How to measure disagreement?*

Enrico Zanardo†

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Abstract

This paper defines a family of functions that measure the distance between opinions. We introduce six axioms that a measure of disagreement should satisfy, and characterize all the functions that satisfy them. The disagreement measures we characterize generalize the Renyi divergences, and include the Kullback-Leibler divergence and the Bhattacharyya distance. We study the properties of our measures, showing how they relate to each other and how they differ from other metrics on opinions.

We analyze two applications. In the first, we find a necessary and sufficient condition under which public information reduces expected disagreement between Bayesian agents. In the second, we show that our measures of disagreement are useful to understand trading under heterogeneous beliefs. Trade volume and gains from trade are increasing in some of our measures of disagreement.

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†ez2197@columbia.edu. PhD candidate in Economics, Columbia University, Department of Economics.

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1 Introduction

Measure what is measurable and make measurable what is not so.

Attributed to Galileo Galilei (1564-1642)

People form opinions about uncertain events in any number of contexts: is global warming happening? Who will win the next elections? Will a business idea be successful? In all these situations, different agents hold different opinions and disagreement is ubiquitous. Mathematically, opinions, or beliefs, are defined as probability distributions. While mathematicians have defined numerous metrics on probability distributions, social scientists have not agreed on a metric that suitably quantifies opinion disagreement.

Besides being useful to social scientists, a quantitative measure of disagreement can be used in several practical problems. Political leaders can use it to measure polarization of opinions in a community; managers can employ disagreement measures to form teams whose opinions are aligned (if a project requires consistent views), or misaligned (if one thinks that different opinions lead to more creative solutions); financial institutions can use it to measure the distance among investors' beliefs.

The goal of this paper is to provide a quantitative measure of disagreement based on first principles, i.e., based on reasonable properties (axioms) one would like such a measure to satisfy. The main theorem characterizes all the functions that satisfy the axioms and the two applications show the usefulness of our measures of disagreement in two models. First, we prove that Bayesian agents expect to disagree less after observing the same piece of public information (under some conditions). Second, we show that trade of contingent assets among agents with heterogeneous beliefs is proportional to disagreement, as measured by some of our functions.

The family of functions we identify as disagreement measures includes several well-known divergence measures such as the Kullback-Leibler divergence and the Bhattacharyya distance. These measures, originally introduced in the information theory literature, are now used in several areas of economic research. The Kullback-Leibler divergence has been used to parametrize the cost of information in rational inattention models (see Sims (2003) and Sims (2010)). Hansen et al. (2014) use the Kullback-Leibler divergence and the Bhattacharyya coefficient to measure similarity in the content of speeches. Eliaz and Spiegler (2016) find a condition on the Bhattacharyya coefficient that implies the existence of a direct mechanism that allocates firms to search pools, in a Bayesian networks.

Let us describe our approach to measuring disagreement with an example. Suppose that three agents, Ann, Bob, and Carl, hold different opinions about four options, 1, 2, 3, and 4. The following table summarizes their opinions, by the probability they ascribe to each option being the best:

	1	2	3	4
Ann (p_A)	8/10	1/10	1/90	8/90
Bob (p_B)	8/10	2/10	0	0
Carl (p_C)	9/10	1/10	0	0

Ann and Bob have different beliefs, more precisely they agree on the probability of state 1 and they disagree on the likelihood of states $2,3,4$. Therefore, Ann and Bob agree on the reduced state space that is obtained by merging states $2,3,4$ into the same event $\{2,3,4\}$. In this new state space, the three beliefs are:

	1	≥ 2
Ann (p_A)	8/10	2/10
Bob (p_B)	8/10	2/10
Carl (p_C)	9/10	1/10

Any metric d (Euclidean distance, Total Variation, etc.[1]) yields that disagreement between Ann and Bob in the original state space is larger than their disagreement in the reduced space, as $d_{\{1,\geq 2\}}(p_A, p_B) = 0 < d_{\{1,2,3,4\}}(p_A, p_B)$. More generally, it is a desirable property of a disagreement measure to decrease whenever some states are merged into the same event, because by doing so any difference of opinions on those states is not measured (the event " ≥ 2" does not capture the fact that Ann and Bob assign different probabilities to states $2,3,4$). Consider then Ann and Carl: do they disagree less after merging states $2,3,4$? Not according to the Euclidean distance, because on the state space $\{1,2,3,4\}$ their Euclidean distance is approximately 0.13 and on the reduced state space $\{1,\{2,3,4\}\}$ it is larger than 0.14. Our Axiom 4 imposes that disagreement between two agents cannot increase if we merge states, so this example shows that the Euclidean distance does not satisfy Axiom 4.

