. Fix some principal utility function, and some rich and PA-aligned set of agent utilities. The optimal probability assignment mechanism gives weakly higher expected payoffs to the principal than does any other aligned delegation mechanism.

This holds for a simultaneous or sequential problem.

Proof. See Appendix A.
3.4 Asymptotic First-Best Payoffs

Above, I show that probability assignment mechanisms are optimal in the class of aligned delegation mechanisms. Here, I show that they provide “high” payoffs (close to the full information first-best) when there are many iid decisions.

We can actually approximate first-best payoffs using a measure which may be considerably easier to calculate than the optimal one. This “naive” measure places a mass on an action proportional to the ex ante probability that the action will be optimal for the principal. If the probability assignment mechanism with the naive measure approximates first-best payoffs, then so too does the one with the optimal measure.

**Definition (Naive Measure).** Let the \( \mu_{\text{naive}} \) be a proper measure defined by

\[
\mu_{\text{naive}}(B) = \sum_{i} \gamma_{i} \text{Prob}[a_{i}^{*}(\theta_{i}) \in B] \quad \text{for } B \subseteq A
\]

I refer to the (simultaneous or sequential) probability assignment mechanism characterized by the naive measure as the naive probability assignment mechanism.

**Definition (Asymptotic first-best payoffs).** Take any \( U_{P} \in \mathcal{U} \), any \( U_{A} \subseteq \mathcal{U} \), and any distribution \( F(\cdot) \) over the set of states \( \Theta \). Fix a sequence of decision problems indexed by \( n = 1, 2, \ldots \). The \( n^{th} \) decision problem will be indicated with superscript \( (n) \); the \( i^{th} \) decision in the \( n^{th} \) problem will have superscript \( (n) \) and subscript \( i \).

Say that a sequence of mechanisms \( \langle D_{(n)} \rangle_{n=1}^{\infty} \) is asymptotically first-best over \( U_{A} \) (i.e., achieves asymptotically close to first-best payoffs for the principal) for this sequence of decision problems if, for each possible agent type in \( U_{A} \), average principal payoffs approach first-best average payoffs.

Formally: For all \( U_{A} \in U_{A} \), there exists a sequence of optimal strategies \( \langle \sigma_{(n)} \in \Sigma_{D_{(n)}(U_{A})} \rangle_{n=1}^{\infty} \) such that for all \( \epsilon > 0 \), there exists \( \bar{n} \in \mathbb{N} \) such that for \( n > \bar{n} \)

\[
\mathbb{E}_{\theta(n),a(n)} \sum_{i=1}^{N_{(n)}} \gamma_{i}^{(n)} \left[ U_{P}(a_{P}^{*}(\theta_{i}^{(n)}))|\theta_{i}^{(n)} \rangle - U_{P}(a_{i}^{(n)}|\theta_{i}^{(n)} \rangle | D_{(n)}, \sigma_{(n)}(U_{A}) \langle \right] < \epsilon
\]

The asymptotic first best payoffs are uniform (with respect to the agent’s types) if we rearrange the order of quantifiers so that the same \( \bar{n} \) works for all types: For all \( \epsilon \), there exists \( \bar{n} \) such that for all \( U_{A} \), there exists a sequence of optimal strategies \( \langle \sigma_{(n)} \rangle_{n=1}^{\infty} \) such that the inequality above holds for \( n > \bar{n} \).

The (uniformly) asymptotic first-best payoffs are strict if we replace “there exists a sequence... such that” with “for all sequences...” in the above definitions.

I will show that we can achieve uniformly asymptotic first-best payoffs when states are iid, as we increase the number of decisions to infinity. It is not necessary that all states be equally significant,
or approach equal significance in the limit. What matters is that the relative significance of any given state vanishes.

**Definition** (Condition (AFB)). Consider a sequence of decisions problems \( n = 1, 2, \ldots \).

Say that these decision problems satisfy Condition (AFB) if

1. The number of decisions in the \( n \)th problem, \( N(n) \), is equal to \( n \).
2. There is some commonly known distribution \( F(\cdot) \) over \( \Theta \) such that \( \theta_i^{(n)} \) is drawn from \( F \), for all \( n \) and for all \( 1 \leq i \leq n \).
3. The significances satisfy

\[
\lim_{n \to \infty} \frac{\max_{i \leq n} \gamma_i^{(n)}}{\Gamma^{(n)}} = 0
\]

where \( \gamma_i^{(n)} \) is the significance of decision \( i \) in problem \( n \), and \( \Gamma^{(n)} = \sum_{i=1}^{n} \gamma_i^{(n)} \).

**Proposition 2.** Fix a principal utility function, a timing for the game (simultaneous or sequential), and a [strictly] PA-aligned set of agent utilities.

Consider a sequence of decision problems satisfying Condition (AFB). The respective naive probability assignment mechanisms are [strictly] uniformly asymptotically first-best, and therefore the optimal probability assignment mechanisms are as well.

**Proof.** Proof outline – for formalization, see Appendix A.

The proof follows the constructions of Jackson and Sonnenschein (2007). I construct a strategy under the naive probability assignment mechanism which gives the principal nearly first-best payoffs when the empirical distribution is close to the theoretical one. As the number of iid decisions increases, a law of large numbers guarantees that these distributions are in fact close. By aligned delegation, the agent’s actual strategy gives the principal at least as high a payoff as does the one I construct.

\[\blacksquare\]

### 3.5 Max-Min Optimality

Under the same richness and PA-alignment assumptions as Proposition 1, the optimal probability assignment mechanism is a **max-min optimal** mechanism. This is the main theorem of the paper.

**Definition** (Max-min Optimality). Fix the principal’s utility function, \( U_P \). Say that a mechanism is **max-min optimal** over a set of agent utilities \( \mathcal{U}_A \subseteq \mathcal{U} \) if it is an arg max of the following problem:

\[
\max_{\text{Mechanisms } D} \left[ \inf_{U_A \in \mathcal{U}_A} \left[ \max_{\sigma \in \Sigma^{*D}(U_A)} \mathbb{E}_{\theta,a} \left[ \sum_i \gamma_i U_P(a_i \mid \theta_i) \mid \sigma, D \right] \right] \right]
\]

In the max-min problem – technically max-inf, since \( \mathcal{U}_A \) need not be compact – the timing is such that the principal first picks a mechanism \( D \). Given this mechanism, an adversary or “devil”
chooses an agent utility type \( U_A \in \mathcal{U}_A \) trying to minimize the principal’s expected payoff. Then states are realized, and the agent plays a strategy \( \sigma \) which is optimal for her type \( U_A \). In the case of multiple optimal strategies, I take the one preferred by the principal (this is the second “max” in the definition).

A max-min optimal mechanism for the principal maximizes this worst-case payoff. “Worst case” is taken over utility realizations, not state realizations.

**Theorem 1.** Fix some principal utility function, and some rich and PA-aligned set of agent utilities. The optimal probability assignment mechanism is max-min optimal.

This holds for a simultaneous or sequential problem.

*Proof.* By Lemma 1, there is some sequence of types \( \langle U^j_A \rangle \) and some measure \( \mu^D_\infty \) such that, under every corresponding sequence of optimal strategies \( \langle \sigma^j \rangle \) and every vector of states \( \theta \), it holds that \( \mu^D_{\sigma^j, \theta} \) weakly converges to \( \mu^D_\infty \).

For any realized states \( \theta \), the payoff to the agent under \( \text{PA}(\mu^D_{\sigma^j, \theta}) \) is weakly larger than the payoff from mechanism \( D \) under strategy \( \sigma^j \) and states \( \theta \) – the agent has the choice to replicate the assignment \( \sigma^j \). (PA is the appropriate simultaneous or sequential probability assignment mechanism). And by continuity of payoffs under probability assignment with respect to the measure (see Claim 2 in the proof of Lemma 4, Appendix A), the payoff of \( \text{PA}(\mu^D_{\sigma^j, \theta}) \) approaches that from \( \text{PA}(\mu^D_\infty) \).

So, as \( j \) goes to \( \infty \), the expected agent payoff under mechanism \( D \) approaches a value bounded above by the (type-independent) payoff under the mechanism \( \text{PA}(\mu^D_\infty) \). Under an aligned strategy, the same holds for the principal’s payoffs. And the principal’s payoff from \( \text{PA}(\mu^D_\infty) \) is weakly below that from the optimal probability assignment mechanism. \[ ]

In general, there may be multiple agent-optimal strategies in an aligned delegation mechanism; we look at an aligned strategy when evaluating max-min payoffs. Under strict aligned delegation, all agent-optimal strategies will be aligned and will provide the principal with an equally high payoff.

### 3.6 Discussion

In this section I introduced the concepts of richness and PA-alignment. Richness states that agent utility sets are “large enough” to include certain extreme functions, while PA-alignment states that these sets are “small enough” that there are no agent types whose play would differ from that of the principal. Proposition 1, Proposition 2, and Theorem 1 all assume PA-alignment, and the first and last also require richness.

For a proposed utility set \( \mathcal{U}_A \), the richness condition can be checked directly by confirming that the set contains an appropriate sequence of functions. The assumption of PA-alignment cannot
be verified mechanically, however. It needs to hold for all \( N \), all possible significance factors, all possible proper measures, etc.

In Sections 4 through 6 I provide various economically meaningful conditions on utility functions which imply PA-alignment. I summarize the applications of the general results to the specific examples in Corollaries 1, 3, and 5. I also give relevant examples of rich utility sets as I proceed, and discuss how probability assignment mechanisms may be implemented through simpler mechanisms.

In Section 7 I show that the conclusions of Proposition 1 and Theorem 1 may fail to hold when the agent’s utility set is not rich. Other aligned delegation mechanisms may strictly improve on probability assignment mechanisms.

4 Increasing Differences: Simultaneous

4.1 PA-Alignment Under Increasing Differences

**Definition** (Increasing Differences). Say that a utility function \( U \in \mathcal{U} \) satisfies increasing differences if for all \( a^2 > a^1 \) in \( \mathcal{A} \) and all \( \theta^2 > \theta^1 \) in \( \Theta \),

\[
U(a^2|\theta^2) - U(a^1|\theta^2) \geq U(a^2|\theta^1) - U(a^1|\theta^1)
\]

The increasing differences are strict if this always holds with strict inequality.

This is a standard condition which implies that a player’s preferred action is increasing in the state of the world – see Topkis (1998), for instance, for applications of increasing differences and similar supermodularity or complementarity conditions to economics. Chakraborty and Harbaugh (2007) consider preferences of this form in a simultaneous cheap talk game.

Increasing differences is easy to check when the function \( U \) is twice differentiable. In that case, \( U \) has increasing differences if and only if it has a nonnegative cross partial derivative: \( \frac{\partial^2 U}{\partial a \partial \theta} \geq 0 \).

Under increasing differences, I will show that the optimal strategy in a probability assignment mechanism is to assign actions “assortatively” – lower actions get assigned to lower states.

**Definition** (Assortative Assignments). Consider the simultaneous environment. Given a probability assignment mechanism \( PA^{sim}(\mu) \) and a state realization \( \theta \), say that an assignment \( m = (m_1, \ldots, m_N) \) is assortative if \( \theta_i < \theta_j \) implies that \( \max[\text{Supp } m_i] \leq \min[\text{Supp } m_j] \) – their supports are fully ordered. If an assignment is not assortative, I call it non-assortative.

There is always at least one assortative assignment. Find some “sorting” permutation \( \pi \) on \( \{1, \ldots, N\} \) for which \( \pi(i) < \pi(j) \) implies that \( \theta_i \leq \theta_j \). One assortative assignment \( m = (m_1, \ldots, m_N) \) is defined by

\[
m_i((-\infty, a]) = \left[ \frac{1}{\gamma_i} \left( \mu((-\infty, a]) - \sum_{j \text{ s.t. } \pi(j) < \pi(i)} \gamma_j \right) \right]
\]
where \([y]\) is 0 if \(y < 0\); \(y\) if \(y \in [0, 1]\); and 1 if \(y > 1\).

In words, think of indexing the actions \(a\) by their “quantile” with respect to \(\mu\), so that the action at quantile \(x\) is above a measure of \(x\) other actions. For the lowest state \(i_1\) such that \(\pi(i_1) = 1\), we assign actions from quantiles 0 through \(\gamma_{i_1}\). For the next lowest state \(i_2\) with \(\pi(i_2) = 2\), we assign quantiles \(\gamma_{i_1}\) through \(\gamma_{i_1} + \gamma_{i_2}\); and so on. Action \(a_i\) is assigned a range \(\gamma_i\) of quantiles from \(\mu\); this measure is scaled up by \(1/\gamma_i\) into a probability distribution from which \(a_i\) is drawn.

If all \(\theta\)’s are distinct, then there is a unique sorting permutation \(\pi\) and the above assignment will be the unique assortative one. When \(\theta_i = \theta_j\) for some \(i \neq j\), then there may be multiple assortative assignments – one for each sorting permutation \(\pi\), as well as a continuum of mixtures of the possible permutations. But even if there are multiple assortative assignments, all are payoff equivalent to any player – they induce the same value of \(\mathbb{E}_\alpha \sum_i \gamma_i U(a_i|\theta_i)\) for any \(U \in \mathcal{U}\). Different assortative assignments simply trade actions across decisions with equal states.

**Lemma 5.** Fix a simultaneous probability assignment mechanism and a vector of states \(\theta\).

1. If the agent has strictly increasing difference utility, an assignment is optimal if and only if it is assortative.

2. If the agent has increasing difference utility, any assortative assignment is optimal.

**Proof.** The proof is formalized in Appendix A. Intuitively, when states are non-assortative, there is (some probability mass of a) low action \(a^1\) played in a high state \(\theta^2\) and a high action \(a^2\) played in a low state \(\theta^1\). Switching these to be assortative changes payoffs by

\[
U(a^1|\theta^1) + U(a^2|\theta^2) - U(a^1|\theta^2) - U(a^2|\theta^1)
\]

And this payoff change is a [strict] improvement under [strictly] increasing differences. (Different significance factors cancel out – payoffs are scaled by \(\gamma_i\) per unit of probability, but the probability on an action per unit of measure is scaled by \(1/\gamma_i\).) ■

**Proposition 3.** Consider a simultaneous environment. If the principal has increasing-difference utility and the agent has [strictly] increasing-difference utility, then preferences are [strictly] PA-aligned.

**Proof.** Follows immediately from Lemma 5. Any assortative strategy is optimal for both parties under increasing differences, and when the agent has strict increasing differences no non-assortative assignment is optimal. ■

**Corollary 1.** Under increasing difference preferences in a simultaneous problem, the optimal probability assignment mechanism satisfies aligned delegation and achieves uniformly asymptotically

\(^{19}\)As a matter of terminology, I say that the agent’s preferences satisfy a condition such as increasing differences if every agent type in \(\mathcal{U}_A\) satisfies the condition.
first-best payoffs. Moreover, when the agent’s set of types is rich, this mechanism is optimal among
aligned delegation mechanisms and is max-min optimal.

The following lemma gives some sufficient conditions on the agent’s preference set which guar-
antee richness.

**Lemma 6** (Sufficient conditions for richness). The agent has rich preferences if $\mathcal{U}_A$ is the set
of all increasing-difference functions in $\mathcal{U}$. Moreover, she has rich preferences if $\mathcal{U}_A$ contains the
set of strict increasing-difference functions which are concave in $a$ for any $\theta$, or the set of strict
increasing-difference functions which are convex in $a$.

**Proof.** See Appendix A.

The set of single-peaked [single-dipped] functions includes the set of concave [convex] functions,
and so if the agent is only known to have single-peaked [single-dipped] increasing difference utility
then the same applies.

### 4.2 Implementation via Ranking Mechanisms

In Lemma 5 I found that assortative assignments were optimal in probability assignment mecha-
nisms, under increasing difference utility. So it is unnecessary for the agent to report full distri-
butions $m_1, \ldots m_N$ of actions. She need only report the relative rankings of the $N$ states, and the
mechanism can assign the probability mass assortatively on its own. I call these mechanisms with
simpler message spaces “generalized ranking mechanisms.” (Under the same increasing-difference
assumption, Chakraborty and Harbaugh (2007) investigated ranking as a cheap talk rather than
delegation protocol).

**Definition.** In a simultaneous environment, a *generalized ranking* mechanism $\text{GR}(\mu)$ is character-
ized by a proper measure $\mu$ on the set of actions.

For the $\text{GR}(\mu)$ mechanism, there is no time 0 message. The message space $\mathcal{M}$ is the set
of permutations on $\{1, \ldots, N\}$ – that is, bijections from the set into itself. For some such reported
permutation $\pi$, $\pi(i) < \pi(j)$ is interpreted as a report that $\theta_i \leq \theta_j$.

Given report $\pi$, action $a_i$ is chosen as a draw from the distribution characterized by cdf

$$\text{Prob}(a_i \leq a) = \left[ \frac{1}{\gamma_i} \left( \mu((\infty, a]) - \sum_{j \text{ s.t. } \pi(j) < \pi(i)} \gamma_j \right) \right]$$

where $[y]$ is defined as 0 if $y < 0$; $y$ if $y \in [0, 1]$; and 1 if $y > 1$.

In a generalized ranking mechanism, say that a reported ranking $\pi$ by the agent is *honest*, given
states $\theta$, if it holds that $\pi(i) < \pi(j)$ implies $\theta_i \leq \theta_j$. 21
The decision with the lowest reported state \( i \) such that \( \pi(i) = 1 \) is assigned the lowest (leftmost) \( \gamma_i \) mass from \( \mu \), “quantiles” 0 through \( \gamma_i \). This continues, up until the highest reported state \( i \) such that \( \pi(i) = N \) is assigned the highest \( \gamma_i \) of mass from \( \mu \), quantiles \( \Gamma - \gamma_i \) to \( \Gamma \). We can then treat the mass assigned to decision \( i \) as a rescaled probability distribution from which action \( a_i \) is drawn, as in a probability assignment mechanism. (See Figure 1).

A generalized ranking mechanism yields deterministic actions if all decisions are equally significant (\( \gamma_i = 1 \) for all \( i \)) and if the measure \( \mu \) is a sum of \( N \) atoms each of size 1. In this case, I call \( \text{GR}(\mu) \) a standard ranking mechanism. The principal gives the agent a list of \( N \) actions, and the decision with the lowest reported state is assigned to the lowest action; the decision with the second-lowest state is assigned the second-lowest action; etc.

If significances are asymmetric, the only possible way for a generalized ranking mechanism to be deterministic is for it to be trivial:

**Observation 2.** Under asymmetric significances, any generalized ranking mechanism necessarily involves some randomization over actions for some report unless the measure \( \mu \) is a single point mass. When \( \mu \) is a single point mass at \( \tilde{a} \), all actions \( a_i \) are taken as \( \tilde{a} \) with probability 1, independent of the report.\(^{20}\)

**Lemma 7.** Suppose that the principal and agent have increasing difference utility. Then the generalized ranking mechanism is an aligned delegation mechanism, with an aligned strategy of honest ranking. Honest ranking under the \( \text{GR}(\mu) \) mechanism is outcome equivalent to some aligned strategy under the \( \text{PA}^{\text{sim}}(\mu) \) mechanism.

The mechanisms are outcome equivalent in the sense that each action \( a_i \) is drawn from identical distributions in the two mechanisms.

*Proof.* Follows from Lemma 5. Any ranking under \( \text{GR}(\mu) \) induces measure \( \mu \), and honest rankings under \( \text{GR}(\mu) \) correspond to assortative assignments under \( \text{PA}(\mu) \).

\[\Box\]

### 4.3 Finding the Optimal Measure

In order to solve for the optimal aligned delegation mechanism, and thus the max-min optimal mechanism, we need to find the principal’s preferred measure \( \mu^{\text{sim}} \) in the generalized ranking or probability assignment mechanism.

**Observation 3.** Consider a simultaneous problem in which the principal and agent have increasing difference utilities.

\(^{20}\)Because I show that honest ranking is optimal, it may still be the case that significances are asymmetric but randomization in a nontrivial GR mechanism only occurs with probability 0. For instance, think of a case with \( \gamma_1 = 1 \) and \( \gamma_2 = 2 \), and a measure placing a mass of 1 on \( \theta \) and a mass of 2 on \( \overline{\theta} \). Randomization only occurs if it is reported that \( \theta_1 > \theta_2 \), but it may be the case that \( \theta_1 < \theta_2 \) almost always.
$N = 2$. Equal Significances: $\gamma_1 = \gamma_2 = 1$. GR($\mu$), with $\mu$ uniform on $[0,1]$.