A distance that satisfies this property is the Hellinger distance, which is defined as: $d(p,q) = \sqrt{1 - \sum_i \sqrt{p_i q_i}}$. Nonetheless, there is another property, which we will impose, that the Hellinger distance does not satisfy. Observe that the agents' opinions in the example can be interpreted as the product of two independent opinions: i) is the best option an odd or an even number? ii) is the best option in $\{1,2\}$ or in $\{3,4\}$?

	Odd	Even
Ann (p_A)	8/9	1/9
Bob (p_B)	8/10	2/10
Carl (p_C)	9/10	1/10

	{1,2}	{3,4}
Ann (p_A)	9/10	1/10
Bob (p_B)	1	0
Carl (p_C)	1	0

We will require that whenever opinions can be written as a product of independent marginals, the disagreement is additively separable in issues (Axiom 5). In the example, the disagreement on the state $\{1,2,3,4\}$ should be the sum of disagreement on the "Odd vs. Even" issue and the "$\{1,2\}$ vs. $\{3,4\}$" issue. This is

[1] For any two beliefs $(p_1, \ldots, p_n)$, $(q_1, \ldots, q_n)$, the Euclidean distance is defined as $d(p,q) = \sqrt{\sum_{j=1}^{n} (p_j - q_j)^2}$, and the Total Variation is defined as $d(p,q) = \frac{1}{2} \sum_{j=1}^{n} |p_j - q_j|$.

not the case for the the Hellinger distance, but if we instead consider the Bhattacharyya distance, $d(p,q) = -\log\left(\sum_i \sqrt{p_i q_i}\right)$, we have that total disagreement can be written as the sum of disagreement across issues.

Neither of the axioms we just described implies that on the "Odd vs. Even" issue, Bob should disagree less with Ann than with Carl (because 8/9, the belief that Ann ascribes to the number being odd, lies between 8/10 and 9/10, respectively Bob's and Carl's belief). This will be assumed in Axiom 3, that implies that whenever an agent's opinion p_A is a convex combination of the opinions of other two agents p_B and p_C, then $d(p_B, p_A) \leq d(p_B, p_C)$.

In addition, we will assume in Axiom 1 that two agents with the same opinion have 0 disagreement (Bob and Carl have 0 disagreement on the "{1,2} vs. {3,4}" issue). We will also impose that the labelling of the state does not affect disagreement: if we permutate the columns of the tables above, disagreement among agents does not change (Axiom 2). We conclude with a separability axiom that restricts the way in which disagreement on state i affects disagreement on other states (Axiom 6).

These axioms allow us to fully characterize the set of functions that measure disagreement (Theorem 1). The axioms we defined are based on geometric properties of opinions, and they are not explicitly tied to economic applications of disagreement. In the last section, then, we consider two models of incomplete information that show how our disagreement measures are useful to understand the interaction of rational agents with heterogeneous beliefs.

We first analyze how disagreement between two Bayesian agents changes when they observe a piece of public information. It will generically be the case that observing a signal realization might increase their disagreement. Nonetheless, disagreement will decrease *on average* if the disagreement measure satisfies a continuity axiom (Axiom 7). In other words, any rational agent expects public information to reduce disagreement with any other agent. In fact something stronger is true: if two experiments are ranked by statistical sufficiency, the more informative one will induce lower expected disagreement than the less informative one. This result matches the intuitive understanding of disagreement: whenever observing more precise public information, agents' opinions should converge. This would not be the case with most metrics on probability distributions, (Euclidean distance, Total Variation, any norm, etc.) whereas it is implied by some of our measures of disagreement.

In the second application, we show that trade in contingent assets among agents with heterogeneous beliefs is increasing in agents' disagreement. The relation between trade and disagreement has been documented extensively, but most metrics on probability distribution would not imply that more disagreement increases volume of trades. In our application, we show that if agents have constant relative risk aversion, the volume of trades is proportional to their disagreement. We conclude by showing that this result holds for any utility function, if the disagreement among agents is small.

1.1 Related Literature

The set of results that are closest to those of this paper come from the information theory literature. Even though those results are not motivated by the problem of measuring disagreement, they define functions that measure the distance between probability distributions, and often characterize them axiomatically. Rényi

(1961) characterizes a family of divergence measures that is similar to our measures of disagreement. Some particular example of our measures are the Kullback-Leibler divergence (Kullback and Leibler (1951)); the Bhattacharyya distance (Bhattacharyya (1946)); and the logarithm of the Chernoff coefficients (Chernoff (1952)). Csiszár (2008) reviews different axiomatizations of some of these divergence measures, and comments on the axioms assumed in the literature. We postpone a detailed analysis of the overlapping between our measures of disagreement and well-known divergence measures to subsection 2.3. In Appendix B, we describe Renyi's axiomatization in detail, and comment on the differences with our setting.

More broadly, our results are related to economic models analyzing the interactions of agents with heterogeneous beliefs. A non-exhaustive list of papers in this literature includes: Mankiw et al. (2003) in the macroeconomics literature; Harrison and Kreps (1978) and Varian (1989) in the finance literature; Piketty (1995) in the political economy literature; Morris (1994) in the trade theory literature; Van Dan Steen (2010) in the firm theory literature; and more generally in the applied theory literature: Yildiz (2004), Che and Kartik (2009), Sethi and Yildiz (2012), Alonso and Camara (2016), Sethi and Yildiz, etc. Our paper provides a way to measure the distance between two opinions, and therefore it can be related to papers studying the interaction of agents with heterogeneous beliefs.

Our application to the effect of public information on disagreement is related to the literature in political polarization (Sunstein (2002), Dixit and Weibull (2007), and Baliga et al. (2013)). Kartik et al. (2015) find conditions under which "information validates the prior", i.e. under which a Bayesian agent expects the posteriors of other agents to approach hers, as more information is observed. They find that this is the case if the experiment satisfies the Monotone Likelihood Ratio Property, and priors are Likelihood Ratio ranked. In our application, we study a similar questions, though instead of comparing expected posteriors, we analyze expected disagreement. Furthermore, we analyze our measures of disagreement, whereas Kartik et al. (2015) consider the expectations of monotone functions. The sufficiency ranking on information structures we consider is defined in Blackwell (1951) and Blackwell (1953). Other authors have investigated the relation between Blackwell's sufficiency and divergence measures (Taneja (1987), Kailath (1967), and Chambers and Healy (2010)).

The second application we study relates disagreement of investors to the volume of trade. The empirical relevance of disagreement on volume of trades has been shown, for example, in Cragg and Malkiel (1982), Kandel and Pearson (1995), Hong and Stein (2007), Cookson and Niessner (2016). Theoretically, many papers have proposed parametric models to explain those regularities: Kim and Verrecchia (1991), Harris and Raviv (1993), Kim and Verrecchia (1994), Kandel and Pearson (1995) Hong and Stein (2007), and Banerjee and Kremer (2010). Their results are related to ours in terms of motivation, but they differ in their setting. The model that is closest to ours is that of Varian (1985), who analyzes asset prices and volume of trades in a market with contingent assets. Similarly to us, he imposes no distributional assumption on the beliefs of the traders, but differently from us, he defines increasing disagreement as a *mean-preserving spread* of agents' beliefs. Owing to our measures of disagreement, we will be able to obtain the effect on trade of *any* change in beliefs, not only mean-preserving spreads.

2 Main Result

In this section, we introduce the model, axioms and main characterization of the paper. We then study the properties of the family of disagreement functions we characterize in Theorem 1. The discussion and motivation of the axioms is postponed to Section 3.

2.1 Model, Axioms and Main Theorem

Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a finite set of unknown states of the world. Define $\Delta(\Theta) := \{p \in \mathbb{R}^n_+ | \sum_j p_j = 1, p_j \geq 0, \forall j\}$ to be the set of beliefs on Θ, and $\Delta^\circ(\Theta) = \{p \in \Delta(\Theta) \mid p_j > 0, \forall j\}$ to be its interior. We will denote by p and q two typical beliefs in $\Delta(\Theta)$, and let p_i or $p(i)$ be p's i-th component. The scope of this paper is defining a function

$$D_\Theta : \Delta(\Theta) \times \Delta(\Theta) \to \mathbb{R}^+ \cup \{+\infty\}$$

that represents disagreement between two beliefs on Θ. Observe that we allow disagreement to be infinite, i.e. the range of D_Θ includes $\{+\infty\}$. We will assume that D_Θ is three times continuously differentiable and finite on $\Delta^\circ(\Theta) \times \Delta^\circ(\Theta)$.