If $\theta_1 \leq \theta_2$ reported:
\[
\begin{align*}
&\begin{cases}
a_1 \text{ uniform over } [0, \frac{1}{2}] \\
a_2 \text{ uniform over } [\frac{1}{2}, 1]
\end{cases}
\end{align*}
\]

If $\theta_2 \leq \theta_1$ reported:
\[
\begin{align*}
&\begin{cases}
a_1 \text{ uniform over } [\frac{1}{2}, 1] \\
a_2 \text{ uniform over } [0, \frac{1}{2}] 
\end{cases}
\end{align*}
\]

\[N = 2\text{. Asymmetric Significances: } \gamma_1 = 1, \gamma_2 = 2\text{. GR}(\mu), \text{ with } \mu \text{ uniform on } [0,1].\]

If $\theta_2 \leq \theta_1$ reported:
\[
\begin{align*}
&\begin{cases}
a_1 \text{ uniform over } [0, \frac{1}{3}] \\
a_2 \text{ uniform over } [\frac{1}{3}, 1]
\end{cases}
\end{align*}
\]

If $\theta_2 \leq \theta_1$ reported:
\[
\begin{align*}
&\begin{cases}
a_1 \text{ uniform over } [\frac{2}{3}, 1] \\
a_2 \text{ uniform over } [0, \frac{2}{3}]
\end{cases}
\end{align*}
\]

Figure 1: Two examples of Generalized Ranking mechanisms, for $N = 2$. 

1. When all decisions are equally significant ($\gamma_i = 1$ for all $i$), there exists an optimal measure $\mu^{\text{sim}}$ comprised of $N$ atoms, each of mass 1. In other words, the optimal generalized ranking mechanism is a standard ranking mechanism.

To calculate $\mu^*$, one can find the distribution of each order statistic of $\theta$ and then place an atom at the principal’s optimal action conditional on this distribution of states.

2. When significances are not all equal, there is an optimal measure $\mu^*$ in which all of the measure is placed on distinct atoms, with the number of atoms at most $2^N - 1$. The atoms are such that in any ranking, each atom is assigned to a single decision – although one decision may be split over a number of atoms, resulting in a randomized action. As in Observation 2, this mechanism involves randomization any time that the atoms are not all on a single point.

One can write out the full maximization problem as a maximization over the placement of the (up to) $2^N - 1$ atoms of appropriate sizes. These sizes are determined by the significances $\langle \gamma_i \rangle$.

(See Figure 2).

Proof. See Appendix A. ■

Corollary 2. Suppose we have a simultaneous problem in which all decisions are equally significant. If the principal and agent each have increasing difference utility, then the optimal probability assignment mechanism can be implemented as a standard ranking mechanism. All results of Corollary 1 apply to the optimal standard ranking mechanism.

Example 1. Let $\gamma_i = 1$ for all $i$. Then, in a simultaneous problem under increasing difference utility, the optimal probability assignment mechanism is a standard ranking mechanism: the principal specifies actions $a^1 \leq \cdots \leq a^N$, and the agent ranks the states from lowest to highest. Then the decision corresponding to the $j^{\text{th}}$ lowest state is assigned action $a^j$.

Let $U_P(a|\theta) = -(a - \theta)^2$, and let $\mathcal{A}$ be large enough so that it contains the convex hull of the states: $[\bar{\theta}, \bar{\theta}] \subseteq \mathcal{A}$. Then the optimal choice of $a^j$ is the ex ante expected value of the $j^{\text{th}}$ lowest state.

For the special case of each $\theta_i$ iid uniform over $\Theta = [0, 1]$, the optimal ranking mechanism assigns the $j^{\text{th}}$ lowest action to $a^j = \frac{j}{N+1}$. This mechanism gives the principal a payoff of $-\frac{1}{6(N+1)}$ per period, compared to $-\frac{1}{12}$ from no delegation (taking $a = 1/2$ each period) and 0 from first-best ($a_i = \theta_i$ each period). In other words, the ranking mechanism gives the principal $\frac{N-1}{N+1}$ of the possible surplus from delegation.

---

21 The $j^{\text{th}}$ order statistic (ie, $j^{\text{th}}$ lowest number) of $N$ uniformly distributed variables is distributed according to a Beta($j, N + 1 - j$) distribution. This has mean $\frac{j}{N+1}$ and variance $\frac{j(N+1-j)}{(N+1)^2(N+2)}$. The principal’s expected lifetime payoff is minus sum of the variances, which can be calculated to be $-\frac{N}{6(N+1)}$. 

---
$N = 2$. Equal Significances: $\gamma_1 = \gamma_2 = 1$

Take any initial measure $\mu$. In this case, we have $\mu$ uniform on $[0,1]$. The low state is assigned an action drawn from $[0, \frac{1}{2}]$ and the high state action is drawn from $[\frac{1}{2}, 1]$.

The principal prefers to “consolidate” the leftmost unit of mass into some single point in $[0, \frac{1}{2}]$, and the rightmost unit of mass into a point in $[\frac{1}{2}, 1]$. This induces a standard, deterministic ranking mechanism.

$N = 2$. Asymmetric Significances: $\gamma_1 = 3$, $\gamma_2 = 2$

Take any initial measure $\mu$. In this case, we have $\mu$ uniform on $[0,1]$. The leftmost two units of mass on $[0, \frac{2}{5}]$ get assigned to the low state. The middle unit of mass on $[\frac{2}{5}, \frac{3}{5}]$ always gets assigned to action $a_1$. The rightmost two units of mass on $[\frac{3}{5}, 1]$ get assigned to the high state.

The principal prefers to “consolidate” the leftmost two units of mass, the middle unit of mass, and the rightmost two units of mass into single points; in this case, to $\frac{1}{5}$, $\frac{1}{2}$, and $\frac{4}{5}$.

If $\theta_1 \leq \theta_2$ then $a_1$ is chosen randomly between $\frac{1}{5}$ (Probability 2/3) or $\frac{1}{2}$ (Prob 1/3), and $a_2$ is taken deterministically at $\frac{4}{5}$. If $\theta_2 \leq \theta_1$ then $a_1$ is chosen randomly between $\frac{1}{5}$ (Probability 1/3) or $\frac{4}{5}$ (Prob 2/3), and $a_2$ is taken at $\frac{1}{5}$.

Figure 2: Illustration of Observation 3.
5 Quadratic Loss: Sequential

5.1 PA-Alignment Under Quadratic Loss Preferences

In a sequential problem, increasing differences no longer guarantees PA-alignment. In this section I look at a subset of increasing differences, the quadratic loss preferences.

**Definition** (Quadratic Loss Preferences). Say that a utility function $U \in \mathcal{U}$ is quadratic loss if there exists a positive constant $\zeta$ and a weakly increasing function $c : \mathcal{A} \to \mathbb{R}$ such that

$$U(a|\theta) = -\zeta \cdot (c(a) - \theta)^2$$

Say that the utility is strict quadratic loss if $c(\cdot)$ is strictly increasing.

This quadratic loss functional form includes the two most common functional forms in the literature, quadratic loss preferences with a constant bias or with linear biases – see, for example, Crawford and Sobel (1982) or Melumad and Shibano (1991).

**Definition** (Quadratic Loss Constant Bias Preferences). Say that a utility function $U \in \mathcal{U}$ is quadratic loss constant bias if there exists $\lambda \in \mathbb{R}$ such that

$$U(a|\theta) = -(a - \theta - \lambda)^2$$

This is a special case of the general quadratic loss form with $c(a) = a - \lambda$ and $\zeta = 1$. A player with these preferences wants actions as close to $a = \theta + \lambda$ as possible.

**Definition** (Quadratic Loss Linear Bias Preferences). Say that a utility function $U \in \mathcal{U}$ is quadratic loss linear bias if there exists $\lambda^{(0)}, \lambda^{(1)} \in \mathbb{R}$ such that $\lambda^{(1)} > 0$ and

$$U(a|\theta) = -\left(a - \lambda^{(1)}\theta - \lambda^{(0)}\right)^2$$

This is a special case of the general quadratic loss form with $c(a) = \frac{a}{\lambda^{(1)}} - \frac{\lambda^{(0)}}{\lambda^{(1)}}$ and $\zeta = (\lambda^{(1)})^2$. A player with these preferences wants actions as close to $a = \lambda^{(1)}\theta + \lambda^{(0)}$ as possible.

The “quadratic losses” are with respect to perturbations of the state, not of the action. Conditional on some distribution of beliefs over the current state, a decisionmaker’s choices depend only on the expected state. This makes these functions natural for cheap talk or delegation problems in which a partially informed principal tries to elicit information on the state of the world from better informed agents.

Under quadratic loss preferences, in state $\theta$ a player wants an action $a$ such that $c(a) - \theta$ is as close as possible to 0. Loosely speaking, $c^{-1}(\theta)$ gives the optimal action for a given state. (There are only a finite number of actions.)
may not be a value of $a$ for which $c(a) = \theta$ if the action set is nonconvex, or if $c(a)$ is everywhere above or below the value $\theta$.) The optimal action must be weakly increasing in the state. But this class of preferences is general enough to model preferences with \textit{any} weakly increasing optimal action function.

Notice that [strict] quadratic loss utilities satisfy [strict] increasing differences: for $a^2 > a^1$ and $\theta^2 > \theta^1$,

$$
(U(a^2|\theta^2) - U(a^1|\theta^2)) - (U(a^2|\theta^1) - U(a^1|\theta^1)) = \zeta (c(a^2) - c(a^1)) (\theta^2 - \theta^1) \geq [>]0
$$

This means that under quadratic loss utilities we can apply all of the analysis from Section 4, if we happen to be in a simultaneous environment. In particular we know that probability assignment mechanisms will be aligned delegation, with all types playing an assortative strategy.

In a sequential problem, I show that agents with these utility functions will play a “sequential-assortative” strategy in a probability assignment mechanism. An assignment $m_i$ at period $i$ will be defined as sequential-assortative if any actions higher than one in the support of $m_i$ will be played in states weakly above $\theta_i$, in expectation, and any lower actions will be played in states below $\theta_i$. All types play this same strategy, so we have aligned delegation.

Before formalizing the concept of sequential-assortativity, I first define functions $\tilde{\theta}_1(\cdot),...,\tilde{\theta}_N(\cdot)$ which will tell us the expected state in which an action will be played, in future periods. The functions $\tilde{\theta}_i : [0, \Gamma_i] \times \Theta^{i-1} \rightarrow \mathbb{R}$ are constructed by backwards induction. I write these functions as $\tilde{\theta}_i(x; \theta_1,\ldots,\theta_{i-1})$ for $x \in [0, \Gamma_i]$, or simply as $\tilde{\theta}_i(x)$ when the earlier states are otherwise implied. See Figure 3 for an illustration of this construction, along with a sequential-assortative strategy.

Constructing $\tilde{\theta}_N(\cdot)$:

For $x \in [0, \gamma_N)$, let $\tilde{\theta}_N(x)$ be constant, equal to the expected value of $\theta_N$ conditional on the previous states:

$$
\tilde{\theta}_N(x; \theta_1,\ldots,\theta_{N-1}) = \mathbb{E}_{\theta_N}[\theta_N | \theta_1,\ldots,\theta_{N-1}]
$$

Let $\tilde{\theta}_N(\gamma_N; \theta_1,\ldots,\theta_{N-1}) = \overline{\theta}$.

Constructing $\tilde{\theta}_i(\cdot)$ given $\tilde{\theta}_{i+1}(\cdot)$, for $1 \leq i < N$:

For any $x \in [0, \Gamma_i]$, let

$$
\tilde{\theta}_i(x; \theta_1,\ldots,\theta_{i-1}) = \mathbb{E}_{\theta_i}
\begin{cases}
\tilde{\theta}_{i+1}(x - \gamma_i) & \text{if } x \geq \gamma_i \text{ and } \theta_i < \tilde{\theta}_{i+1}(x - \gamma_i) \\
\tilde{\theta}_{i+1}(x) & \text{if } x < \Gamma_{i+1} \text{ and } \theta_i > \tilde{\theta}_{i+1}(x) \\
\theta_i & \text{otherwise}
\end{cases}
\begin{bmatrix}
\theta_1,\ldots,\theta_{i-1}
\end{bmatrix}
$$

It can shown by backwards induction that $\tilde{\theta}_i(\Gamma_i)$ is equal to $\overline{\theta}$ for each $i$, that $\tilde{\theta}_i(x)$ is weakly
increasing in $x$, and that $\tilde{\theta}_i$ is right-continuous.

The interpretation of $\tilde{\theta}_{i+1}(x) = \theta$ is that at the end of period $i$, prior to the observation of $\theta_{i+1}$, whatever action is at quantile $x$ out of $\Gamma_{i+1}$ in measure $\mu_{i+1}$ will eventually be played at a state with expected value of $\theta$. (This holds for all $x < \Gamma_{i+1}$; at $x = \Gamma_{i+1}$, $\tilde{\theta}_{i+1}(x)$ is set at $\bar{\theta}$.)

**Definition** (Sequential-Assortativity). Take some sequential probability assignment mechanism with remaining measure $\mu_i$ at period $i < N$. Given $\theta_i$, say that the feasible distribution $m_i$ is sequential-assortative if it holds that

1. $\forall a$ s.t. $\tilde{\theta}_{i+1}\left((\mu_i - \gamma_i m_i)((-\infty, a])\right) < \theta_i$, it holds that $a \leq \min[\text{Supp } m_i]$; and
2. $\forall a$ s.t. $\tilde{\theta}_{i+1}\left((\mu_i - \gamma_i m_i)((-\infty, a])\right) > \theta_i$, it holds that $a \geq \max[\text{Supp } m_i]

where $(\mu_i - \gamma_i m_i)$ is the measure $\mu_{i+1}$ remaining at the start of the period $i + 1$.

A strategy is said to be sequential-assortative if it selects a sequential-assortative distribution at every history on the equilibrium path.

Just as multiple simultaneous assortative assignments could exist when two states were equal, multiple sequential-assortative distributions might exist when the current state $\theta_i$ is equal to $\tilde{\theta}_{i+1}(x)$ for multiple values of $x$. But $\tilde{\theta}$ is increasing, so there can only be countably many such points. In either case, assortative strategies are “generically” unique in the sense that, if $\theta_i$ is drawn from a continuous distribution, the strategy is uniquely defined almost always.

Here is how to define the sequential-assortative assignment which takes the lowest actions in the current period, if multiple such assignments are possible. At period $i$ with current state $\theta_i$, let $w_i = \min\{x \mid \tilde{\theta}_{i+1}(x) \geq \theta_i\}$. Then choose the assignment $m_i$ as the quantiles $w_i$ through $w_i + \gamma_i$ from measure $\mu_i$, rescaled by $\frac{1}{\gamma_i}$. This gives

$$m_i((-\infty, a]) = \left[\frac{1}{\gamma_i} (\mu_i((-\infty, a]) - w_i)\right]$$

where $[y]$ is 0 if $y < 0$; $y$ if $y \in [0, 1]$; and 1 if $y > 1$. This is depicted in Figure 3.

**Lemma 8.** Fix a sequential probability assignment mechanism.

1. If the agent has strict quadratic loss utility, an assignment strategy is optimal if and only if it is sequential-assortative.
2. If the agent has quadratic loss utility, any sequential-assortative strategy is optimal.

**Proof.** See Appendix A. $lacksquare$

**Proposition 4.** Consider a sequential environment. If the principal has quadratic loss utility and the agent has [strict] quadratic loss utility, then preferences are [strictly] PA-aligned.
The horizontal axis is in units of “quantiles” of measure over actions. Increasing values of $x$ correspond to weakly increasing actions $a$. The vertical axis is over states $\theta$.

Given the current state $\theta_i$ and the weakly increasing function $\tilde{\theta}_{i+1}(x)$ on $[0, \Gamma_{i+1}]$, find $w_i$ as the lowest value such that $\tilde{\theta}_{i+1}(w_i) \geq \theta_i$. The agent then plays actions corresponding to the quantiles $w_i$ through $w_i + \gamma_i$ of measure $\mu_i$.

In the future, an action at quantile $x < w_i$ is expected to be played at state $\theta = \tilde{\theta}_{i+1}(x)$. An action at quantile $x > w_i + \gamma_i$ is expected to be played at state $\theta = \tilde{\theta}_{i+1}(x - \gamma_i)$.

The function $\tilde{\theta}_i(x)$ is constructed from $\tilde{\theta}_{i+1}$ by backwards induction, taking the expectation over $\theta_i$ of the lower-envelope of the graphed curve. This gives us the expected state at which quantile $x$ is played, prior to the realization of state $\theta_i$.

Figure 3: Construction of Sequential-Assortative Strategies

**Proof.** Follows immediately from Lemma 8. ■

**Corollary 3.** Suppose the principal and agent have quadratic loss preferences in a sequential environment. Then the optimal probability assignment mechanism satisfies aligned delegation and achieves uniformly asymptotically first-best payoffs. Moreover, when the agent’s set of types is rich within this class, this mechanism is optimal among aligned delegation mechanisms and is max-min optimal.

The same holds for the quadratic loss preferences in a simultaneous environment, as covered in Corollary 1.

**Lemma 9** (Sufficient conditions for richness). Suppose the agent has quadratic loss preferences with $\zeta$ normalized to 1: $U_A(a|\theta) = -(c(a) - \theta)^2$. Consider the set of all $c(\cdot)$ functions of all agent types in $\mathcal{U}_A$. The agent has rich preferences if this set contains all increasing concave functions; or all increasing convex functions.

**Proof.** See Appendix A. ■
5.2 Implementation and Optimal Measures: Sequential Quotas

It is actually easier to express the optimal measure in a sequential probability assignment mechanism than a simultaneous one, given the notation we have defined for the sequential case. The function \( \tilde{\theta}_1(x) \) tells us the ex ante expected state at which the action at quantile \( x \) will be played. The optimal measure will place action \( a_p^*(\tilde{\theta}_1(x)) \) at quantile \( x \).

**Lemma 10.** For a sequential problem in which the principal and agent have quadratic loss utilities, there is an optimal measure \( \mu^{\text{seq}} \) satisfying

\[
\mu^{\text{seq}}(B) = \left\{ x \in [0, \Gamma] \middle| a_p^* \left( \tilde{\theta}_1(x) \right) \in B \right\}
\]

for \( B \subseteq A \) where \( | \cdot | \) denotes the Lebesgue measure of a set.

In fact, this uniquely defines the optimal measure if \( a_p^*(\theta) \) is everywhere uniquely defined.

**Proof.** See Appendix A. \( \blacksquare \)

Having established this result on the optimal measures, I now consider implementation of sequential probability assignment mechanisms.

Under quadratic loss preferences, in a sequential environment, agents play sequential-assortative strategies in probability assignment mechanisms (Lemma 8). As illustrated in Figure 3, these strategies can be implemented by having the agent report a single real number in each period – i.e., reporting that the interval should start at quantile \( w_i \) – rather than reporting a full distribution over actions.\(^{23}\) Call the mechanism which elicits these starting points an interval assignment mechanism.

**Definition** (Interval Assignment Mechanism). In a sequential environment, an interval assignment mechanism is characterized by a proper measure \( \mu \). There is no time 0 message. The message space \( \mathcal{M}_i \) is the interval \([0, \Gamma_{i+1}] \subset \mathbb{R} \).

Given report \( w_i \in [0, \Gamma_{i+1}] \) at period \( i \), action \( a_i \) is chosen as a draw from the distribution \( m_i \) characterized by cdf

\[
m_i((-\infty, a]) = \left[ \frac{1}{\gamma_i} \left( \mu_i((-\infty, a]) - w_i \right) \right]
\]

where \( \mu_i \) is defined inductively as \( \mu_1 = \mu, \mu_{i+1} = \mu_i - \gamma_i m_i \).

**Lemma 11.** Consider a sequential problem in which the principal and agent have quadratic loss utilities. Then any interval assignment mechanism is an aligned delegation mechanism, and is outcome equivalent to some aligned strategy of the probability assignment mechanism characterized by the same measure.

\(^{23}\)By the revelation principle, we already knew that it would be possible to implement this through a direct mechanism in which the agent reported the one-dimensional state variable in each period. But in this indirect implementation, the principal can “program” the mechanism very simply, offloading the computation of strategies onto the agent.
Proof. Follows from Lemma 8 – the agent can replicate some sequential-assortative strategy in the probability assignment mechanism through the interval assignment mechanism.

As in the simultaneous problem, randomization is unavoidable under asymmetric significances. What if all decisions are equally significant: under a probability assignment measure characterized by a measure with \( N \) atoms of mass one, will a sequential-assortative strategy induce deterministic actions? If so, then we can implement the mechanism through a sequential quota. At the beginning of the game the principal lists \( N \) actions. Then in each period the agent chooses one of the actions from the list, without replacement.

The following lemma shows that the agent will choose strategies inducing deterministic actions when we give atomic measures to probability or interval assignment mechanisms. Moreover, just as in the simultaneous case (Observation 3), the optimal measure will in fact be of this form. So the optimal probability assignment mechanism can indeed be implemented as a sequential quota when significances are all equal.

Lemma 12. Consider a sequential problem in which the principal and agent have quadratic loss utilities, and all decisions are equally significant: \( \gamma_i = 1 \) for all \( i \).

1. In any interval assignment mechanism, there is an optimal strategy for the agent in which she only makes reports \( w_i \in \{0, 1, \ldots, N-i\} \) in period \( i \).

2. The optimal measure \( \mu^{seq} \) places a mass of size 1 at each of \( a^*_P(\tilde{\theta}_1(0)), a^*_P(\tilde{\theta}_1(1)), \ldots \) and \( a^*_P(\tilde{\theta}_1(N-1)) \).