We say that a function D_Θ is a disagreement function if it satisfies the following axioms:

Axiom 1 (Zero Disagreement)**.** *For all Θ and $p, q \in \Delta(\Theta)$:*

$$D_\Theta(p,q) = 0 \Leftrightarrow p = q.$$

Plainly, two agents p, q have zero disagreement $D(p,q) = 0$ if and only if they have the same opinion, $p = q$.

Axiom 2 (Anonymity of the State Space)**.** *Consider two state spaces Θ_1, Θ_2. If $\gamma : \Theta_1 \to \Theta_2$ is a bijection, then for all $p, q \in \Delta(\Theta_1)$:*

$$D_{\Theta_2}(p \circ \gamma^{-1}, q \circ \gamma^{-1}) = D_{\Theta_1}(p,q),$$

where $p \circ \gamma^{-1}$ is the distribution on Θ_2 defined by $p \circ \gamma^{-1}(\theta_2) := p(\gamma^{-1}(\theta_2))$.

Axiom 2 implies that disagreement only depends on the cardinality of Θ, $n = |\Theta|$ (see Lemma 2). Therefore, in the next axioms, we will denote by D_n a disagreement function on a simplex of dimension n, $\Delta_n := \{x \in \mathbb{R}_n \mid \sum_i x_i = 1, x_j \geq 0\}$, regardless of the underlying state space Θ.

If we consider permutations of the state space Θ ($\gamma : \Theta \to \Theta$), we obtain that Axiom 2 implies that disagreement D_Θ does not depend on the structure of the state space (order of the states, metric on the states, etc.). We discuss the implications of this axiom in detail in Subsection 3.2. Observe that the metrics that are typically used in $\mathbb{R}^n$ (Euclidean distance, Total Variation, p-norms) all satisfy Axiom 2.

Axiom 3 (In Betweenness)**.** *For all $p^1, p^2, q^1, q^2 \in \Delta_n$:*

$$D_n(\lambda p^1 + (1-\lambda)p^2, \lambda q^1 + (1-\lambda)q^2) \leq \max\{D_n(p^1,q^1), D_n(p^2,q^2)\},$$

for all $\lambda \in [0,1]$.

Axiom 3 implies that the distance of two beliefs in the convex combination of four beliefs p^1, q^1, p^2, q^2 cannot be larger than the maximum pairwise disagreement. As a special case, we obtain some natural properties of disagreement: if $p^1 = p$ and $p^2 = q^1 = q^2$, Axiom 3 implies that $D_n(\lambda p + (1-\lambda)q, q) \le D_n(p,q)$. If $p = p^1 = p^2$, we find that $D_n(p, \lambda q^1 + (1-\lambda)q^2) \le \max\{D_n(p,q^1), D_n(p,q^2)\}$, which is equivalent to assuming that the balls in the topology induced by D_n are convex.

Axiom 4 (Coarsening)**.** *For all $p,q \in \Delta_n$:*

$$D_{n-1}((p_1+p_2,\dots,p_n),(q_1+q_2,\dots,q_n)) \le D_n((p_1,p_2,\dots,p_n),(q_1,q_2,\dots,q_n)).$$

This axiom implies that disagreement cannot increase after two states are merged. As observed in the introduction, this axiom is not satisfied by the Euclidean distance (unlike the previous axioms).

For any $n,m \in \mathbb{N}$, and for any pair of beliefs $p \in \Delta_n$ and $q \in \Delta_m$, we denote by $p * q \in \Delta_{nm}$ the belief, on a state space with nm elements, with independent marginals p and q. Formally:

$$p * q = (p_1q_1,\dots,p_1q_m,\dots,p_jq_1,\dots,p_jq_m,\dots,p_nq_1,\dots,p_nq_m).$$

Axiom 5 (Independence)**.** *For all $n,m \in \mathbb{N}$, and all $p^{(1)},p^{(2)} \in \Delta_n$ and $q^{(1)},q^{(2)} \in \Delta_m$:*

$$D_{nm}(p^{(1)} * q^{(1)}, p^{(2)} * q^{(2)}) = D_n(p^{(1)},p^{(2)}) + D_m(q^{(1)},q^{(2)}).$$

This axiom means that if two agents consider two issues independent, then their disagreement on the product space is the sum of the disagreement across issues. This axiom is a departure from the previous ones in that it imposes a cardinal property of disagreement. In Subsection 3.5, we comment on how to relax this axiom to an ordinal property.