Proof. See Appendix A.

Corollary 4. Suppose we have a sequential problem in which all decisions are equally significant. If the principal and agent each have quadratic loss utility, then the optimal probability assignment mechanism can be implemented as a sequential quota. All results of Corollary 3 apply to the optimal sequential quota.

Example 2. Let \( \gamma_i = 1 \) for all \( i \). Under quadratic loss utility, the optimal probability assignment mechanism corresponds to a sequential quota. The principal gives a list of actions \( a^1 \leq \cdots \leq a^N \). In each period the agent chooses one action to play from the list, then that action is removed.

Let \( U_P(a|\theta) = -(a - \theta)^2 \), let each \( \theta_i \) be iid uniformly distributed over \( \Theta = [0,1] \), and let \([0,1] \subseteq A\).

The optimal choice of \( a^j \) can be found by solving for \( \tilde{\theta}_1(j-1) \) through backwards induction – I do not have a general closed-form expression for these values.\(^{24}\)

\(^{24}\)At period \( N \), \( \tilde{\theta}_N(x) \) is constant over \( 0 \leq x < 1 \) at the expected value of \( \theta_N, 1/2 \). At \( N-1 \), for \( 0 \leq x < 1 \) and \( 1/2 \) if \( \theta_{N-1} < 1/2 \), \( \tilde{\theta}_{N-1}(x) \) is the expected value \((\theta_{N-1} < 1/2, 1/2 \text{ otherwise}) \), which is \( 3/8 \); for \( 1 \leq x < 2 \), the expectation is \( 1/2 \). For \( \theta_{N-1} < 1/2 \), \( \tilde{\theta}_{N-1}(x) \) is the expectation \((1/2 \text{ otherwise}) \), which is \( 5/8 \). At \( N-2 \), for \( 0 \leq x < 1 \) and \( 1/2 \) if \( \theta_{N-2} < 3/8 \), \( \tilde{\theta}_{N-2}(x) \) is the expectation \((3/8 \text{ otherwise}) \), which is \( 39/128 \); for \( 1 \leq x < 2 \), the expectation is \( 3/8 \text{ if } 3/8 < 3/8 \). If \( \theta_{N-2} < 3/8 \), \( \tilde{\theta}_{N-2}(x) \) is
If $N = 1$, the optimal sequential quota corresponds to the principal’s requiring that action $a^1 = \frac{1}{2}$ be taken. The principal gets the no-delegation payoff of $-\frac{1}{12} = -0.0875$ (per period).

If $N = 2$, the principal gives the agent a list of actions $a^1 = \frac{3}{8}$ and $a^2 = \frac{5}{8}$. At period 1, action $\frac{3}{8}$ is taken if $\theta_1 < \frac{1}{2}$, and action $\frac{5}{8}$ is taken if $\theta_1 > \frac{1}{2}$. At period 2, the remaining action is taken. This would give the principal an expected payoff of $-\frac{13}{192} \approx 0.068$ per period, or 18.75% of the possible surplus gain from no delegation to first-best. (In the simultaneous problem of Example 1, the principal instead chose actions $a^1 = \frac{1}{3}$ and $a^2 = \frac{2}{3}$, and got $\frac{1}{3}$ of the surplus).

If $N = 3$, the optimal list of actions is $a^1 = \frac{39}{128} \approx 0.305$, $a^2 = \frac{1}{2}$, and $a^3 = \frac{89}{128} \approx 0.695$. At period 1 action $a^1$ is taken if $\theta_1 < \frac{3}{8}$; action $a^2$ is taken if $\frac{3}{8} \leq \theta_1 < \frac{5}{8}$; and action $a^3$ is taken if $\theta_1 \geq \frac{5}{8}$. At period 2, the lower of the remaining two actions is taken if $\theta_2 < \frac{1}{2}$ and the higher is taken if $\theta_2 \geq \frac{1}{2}$. At period 3 the last remaining action is taken. This would give the principal an expected payoff of about $-0.058$ per period, corresponding to approximately 30.5% of the possible surplus. (This compares to actions $a^1 = \frac{1}{3}$, $a^2 = \frac{1}{2}$, and $a^3 = \frac{3}{4}$ in the otherwise equivalent simultaneous problem, yielding $\frac{1}{2}$ of the surplus).

### 5.3 Further Generalizations

When the principal and agent have quadratic loss utility, we can rewrite utilities as

$$
\begin{align*}
U_P(a|\theta) &= -\zeta P c_P(a)^2 - \zeta P \theta^2 + 2\zeta P c_P(a)\theta \\
U_A(a|\theta) &= -\zeta A c_A(a)^2 - \zeta A \theta^2 + 2\zeta A c_A(a)\theta
\end{align*}
$$

where the principal has no information about $c_A(\cdot)$ or $\zeta A$ except that they are, respectively, increasing and positive.

Consider the more general class of utility functions,

$$
\begin{align*}
U_P(a|\theta) &= d_P(a) + b_P(\theta) + s_P(a)t(\theta) \\
U_A(a|\theta) &= d_A(a) + b_A(\theta) + s_A(a)t(\theta)
\end{align*}
$$

where $s_A$, $s_P$, and $t$ are weakly increasing. The function $t(\theta)$ is common knowledge, but the principal does not necessarily have information on $d_A$, $b_A$, or $s_A$.

In this case, the agent’s strategies in a probability assignment mechanism are no longer sequential-assortative with respect to the expectation of $\theta$, but rather the expectation of $t(\theta)$. Aside from that, everything follows as in the quadratic loss case. To construct the strategies we would define new $\tilde{\theta}_i$ functions which kept track of the expected value of $t(\theta)$ on which an action would be played in the future.

---

$3/8 < \theta_{N-2} < 5/8$, and $5/8$ if $\theta_{N-2} > 5/8$ which is 1/2; and for $2 \leq x < 3$, the expectation of (5/8 if $\theta_{N-2} < 5/8$, and $\theta_{N-2}$ otherwise) which is 89/128.
6 Altruism with Private Costs: Simultaneous or Sequential

6.1 PA-Alignment Under Altruism With Private Costs

I say that an agent is “altruistic with private costs” if her utility is a mixture between the principal’s utility function and some state-independent cost or benefit of actions.

Definition (Altruism with Private Costs). Given a principal utility function $U_P \in \mathcal{U}$, say that an agent with utility $U_A$ is altruistic with private costs if there exists a constant $\zeta \geq 0$ and functions $b : \Theta \to \mathbb{R}$ and $c : \mathcal{A} \to \mathbb{R}$ such that for all $a$ and $\theta$,

$$U_A(a|\theta) = \zeta U_P(a|\theta) + c(a) + b(\theta)$$

Say that the altruism is strict if $\zeta > 0$, and say that the agent has state-independent preferences if $\zeta = 0$.

$\zeta U_P$ is the “altruistic” term, and $c(a)$ represents the “private costs.” We can also allow a $b(\theta)$ term which shifts payoffs up or down, but is irrelevant to the preferences over actions. When I say that the agent is altruistic with private costs, different types may have different $\zeta$’s, $b$’s, and $c$’s.\(^{25,26}\)

In general, preferences can satisfy increasing differences without satisfying altruism with private costs, and vice versa. Under strict altruism with private costs, the agent has increasing difference preferences if and only if the principal does.

**Proposition 5.** Fix any principal utility $U_P \in \mathcal{U}$. If the agent is [strictly] altruistic with private costs, then her utility is [strictly] PA-aligned.

This holds for a simultaneous or a sequential problem.

**Proof.** If $U_A(a|\theta) = \zeta U_P(a|\theta) + c(a) + b(\theta)$ then the agent’s payoff can be written as

$$\sum_i \gamma_i U_A(a_i|\theta_i) = \zeta \sum_i \gamma_i U_P(a_i|\theta_i) + \sum_i \gamma_i (c(a_i) + b(\theta_i))$$

The second summation on the right-hand side is constant in expectation across all agent strategies in a probability assignment mechanism. So the agent’s problem is equivalent to maximizing $\zeta$ times
the principal’s payoff. If \( \zeta > 0 \), then the agent’s payoff is maximized by a strategy if and only if the principal’s is as well. If \( \zeta = 0 \), then the agent is indifferent across all strategies.

The proposition implies that if the agent is known to have state-independent preferences, then probability assignment mechanisms will be aligned delegation.

**Corollary 5.** Suppose that the agent has altruistic with private cost preferences in a simultaneous or sequential environment. Then the optimal probability assignment mechanism satisfies aligned delegation and achieves uniformly asymptotically first-best payoffs. Moreover, when the agent’s set of types is rich within this class, this mechanism is optimal among aligned delegation mechanisms and is max-min optimal.

**Lemma 13** (Sufficient conditions for richness). Suppose the agent is altruistic with private costs, \( U_A(a|\theta) = \zeta U_P(a|\theta) + c(a) + b(\theta) \), and normalize all \( \zeta \)'s to either 0 or 1. Consider the set of all \( c(\cdot) \) functions of all agent types in \( U_A \).\(^{27}\) If this set contains all continuous functions, then the agent’s type set is rich. The same holds if the set contains all increasing continuous functions; all decreasing continuous functions; all concave continuous functions; or all convex ones.

*Proof.* See Appendix A.

### 6.2 Implementation

In previous sections I solved for the aligned strategies under probability assignment mechanisms, and then showed that these strategies could be replicated by equivalent but simpler mechanisms. It is harder to generally characterize strategies under altruism with private costs because the principal may have any utility function, and different types of principal may play differently. Without solving for the principal’s strategy, I only know that altruistic with private cost agents will play the same strategy.

In those special cases where I can solve for the principal’s strategy, though, I can consider implementation. In particular, I know the principal’s strategy when he has increasing difference preferences in a simultaneous environment, or quadratic loss preferences in a sequential environment. In either of these cases, probability assignment mechanisms can be implemented as in Sections 4 and 5, and optimal measures can be solved for using the previous techniques.\(^{28}\)

\(^{27}\)Any utility function with \( \zeta > 0 \) can be rescaled to an equivalent one with \( \zeta = 1 \). When \( \zeta = 0 \), the normalization is consistent with any rescaling of \( c(\cdot) \). So we could expand the set of considered costs to allow for any possible rescaled \( c(\cdot) \) functions as well.

\(^{28}\)When the principal has increasing difference preferences and the agent is altruistic with private costs, then the agent does as well. So the simultaneous environment under these assumptions is explicitly covered in Section 4. The current section merely adds to the earlier one by providing additional sufficient conditions for richness.

When the principal has quadratic loss preferences, though, an altruistic with private cost agent need not have quadratic loss preferences. As discussed in Section 5.3, the approach of Section 5 can still be used to solve for strategies and implement the mechanism via interval assignment or quotas.
7 Preferences without Richness

In this section I consider two PA-aligned classes of utility functions which do not satisfy richness: quadratic loss constant bias and quadratic loss linear bias utilities. Without richness, Lemma 2 no longer applies – there may be aligned delegation mechanisms in which the induced measure varies across reports, which do better than probability assignment mechanisms. For these two classes of utilities I derive intuitive mechanisms which improve upon probability assignment, are optimal in the class of aligned delegation mechanisms, and are max-min optimal.

7.1 Quadratic Loss Constant Bias

Suppose the principal and agent each have quadratic loss constant bias utility. Each party’s utility function is of the form \( U_A(a|\theta) = -(a - \theta - \lambda)^2 \) for some bias term \( \lambda \), where the principal is uncertain about the agent’s bias.

The agent’s objective is to maximize

\[
-\mathbb{E} \sum_i \gamma_i (a_i - \theta_i - \lambda)^2 = -\mathbb{E} \sum_i \gamma_i (a_i^2 - 2a_i\theta_i - 2\lambda a_i + [\theta^2 + 2\theta_i \lambda + \lambda^2])
\]

The bracketed terms are independent of any actions that are taken. So changing the bias \( \lambda \) only affects preferences through an additive \( 2\lambda \sum_i \gamma_i \mathbb{E} a_i \) term. As \( \lambda \) goes to positive infinity, the agent seeks to maximize this term; as \( \lambda \) goes to negative infinity, the agent minimizes it.

**Definition** (Unboundedness). Suppose that the agent has quadratic loss constant bias utilities, so that \( U_A(a|\theta) = -(a - \theta - \lambda)^2 \) for any \( U_A \in \mathcal{U}_A \). Say that the agent’s utility set is *unbounded* if there is a sequence of types in \( \mathcal{U}_A \) with \( |\lambda| \to \infty \).

Unboundedness is to quadratic loss constant bias preferences, as richness is to more general preferences (recall Lemmas 1 and 2):

**Lemma 14.** Let the agent have unbounded quadratic loss constant bias utilities. Fix any mechanism \( D \).

1. There exists a value \( K \) and a sequence of types \( \left\{ U_A^j \right\}_{j=1}^{\infty} \) in \( \mathcal{U}_A \) such that for all states \( \theta \) and all corresponding sequences of optimal agent strategies \( \left\{ \sigma^j \right\} \),

\[
\mathbb{E} \left[ \sum_i \gamma_i a_i | \sigma^j, \theta \right] \to K \text{ as } j \to \infty
\]

2. If \( D \) is aligned delegation, it holds that \( \mathbb{E}[\sum_i \gamma_i a_i | \sigma^*, \theta] \) is constant across all \( \theta \) in \( \Theta^N \) and all aligned strategies \( \sigma^* \).

This holds for a simultaneous or sequential problem.
Proof. See Appendix A.

Suppose we have a mechanism which fixes the sum \( \sum \gamma_i E a_i \) at a level \( K \) over all reports. Ignoring the action-independent terms, the agent’s problem now is to maximize

\[
-\mathbb{E} \left[ \sum \gamma_i (a_i^2 - 2a_i \theta_i) \right] + 2\lambda K
\]

The bias term \( \lambda \) no longer interacts with the actions, and so all agent types will have identical preferences in this mechanism.

In particular, consider the budget mechanism characterized by a budget \( K \), which gives the agent freedom to choose any actions for which \( \sum \gamma_i a_i = K \). (When the action set is not convex, certain choices yield mixed actions).

Under probability assignment mechanisms, only proper measures \( \mu \) placing mass \( \Gamma \) on the set of actions were achievable. Correspondingly, I call \( K \) a "proper" budget if it is neither so high nor so low as to make it impossible to hit the budget constraint: a budget \( K \in \mathbb{R} \) is proper if \( K \in [\Gamma_\mathbb{A}, \Gamma_\mathbb{A}] \).

**Definition** (Simultaneous Budget Mechanism). In a simultaneous environment, a budget mechanism \( B^{\text{sim}}(K) \) is characterized by a proper budget \( K \).

For the \( B^{\text{sim}}(K) \) mechanism, there is no time 0 message. The interim message space \( \mathcal{M} \) is the subset of \([a, \overline{a}]^N \) (vectors of \( N \) numbers) for which \( r = (r_1, ..., r_N) \in [a, \overline{a}]^N \) satisfies \( \sum \gamma_i r_i = K \). The message \( (r_1, ..., r_N) \) is interpreted as a report that action \( a_i \) should be taken “as close to” \( r_i \) as possible.

If \( r_i \in \mathcal{A} \), then action \( a_i \) is chosen as \( r_i \) (deterministically).

If \( r_i \notin \mathcal{A} \), we choose the action \( a_i \) from the lowest variance distribution which has expectation \( r_i \). In particular, let \( |r_i| = \max\{a_i \in \mathcal{A} \mid a_i < r_i \} \) and let \( |r_i| = \min\{a_i \in \mathcal{A} \mid a_i > r_i \} \). Action \( a_i \) is chosen stochastically as \( |r_i| \) with probability \( \frac{r_i - |r_i|}{|r_i| - |r_i|} \), and \( |r_i| \) with the complementary probability.

**Definition** (Sequential Budget Mechanism). In a sequential environment, a budget mechanism \( B^{\text{seq}}(K) \) is characterized by a proper budget \( K \).

For the \( B^{\text{sim}}(K) \) mechanism, there is no time 0 message. The period \( i \) message space \( \mathcal{M}_i \) is \([a, \overline{a}]\), and a message \( r_i \) at time \( i \) is feasible if

\[
\gamma_i r_i \in \left[ K - \sum_{j<i} \gamma_j r_j - \Gamma_{i+1} \mathcal{A}, K - \sum_{j<i} \gamma_j r_j - \Gamma_{i+1} a \right]
\]

Call \( K_i = K - \sum_{j<i} \gamma_j r_j \) the “remaining budget.”\(^{29}\)

\(^{29}\)A report in one period is feasible if it allows for future reports to sum to \( K_{i+1} \); at period \( N \), the only feasible report is \( r_N = K_N/\gamma_N \).
After feasible message $r_i$ is sent, action $a_i$ is determined analogously to the simultaneous mechanism. If $r_i \in A$, then $a_i = r_i$. Otherwise, $a_i$ is chosen as $\lceil r_i \rceil$ with probability $\frac{r_i - \lfloor r_i \rfloor}{\lceil r_i \rceil - \lfloor r_i \rfloor}$, and otherwise is $\lfloor r_i \rfloor$.

When the action space is convex, actions are deterministic and so we can think of the agent as choosing actions $a_i$ directly.

Budget mechanisms give more freedom to the agent than do probability assignment mechanisms, because they only constrain the agent’s choice over one moment of the measure rather than all moments.\(^{30}\) Under quadratic loss constant bias utility, budget mechanisms satisfy aligned delegation so the agent uses this additional freedom in a way that benefits the principal.

**Lemma 15.** Let the principal and agent have quadratic loss constant bias utility. Then any budget mechanism is strict aligned delegation. This holds for a simultaneous or sequential problem.

*Proof.* See Appendix A.

**Lemma 16.** When the principal and agent have quadratic loss constant bias utility, an optimal budget exists. That is, in a simultaneous problem, there is some proper budget $K^{\ast\text{sim}}$ which maximizes the principal’s expected payoff from $B^{\text{sim}}(K)$ over budgets $K$. In a sequential problem, there is some $K^{\ast\text{seq}}$ which maximizes the principal’s payoff from $B^{\text{seq}}(K)$.

*Proof.* See Appendix A.

Call the budget mechanism characterized by the optimal measure the *optimal budget mechanism*.

**Proposition 6** (Optimality of Budget Mechanisms). Let the principal and agent have quadratic loss constant bias utility, and let the agent’s utility set be unbounded. Then, for a simultaneous or sequential problem,

1. The optimal budget mechanism is aligned delegation, and gives weakly higher expected payoffs to the principal than any other aligned delegation mechanism.
2. The optimal budget mechanism is max-min optimal.

*Proof.* See Appendix A.

In Appendix B I examine the agent’s strategy in a budget mechanism in more detail, and show how to solve for the optimal budget levels. Suppose that the set of actions is convex and “big enough” so that the only binding constraints on the agent’s action choices come from the budget

\(^{30}\) As defined below, budget mechanisms force the agent to choose a probability distribution for each action which is concentrated on, or about, a single point. But if the agent had the freedom to choose any distributions subject to a budget constraint on the first moment, she would choose exactly these distributions – this is formalized in the proof of Proposition 6. So budget mechanisms are outcome equivalent to mechanisms which do give strictly more freedom than under probability assignment.
constraint rather than the set of available actions. Then the agent optimally chooses actions so as to keep $a_i - \theta_i$ constant across decisions. (In the sequential case, the agent keeps $a_i - \theta_i$ constant in expectation). Normalizing the principal’s bias to 0, the optimal budget $K^*$ for a simultaneous or sequential problem is the ex ante expected value of $\sum_i \gamma_i \theta_i$. (See Lemma 17 of Appendix B).

**Example 3.** Let $\gamma_i = 1$ for all $i$, let $U_p(a|\theta) = -(a - \theta)^2$, let the agent have quadratic loss constant bias utility, let each $\theta_i$ be iid uniformly distributed over $\Theta = [0, 1]$. A budget mechanism will let the agent choose any actions, subject to a constraint $\sum_i a_i = K$.

**Simultaneous Problem:** Suppose that $[-\frac{1}{2}, \frac{1}{2}] \subseteq A$. The optimal budget level is $K^{*\text{sim}} = N/2$.

Given states $\theta$, the agent chooses action $a_i = \theta_i + \frac{1}{2} - \frac{1}{N} \sum_{j=1}^{N} \theta_j$ in the optimal budget mechanism. The principal’s expected per-period payoff can be shown to be minus the variance of $\left(\frac{1}{N} \sum_{j=1}^{N} \theta_j\right)$, which is $-\frac{1}{12N}$. Relative the no-delegation payoff of $-\frac{1}{12}$ per period and the first-best payoff of 0, the budget mechanism achieves $\frac{N-1}{N}$ of the possible surplus.

If the principal knew that the agent had increasing difference or quadratic loss utility (but didn’t know she had a constant bias) and used the optimal ranking mechanism, he would get only $\frac{N-1}{N+1}$ of the surplus (see Example 1).

**Sequential Problem:** Suppose that $[\frac{1}{2} - \sum_{j=2}^{N} \frac{1}{2}, \frac{1}{2} + \sum_{j=2}^{N} \frac{1}{2}] \subseteq A$.

The optimal budget level is $K^{*\text{seq}} = N/2$.

At period $i$, with remaining budget $K_i = \frac{N}{2} - \sum_{j<i} a_j$, the agent observes $\theta_i$ and then will choose action $a_i = \theta_i + \frac{K_i - \theta_i - (N-i)/2}{N-i+1}$ in the optimal budget mechanism.

If $N = 1$, this replicates the no-delegation outcome of $a_1 = 1/2$.

If $N = 2$, the agent chooses action $a_1 = \frac{1}{4} + \frac{\theta_1}{2}$, then action $a_2 = 1 - a_1$. This gives the principal an expected payoff of $-\frac{1}{16}$ per period, i.e., 25% of the possible surplus. If the principal used the optimal sequential quota mechanism, he would get only 18.75% of the surplus (Example 2).

In the $N = 3$ case, the principal gets an expected payoff of about $-0.051$ per period from the optimal budget mechanism, which is 38.9% of the possible surplus from delegation. The optimal sequential quota would give 30.5% of the surplus.

### 7.2 Quadratic Loss Linear Bias

Suppose the principal and agent each have quadratic loss linear bias utility. Each party’s utility function is of the form $U(a|\theta) = -(a - \lambda(1)\theta - \lambda(0))^2$ for some constants $\lambda(0) \in \mathbb{R}$ and $\lambda(1) \in \mathbb{R}^+$. With the principal uncertain about the agent’s $\lambda$'s.

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31In Appendix B, Example 4 and the discussion of a “big enough” action space explain why the principal and agent are generally better off if actions may be taken in an interval larger than the state space $[0, 1]$. Observation 4 explains how the bounds on $A$ in the simultaneous and sequential example here are derived. Lemma 17 shows how to calculate the optimal budget level and the agent’s strategy.
This functional form is a generalization of the constant bias utilities. The \( \lambda^{(0)} \) term shifts preferences uniformly to the left or right, while the \( \lambda^{(1)} \) term allows us to vary the “sensitivity” of a player to changes in the state. Melumad and Shibano (1991) provides an early characterization of optimal delegation sets when utilities are of this form, assuming a single decision (\( N = 1 \)) with common knowledge of the agent’s utility.

When there were rich utilities, the optimal aligned delegation and max-min optimal mechanism fixed the induced measure over actions – that is, fixed every moment of the measure. When we restricted to quadratic loss constant bias utilities, the corresponding optimal aligned delegation mechanism fixed only the first moment of the measure. It turns out that in the quadratic loss linear bias case, the optimal aligned delegation mechanism fixes the first two moments of the measure but otherwise gives complete freedom. I call these “two-moment” mechanisms.\(^{32,33}\)

Say that a pair \( (K^{(1)}, K^{(2)}) \) of proposed first and second moments is \textit{proper} at period \( i \leq N \) if there exist distributions \( m_i, \ldots, m_N \) such that

\[
\sum_{j=i}^{N} E_{a_j \sim m_j} \gamma_j a_j = K^{(1)}
\]

\[
\sum_{j=i}^{N} E_{a_j \sim m_j} \gamma_j \cdot (a_j)^2 = K^{(2)}
\]

Say that \( (K^{(1)}, K^{(2)}) \) is proper at \( i = N + 1 \) if \( K^{(1)} = K^{(2)} = 0 \).

If \( \mathcal{A} \) is convex – that is, \( \mathcal{A} \) is an interval \( [\underline{a}, \overline{a}] \) – then a pair of moments \( (K^{(1)}, K^{(2)}) \) is proper at period \( i \) if and only if \( K^{(1)} \in [\Gamma_i \underline{a}, \Gamma_i \overline{a}] \) and

\[
K^{(2)} \in \left[ \left( K^{(1)} \right)^2, \left( K^{(1)} \right)^2 + \frac{\left( \Gamma_i \overline{a} - K^{(1)} \right) \left( K^{(1)} - \Gamma_i \underline{a} \right)}{\Gamma} \right]
\]

When the action space is nonconvex the same values of \( K^{(1)} \) are feasible but, for a given \( K^{(1)} \), only some subset of \( K^{(2)} \) values may be possible. (The possible values of \( K^{(2)} \) are those above some cutoff).\(^{34}\)

**Definition** (Simultaneous Two-Moment Mechanism). In a simultaneous environment, a \textit{two-moment} mechanism\(^{32}\) proposes a multiplayer social choice mechanism in which each agent reports her values for each decision, with a similar restriction on the first two moments of the reports. It replicates the asymptotic efficiency properties of the mechanism in Jackson and Sonnenschein (2007), but can be implemented with less information about the ex ante distribution of preferences.

\(^{33}\)Any two of the first moment, the second moment, and the second central moment determine the third. And the mean and variance are simply rescalings of the first moment and the second central moment. I work with moments so as to parallel the previous sections, but it would be equivalent to think of these mechanisms as fixing the significance-weighted mean and variance of the chosen actions.

\(^{34}\)For a given sum of actions \( K^{(1)} \), the minimum possible sum-squared \( K^{(2)} \) arises if each action is taken at \( a_i = K^{(1)}/\Gamma_i \) and the maximum arises if each is taken as an appropriate mixture between \( \underline{a} \) and \( \overline{a} \).
mechanism $\text{TM}^{\text{sim}}(K^{(1)}, K^{(2)})$ is characterized by a pair of proposed moments $(K^{(1)}, K^{(2)})$ which are proper at period 1.

For the $\text{TM}^{\text{sim}}(K^{(1)}, K^{(2)})$ mechanism, there is no time 0 message. The interim message space $\mathcal{M}$ is the subset of $\Delta(\mathcal{A})^N$ for which $(m_1, \ldots, m_n) \in \Delta(\mathcal{A})^N$ satisfies

$$\sum_{j=i}^{N} \mathbb{E}_{a_j \sim m_j} \gamma_j a_j = K^{(1)}$$

$$\sum_{j=i}^{N} \mathbb{E}_{a_j \sim m_j} \gamma_j a_j^2 = K^{(2)}$$

Given message $(m_1, \ldots, m_n)$, action $a_i$ is drawn according to distribution $m_i$.

**Definition** (Sequential Two-Moment Mechanism). In a sequential environment, a two-moment mechanism $\text{TM}^{\text{seq}}(K^{(1)}, K^{(2)})$ is characterized by a pair of proposed moments $(K^{(1)}, K^{(2)})$ which are proper at period 1.

For the $\text{TM}^{\text{seq}}(K^{(1)}, K^{(2)})$ mechanism, there is no time 0 message. The message space $\mathcal{M}_i$ at time $i$ is some feasible subset of $\Delta(\mathcal{A})$. A message $m_i$ is feasible at period $i$ if the pair of moments

$$\left( K^{(1)} - \sum_{j \leq i} \mathbb{E}_{a_j \sim m_j} \gamma_j a_j, K^{(2)} - \sum_{j \leq i} \mathbb{E}_{a_j \sim m_j} \gamma_j a_j^2 \right)$$

is proper at period $i + 1$.

If feasible message $m_i \in \Delta(\mathcal{A})$ is chosen, then action $a_i$ is drawn according to the distribution $m_i$.

If we knew that the player would always choose deterministic actions (ie, degenerate distributions), then we could think of the first moment constraint as restricting vectors of actions to a hyperplane in the set of all actions, and the second moment constraint as restricting actions to an ellipsoid. Each of these is an $N-1$ dimensional surface, and their intersection – where both constraints are satisfied – is $N-2$ dimensional. So a budget mechanism limits the agent to an $N-1$ dimensional subset of $N$ actions, and a two-moment mechanism limits the agent to an $N-2$ dimensional subset. (A ranking or quota mechanism would restrict actions to a 0-dimensional set consisting of $N!$ possible vectors.)

For a sequential mechanism, this means that in a budget mechanism the agent has freedom to choose actions from a nondegenerate interval in all periods but the last. In period $N$, her action is predetermined. In a two-moment mechanism, if the agent plays deterministic actions, then she can choose from an interval of actions in all periods but the last two. In the second-to-last period, the agent has a choice of two actions; in the last period, her action is predetermined.

The agent may choose not to play deterministically. The action set may be nonconvex, or
– depending on the proposed moments – deterministic actions may be impossible. For instance, suppose that \( N = 1, \gamma_1 = 1, \) and \( K^{(2)} > (K^{(1)})^2. \) Then the agent must play a distribution with mean \( K^{(1)} \) and with positive variance. But the agent “tries” to play deterministically, if such play is possible and if the action set is “big enough”.

To see why it is that two-moment mechanisms will be aligned delegation, note that a player’s payoffs can be expanded to

\[
\sum_i \gamma_i U(a|\theta) = - \sum_i \gamma_i (a_i - \lambda^{(1)} \theta_i - \lambda^{(0)})^2 \\
= 2 \sum_i \gamma_i \lambda^{(0)} a_i - \sum_i \gamma_i a_i^2 - \sum_i \gamma_i (\lambda^{(1)} \theta_i + \lambda^{(0)})^2 + 2 \sum_i \gamma_i \lambda^{(1)} a_i \theta_i
\]

By fixing the first moment on actions, we fix the first sum across all strategies. By fixing the second moment on actions, we fix the second sum. The third sum is necessarily constant across strategies. So any player with quadratic loss linear bias utilities facing a two-moment mechanism chooses actions to maximize \( 2 \sum_i \gamma_i \lambda^{(1)} a_i \theta_i \) subject to the moment constraints. But the choice of actions which maximizes this is independent of the coefficient \( \lambda^{(1)} \) (which only enters as a scalar), and also of the coefficient \( \lambda^{(0)} \) (which does not appear at all). Therefore, all types play identically.

**Definition (Unboundedness).** Suppose that the agent has quadratic loss linear bias utilities, so that \( U_A(a|\theta) = -(a - \lambda^{(1)} \theta - \lambda^{(0)})^2 \) with \( \lambda^{(0)} \in \mathbb{R} \) and \( \lambda^{(1)} \in \mathbb{R}_{++} \) for any \( U_A \in \mathcal{U}_A \). Say that the agent’s utility set is **unbounded** if there is a sequence of types with \( |\lambda^{(0)}| \to \infty \) while \( \lambda^{(1)} \to 0.\)

**Proposition 7 (Optimality of Two-Moment Mechanisms).** Let the principal and agent have quadratic loss linear bias utility, and let the agent’s utility set be unbounded. Then, for a simultaneous or sequential problem,

1. The optimal two-moment mechanism is strictly aligned delegation, and gives weakly higher expected payoffs to the principal than does any other aligned delegation mechanism.
2. The optimal two-moment mechanism is max-min optimal.

**Proof.** See Appendix A. ■

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35That is, for any proper moments and for any distribution of states, we could suppose that the action set were equal to \( \mathbb{R} \) and solve for the optimal strategy. It would be deterministic so long as \( N \geq 2. \) Across all state realizations the chosen actions would all lie in some compact interval of \( \mathbb{R} \). So if the true action set \( \mathcal{A} \) is “big enough” that it contains this compact interval, then the agent will choose these same deterministic actions. If the action set is convex but “too small,” then there are examples where deterministic actions are possible but are not chosen.

36The unboundedness condition for the quadratic loss constant bias case was essentially “tight” – we needed the magnitude of the bias to go to infinity to get the max-min result. On the other hand, the richness condition for the general case was merely sufficient – other similar conditions could have guaranteed the same results. This unboundedness condition is in the latter category rather than the former. It will give a single sequence of types for which the agent lexicographically cares about the first moment and then the second, in the limit. Another condition could give a sequence for which the agent cared about the second and then the first. Or there could be two different sequences which each induce the agent to care about only one of the moments at a time.
7.3 General Sets of Budgets

Under unbounded quadratic loss constant bias, the max-min optimal mechanism fixes the first moment – the weighted sum of actions. Under unbounded linear bias, the max-min optimal mechanism fixes the first and second moments – the weighted sums of actions and actions squared. Under richness, the max-min optimal mechanism fixes the weighted sums of actions to every positive integer power. In fact, for any “moment conditions” – weighted sums of specified functions of actions – we could find an environment in which the max-min optimal mechanism fixed the set of such moments.

To make this statement a little clearer, take some set of functions \( \langle s^{(j)} : A \rightarrow \mathbb{R} \rangle_{j=1}^{J} \). I can come up with an environment in which the max-min optimal mechanism gives the agent freedom to take any action distributions, subject to the constraints that \( \sum_{i} \gamma_{i} s^{(j)}(a_{i}) = K^{(j)} \) for each \( j \), for some sequence of real values \( \langle K^{(j)} \rangle_{j=1}^{J} \).

One way to construct such an environment is to let \( U_{A}(a|\theta) = U_{P}(a|\theta) + \lambda^{(1)} s^{(1)}(a) + \cdots + \lambda^{(J)} s^{(J)}(a) \) for unknown and unbounded constants \( \langle \lambda^{(j)} \rangle_{j=1}^{J} \) in \( \mathbb{R}^{37} \).

For instance, if \( U_{A}(a|\theta) = U_{P}(a|\theta) + c(a) \) for \( c(\cdot) \) an unknown polynomial of degree \( n \), then the principal’s optimal max-min mechanism fixes \( \sum_{i} \gamma_{i}(a_{i})^{j} \) for each \( j = 1, \ldots, n \).

8 Discussion and Extensions

8.1 Consensus Decisionmaking

Any of the aligned delegation mechanisms from the paper can easily be extended to an environment with multiple agents who each have noisy signals of the states of the world, or an environment where the principal also has a signal. Simply add a communication stage after the players observe their signals. The players share their information through some process, and then one sends the “official” message to the mechanism.\(^{38}\)

\(^{37}\)This construction generalizes that of the quadratic loss constant bias case, but the quadratic loss linear bias case is actually not of this form.

\(^{38}\)Mylovanov and Zapechelnyuk (2008) considers a one-shot delegation game with multiple experts and transfer payments, and assumes that expert preferences are independent of the underlying state. A number of papers also consider more standard cheap talk models (no transfers, preferences depend on the state) with multiple experts; for example, see Krishna and Morgan (2001), Battaglini (2002), Ambrus and Takahashi (2008), and Ambrus and Lu (2010). Payoffs in a cheap talk equilibrium provide a lower bound on delegation payoffs.

These papers show that agents with different biases can often be “played against each other” to induce them to reveal all or nearly all of their private information to the principal, who then takes an approximately optimal action. For the most part these papers focus on the strategic interactions between the experts, rather than the informational benefits of observing more signals – they tend to assume that agents are perfectly informed, or nearly so. This is not just a simplifying assumption, and indeed is often crucial for the equilibria – many of the constructions involve the threat of “bad” actions when agents’ reports disagree.

The results of these papers depend in different ways on the functional forms of the utilities, on the information structure, and on the precise set of possible actions. Moreover, they all differ from the current analysis in assuming that the principal knows the agents’ utility functions exactly. Some of the constructions from Krishna and Morgan (2001) or Ambrus and Lu (2010) might generalize to the case of unknown types with bounded biases.
The mechanisms are no longer necessarily max-min optimal. Even so, aligned delegation mechanisms are appealing in this environment for a new reason – they encourage consensus decision-making. Each player has her own bias but, by construction, preferences are fully aligned by the mechanism. In a ranking mechanism, for instance, everyone wants to get the final ranking right. One’s incentive in the communication stage is to share her private information as fully and honestly as possible, and after all information is aggregated they collectively agree on the message to be sent. This yields benefits when any one agent is poorly informed, but the group as a whole is well informed.

Solving for any sort of optimal mechanism – max-min or otherwise – would depend on the details of the information structure and of the agents’ beliefs, and would likely rely on an assumption that agents could not collude or make side-agreements. But aligned delegation mechanisms induce agents to act nonstrategically, and do not require the mechanism designer to account for these informational details.

8.2 Relaxed Mechanisms

If it is common knowledge that the principal prefers lower actions than the agent, then intuitively the principal should not push the agent to take higher actions than she prefers.

For instance, suppose that teachers have quadratic loss constant bias preferences over their students’ grades, and all teachers have a positive bias – they want to give their students higher grades than does the school. Rather than mandating a GPA of exactly 3.0 for the class (an example of a budget mechanism), why not let the teacher give out any GPA of at most 3.0? If the class did well, the cap would bind and the grades would be as before. If the class underperformed though, the teacher might prefer a GPA below 3.0. Letting her do this would be good for the school, too – when the teacher wants low grades, the school wants even lower grades. This new contract would be a “relaxed” budget constraint.

In Appendix C.1, I show how we can relax probability assignment or budget mechanisms in this manner, taking the constraints from equalities to inequalities. When the agent is known to have a positive bias – formalized in the sense that the agent has altruistic with private cost preferences, with a state-independent cost term \( c(a) \) increasing in \( a \) – then relaxing the constraints improves the principal’s payoff for some states and some agent types, and never hurts performance. (This

\[ ^{39} \text{After the information is aggregated, the agents all share a posterior on the distribution of each state. It is actually no longer enough to assume that agents agree about actions for every realized state, but also for every distribution over realized states.} \]

If agents each have altruistic with private cost preferences, then probability assignment mechanisms continue to be aligned over all distributions of states. There is also no conflict with quadratic loss preferences – agents want to assortatively match actions and states, where they agree that distributions of states should be ranked by their means.

But in a simultaneous probability assignment mechanism in which agents have general increasing difference utilities, the principal and each of the agents do not necessarily agree unless the posterior state distributions can be ranked in the sense of first order stochastic dominance. This can be guaranteed by reasonable assumptions on the information structure.
covers the teacher example, above, with quadratic loss constant bias preferences).

In the language of Jackson and Barbera (1988), the relaxed mechanisms dominate the original mechanisms according to the “protective criterion.” The relaxed mechanisms are equivalent according to the max-min criterion – they are still max-min optimal – and they are weakly better for any possible priors. (The relaxed mechanisms no longer satisfy aligned delegation.)

In Appendix C.2, I look for conditions which guarantee that aligned delegation mechanisms are undominated by other max-min optimal mechanisms.

8.3 Alternate Information Structures

In this paper I assumed that the principal and agent share a prior belief on the distribution of states, and that as the game progresses they only update their beliefs about the distribution of future states by observing past ones.

In a sequential problem where the principal had private information about the distribution of future states, we should no longer expect aligned delegation contracts to be max-min optimal. This is because aligned delegation ensures that agents act as if they shared the principal’s utility function, but not necessarily the principal’s beliefs. Likewise if the two parties have noncommon priors – the agent’s choices no longer maximize the principal’s subjective payoffs.

If the principal and agent do share a common prior, and if at every decision node the agent is at least as well informed as the principal about the distribution of future states, then the results of the paper do go through – aligned delegation contracts remain max-min optimal. (At least as well informed means that if the principal and agent were to share information, both parties would converge to the agent’s beliefs). For example the agent might observe private or public signals about future states over the course of the game, or might initially know more about the joint distribution of states.

8.4 Approximate Bayes Optimality

The asymptotic first-best payoff results give conditions under which the max-min optimal mechanisms I introduce will give payoffs approximately equal to the Bayesian optimal mechanism, when there are many decisions. Even for a fixed number of decisions, there is another sense in which max-min optimal mechanisms may be approximately Bayesian optimal. The intuition of standard “saddle-point” theorems suggests that max-min mechanisms should be min-max as well – that they should be Bayes optimal against some worst possible prior belief over types.

Because the agent’s utility space is not necessarily compact, there is no single type or distribution of types against which aligned delegation mechanisms are guaranteed to be exactly Bayesian optimal. But we can find a sequence of “extreme” types (the same sequences implied by richness or unboundedness) for which the Bayes optimal payoff approaches the aligned delegation payoff, which does not vary with the prior.
9 Conclusion

This paper looks at delegation contracts which restrict the actions that an agent may choose. The agent has private information related to each of a number of decisions and also has persistent private information on her own preferences. I show how to derive max-min optimal contracts, contracts which maximize the principal’s payoff against the worst possible agent types.

These contracts often take surprisingly simple and familiar forms. I give conditions under which simultaneous ranking mechanisms, sequential quotas, or budget mechanisms may be max-min optimal. Other sets of assumptions suggest the use of novel mechanisms – probability assignment, for instance, or two-moment mechanisms.

The max-min optimal contracts I derive all have another special feature – they are also optimal in the class of aligned delegation mechanisms, mechanisms for which every agent type acts as if she is maximizing the principal’s payoff. I believe that the property of aligned delegation may be useful in other contracting contexts where a principal wants to protect himself against extreme agent types.