For all $i = 1,\dots n-1$ and any two agents $p,q \in \Delta_n^\circ$ define the derivative of disagreement in the direction i to $i+1$:

$$\partial_i D_n(p,q) := \lim_{\epsilon\to 0} \frac{D_n(p,(q_1,\dots,q_i-\epsilon,q_{i+1}+\epsilon,\dots,q_n)) - D_n(p,q)}{\epsilon}.$$

In words, $\partial_i D_n(p,q)$ denotes the marginal change in disagreement as the beliefs q increase the probability of state $i+1$ and decrease the probability of state i. For any two states i and j the quantity $\frac{\partial_i D_n(p,q)}{\partial_j D_n(p,q)}$ then denotes the Marginal Rate of Substitution (MRS) of disagreement.

Axiom 6 (MRS of Disagreement)**.** *Consider any pair of states $i,j \in \{1,\dots,n-1\}$. If $\partial_j D_n(p,q) \neq 0$, then:*

$$\frac{\partial_i D_n(p,q)}{\partial_j D_n(p,q)} = g(p_i,p_{i+1},p_j,p_{j+1};q_i,q_{i+1},q_j,q_{j+1}), \tag{1}$$

that is, such ratio does not depend on the belief on any other state.

This axiom can be interpreted as imposing a separability property across states. The effect of a change in states $i, i+1, j, j+1$ on disagreement is independent of the beliefs on other states. Also, observe that Axiom 6 imposes a local condition. In Subsection 3.6, we provide a global condition that implies our axiom, and explains why Axiom 6 can be thought of as a local separability condition.

The following theorem, which is our main result, characterizes the functions that satisfy the above axioms.

Theorem 1. *The only functions D_Θ that satisfy Axioms 1, 2, 3, 4, 5, 6 have the following functional form (for all Θ finite, and all $p, q \in \Delta(\Theta)$):*[2]

1. *either:*
$$D_\Theta(p,q) = a\sum_{\theta\in\Theta} p(\theta)\log\left(\frac{p(\theta)}{q(\theta)}\right) + b\sum_{\theta} q(\theta)\log\left(\frac{q(\theta)}{p(\theta)}\right), \tag{2}$$
for some $a, b \geq 0$ (not both zero);

2. *or:*
$$D_\Theta(p,q) = a\log\left(\sum_{\theta} p(\theta)\left(\frac{p(\theta)}{q(\theta)}\right)^{z-0.5}\right) \quad where \quad \begin{cases} a > 0 & if\ |z| > 0.5 \\ a < 0 & if\ |z| < 0.5 \end{cases} \tag{3}$$
for $z \in \mathbb{R} \setminus \{-0.5, 0.5\}$.

Proof of Theorem 1. The proof is in Appendix A.1.

□

Remark 1. Notice that the functions $D(p,q)$ are well-defined and finite for $p, q \in \Delta_n^\circ$. There is a unique way to extend these functions to Δ_n without violating any of the axioms. This extension uses the following conventions: if $p_j = q_j = 0$ then $0^\alpha/0^\beta = 0$ and $0\log(0/0) = 0$. If $p_j > 0 = q_j$ then for all $\alpha, \beta > 0$, $p_j^\alpha/0 = +\infty$; $0/p_j^\beta = 0$; $p_j\log(p_j/0) = +\infty$; $0\log(0/p_j) = 0$.

2.2 Analysis of the Disagreement Functions

We will denote by $Supp(p)$ the support of the distribution p: $Supp(p) = \{\theta \in \Theta \mid p_\theta > 0\}$. Unless specified otherwise, we will consider $p, q \in \Delta_n^\circ$, which implies that for all j, p_j/q_j is finite (and then so is $D_n(p,q)$). Since in this section we will fix Θ, we will drop the index n writing simply $D(p,q)$ whenever unambiguous.