A Omitted Proofs

Proof of Lemma 1. For a measure $\mu$ over the set of actions $\mathcal{A}$, for $k = 0, 1, 2, \ldots$, define the $k^{th}$ moment of $\mu$ as

$$\text{Mom}^k(\mu) \equiv \int_a a^k d\mu(a)$$

In order to show that two measures are identical, it suffices to show that all of their moments are equal. This follows from the compactness of $\mathcal{A}$ and $\Theta$; see, e.g., Billingsley (1995) Theorem 30.1. And to show that a sequence of measures approaches some limiting measure (in the sense of weak convergence), it suffices to show that each fixed moment converges to the limiting moment; see Billingsley (1995) Theorem 30.2.40

If the agent has a rich set of preferences then we can consider four (non-exclusive) cases, at least one of which holds. There is a sequence of $n$’s going to infinity such that for each of these $n$’s, we can find a sequence of $U_A$’s in $\mathcal{U}_A$ for which

Richness Case 1: $U_A(a|\theta) = \psi(a|\theta; \lambda, n) - (a - \lambda)^{2n}$, with $\lambda \to -\infty$;

Richness Case 2: $U_A(a|\theta) = \psi(a|\theta; \lambda, n) - (a - \lambda)^{2n}$, with $\lambda \to \infty$;

Richness Case 3: $U_A(a|\theta) = \psi(a|\theta; \lambda, n) + (a - \lambda)^{2n}$, with $\lambda \to -\infty$; or

Richness Case 4: $U_A(a|\theta) = \psi(a|\theta; \lambda, n) + (a - \lambda)^{2n}$, with $\lambda \to \infty$.

The cited results are stated for distributions rather than general measures, using the first moment and above. But the $0^{th}$ moment is just the total mass placed on $\mathcal{A}$, so fixing this as well as the higher moments from 1 through $\infty$ fixes the measure. In any proper measure, the $0^{th}$ moment is $\Gamma$. 40
In cases 1 and 2, we can expand the polynomial in $U_A$ to get

$$U_A(a|\theta) = \left[ \psi(a|\theta; \lambda, n) - \binom{2n}{0} a^{2n} + \binom{2n}{1} \lambda a^{2n-1} - \cdots \pm \binom{2n}{n} \lambda^n a^n \right]$$

\[ + \left[ \mp \binom{2n}{n+1} a^{n-1} + \cdots + \left( \frac{2n}{2n-1} \right) \lambda^{2n-1} a - \left( \frac{2n}{2n} \right) \lambda^{2n} \right] \tag{1} \]

In cases 3 and 4, we can expand the polynomial in $U_A$ to get

$$U_A(a|\theta) = \left[ \psi(a|\theta; \lambda, n) + \binom{2n}{0} a^{2n} - \binom{2n}{1} \lambda a^{2n-1} + \cdots \mp \binom{2n}{n} \lambda^n a^n \right]$$

\[ + \left[ \pm \binom{2n}{n+1} a^{n-1} \mp \cdots - \left( \frac{2n}{2n-1} \right) \lambda^{2n-1} a + \left( \frac{2n}{2n} \right) \lambda^{2n} \right] \tag{2} \]

By the assumption of richness $|\psi|$ is bounded by $C\lambda^n \phi(a|\theta)$ for some $C$ and $\phi$, and so the first bracketed expression in (1) or (2) has a coefficient of order $\lambda^n$. As the absolute value of $\lambda$ goes to infinity, the second expression dominates. And the second bracketed expression is independent of $\theta$; its significance-weighted sum across decisions depends only on the first $n-1$ moments of the measure, each of which is of order greater than $\lambda^n$. So as we take $|\lambda|$ large, the agent only lets $\theta$ affect moments $n$ and greater. Going through each of the cases:

In richness case 1, in the limit as $\lambda \to -\infty$ in (1) the agent lexicographically prefers to minimize the first moment, then the second moment, and so on through the $n-1^{st}$ moment. Only after fixing these first $n-1$ moments does the agent consider the states $\theta$. In richness case 2, as $\lambda \to \infty$ in (1) the agent lexicographically maximizes the first moment, minimizes the second, maximizes the third, etc., before considering the states. In richness case 3, as $\lambda \to -\infty$ in (2) the agent maximizes the first $n-1$ moments; in richness case 4, as $\lambda \to \infty$ in (2) the agent minimizes the first, then maximizes the second, etc, through the $n-1^{st}$ moment.

Taking $n$ to infinity, all moments are fixed – that is, the measure does not depend on the states $\theta$.

Formalizing this argument, consider a public history – a list of all past messages and actions. At the initial message stage, call this history $h_0 = (\emptyset)$. For the simultaneous problem there are two more subsequent histories, $h_1 = (m_0)$ and $h_{N+1} = (m_0, m, a)$. For the sequential problem there are $N+1$ further histories $h_1 = (m_0)$, $h_2 = (m_0, m_1, a_1), ..., h_{N+1} = (m_0, m_1, a_1, ..., m_N, a_N)$. There are no additional reports or actions at history $h_{N+1}$ – at this history, the game is over.

Let $\mu^D_{\sigma, \theta}(h_i)$ be the measure of remaining actions from period $i$ onward, given public history $h_i$, if past and future states are given by $\theta$.\footnote{The notation is slightly redundant – past $\theta$’s are included in both the history and the vector $\theta$.} This measure has a total mass of $\Gamma_i = \sum_{j=1}^N \gamma_i$, where $\Gamma_0 = \Gamma_1 = \Gamma$ and $\Gamma_{N+1} = 0$. For instance, the measure $\mu^D_{\sigma, \theta}(h_0)$ is equal to $\mu^D_{\sigma, \theta}$ and the measure $\mu^D_{\sigma, \theta}(h_{N+1})$ places a mass of 0 on any set.

For each of the richness cases, starting from any history $h_i$, I will define the infinite sequence
of moments \((\alpha^0(h_0), \alpha^1(h_0), \ldots)\) that are “most desirable” for the agent as we take \(n\) and then \(|\lambda|\) to infinity, according to Equations (1) and (2).

Let \(\alpha^0(h_0) = \Gamma_i\) in all of the richness cases, for any history \(h_i\).

To define \(\alpha^k(h_i)\) for \(k \geq 1\), first let \(Z(k, \epsilon, h_i)\) be the set of \(k^{th}\) moments for which all lower moments \(l < k\) are within \(\epsilon\) of \(\alpha^l(h_i)\).

\[
Z(k, \epsilon, h_i) = \left\{ \begin{array}{l}
\text{Mom}^k (\mu_{\sigma, \theta}^D(h_i)) \quad \text{s.t.} \quad \sigma \in \Sigma^D, \theta \in \Theta_N \text{ consistent with } h_i, \\
& |\text{Mom}^l (\mu_{\sigma, \theta}^D(h_i)) - \alpha^l(h_i)| < \epsilon \text{ for each } l < k
\end{array} \right\}
\]

Now define \(\alpha^k(h_i)\) inductively, given \(\alpha^0(h_i), \ldots, \alpha^{k-1}(h_i)\).\(^{42}\)

Richness Case 1: \(\alpha^k(h_i) = \lim_{\epsilon \to 0^+} \inf Z(k, \epsilon, h_i)\)

Richness Case 2: \(\alpha^k(h_i) = \left\{ \begin{array}{l}
\lim_{\epsilon \to 0^+} \sup Z(k, \epsilon, h_i) \quad \text{if } k \text{ is odd} \\
\lim_{\epsilon \to 0^+} \inf Z(k, \epsilon, h_i) \quad \text{if } k \text{ is even}
\end{array} \right\}
\)

Richness Case 3: \(\alpha^k(h_i) = \lim_{\epsilon \to 0^+} \sup Z(k, \epsilon, h_i)\)

Richness Case 4: \(\alpha^k(h_i) = \left\{ \begin{array}{l}
\lim_{\epsilon \to 0^+} \inf Z(k, \epsilon, h_i) \quad \text{if } k \text{ is odd} \\
\lim_{\epsilon \to 0^+} \sup Z(k, \epsilon, h_i) \quad \text{if } k \text{ is even}
\end{array} \right\}
\)

For each of these cases, the list of moments \((\alpha^0(h_i), \alpha^1(h_i), \alpha^2(h_i), \ldots)\) uniquely defines a measure over \(\mathcal{A}\). (The moments define some measure because they were found as a limiting sequence of the moments of other measures). Let the measure \(\mu_D^\infty\) be the one implied by history \(h_0\).

Claim 1. For any public history \(h_i\), any \(k \geq 0\), and any \(\epsilon > 0\), fix an exponent \(n > k\) and a vector of states \(\theta\). For each of the four richness cases with \(U_A = \psi(a|\theta; \lambda, n) \pm (a - \lambda)^{2n}\), if \(|\lambda|\) is large enough then

\[
\left| \text{Mom}^k (\mu_{\sigma, \theta}^D(h_i)) - \alpha^k(h_i) \right| < \epsilon
\]

for any \(\sigma \in \Sigma^D(U_A)\).

This holds for a simultaneous or sequential problem.

This claim implies the result. For each of the richness cases, taking \(n\) to infinity and then \(|\lambda|\) to infinity, we can find a sequence of utility functions \(\langle U_A^j \rangle\) for which each fixed \(k^{th}\) moment converges to \(\alpha^k\) for every state vector and every optimal strategy at the null history \(h_0\). Therefore

\(^{42}\)I do not include current and past states in the construction of \(\alpha^k(h_i)\) even though they may affect an agent’s strategy, because they do not affect the set of possible action distributions going forward. The agent can always play as if the states had been something else.
the sequence of agent strategies \( \langle \sigma^j \rangle \) takes \( \mu^{D}_{\sigma^j, \theta} \) to \( \mu^D_{\infty} \).

**Proof of Claim 1.** I prove this by backwards induction on the period \( i \). I focus on the first richness case, with \( U_A = \psi(a|\theta; \lambda, n) - (a - \lambda)^{2n} \) and \( \lambda \to -\infty \); the proof proceeds similarly for the other cases, with some switched signs as suggested by the different definitions of \( \alpha \)'s.

The claim holds for any history \( h_i \) with \( i = N + 1 \) because \( \mu^D_{\sigma, \theta}(h_{N+1}) \) is always the 0-measure, for which every moment is 0.

Inductive hypothesis: Suppose the claim holds for all histories \( h_i \) with \( i > i' \). I seek to show that under the inductive hypothesis the claim also holds for any history \( h_{i'} \) as well.

At history \( h_{i'} \), after observing any states revealed in that period, the agent's expected future payoff given strategy \( \sigma \) can be written as

\[
\begin{align*}
\mathbb{E} & \sum_{s=s'}^N \gamma_s \left[ \psi(a_s|\theta_s; \lambda, n) - \binom{2n}{0} a_s^{2n} + \binom{2n}{1} a_s^{2n-1} - \cdots \pm \binom{2n}{n} \lambda^n a_s^n \right] \\
& + \mathbb{E} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2n-k} \lambda^{2n-k} \text{Mom}^k \left( \mu^D_{\sigma, \theta}(h_{i'}) \right)
\end{align*}
\]

where the expectations are over future states as well as current and future actions.

Now, take a sequence of utility functions, indexed by \( j \), which each have the same fixed \( n \) but for which \( \lambda \to -\infty \), and a respective sequence of optimal strategies \( \sigma^j \). Suppose there is some \( k < n \) such that the \( k^{th} \) moment \( \mathbb{E} \left[ \text{Mom}^k \left( \mu^D_{\sigma^j, \theta}(h_{i'}) \right) \right] \) does not approach \( \alpha^k(h_{i'}) \); without loss of generality, assume it is bounded away from (above) this value. Well, by construction of \( \alpha^k(h_{i'}) \) we can construct an alternate strategy \( \sigma' \) for which all of the moments \( \mathbb{E} \left[ \text{Mom}^l \left( \mu^D_{\sigma', \theta}(h_{i'}) \right) \right] \) are arbitrarily close to \( \alpha^l(h_{i'}) \) for each \( l < n \). (This strategy \( \sigma' \) may be taken to be state-independent, so that the expectation over future states is irrelevant.)

As \( \lambda \) goes to \( -\infty \), we can find such a \( \sigma' \) which must eventually be strictly preferred to the proposed optimal strategy \( \sigma^j \). That’s because the first sum in (3) is bounded by a constant expression times \( \lambda^n \), so the difference between \( \sigma' \) and \( \sigma^j \) is as well. But the difference in the second sums goes as at least \( |\lambda|^{2n-k>n} \) times the difference in \( k^{th} \) moments, which is positive. (Each term in this difference of moments is nonnegative, so the difference in \( k^{th} \) moments is a lower bound for the total difference).

\[\square\]

**Proof of Lemma 3.** Fix an agent utility function \( U_A \in \mathcal{U} \) and a proper measure \( \mu \).

For the simultaneous problem, consider the \( PA^{sim}(\mu) \) mechanism. Take any realized states \( \theta \). The agent’s payoff from an assignment \( m = (m_1, \ldots, m_N) \) is

\[
\sum_i \gamma_i \mathbb{E}_{a_i \sim m_i}[U_A(a_i|\theta_i)]
\]

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which is bounded above by $\Gamma \cdot \max_{a,\theta} U_A(a|\theta)$. So we can find a sequence of assignments $(m_n^\infty)_{n=1}^\infty$ for which the payoffs of $PA^{sim}(\mu)$ approach the supremum over measures of probability assignment payoffs.

The set of assignments $(m_1, \ldots, m_N)$ satisfying $\sum_i \gamma_i m_i = \mu$ is compact with respect to weak convergence of measures, and the agent’s payoff is continuous with respect to the same.\footnote{A vector of distributions converges weakly to a limiting vector if it does so component-by-component.} So some optimal assignment $m$ exists.

For the sequential problem, consider the $PA^{seq}(\mu)$ mechanism. Working backwards, at period $N$ with any remaining measure $\mu_N$ and state $\theta_N$ there is some optimal assignment $m_N$ by the argument above. This induces a continuation payoff at time $N$, prior to the realization of the state, which is again continuous in the measure $\mu_N$. Therefore at period $N-1$, after observing $\theta_{N-1}$, expected lifetime payoffs are continuous in the assignment $m_{N-1}$ and the set of possible assignments is compact. So some optimal choice $m_{N-1}$ exists. By backwards induction, an optimal report exists at every history. $\blacksquare$

**Proof of Lemma 4.** The payoff from any mechanism is bounded above by $\Gamma \cdot \max_{a,\theta} U_P(a|\theta)$, and so we can find a sequence of proper measures $(\mu_n^\infty)_{n=1}^\infty$ for which the payoffs of $PA(\mu_n)$ approach the supremum over measures of probability assignment payoffs. (PA represents the appropriate simultaneous or sequential mechanism). There is a measure achieving this limiting payoff by the compactness of the set of measures with respect to weak convergence combined with the continuity of payoffs as guaranteed by Claim 2.

**Claim 2.** In a simultaneous or sequential problem, suppose that utilities are PA-aligned. Take a sequence of proper measures $(\mu_n^\infty)_{n=1}^\infty$ converging weakly to $\mu$. Any player’s expected payoff from $PA(\mu_n)$ approaches that from $PA(\mu)$, where PA represents the respective probability assignment mechanism.

**Proof of Claim 2.** The proof is identical for the simultaneous and sequential cases.

As $n$ goes to infinity, the agent can choose distributions under $PA(\mu_n)$ which weakly approach those from optimal play under $PA(\mu)$ for each action $a_i$. So payoffs of $PA(\mu_n)$ are eventually bounded below by any payoff arbitrarily smaller than that from $PA(\mu)$ (for the agent and, by aligned delegation, for the principal). By a symmetric argument, the payoff from $PA(\mu)$ is bounded below by any payoff arbitrarily smaller than that from $PA(\mu_n)$, for large enough $n$. So the $PA(\mu_n)$ payoffs approach those of $PA(\mu)$. $\blacksquare$

**Proof of Proposition 1.** Take some aligned delegation mechanism $D$. By Lemma 2, there is a constant measure $\mu^D$ across all aligned strategies and all states.
Under the probability assignment mechanism PA(µD), the agent has the option of replicating the action distributions that would result from her optimal strategy in D. (Here PA is the respective simultaneous or sequential mechanism). So her payoff under the optimal (aligned) strategy in this probability assignment mechanism must be at least as high as her payoff under D.

Likewise, if the principal were playing the mechanism with the agent’s information, his optimal strategy under the probability assignment mechanism would yield a weakly higher payoff than his optimal strategy under D. By aligned delegation, in each case the agent’s strategy is optimal for the principal. So the principal’s expected payoff from PA(µD) is weakly higher than his payoff from D.

Finally, the principal’s expected payoff from the optimal probability assignment mechanism is weakly higher than his payoff from PA(µD).

Proof of Proposition 2. By aligned delegation, it suffices to show that there exists some strategy of the agent for which the principal’s weighted expected per-period payoff loss \( U_P(a^*_P(θ) | θ) - U_P(a(θ) | θ) \) is arbitrarily close to 0. An aligned strategy, which will be optimal for the agent, gives the principal a weakly smaller payoff loss. And under strict aligned delegation, every optimal strategy for an agent is aligned.

I propose a strategy which can be played in a sequential problem, and so can be replicated for a simultaneous problem.

For any positive integer L, divide the state space Θ into L “bins” in the following manner: For \( 1 ≤ l < L \), bin \( l \) is the set \( Θ^{l,L} = Θ ∩ [(\bar{a} + (\bar{a} - a) \frac{l-1}{L}, \bar{a} + (\bar{a} - a) \frac{l}{L})] \). Bin \( L \) is the set \( Θ^{L,L} = Θ ∩ [(a + (π - a) \frac{L-1}{L}, π)] - closed on the right.

For any \( l \) with \( Θ^{l,L} \) nonempty, let \( A^{l,L} = \{ a^*_P(θ) | θ ∈ Θ^{l,L} \} \).

Given a remaining measure \( µ_i = µ - ∑_{j<i} γ_j m_j \) with \( µ_i(A^{l,L}) ≥ γ_i \), say that the agent “places \( a_i \) in bin \( l \) (of \( L \))” if she plays actions proportionally from the remaining measure of \( µ_i \) on the support \( A^{l,L} \). Formally, she chooses distribution \( m_i \) as the measure defined by

\[
m_i(B) = \frac{µ_i(A^{l,L} ∩ B)}{µ_i(A^{l,L})} \quad \text{for } B ⊆ A
\]

Tweaking the terminology of Jackson and Sonnenschein (2007), fix \( L \) and say that a strategy in the probability assignment mechanism is \textit{approximately truthful} with respect to these bins if, whenever \( θ_i ∈ Θ^{l,L} \) and \( µ_i(Θ^{l,L}) ≥ γ_i \), the agent places action \( a_i \) in the “appropriate” bin, bin \( l \). When \( µ_i(Θ^{l,L}) < γ_i \) the action cannot be placed in the appropriate bin and so the agent’s choice \( m_i \) may be arbitrary, subject to the feasibility constraints.

Given states \( θ \), a number of bins \( L \), and the naive initial measure, let \( z(θ) \) be the last period for which every action from 1 through \( z \) has been placed in its appropriate bin under an approximately truthful strategy. By continuity of \( U_P \) and compactness of \( A \) and \( Θ \), the result follows from showing
that for any fixed $L$, as $n \to \infty$, $E_{\theta} \frac{\sum_{i=1}^{n} (\theta) \gamma_i^{(n)}}{\Gamma(n)} \to 1.44$

Suppose not. Suppose instead that $E_{\theta} \frac{\sum_{i=1}^{n} (\theta) \gamma_i^{(n)}}{\Gamma(n)}$ has lim inf strictly less than 1. Since the expression is bounded between 0 and 1, and since there are finitely many bins, this implies that there is some $l$ such that $\Theta^{l,L}$ is realized by $F(\cdot)$ with probability $p^{l,L} > 0$; there is some infinite subsequence of $n$ values; and some sequence $z^{(n)}(\theta)$ for which

- $E_{\theta} \frac{\sum_{i=1}^{n} (\theta) \gamma_i^{(n)}}{\Gamma(n)}$ is bounded away from 1 – say, is at most $1 - \xi$
- the probability over realizations of $\theta$ that $\frac{\sum_{i=1}^{n} (\theta) \gamma_i^{(n)}}{\Gamma(n)} \geq 1 - \gamma_i^{(n)} \in \Theta^{l,L}$ is bounded away from 0

But the weak law of large numbers on a triangular array (see Dembo (2010), Theorem 2.1.11), says that $\frac{\sum_{i=1}^{n} (\theta) \gamma_i^{(n)}}{\Gamma(n)}$ approaches its expectation – something at most $(1 - \xi) p^{l,L} < p^{l,L}$ – in probability, contradicting the second bullet.45

\[\square\]

**Proof of Lemma 5.**

1. It suffices to show that any non-assortative strategy is not optimal, because all assortative assignments are payoff equivalent.

Take some non-assortative assignment $m$. There are some decisions $i \neq j$ and some action $\hat{a} \in A$ for which $\theta_i > \theta_j$, and for which $m_i$ places positive probability on the set $\{a | a < \hat{a}\}$ while $m_j$ places positive probability on the set $\{a | a > \hat{a}\}$.

Find measures $\nu_i$ and $\nu_j$ such that each places the same mass $\epsilon > 0$ on $A$; such that $\frac{1}{\gamma_i} \nu_i \leq m_i$, and $\frac{1}{\gamma_j} \nu_j \leq m_j$; and $m_i$ places 0 mass on $\{a | a > \hat{a}\}$ while $m_j$ places 0 mass on $\{a | a \leq \hat{a}\}$.

Now we can consider “swapping” the measures $\nu_i$ and $\nu_j$, removing their weighted mass from the distributions $m_i$ and $m_j$ and applying them instead to $m_j$ and $m_i$. Let $m' = (m'_1, ..., m'_N)$

44For any $\epsilon > 0$, we can find $L$ large enough such that for any $l$ with $\Theta^{l,L}$ nonempty, for any $\theta' \in \Theta^{l,L}$, and for any $a' \in A^{l,L}$, it holds that $U_{\theta'}(a') \geq 1 - \epsilon$. Moreover, the principal’s worst possible stage utility level $U_{\theta'}(a') \in A, \theta' \in \Theta$ is a finite value. So if a weighted proportion of actions approaching 1 are placed in their appropriate bins, then the principal’s payoff is close to his optimal payoff.

45In the notation of Dembo (2010), let the variable $X_{n,i}$ be $\gamma_i^{(n)} \chi_{\theta \in \Theta^{l,L}}$ and let $b_n = \sum_{i=1}^{n} \gamma_i^{(n)}$. By construction, $|X_{n,i}| \leq b_n$. So to satisfy Theorem 2.1.11 we only need to show that $b_n^{-2} \sum_{i=1}^{n} \text{Var}(X_{n,i}) \to 0$. Var$(X_{n,i}) = p^{l,L}(1 - p^{l,L}) \left(\gamma_i^{(n)}\right)^2$, so it suffices to show that $\sum_{i=1}^{n} \gamma_i^{(n)} \to 0$. Letting $\bar{\gamma}_n = \max_{i \leq n} \gamma_i^{(n)}$,

\[\sum_{i=1}^{n} \left(\gamma_i^{(n)}\right)^2 \leq \sum_{i=1}^{n} \bar{\gamma}_n \gamma_i^{(n)} \leq \sum_{i=1}^{n} \gamma_i^{(n)} \to_{n} 0\]
be the feasible assignment defined by

\[ m'_i = m_i - \frac{1}{\gamma_i} \nu_i + \frac{1}{\gamma_j} \nu_j \]
\[ m'_j = m_j - \frac{1}{\gamma_j} \nu_j + \frac{1}{\gamma_i} \nu_i \]
\[ m'_l = m_l \quad \text{for } l \neq i, j \]

The payoff change to a player with utility \( U \in \mathcal{U} \) from this swap is

\[
\gamma_i \left[ \int_{\mathcal{A}} U(a|\theta_i) d\frac{1}{\gamma_i} \nu_j(a) - \int_{\mathcal{A}} U(a|\theta_i) d\frac{1}{\gamma_i} \nu_i(a) \right] + \gamma_j \left[ \int_{\mathcal{A}} U(a|\theta_j) d\frac{1}{\gamma_j} \nu_j(a) - \int_{\mathcal{A}} U(a|\theta_j) d\frac{1}{\gamma_j} \nu_i(a) \right] = \int_{\mathcal{A}} (U(a|\theta_i) - U(a|\theta_j)) d\nu_j(a) - \int_{\mathcal{A}} (U(a|\theta_i) - U(a|\theta_j)) d\nu_i(a)
\]

Now let \( H^{\nu_i} : [0, \epsilon] \to \mathcal{A} \) be the inverse cumulative mass function of \( \nu_i \), defined (for concreteness) by \( H^{\nu_i}(x) = \min\{a \in \mathcal{A} | \nu_i(\{a, a \cap \mathcal{A}\} = x) \}. \)

Likewise define \( H^{\nu_j} \) as the inverse cmf of \( \nu_j \). Notice that \( H^{\nu_i}(x) < \hat{a} < H^{\nu_j}(x) \) for all \( x \in (0, \epsilon) \). We can now rewrite the payoff change as

\[
\int_0^\epsilon \left[ (U(H^{\nu_i}(x)|\theta_i) - U(H^{\nu_i}(x)|\theta_j)) - (U(H^{\nu_j}(x)|\theta_j) - U(H^{\nu_j}(x)|\theta_j)) \right] dx
\]

And this is strictly positive if \( U \) satisfies strict increasing differences. Hence, \( m \) could not have been optimal for a player with strict increasing differences.

2. Take an increasing-difference utility function \( U \), not strictly increasing, and suppose for the sake of contradiction that a non-assortative assignment \( m' \) gives strictly higher payoff than an assortative assignment \( m \) to a player with utility \( U \).

Consider a perturbation \( U^\epsilon \) of \( U \), where \( U^\epsilon(a|\theta) = U(a|\theta) + \epsilon(a - \theta)^2 \). The function \( U^\epsilon \) has strict increasing differences for all \( \epsilon > 0 \), so under \( U^\epsilon \) the payoff from \( m \) is always larger than the payoff from \( m' \) (by Part 1). But as \( \epsilon \) goes to 0 the payoff under \( U^\epsilon \) from any assignment approaches that under utility \( U \). This contradicts the assumption that \( U \) has strictly higher payoff from \( m' \) than \( m \).

\[ \blacksquare \]

Proof of Lemma 6. Fix some function \( U : \mathcal{A} \times \Theta \to \mathbb{R} \). If \( \mathcal{U}_\mathcal{A} \) contains all functions of the form \( U(a|\theta) - (a - \lambda)^{2n} \), for \( n \to \infty \) and for \( \lambda \) going to plus or minus infinity, then richness is satisfied with \( \psi = U \).

If \( U \) satisfies strict increasing differences, or concavity, then the function \( U(a|\theta) - (a - \lambda)^{2n} \) does as well. For convexity, look at functions of the form \( U(a|\theta) + (a - \lambda)^{2n} \).

\[ \blacksquare \]

\[ \text{The inverse function is in general uniquely defined everywhere but a measure 0 of points; this construction chooses the lowest possible points whenever there is freedom.} \]
Proof of Observation 3. Given a proper measure \( \mu \) and a ranking \( \pi \), the mechanism assigns quantiles \( \sum_j \pi(j) < \pi(i) \gamma_j \) through \( \gamma_i + \sum_j \pi(j) < \pi(i) \gamma_j \) to decision \( i \):

\[
\text{Prob}(a_i \leq a) = \left\lfloor \frac{1}{\gamma_i} \left( \mu \left( \left[ -\infty, a \right] \right) - \sum_{j \text{ s.t. } \pi(j) < \pi(i)} \gamma_j \right) \right\rfloor
\]

If under every possible ranking \( \pi \) the interval of quantiles \( (x, x + \delta) \) are assigned to the same decision as one another, then the principal can improve his payoff by altering the measure \( \mu \) so as to optimally “consolidate” the quantiles from \( (x, x + \delta) \) onto a mass point of size \( \delta \). Find the expected distribution of underlying states to which this interval is assigned, and place the mass on the preferred action from the \( x^{th} \) through \( x + \delta^{th} \) quantiles of \( \mu \) given this distribution.

When all significances are equal, for any \( k = 1, \ldots, N \) the interval \( (k - 1, k) \) is always assigned to the decision \( i \) for which \( \pi(i) = k \). I say that the points \( 1, 2, \ldots, N \) are the “upper cutoffs” of quantiles at which the decisions switch. We can weakly improve any measure \( \mu \) by consolidating the mass from each of the \( N \) intervals between upper cutoffs onto a point mass, giving us a sum of \( N \) point masses of size 1.

In an asymmetric problem, given a ranking \( \pi \), the first (i.e., lowest) upper cutoff is at \( \gamma_i \) such that \( \pi(i) = 1 \). There are at most \( N \) such distinct first cutoffs over all possible rankings. The second cutoff is at \( \gamma_{i_1} + \gamma_{i_2} \) such that \( \pi(i_1) = 1 \) and \( \pi(i_2) = 2 \); there are at most \( \binom{N}{2} \) such points. There are \( \binom{N}{k} \) possible \( k^{th} \) cutoffs, for \( k \) ranging from 1 to \( N \) (where there is one possible \( N^{th} \) cutoff, at \( \Gamma \)). This gives us a maximum of \( \sum_{k=1}^{N} \binom{N}{k} = 2^N - 1 \) distinct upper cutoffs. We can weakly improve the principal’s payoff from any measure by consolidating the mass between quantiles 0 and the lowest possible upper cutoff onto a single point; the lowest possible and the second lowest possible; etc.

Proof of Lemma 8. For a probability distribution \( G(\cdot) \) over \( A \) and \( F(\cdot) \) over \( \Theta \), define \( Q(G, F) \) to be the set of joint distributions over \( A \times \Theta \) with marginal distributions \( G \) over \( A \) and \( F \) over \( \Theta \). For \( q \in Q(G, F) \) and \( U \in U \), let \( V(q|U) = \mathbb{E}[U(a|\theta)|q(a, \theta) \sim q] \). \( V(q|U) \) can be interpreted as the “stage payoff” for a player with utility \( U \) if \( (a, \theta) \) is drawn from \( q \).

Let \( q^A(\cdot|\theta) \) denote the conditional distribution over \( A \), given joint distribution \( q \) and state \( \theta \).

One element in \( Q(G, F) \) is of particular interest: let \( q^{\text{assort}}(G, F) \in Q(G, F) \) be the “assortative” joint distribution. This is the distribution which for which the support of \( q^A(\cdot|\theta) \) is weakly increasing in \( \theta - \) for any \( \theta < \theta' \) in the support of \( F \), if \( a \in \text{Supp} q^A(\cdot|\theta) \) and \( a' \in \text{Supp} q^A(\cdot|\theta') \), then \( a \leq a' \).\(^{47}\)

\(^{47}\)One can define \( q^{\text{assort}} \) more explicitly as follows. Letting \( H^G \) and \( H^F \) be the respective inverse distributions taking \([0, 1]\) into \( A \) or \( \Theta \), the probability of a measurable set \( B \subseteq A \times \Theta \) is

\[
\text{Prob}[B|(a, \theta) \sim q^{\text{assort}}(G, F)] = \{x| \left( H^G(x), H^F(x) \right) \in B\}
\]
Claim 3. For any $U$ satisfying increasing differences, 

$$q^{assort}(G, F) \in \arg\max_{q \in \mathcal{Q}(G, F)} V(q|U)$$

If $U$ satisfies strict increasing differences then $q^{assort}(G, F)$ is the unique maximizer.

This result is a mild generalization of Lemma 5, which says that assortative assignments are optimal in a simultaneous problem with increasing differences. The lemma applies to the observed empirical distribution of states, known to be composed of $N$ point masses, while this applies to any distribution.

Given the state $\theta_i$ and the function $\tilde{\theta}_{i+1} : [0, \Gamma_i] \rightarrow \Theta$, let $\tilde{F}_{\theta_{i,i+1}}$ be the probability distribution which takes the value $\theta_i$ with probability $\gamma_i/\Gamma_i$; and, with probability $\Gamma_{i+1}/\Gamma_i$, takes the value of $\tilde{\theta}_{i+1}(x)$ for a value $x$ uniformly drawn on $[0, \Gamma_{i+1}]$:

$$\text{Prob}_{\theta_i \sim \tilde{F}_{\theta_{i,i+1}}}[\theta \in B] = \frac{1}{\Gamma_i} \cdot \left( \left| \left\{ x \in [0, \Gamma_{i+1}] : \tilde{\theta}_{i+1}(x) \in B \right\} \right| + \gamma_i \cdot \chi_{\{\theta_i \in B\}} \right)$$ for $B \subseteq \Theta$

where $\chi_S$ is the indicator variable for event $S$ and $|S|$ denotes the Lebesgue measure of a set $S$. Loosely speaking, $\tilde{\theta}_{i+1}$ tells us the expected future states at which actions will be played in a sequential-assortative strategy, and $\theta_i$ tells us the current state. Then $\tilde{F}_{\theta_{i,i+1}}$ puts these together into a single probability distribution of current and future expected states to which actions will be assigned.

Claim 4. Take a sequential probability assignment mechanism with remaining measure $\mu_i$ at period $i$, and fix some $U_A$ satisfying quadratic loss. After observing the state $\theta_i$,

1. Suppose the agent plays a sequential-assortative strategy from period $i$ onwards. Then the expected payoff to an agent of type $U_A$ from period $i$ onward is

$$\Gamma_i \cdot V \left( q^{assort} \left( \frac{1}{\Gamma_i} \mu_i, \tilde{F}_{\theta_{i,i+1}} \right) \Big| U_A \right) + C$$

where $C$ is constant with respect to $\mu_i$, and depends the history only through the beliefs over the distribution of states $\theta_{i+1}$ through $\theta_N$.

2. Suppose the agent chooses a non sequential-assortative assignment at period $i$ but plays a sequential-assortative strategy at all periods after $i$. Then the expected payoff to an agent of type $U_A$ from period $i$ onward is

$$\Gamma_i \cdot V \left( q | U_A \right) + C$$

for the same $C$, and for some $q \in \mathcal{Q} \left( \frac{1}{\Gamma_i} \mu_i, \tilde{F}_{\theta_{i,i+1}} \right)$ with $q \neq q^{assort} \left( \frac{1}{\Gamma_i} \mu_i, \tilde{F}_{\theta_{i,i+1}} \right)$. 

where $| \cdot |$ denotes Lebesgue measure.
The key to this claim is that a quadratic loss utility function can be written (up to a scalar) as $-c(a)^2 + 2c(a)\theta - \theta^2$. States interact with action assignments through the $2c(a)\theta$ term, which is linear in $\theta$. This linearity lets us replace distributions of future states with expected future states when considering different assignment strategies. The $\hat{\theta}_{i+1}$ function tells us the expected future state at which an action will be played under a sequential-assortative strategy.

The sequential-assortative strategy assortatively assigns a measure of actions $\mu_i$ into a measure of current and future expected states given by $\Gamma_i \cdot \tilde{F}_{\theta_i,i+1}$, yielding the $\Gamma_i \cdot V$ payoff term (after dividing through and then multiplying by $\Gamma_i$). This would be the payoff if the distribution of future states were known with certainty to be the levels specified by $\tilde{\theta}$. The $C$ term adjusts this for the variance in future states, where variance lowers payoffs through the $-\theta^2$ term.

Deviating in period $i$ gives some alternative assignment, which is not fully assortative.

With these two claims, we can now prove part 1 of the Lemma by backwards induction. Let the agent have strict quadratic loss utility, which therefore satisfies strict increasing differences.

- **Base Case:** The only feasible period $N$ assignment is sequential-assortative.
- **Inductive Case:** Suppose that at any history starting at period $i + 1$, a strategy is optimal if and only if it is sequential-assortative. I seek to show that an assignment in period $i$ is optimal if and only if it is sequential-assortative.

This follows from Claims 3 and 4. According to the inductive hypothesis future play is sequential-assortative, and so the two claims imply that payoffs today can only be maximized by a sequential-assortative assignment.

Part 2 – that sequential-assortative strategies are optimal under non-strict quadratic loss utility – follows from the same argument as the proof of part 2 of Lemma 5, replacing $U^\epsilon(a|\theta)$ with (for example) $-\zeta(c(a) + \epsilon a - \theta)^2$.

**Proof of Claim 3.** Take $q \in Q(G, F)$ with $q \neq q^{assort}$; I will show that there exists $q' \in Q(G, F)$ for which $V(q'|U) > V(q|U)$ if $U$ satisfies strict increasing differences. So $q^{assort}$ must be the unique maximizer.

Informally, nonassortativity implies that we can find a positive measure of pairs of points $(a^1, \theta^2)$ and $(a^2, \theta^1)$ in $q$ such that $a^2 > a^1$ and $\theta^2 > \theta^1$. We will then “swap” these pairs to get $q'$ which is otherwise identical but has sorted these pairs, taking $(a^1, \theta^1)$ and $(a^2, \theta^2)$. Then I show that $q'$ is a strict improvement over $q$.

Formalizing this argument, for some $\epsilon > 0$ and for any measurable functions $a : [0, \epsilon] \to A$ and $\theta : [0, \epsilon] \to \Theta$, define a measure $\nu^{a,\theta}$ over the set $A \times \Theta$ as

$$
\nu^{a,\theta}(B) = \left| \{ x \text{ s.t. } (a(x), \theta(x)) \in B \} \right|
$$

for any measurable $B \subseteq A \times \Theta$. The notation $| \cdot |$ denotes Lebesgue measure.
Now, by nonassortativity we can find functions $a^1, a^2 : [0, \epsilon] \rightarrow A$ and $\theta^1, \theta^2 : [0, \epsilon] \rightarrow \Theta$ so that

- $\theta^2(x) > \theta^1(x)$ for each $x \in [0, \epsilon]$
- $a^2(x) > a^1(x)$ for each $x \in [0, \epsilon]$
- $\nu^{a^1, \theta^2} + \nu^{a^2, \theta^1} \leq q$ (as an inequality of measures)

Define $q'$ as

$$q' = q - \nu^{a^1, \theta^2} - \nu^{a^2, \theta^1} + \nu^{a^1, \theta^1} + \nu^{a^2, \theta^2}$$

The marginals of $q'$ are identical to the marginals of $q$, so $q' \in Q(G, F)$. And $q'$ is preferred to $q$:

$$V(q' | U) - V(q | U) = \int_0^\epsilon \left( [U(a^1(x), \theta^1(x)) + U(a^2(x), \theta^2(x))] - [U(a^1(x), \theta^2(x)) + U(a^2(x), \theta^1(x))] \right) dx$$

$$> 0 \quad {\text{by strict increasing differences of}} \ U$$

So the only maximizer of $V(\cdot | U)$ is $q^{assort}$ when $U$ has strict increasing differences.

And one can show that $q^{assort}$ maximizes $V(q | U)$ when $U$ satisfies (weak) increasing differences following a continuity argument as in the proof of Lemma 5.

Proof of Claim 4. **Step 1:** First I establish two equalities.

For any $a$ and any distribution over $\theta$, if $U_A$ is of the quadratic loss form then we can express $\mathbb{E}_q U_A(a | \theta)$ as

$$\mathbb{E}_q U_A(a | \theta) = -\zeta \mathbb{E}_q (c(a) - \theta)^2$$

$$= -\zeta (c(a) - \mathbb{E}_q \theta)^2 + \zeta \left( (\mathbb{E}_q [\theta])^2 - \mathbb{E}_q [\theta^2] \right)$$

$$= U_A(a | \mathbb{E}_q \theta) + \zeta \left( (\mathbb{E}_q [\theta])^2 - \mathbb{E}_q [\theta^2] \right) \quad (4)$$

We can also expand the expression $V \left( q^{assort} \left( \frac{\mu}{\Gamma}, \tilde{F}_{\theta,i+1} \right) | U_A \right)$ as an integral in the following way, with $H^{\mu_i}$ the inverse cdf of $\mu_i$:

$$\Gamma_i V \left( q^{assort} \left( \frac{\mu_i}{\Gamma_i}, \tilde{F}_{\theta,i+1} \right) | U_A \right) = \int_0^{\Gamma_i} U_A \left( H^{\mu_i}(x) \left\{ \begin{array}{ll}
\tilde{\theta}_{i+1}(x - \gamma_i) & \text{if } x \geq \gamma_i \text{ and } \theta_i < \tilde{\theta}_{i+1}(x - \gamma_i) \\
\tilde{\theta}_i(x) & \text{if } x < \Gamma_{i+1} \text{ and } \theta_i > \tilde{\theta}_{i+1}(x) \\
\theta_i & \text{otherwise}
\end{array} \right. \right) dx \quad (5)$$

**Step 2:** Backwards Induction Base Case. In this step I show that the claim holds at period $N$.

At period $N$, conditional on any $\theta_N$, the only possible assignment is to choose $a_N$ from the distribution $\frac{1}{\gamma_N} \mu_N$. This assignment is sequential-assortative, so part 2 holds vacuously. The
agent’s payoff at this period can be written as

$$\gamma_N \mathbb{E}_{(a,\theta) \sim \tilde{q}} [U_A(a|\theta)]$$

for $q$ equal to the joint distribution over $A \times \Theta$ which chooses $a$ from the distribution $\frac{1}{\gamma_N} \mu_N$ over $A$, and places a probability 1 on $\theta = \theta_N$. Noting that $\gamma_N = \Gamma_N$ and that $\tilde{F}_{\theta_N,N+1}$ is the distribution over $\Theta$ which places a probability 1 on $\theta_N$, this $q$ is exactly equal to $q^{assort}(\frac{1}{\gamma_i} \mu_i, \tilde{F}_{\theta_i,i+1})$ – indeed, this is the unique element of $Q\left(\frac{1}{\gamma_i} \mu_i, \tilde{F}_{\theta_i,i+1}\right)$. So part 1 holds at period $N$ with $C = 0$.

**Step 3:** Inductive Step. Now, suppose that the Claim holds for all periods $i + 1$ and beyond. I seek to show that it holds for period $i$ as well.

Let $C_{i+1}(\theta_{i+1})$ be the appropriate constant at period $i + 1$, with the beliefs about the distributions of $\theta_i + 2, \ldots, \theta_N$ as determined by the observations of $(\theta_1, \ldots, \theta_{i+1})$.