Notice that we have not assumed the disagreement functions to be symmetric. The next proposition shows which measures are symmetric in p and q:

Proposition 1. *The only disagreement functions that satisfy Axioms 1–6 and are such that $D(p,q) = D(q,p)$ for all p, q, are proportional to:*

- *the symmetric divergence:*
$$D(p,q) = \sum_j (p_j - q_j)\log\left(\frac{p_j}{q_j}\right) = \sum_j p_j\log\left(\frac{p_j}{q_j}\right) + \sum_j q_j\log\left(\frac{q_j}{p_j}\right);$$

- *the Bhattacharyya distance:*
$$D(p,q) = -\log\left(\sum_j \sqrt{p_j q_j}\right).$$

[2] The expression is not well-defined for non fully-mixed beliefs. See Remark 1 after the Theorem for a clarification.

Furthermore, the only symmetric disagreement function that is additively separable in the states[3] *is the symmetric divergence, and the only symmetric disagreement function such that* $D(p,q) < +\infty$ *if and only if* $Supp(p) \cap Supp(q) \neq \emptyset$ *is the Bhattacharyya distance.*

The symmetric divergence is also called J-divergence, or Jeffreys divergence, after Harold Jeffreys who first introduced it in Jeffreys (1946). It is also sometimes referred to as symmetrized Kullback-Leibler divergence, as it can be obtained as $D_{KL}(p\|q) + D_{KL}(q\|p)$, where $D_{KL}(p\|q) = \sum_j p_j \log\left(\frac{p_j}{q_j}\right)$ is the Kullback-Leibler divergence.

We allow the disagreement functions to be asymmetric because disagreement between agent p and q need not be the same as disagreement between agent q and p. Symmetry is assumed for *metrics*, as metrics capture the distance between objects, an objective property of the geometry of the space. On the other hand, disagreement measures involve beliefs and thus are subjective evaluations. While we do not model the subjective process behind a definition of disagreement, we do allow for asymmetric disagreement measures. Furthermore, in the applications of Section 4, expected disagreement decreases in information only for asymmetric disagreement measures; and expected volume of trades are also captured by asymmetric disagreement measures. Namely, an agent p who trades with agent q might expect to receive a larger amount good than q expects to give her.

The next proposition illustrates the relation between the Likelihood Ratio (LR) order and our disagreement functions. Given an order $\preceq$ on Θ we say that q likelihood ratio dominates p (and write $p \leq_{LR} q$) if:

$$p(\theta)q(\theta') \leq p(\theta')q(\theta), \quad \forall \theta \preceq \theta'.$$

Recall that our state space Θ is not endowed with an order, so in the following theorem we write that $p \leq_{LR} q \leq_{LR} r$ meaning that *there exists an order* $\succeq$ *on* Θ *such that*:

$$p_i q_j \leq q_i p_j \quad \text{and} \quad q_i r_j \leq r_i q_j, \quad \forall i \succeq j.$$

Proposition 2. *Let* $p,q,r \in \Delta_n^\circ$ *be beliefs ranked by Likelihood Ratio:*

$$p <_{LR} q <_{LR} r,$$

then for any disagreement function we have that:

$$D(p,q) \leq D(p,r).$$

Therefore, our measures of disagreement are compatible with the LR order on beliefs, regardless of the underlying order on Θ. Since the likelihood ratio order is often related to First Order Stochastic Dominance (FOSD),[4] it is useful to notice that in general it is *not* true that $p <_{FOSD} q <_{FOSD} r$ implies $D(p,q) \leq D(p,r)$.

[3] A divergence metric is said to be additively separable in the states if $D(p,q) = \sum_j f_j(p_j,q_j)$, for some functions $f_j : [0,1] \times [0,1] \to \mathbb{R}$.

[4] We say that q first order stochastic dominates p if:

$$\sum_{i=1}^{j} q_i \leq \sum_{i=1}^{j} p_i, \quad \forall j,$$

In order to study the disagreement functions characterized in Theorem 1, let us rewrite them in order to avoid redundancy. If a function D satisfies Axioms 1–6, then so does αD, for any $\alpha > 0$, and these functions are ordinally equivalent. Let us define a representative in each of these classes of equivalence:

$$D^z(p,q) := \begin{cases} \log\left(\sum_j p_j \left(\frac{p_j}{q_j}\right)^{z-0.5}\right) & \text{if } |z| > 0.5, \\ -\log\left(\sum_j p_j \left(\frac{p_j}{q_j}\right)^{z-0.5}\right) & \text{if } |z| < 0.5. \end{cases} \tag{4}$$