Starting at period $i$, and prior to the observation of $\theta_{i+1}$, the agent’s expected continuation payoff from playing sequential-assortatively from period $i + 1$ onward is given by

$$\mathbb{E}_{\tilde{\theta}_{i+1}} \left[ \Gamma_{i+1} V( q^{assort}(\frac{\mu_{i+1}}{\Gamma_{i+1}}, \tilde{F}_{\theta_{i+1},i+2}) U_A(i) + C_{i+1}(\theta_{i+1}) \right]$$

$$\text{Eq (5)} = \mathbb{E}_{\tilde{\theta}_{i+1}} \int_0^{\Gamma_{i+1}} U_A \left( H^{\mu_{i+1}}(x) \begin{cases} \tilde{\theta}_{i+2}(x - \gamma_{i+1}) & \text{if } x \geq \gamma_{i+1} \text{ and } \theta_{i+1} \leq \tilde{\theta}_{i+2}(x - \gamma_{i+1}) \\ \tilde{\theta}_{i+1}(x) & \text{if } x < \Gamma_{i+2} \text{ and } \theta_{i+1} > \tilde{\theta}_{i+2}(x) \\ \theta_{i+1} & \text{otherwise} \end{cases} \right) dx$$

$$+ \mathbb{E}_{\tilde{\theta}_{i+1}} \left[ C_{i+1}(\theta_{i+1}) \right]$$

Eq (4), Defn of $\tilde{\theta}_{i+1}$

$$\int_0^{\Gamma_{i+1}} U_A \left( H^{\mu_{i+1}}(x) \tilde{\theta}_{i+1}(x) dx + \mathbb{E}_{\tilde{\theta}_{i+1}} \left[ C_{i+1}(\theta_{i+1}) \right] \right)$$

$$+ \zeta \cdot \int_0^{\Gamma_{i+1}} \left( \tilde{\theta}_{i+1}(x) \right)^2 - \mathbb{E}_{\tilde{\theta}_{i+1}} \left[ \begin{cases} \tilde{\theta}_{i+2}(x - \gamma_{i+1}) & \text{if } x \geq \gamma_{i+1} \text{ and } \theta_{i+1} \leq \tilde{\theta}_{i+2}(x - \gamma_{i+1}) \\ \tilde{\theta}_{i+1}(x) & \text{if } x < \Gamma_{i+2} \text{ and } \theta_{i+1} > \tilde{\theta}_{i+2}(x) \\ \theta_{i+1} & \text{otherwise} \end{cases} \right] \right) dx$$

$$= \int_0^{\Gamma_{i+1}} U_A \left( H^{\mu_{i+1}}(x) \tilde{\theta}_{i+1}(x) dx + C \right) \quad \text{(6)}$$

where $C$ captures the two last terms of the previous line, both independent of $\mu_{i+1}$.

So after observing $\theta_i$, the payoff from period $i$ of choosing assignment $m_i$ then reverting to a sequential-assortative strategy is

$$\int_0^{\gamma_i} U_A \left( H^{\gamma_i m_i}(x) | \theta_i \right) dx + \int_0^{\Gamma_{i+1}} U_A \left( H^{\mu_{i+1}}(x) \tilde{\theta}_{i+1}(x) \right) dx + C$$

For any chosen $m_i$, the sum of the two integrals corresponds to $\Gamma_i V(q | U_A)$ for some $q \in Q \left(\frac{1}{\gamma_i} \mu_i, \tilde{F}_{\theta_i,i+1}\right)$. It corresponds to $q^{assort}$ if and only if $m_i$ is a sequential-assortative assignment.  

\[\square\]
Proof of Lemma 9. Write $U_A$ as $U_A(a|\theta) = -(c(a))^2 + 2c(a)\theta - \theta^2$.

The function $c(a) = (a - \lambda)^n$ is increasing and convex if $n$ is even and $\lambda \leq a$. The function $c(a) = -(a - \lambda)^n$ is increasing and concave if $n$ is even and $\lambda \geq a$. For the first case, fix $\zeta$ and take $\lambda$ to $-\infty$ for arbitrarily large even $n$; for the second case, fix $\zeta$ and take $\lambda$ to $+\infty$ for large even $n$.

In either case, let $\psi(a|\theta; \lambda, n) = 2c(a)\theta - \theta^2$, which is an $n^{th}$ degree polynomial in $\lambda$. This gives us an appropriate sequence of utility functions of the form $\psi(a|\theta; \lambda, n) \pm (a - \lambda)^{2n}$.

Proof of Lemma 10. Follows from Equation (6) in the proof of Lemma 8. The principal’s expected payoff from a probability (or interval) assignment mechanism at time 0 with measure $\mu$ can be expressed as

$$\int_0^\Gamma U_P \left( H^\mu(x) | \tilde{\theta}_1(x) \right) dx + C$$

This is maximized over measures $\mu$ if $H^\mu(x) = a_P \left( \tilde{\theta}_1(x) \right)$ for almost every $x$.

Proof of Lemma 12. Step 1: I show by induction that for any $i < N$ and $w \in \{0, ..., N - i - 1\}$, the function $\tilde{\theta}_{i+1}(x)$ is constant on $[w, w + 1)$ when all significances are equal to 1.

Base Case: For $i = N - 1$, this holds by construction of $\tilde{\theta}_N(\cdot)$.

Inductive Case: Suppose that $\tilde{\theta}_{i+1}$ is constant over $[w, w + 1)$ for integers $w \leq i + 1$. Then $\tilde{\theta}_i$ is constant over each integer step as well; its value can only vary at points $x$ where $\tilde{\theta}_{i+1}(x)$ or $\tilde{\theta}_{i+1}(x - 1)$ changes value, or at $x = 1$ or $x = N - i + 1$.

Step 2: The above result implies part 1 because it is an optimal strategy to choose $w_i$ as the minimum value $w \in [0, \Gamma_{i+1}]$ such that $\tilde{\theta}_{i+1}(w) \geq \theta_i$. This implies choosing $w_i$ as an integer between 0 and $N - i$.

The same result implies part 2 by Lemma 10.

Proof of Lemma 13. Consider a cost function $c(a) = -(a - \lambda)^{2n}$. This cost function is strictly concave for any $n > 1$ and any $\lambda > 0$, is everywhere increasing if $\lambda \geq \bar{a}$, and is decreasing if $\lambda \leq \underline{a}$. If the possible cost functions include all of these costs for $n$ arbitrarily large and $\lambda$ going to either plus or minus infinity, then the agent’s utility set is rich. (Let $\psi(a|\theta; \lambda, n)$ be equal to either $U_P(a|\theta)$ or 0, depending on whether these limiting costs exist for $\zeta = 1$ or $\zeta = 0$.)

Likewise, if $c(a) = (a - \lambda)^{2n}$, then the cost function is convex. If possible costs include these for $n$ large and $\lambda \to \pm\infty$ then again the utility set is rich.

Proof of Lemma 14.

1. We can rewrite the agent’s utility as

$$-(a - \theta - \lambda)^2 = (-\theta^2 + 2\theta(a - \lambda)) - (a - \lambda)^2$$

$$= \psi(a|\theta; \lambda) - (a - \lambda)^{2n} \text{ for } n = 1, \quad \psi(a|\theta; \lambda) = (-\theta^2 + 2\theta(a - \lambda))$$
The proof now follows that of Lemma 1 exactly, considering only the first moment $\mathbb{E}[\sum_i \gamma_i a_i | \sigma^*, \theta]$ of the induced measure $\mu^D_{\sigma^*, \theta}$.

2. Follows from part 1, as in the proof of Lemma 2.

**Proof of Lemma 15.** It is enough to show that an agent with stage utility $-(a - \theta - \lambda)^2$ faces an identical maximization problem over all values of $\lambda$. From any history, the agent maximizes the payoff

$$\mathbb{E} \sum_i -\gamma_i (a_i - \theta_i - \lambda)^2 = \mathbb{E} \left[ -\sum_i \gamma_i (a_i - \theta_i)^2 \right] + \mathbb{E} \left[ 2\lambda \sum_i \gamma_i a_i - \sum_i \gamma_i (2\lambda \theta_i + \lambda^2) \right]$$

In a budget mechanism, $\mathbb{E}2\lambda \sum_i \gamma_i a_i = 2\lambda K$. So this problem is equivalent to maximizing the first bracketed expression, which is independent of $\lambda$.

**Proof of Lemma 16.** Follows from compactness of the set of proper budgets $[\Gamma a, \Gamma a]$, and continuity of the principal’s payoff with respect to $K$ (as seen by an argument like that in Claim 2 in the proof of Lemma 4, or by an application of Berge’s Theorem of the Maximum).

**Proof of Proposition 6.**

1. As written, the budget mechanism allows the agent to report one-dimensional numbers $r_i \in [a, a]$ which each induce a distribution over actions, subject to the weighted sum of reports’ being $K$. Suppose instead we allowed the agent to report full action-distributions $m_i \in \Delta(A)$, as in a probability assignment mechanism, subject to the constraint that the first moment of the induced measure (ie, the weighted sum of expected actions) was $K$. Call this a “modified” budget mechanism. This modified mechanism would maximize the agent’s payoffs subject to the moment constraint, and by aligned delegation (Lemma 15) would maximize the principal’s payoffs as well.

   By Lemma 14 part 2, any alternate aligned delegation mechanism could be improved upon by such a modified budget mechanism. The principal-optimal aligned delegation mechanism would therefore be the optimal modified budget mechanism.

   So it suffices to show that the modified budget mechanism with first moment fixed at $K$ is outcome equivalent to the budget mechanism as originally defined with budget $K$. In other words, for any feasible distribution $m_i$ chosen by the agent in the modified mechanism, there exists some feasible report $r_i$ in the original mechanism which yields the same distribution.

   By aligned delegation, it is without loss of generality to look at the strategy of an agent with utility $U_A(a | \theta) = -(a - \theta)^2$. Given a state $\theta_i$ and a distribution $m_i$ over actions, we can
decompose the stage payoff as a function of $\theta_i$, the mean of $a_i$, and the variance of $a_i$:\footnote{Expanding, $\mathbb{E}_{a_i \sim m_i}[-(a_i - \theta_i)^2] = -(\mathbb{E}_{a_i \sim m_i} a_i - \theta_i)^2 - \mathbb{E}_{a_i \sim m_i}[(a_i - \mathbb{E}_{a_i \sim m_i} a_i)^2]$.}

$$\mathbb{E}_{a_i \sim m_i}[-(a_i - \theta_i)^2] = -(\mathbb{E}_{a_i \sim m_i} a_i - \theta_i)^2 - \text{Var}(a_i \mid a_i \sim m_i)$$

So for a given mean action $\mathbb{E}a_i$, the agent always prefers the distribution $m_i$ which minimizes variance. This is exactly the distribution over actions in the budget mechanism if $r_i = \mathbb{E}a_i$ is reported. And if the budget is set at the same level as the first moment of the modified mechanism, then the report $r_i$ is feasible in the original mechanism whenever the distribution $m_i$ with mean $r_i$ is feasible in the modified one.

2. The proof is nearly identical to that of Theorem 1, focusing on a sequence of utilities for which the expected weighted sum of actions (the first moment of the measure) converges, rather than one for which the measure converges. We apply Lemma 14 part 1 rather than Lemma 1.

Proof of Proposition 7. The argument follows that of Section 7.1 nearly exactly. I go through the equivalent of Lemma 14 part 1 here. The rest is straightforward.

Claim 5. Let the agent have unbounded quadratic loss linear bias utilities, and fix a mechanism $D$. There exist values $K^{(1)}$ and $K^{(2)}$ and a sequence of types $\langle U_j^j \rangle$ in $U_A$ such that for all states $\theta$ and all corresponding sequences of optimal agent strategies $\langle \sigma^j \rangle$,

$$\mathbb{E} \left[ \sum_i \gamma_i a_i \mid \sigma^j, \theta \right] \rightarrow K^{(1)} \text{ as } j \rightarrow \infty$$

$$\mathbb{E} \left[ \sum_i \gamma_i a_i^2 \mid \sigma^j, \theta \right] \rightarrow K^{(2)} \text{ as } j \rightarrow \infty$$

Proof of Claim 5. The agent’s utility can be written as

$$U_A(a|\theta) = 2\lambda^{(0)}a - a^2 + 2\lambda^{(1)}a\theta - (\lambda^{(1)}\theta + \lambda^{(0)})^2$$

The last term can never be affected by an agent’s strategy, and so is irrelevant.

As $|\lambda^{(0)}| \rightarrow \infty$ and $\lambda^{(1)} \rightarrow 0$, the agent lexicographically maximizes (if $\lambda^{(0)} \rightarrow \infty$) or minimizes (if $\lambda^{(0)} \rightarrow -\infty$) the first moment of the measure $\mathbb{E} \left[ \sum_i \gamma_i a_i \right]$; then minimizes the second moment of the measure $\mathbb{E} \left[ \sum_i \gamma_i a_i^2 \right]$; and only after these are fixed looks at the term $\mathbb{E} \left[ \sum_i \gamma_i a_i \theta_i \right]$ which depends on the states.

From here, the argument follows the proof of Lemma 1. □
### B Strategies In Budget Mechanisms

In this section, I characterize optimal strategies and budget levels under certain regularity conditions.

If the principal’s utility were normalized to $U_P(a|\theta) = -(a - \theta)^2$ and he had every action in $\mathbb{R}$ available, his unconstrained preferred action would always lie in the convex hull of the state space $\Theta$. But in a budget mechanism, his constrained optimal choices might lie outside of this convex hull. As I illustrate with the following example, we may face corner solutions unless we expand the diameter of the set of actions, $\pi - a$, to be larger than that of the set of states, $\overline{\theta} - \overline{\theta}$.

**Example 4.** Take a simultaneous problem with $\Theta = [0, 1]$, and let there be $N = 4$ equally significant decisions ($\gamma_i = 1$ for each $i$). Let the principal and agent have quadratic loss constant bias utilities, with $U_P(a|\theta) = -(a - \theta)^2$.

First suppose that $A = \Theta = [0, 1]$, and take the budget mechanism with $K = 2$ – that is, an average action of $\frac{1}{2}$ in each of four periods. (This would be the optimal budget level if the ex ante expected state in each period were $E[\theta_i] = \frac{1}{2}$).

If the realized states are $\theta = (0, 1, 1, 1)$ then the agent optimally chooses actions $a = (0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, giving the principal a lifetime payoff of $-3 \ast (\frac{1}{3})^2 = -\frac{1}{3}$.

Now expand the action space so that it contains $[-\frac{1}{4}, 1\frac{1}{4}]$. Under the same budget mechanism and the same realized states, the agent now chooses actions $(-\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ and the principal gets a higher payoff of $-4 \ast (\frac{1}{4})^2 = -\frac{1}{4}$.

To avoid these corner solutions, the action space needs to be “big enough”. Being big enough depends on the state space as well as the given budget level.

**Definition (Big Enough).**

- **Simultaneous:** Given mechanism $B^{\text{sim}}(K)$, say that $A$ is big enough if
  \[
  \left[ \frac{K}{\Gamma} - (\overline{\theta} - \overline{\theta}) \left( 1 - \min_i \frac{\gamma_i}{\Gamma} \right), \frac{K}{\Gamma} + (\overline{\theta} - \overline{\theta}) \left( 1 - \min_i \frac{\gamma_i}{\Gamma} \right) \right] \subseteq A
  \]
  When all decisions are equally significant ($\gamma_i = 1$ for all $i$), this reduces to
  \[
  \left[ \frac{K}{N} - (\overline{\theta} - \overline{\theta}) \frac{N - 1}{N}, \frac{K}{N} + (\overline{\theta} - \overline{\theta}) \frac{N - 1}{N} \right] \subseteq A
  \]

- **Sequential:** Given mechanism $B^{\text{seq}}(K)$, say that $A$ is big enough if
  \[
  \left[ \frac{K}{\Gamma} - (\overline{\theta} - \overline{\theta}) \cdot \max_i \left\{ \frac{\Gamma_{i+1}}{\Gamma_i} + \sum_{j=1}^{i-1} \frac{\gamma_j}{\Gamma_j} \right\}, \frac{K}{\Gamma} + (\overline{\theta} - \overline{\theta}) \cdot \max_i \left\{ \frac{\Gamma_{i+1}}{\Gamma_i} + \sum_{j=1}^{i-1} \frac{\gamma_j}{\Gamma_j} \right\} \right] \subseteq A
  \]
When all decisions are equally significant, this reduces to\(^{49}\)

\[
\left[ \frac{K}{N} - (\bar{\theta} - \theta) \sum_{j=2}^{N} \frac{1}{j}, \quad \frac{K}{N} + (\bar{\theta} - \theta) \sum_{j=2}^{N} \frac{1}{\gamma_j} \right] \subseteq A
\]

In general, a larger set of actions is required to make \(A\) big enough in a sequential problem.

These are tight bounds, given \(N\) and \(\langle \gamma_i \rangle\) – if we take a convex action space smaller than this, then there are state realizations (and in the sequential problem, beliefs) for which we get corner solutions. The agent takes actions on the boundary, and would take actions beyond the boundaries of \(A\) if possible. These tight bounds imply looser but more straightforward sufficient conditions to be big enough:

**Observation 4** (Sufficient Conditions to be Big Enough).

- In a simultaneous problem, \(A\) is guaranteed to be big enough if
  \[
  \left[ \frac{K}{\Gamma} - (\bar{\theta} - \theta), \quad \frac{K}{\Gamma} + (\bar{\theta} - \theta) \right] \subseteq A
  \]

- In a sequential problem, \(A\) is guaranteed to be big enough if\(^{50}\)
  \[
  \left[ \frac{K}{\Gamma} - (\bar{\theta} - \theta)(N - 1), \quad \frac{K}{\Gamma} + (\bar{\theta} - \theta)(N - 1) \right] \subseteq A
  \]

**Lemma 17.** Let the principal and agent have quadratic loss constant bias preferences, with \(U_P(a|\theta) = -(a - \theta)^2\).

1. Take a \(B^{\text{sim}}(K)\) mechanism in the simultaneous environment with \(A\) big enough. Given \(\theta\), the agent chooses \(a\) so that \(a_i - \theta_i\) is constant across \(i\):
   \[
a_i = \theta_i + \frac{K - \sum_{j=1}^{N} \gamma_j \theta_j}{\Gamma}
   \]

2. Take a \(B^{\text{seq}}(K)\) mechanism in the sequential environment, with \(A\) big enough. Given remaining budget \(K_i\) and current state \(\theta_i\), the agent chooses \(a_i\) so that \(a_i - \theta_i\) is equal to the weighted average of \(a_j - \theta_j\) in the future:
   \[
a_i = \theta_i + \frac{K_i - \gamma_i \theta_i - \sum_{j>i} \gamma_j \theta_{j+1}}{\Gamma_i}
   \]

\(^{49}\)Under equal significances there are two argmaxes, \(i = N - 1\) and \(i = N\). Plugging in either value for \(i\) along with \(\gamma_j = 1\), \(\Gamma_j = N - j + 1\) yields the result.

\(^{50}\)The result here follows so long as \(\max \left\{ \frac{\Gamma_i}{\Gamma_i + \sum_{j=1}^{i-1} \frac{\gamma_i}{\Gamma_j}} \right\} \leq N - 1\). This is a sum of \(i\) terms, each less than or equal to 1. If \(i = N\), then \(\Gamma_{i+1} = 0\) so the first term is 0 and each remaining term is bounded above by 1.
where the operator $\mathbb{E}^{i+1}$ denotes the expectation conditional on $\theta_1, \ldots, \theta_i$.\footnote{This expression for $a_i$ solves $a_i - \theta_i = \mathbb{E}^{i+1} \sum_{j \geq i} \gamma_j (a_j - \theta_j)$.}

3. In a simultaneous or sequential environment, when $\mathcal{A}$ is big enough, the optimal budget level is

$$K^{* \text{sim}} = K^{* \text{seq}} = \mathbb{E} \sum_i \gamma_i \theta_i$$

(Being big enough is evaluated with respect to this proposed budget level).

Proof of Lemma 17. \textbf{Step 1:} I first confirm that the proposed actions are in $\mathcal{A}$.

For the simultaneous case,

$$a_i = \theta_i + \frac{K - \sum_{j=1}^N \gamma_j \theta_j}{\Gamma}$$

This is minimized if $\theta_i = \bar{\theta}$ and $\theta_j = \bar{\theta}$ for each $j \neq i$; it is maximized if $\theta_i = \underline{\theta}$ and $\theta_j = \underline{\theta}$ for each $j \neq i$. This yields the following bounds for $a_i$:

$$a_i \in \left[ \frac{K}{\Gamma} - (\bar{\theta} - \underline{\theta}) \left(1 - \frac{\gamma_i}{\Gamma}\right), \frac{K}{\Gamma} + (\bar{\theta} - \underline{\theta}) \left(1 - \frac{\gamma_i}{\Gamma}\right) \right]$$

If $\mathcal{A}$ is big enough then this holds for the decision with the minimum $\gamma_i$, and so holds for all decisions.

For the sequential case, at period $i$ with remaining sum $K_i$, the argument above shows that the proposed $a_i$ is contained in

$$K_{i+1} = \left[ \frac{K_i - (\bar{\theta} - \underline{\theta}) \left(1 - \frac{\gamma_i}{\Gamma}\right)}{\Gamma_i}, \frac{K_i + (\bar{\theta} - \underline{\theta}) \left(1 - \frac{\gamma_i}{\Gamma}\right)}{\Gamma_i} \right]$$

(7)

where I have rewritten $\left(1 - \frac{\gamma_i}{\Gamma}\right)$ as $\frac{\Gamma_{i+1}}{\Gamma_i}$. Given $K_i$, we can find the range of possible $K_{i+1}$ values by taking $K_i$ minus $\gamma_i$ times the maximum or minimum possible $a_i$. This evaluates to

$$K_{i+1} = \left[ (K_i - \gamma_i (\bar{\theta} - \underline{\theta}) \frac{\Gamma_{i+1}}{\Gamma_i}, (K_i + \gamma_i (\bar{\theta} - \underline{\theta}) \frac{\Gamma_{i+1}}{\Gamma_i} \right]$$

By induction, we can then find bounds for the minimum and maximum possible $K_i$ as a function of $K_1$:

$$K_i \in \left[ \Gamma_i \left( \frac{K_1}{\Gamma_1} - \sum_{j=1}^{i-1} (\bar{\theta} - \underline{\theta}) \frac{\gamma_j}{\Gamma_j} \right), \Gamma_i \left( \frac{K_1}{\Gamma_1} + \sum_{j=1}^{i-1} (\bar{\theta} - \underline{\theta}) \frac{\gamma_j}{\Gamma_j} \right) \right]$$

We now plug this back into (7) to find the possible range of values for $a_i$:

$$a_i \in \left[ \frac{K}{\Gamma} - (\bar{\theta} - \underline{\theta}) \cdot \left( \frac{\Gamma_{i+1}}{\Gamma_i} + \sum_{j=1}^{i-1} \frac{\gamma_j}{\Gamma_j} \right), \frac{K}{\Gamma} + (\bar{\theta} - \underline{\theta}) \cdot \left( \frac{\Gamma_{i+1}}{\Gamma_i} + \sum_{j=1}^{i-1} \frac{\gamma_j}{\Gamma_j} \right) \right]$$
If $\mathcal{A}$ is big enough, then it contains the minimum and maximum such values.

**Step 2:** Having confirmed that the proposed actions are in $\mathcal{A}$, it is enough to solve a “relaxed problem” in which we optimize the agent’s action choices over all of $\mathbb{R}$. If the proposed actions solve the relaxed problem, then they also solve the true problem of maximizing over actions in $\mathcal{A}$.

In the relaxed problem, we can write a budget mechanism as one with a single constraint on actions, that $\sum_j \gamma_j a_j = K$.

Say that agent has utility $U_A(a|\theta) = -(a - \theta - \lambda)^2$.

1. Simultaneous Case:
   The agent observes $\theta$, and her maximization problem can be written via the Lagrangian
   \[
   \mathcal{L} = -\sum_{i=1}^{N} \gamma_i (a_i - \theta_i - \lambda)^2 - \xi \left( K - \sum_i \gamma_i a_i \right)
   \]
   where $\xi$ is the multiplier. This has FOC with respect to $a_i$ of
   \[
   2\gamma_i (a_i - \theta_i - \lambda) = \xi \gamma_i \\
   \implies a_i = \theta_i + \left( \lambda + \frac{\xi}{2} \right)
   \]
   The parenthetical expression is constant across $i$, and so to satisfy $\sum_i \gamma_i a_i = K$ it must hold that $\left( \lambda + \frac{\xi}{2} \right) = \frac{K - \sum_i \gamma_i \theta_i}{\sum_j \gamma_j}$. This gives us the strategy.

2. Sequential Case:
   In the relaxed problem, the budget constraint can be written as a constraint on the final action: $a_N = \frac{K_i - \sum_{j=1}^{N-1} \gamma_j a_j}{\gamma_N}$. Given that state $\theta_i$ has been observed and there is remaining sum $K_i$, the agent’s problem has a Lagrangian of
   \[
   \mathcal{L} = -\mathbb{E}_{i+1} \left[ \left( \sum_{j \geq i} \gamma_j (a_j - \theta_j - \lambda)^2 \right) - \xi \left( K_i - \sum_{j \geq i} \gamma_j a_j \right) \right]
   \]
   where $a_j$ is restricted to be a function of states $\theta_1, \ldots, \theta_j$, and the operator $\mathbb{E}_{i+1}$ indicates the expectation conditional on states $\theta_1, \ldots, \theta_i$. The multiplier $\xi$ is a random variable which depends on all states. This has an FOC on $a_j$ of
   \[
   2\gamma_j (a_j - \theta_j - \lambda) = \gamma_j \mathbb{E}_{j+1} [\xi] \\
   \]
   Rearranging,
   \[
   a_j = \theta_j + \lambda + \frac{\mathbb{E}_{j+1} [\xi]}{2}
   \]
Substituting this into the constraint \( \sum_{j \geq i} \gamma_j a_j = K_i, \)
\[
\sum_{j \geq i} \gamma_j \left[ \theta_j + \lambda + \frac{E_{j+1}[\xi]}{2} \right] = K_i
\]

Taking the expectation \( E_{i+1} \) of both sides, and applying the law of iterated expectations,
\[
\gamma_i \theta_i + \left( \sum_{j > i} \gamma_j E_{i+1}^j \theta_j \right) + \left( \lambda + \frac{E_{i+1}[\xi]}{2} \right) \Gamma_i = K_i
\]

And \( a_i \) is equal to this last expression plus \( \theta_i \).

**Step 3:** I will use the strategies from parts 1 and 2 to calculate the principal’s expected payoff for a given value of \( K \). Then I maximize the payoffs over choice of \( K \) to show part 3 of the Lemma.

For the simultaneous case, plugging the values of \( a_i \) from part 1 of the Lemma into the principal’s utility function, we can calculate the principal’s expected payoffs as
\[
\mathbb{E} \left[ \sum_{i=1}^{N} \gamma_i (a_i - \theta_i)^2 \right] = -\Gamma \cdot \mathbb{E} \left[ \left( \frac{K - \sum_{i=1}^{N} \gamma_i \theta_i}{\Gamma} \right)^2 \right]
\]

And this is maximized by \( K^* = \mathbb{E} \sum_{i=1}^{N} \gamma_i \theta_i \).

The sequential case is more involved, and requires the following claim.

**Claim 6.** In a sequential budget mechanism with remaining budget \( K_i \), the principal has expected payoff from period \( i \) onwards of
\[
\mathbb{E}^i \left[ \sum_{j \geq i} \gamma_j U_P(a_j | \theta_j) \right] = \frac{\mathbb{E}^i \left[ \left( K_i - \gamma_i \theta_i - \sum_{j > i} \gamma_j E_{i+1}^j \theta_j \right)^2 \right]}{\Gamma_i} + R_i
\]

where \( R_i \) depends on the distributions of \( \theta_i, ..., \theta_N \) but is independent of \( K_i \). (\( R_i \) is affected by past values of \( \theta \) if these affect the distributions of future states).

This claim implies the result – ex ante payoffs are maximized by \( K^*_1 = \mathbb{E} \sum_{i=1}^{N} \gamma_i \theta_i \).

**Proof of Claim 6.** **Step 1:** I will show that for any \( j \geq i \), the action \( a_j \) can be expressed in equilibrium as \( \theta_j + p(\theta_i, ..., \theta_j) + \frac{K_i}{\Gamma_i} \) for some function \( p(\cdot) \) which is independent of \( K_i \) (over the
relevant range for which the action space is big enough). Moreover,

\[ \sum_{j=i}^{N} \gamma_j E^i [p(\theta_i, \ldots, \theta_j)] = \Gamma_i E^i [p(\theta_i)] \]

To show this, backwards induct on \( i \). This holds for \( i = N \) with \( p(\theta_i) = -\theta_i \), because the action \( a_N \) is \( K_N / \gamma_N \) with certainty.

Now, suppose that it holds for \( i + 1 \) – i.e., that \( \frac{da_i}{dK_{i+1}} = 1 / \Gamma_{i+1} \) for \( j > i \). From the strategy determined above, \( a_i = p(\theta_i) + K_i / \Gamma_i \), for \( p(\theta_i) = -\gamma_1 \sum_{j=1}^{i} \gamma_j E^{i+1} \theta_j \). And conditional on \( \theta_i \), action \( a_i \) and remaining budget \( K_i \) affect actions \( a_{j<i} \) only through their effect on \( K_{i+1} \), where \( K_{i+1} = K_i - \gamma_i a_i = K_i - \gamma_i (p(\theta_i) + K_i / \Gamma_i) = \frac{K_{i+1} K_i}{\Gamma_i} - \gamma_i p(\theta_i) \). So

\[ \frac{da_j}{dK_i} = \frac{da_j}{dK_{i+1}} \frac{dK_{i+1}}{dK_i} = 1 / \Gamma_{i+1} \frac{\Gamma_i}{\Gamma_i} = 1 / \Gamma_i \]

This establishes that \( a_j = \theta_j + p(\theta_i, \ldots, \theta_j) + \frac{K_i}{\Gamma_i} \).

And the fact that

\[ \sum_{j=i}^{N} \gamma_j E^i [p(\theta_i, \ldots, \theta_j)] = \Gamma_i E^i [p(\theta_i)] \]

holds because \( p(\theta_i, \ldots, \theta_j) = a_j - \theta_j \), and we showed above that \( a_j - \theta_j = \frac{\sum_{k>i} \gamma_k E_{j+1}(a_k - \theta_k)}{\gamma_j} \).

Therefore

\[ \sum_{j=i}^{N} \gamma_j E_{i+1} [p(\theta_i, \ldots, \theta_i)] = \Gamma_i p(\theta_i) \]

for every \( \theta_i \), and so it holds in expectation when we apply \( E^i \) to each side.

**Step ii:** Therefore, the expected payoff going forward is

\[ -E^i \left[ \sum_{j \geq i} \gamma_j \left( p(\theta_i, \ldots, \theta_j) + \frac{K_i}{\Gamma_i} \right)^2 \right] \]

\[ = \sum_{j \geq i} \gamma_j \left( E^i [p(\theta_i)^2] + 2 E^i [p(\theta_i, \ldots, \theta_j)] \frac{K_i}{\Gamma_i} + \left( \frac{K_i}{\Gamma_i} \right)^2 \right) + \sum_{j \geq i} \gamma_j E^i [p(\theta_i, \ldots, \theta_j)^2 - p(\theta_i)^2] \]

\[ = \sum_{j \geq i} \gamma_j \left( E^i [p(\theta_i)^2] + 2 E^i [p(\theta_i)] \frac{K_i}{\Gamma_i} + \left( \frac{K_i}{\Gamma_i} \right)^2 \right) \]

\[ + \sum_{j \geq i} \gamma_j E^i [p(\theta_i, \ldots, \theta_j)^2 - p(\theta_i)^2] \]

\[ = E^i \left[ \sum_{j \geq i} \gamma_j \left( p(\theta_i) + \frac{K_i}{\Gamma_i} \right)^2 \right] + \sum_{j \geq i} \gamma_j E^i [p(\theta_i, \ldots, \theta_j)^2 - p(\theta_i)^2] \]

Call the second summation \( R_i \) (which is independent of \( K_i \)). Plug in \( p(\theta_i) + \frac{K_i}{\Gamma_i} = \frac{K_i - \gamma_i \theta_i - \sum_{k>i} \gamma_k E^{i+1} \theta_k}{\Gamma_i} \).
C Elaborations on Extensions

C.1 Relaxed Constraints

Say that $\tilde{U}$ is biased upwards relative to $U$ if there exists a weakly increasing function $c : A \to \mathbb{R}$ for which the function $U_A(a|\theta)$ is equivalent to $U_P(a|\theta) + c(a)$.

When preferences are quadratic loss constant bias, taking $U_P(a|\theta) = -(a - \theta)^2$ and $U_A(a|\theta) = -(a - \theta - \lambda)^2$, the agent is biased upwards for any $\lambda \geq 0$.

Define the relaxed mechanisms (informally) as follows.

A downward relaxed budget mechanism characterized by $K$ is otherwise identical to the original budget mechanism, with the relaxed constraint that $\sum_i r_i \leq K$ instead of $\sum_i r_i = K$. Denote the appropriate simultaneous or sequential relaxed mechanism by $B_{\text{down}}(K)$, as compared to the original mechanism $B(K)$.

A downward relaxed probability assignment mechanism characterized by measure $\mu$ is otherwise identical to the original probability assignment mechanism, with the relaxed constraint that the measure over all actions induced by the sequence $(m_1, ..., m_N)$ is weakly first-order stochastically dominated by $\mu$ rather than equal to $\mu$. Call the appropriate simultaneous or sequential relaxed mechanism $PA_{\text{down}}(\mu)$, compared to $PA(\mu)$ for the original one.

**Proposition 8.** 1. Take any principal utility function $U_P$; any biased-upwards set of agent utilities; and any proper measure $\mu$. Given any $U_A \in \mathcal{U}_A$, the mechanism $PA(\mu)$ gives weakly lower expected payoffs to the principal than does $PA_{\text{down}}(\mu)$.

2. Let the principal and agent have quadratic loss constant bias utility, with the agent biased upwards, and take any proper budget $K$. Given any $U_A \in \mathcal{U}_A$, the mechanism $B(K)$ gives weakly lower expected payoffs to the principal than does $B_{\text{down}}(K)$.

As the proof shows, if the agent ever plays differently in the relaxed mechanism than the original ones, it can only help the principal.

**Proof.** 1. Consider the history $h_1$ just prior to the first report in the relaxed mechanism, where the agent has observed $\theta_1$ (in the sequential game) or $\theta$ (in the simultaneous game). In the original PA mechanism, the agent’s reporting strategy maximizes $\mathbb{E} \left[ \sum_i \gamma_i (U_P(a|\theta) + c(a)) \mid h_1 \right]$ subject to the constraint that the measure over actions is $\mu$. Call this strategy $\sigma_{h_1}^{\text{old}}$.

This same strategy is available in the relaxed mechanism. But suppose the agent chooses some
alternative strategy $\sigma_{h_1}^{new}$ which induces expected measure $\mu'$ dominated by $\mu$. Then

$$\mathbb{E}\left[ \sum_i \gamma_i (U_P(a|\theta) + c(a)) \mid \sigma_{h_1}^{new}, h_1 \right] \geq \mathbb{E}\left[ \sum_i \gamma_i (U_P(a|\theta) + c(a)) \mid \sigma_{h_1}^{old}, h_1 \right]$$

But

$$\mathbb{E}\left[ \sum_i \gamma_i c(a) \mid \sigma_{h_1}^{new}, h_1 \right] \leq \mathbb{E}\left[ \sum_i \gamma_i c(a) \mid \sigma_{h_1}^{old}, h_1 \right]$$

by monotonicity of $c$ combined with the domination of $\mu'$ by $\mu$, because $\mathbb{E}\left[ \sum_i \gamma_i c(a) \right]$ is exactly just the integral of $c(a)$ with respect to the respective measure over actions. And so

$$\mathbb{E}\left[ \sum_i \gamma_i U_P(a|\theta) \mid \sigma_{h_1}^{new}, h_1 \right] \geq \mathbb{E}\left[ \sum_i \gamma_i U_P(a|\theta) \mid \sigma_{h_1}^{old}, h_1 \right]$$

which is the result we set out to prove.

2. The proof follows the one above, replacing $c(a)$ with $2a\lambda - \lambda^2 - 2\theta \lambda$; measure $\mu$ with budget $K$; and measure $\mu'$ with budget $K' < K$. Then

$$\mathbb{E}\left[ \sum_i \gamma_i (2a\lambda - \lambda^2 - 2\theta \lambda) \mid \sigma_{h_1}^{new}, h_1 \right] \leq \mathbb{E}\left[ \sum_i \gamma_i (2a\lambda - \lambda^2 - 2\theta \lambda) \mid \sigma_{h_1}^{old}, h_1 \right]$$

because, removing the identical $-\lambda^2 - 2\theta \lambda$ terms from both sides, the LHS becomes $2\lambda K'$ and the RHS becomes $2\lambda K$. 

C.2 Are Aligned Delegation Mechanisms Undominated?

In an aligned delegation mechanism, all types play identically to the principal’s least preferred type. So any other max-min optimal mechanism might do better, and cannot do worse, for a given type. When is it the case that no improvements over aligned delegation mechanisms are possible?

In the case of quadratic loss constant bias preferences, a strict budget will be undominated as long as there are types with biases that are arbitrarily high ($\lambda \to \infty$) as well as biases that are arbitrarily low ($\lambda \to -\infty$), and the action space $A$ is convex.\(^{52}\)

\(^{52}\)Here is an illustration of why a convex action space is important. Suppose the principal has utility $U_P(a|\theta) = -(a - \theta)^2$ and the agent has utility $U_P(a|\theta) = -(a - \theta - \lambda)^2$, where the agent may have any bias $\lambda$ in $\mathbb{R}$. Consider the discrete action space $A = \{0, 1\}$. Let there be a single decision ($N = 1$), and suppose that the expected state $\mathbb{E}\theta_1$ is $\frac{1}{2}$.

Any budget $K \in [0, 1]$ is equally good for the principal; in particular, $K = 0$ or $K = 1$ are both optimal.

Consider the mechanism which gives the agent complete freedom. The agent will choose $a_1 = 0$ when $\theta + \lambda \leq 1/2$, and $a_1 = 1$ when $\theta + \lambda \geq 1/2$. Some extreme types with very negative biases will choose $a_1 = 0$ for any state $\theta_1$, and ones with very positive biases will always choose $a_1 = 1$. So this has the same worst-case properties as the budget mechanisms with $K = 0$ or $K = 1$. But agents with small biases will condition their play on the state, and for these types the principal is strictly better off than under any budget.
Probability assignment mechanisms can be shown to be undominated under two conditions.

1. There is a single optimal measure $\mu^*$ – the principal strictly prefers $\text{PA}(\mu^*)$ to $\text{PA}(\mu)$ whenever $\mu \neq \mu^*$.

2. Given any mechanism $D$, if two distinct induced measures are possible, then there exist proper measures $\mu^1, \mu^2$ and sequences of types $\langle U_{A1}^1, j \rangle_{j=1}^{\infty}$, $\langle U_{A2}^2, j \rangle_{j=1}^{\infty}$ such that for any sequences $\sigma^{1,j} \in \Sigma^*(U_{A1}^1)$ and $\sigma^{2,j} \in \Sigma^*(U_{A2}^2)$,

$$
\mu^{D_{\sigma^{1,j}}, \theta} \text{ weakly converges to } \mu^1 \text{ for all } \theta \text{ as } j \to \infty
$$

$$
\mu^{D_{\sigma^{2,j}}, \theta} \text{ weakly converges to } \mu^2 \text{ for all } \theta \text{ as } j \to \infty
$$

Suppose Condition 1 did not hold, and there were multiple optimal measures $\mu^{*1}$ and $\mu^{*2}$. Then one mechanism which might improve on $\text{PA}(\mu^{*1})$ or on $\text{PA}(\mu^{*2})$ is the mechanism which allows any assignments such that the induced measure is either $\mu^{*1}$ or $\mu^{*2}$. The old worst-case extreme types will always play one or the other, so the max-min properties of this new mechanism may be no worse than the PA mechanisms. But other types may take advantage of the added flexibility in ways that benefit the principal.

Condition 2 is a strengthening of the conclusion of Lemma 1. Lemma 1 implied that some type (in the limit) would choose a constant measure; this condition says that if two measures are possible, there are types which choose different constant measures.

What assumptions on the primitives of the problem guarantee that these two conditions hold?

The first can be shown to hold when $A$ is convex and $U_P$ is strictly concave, for the cases where I can characterize optimal measures: when the principal has increasing difference utility in the simultaneous case, or quadratic loss utility in the sequential case.

To guarantee the second condition, we need a strengthening or a modification of the richness assumption which implied Lemma 1. Richness required the agent’s type set to contain a sequence satisfying one of four possibilities, which can be listed as $-(a-\lambda)$ for $\lambda \to -\infty$; $-(a-\lambda)$ for $\lambda \to \infty$; $(a-\lambda)$ for $\lambda \to -\infty$; or $(a-\lambda)$ for $\lambda \to \infty$. One strengthening of richness which guarantees Condition 2 is that the agent’s type set contains distinct sequences satisfying all four possibilities, rather than just one. In other words, the agent has possible utilities which are arbitrarily convex or arbitrarily concave, and also may prefer very high or very low actions. This holds for the full set of altruistic with private cost preferences, and the full set of increasing difference preferences.

I do not currently have a proof that probability assignment mechanisms are undominated under quadratic loss preferences.

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53 That is, there exist states $\theta, \theta'$ and strategies $\sigma, \sigma'$ such that $\mu^{D_{\sigma}, \theta} \neq \mu^{D_{\sigma'}, \theta'}$.

54 This assumption guarantees that for any distribution of $\theta$, there is a unique maximizer of $\max_a E_\theta[U_P(a|\theta)]$. My constructions of the optimal measures are therefore unique.
References