Observation 1. *For all $z \neq \tilde{z}$, the disagreement functions D^z and $D^{\tilde{z}}$ are* not *ordinally equivalent, i.e. there exist $p,q,r \in \Delta_n$ such that:*

$$D^z(p,q) < D^z(p,r) \quad \text{and} \quad D^{\tilde{z}}(p,q) > D^{\tilde{z}}(p,r).$$

There is a natural graphical interpretation of the functions D^z: the argument of the logarithm can be written as an average of a function of the likelihood ratio,

$$\sum_j p_j \left(\frac{p_j}{q_j}\right)^{z-0.5} = \sum_j p_j \phi_z\left(\frac{q_j}{p_j}\right), \quad \text{where} \quad \phi_z(x) = x^{0.5-z}.\text{[5]} \tag{5}$$

Our disagreement functions can then be interpreted as average dispersion of the likelihood ratio between p and q, as Figure 1 shows.[6] This implies that the key statistic for our measures of disagreement is the vector of likelihood ratios $(p_1/q_1, \ldots, p_n/q_n)$, and this property sets our measures apart from other metrics on the space of beliefs (all norms, Euclidean distance, Total Variation, etc.). Figure 2 shows graphically the function ϕ_z, for $z \neq 0.5, -0.5$. Observe that if $|z| > 0.5$, $\phi_z(\cdot)$ is a convex function; while if $|z| < 0.5$, $\phi_z(\cdot)$ is concave. This explains the different sign in the definition of D^z, equation (4).

The following proposition summarizes some noteworthy properties of the functions $(D^z)_{z \in \mathbb{R}\setminus\{-0.5,0.5\}}$.

Proposition 3. *For all $p,q \in \Delta_n^\circ$ and for all $z \neq 0.5, -0.5$:*

1. *$D^z(p,q) = D^{-z}(q,p)$;*
2. *for all $z \in \mathbb{R}$, $z \neq 0.5$:*
$$\frac{1}{z-0.5} D^z(p,q) = \log(\|p(\theta)/q(\theta)\|^p_{z-0.5}),$$
where the w-norm of a function $f : \Theta \to \mathbb{R}^+$, is defined by $\|f(\theta)\|^p_w := (\sum_i |f(\theta_i)|^w p(\theta_i))^{1/w}$;

and write $p \leq_{FOSD} q$. It is well-known that LR implies FOSD, see e.g. Shaked and Shanthinkumar (2006).

[5] Observe that this average is computed with respect to p, but analogously we could have written it with respect to q, after rescaling the parameter z accordingly. Formally:

$$\sum_j p_j \phi_z(q_j/p_j) = \sum_j p_j \left(\frac{p_j}{q_j}\right)^{z-0.5} = \sum_j q_j \left(\frac{q_j}{p_j}\right)^{-z-0.5} = \sum_j q_j \phi_{-z+1}(p_j/q_j).$$

[6] Notice also that having defined ϕ_z on q_j/p_j, for all p,q, $\sum_j p_j \frac{q_j}{p_j} = 1$, so we *normalized* the mean of the likelihood ratios. Therefore if $\frac{q_j}{p_j}$ is a "spread" of $\frac{q'_j}{p_j}$, it will be a *mean preserving spread*.

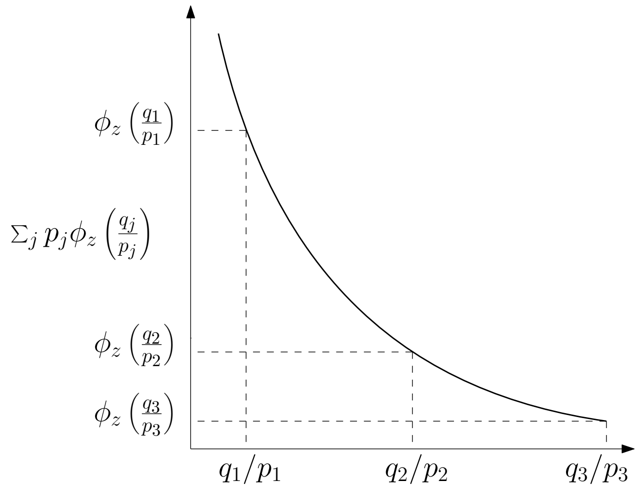
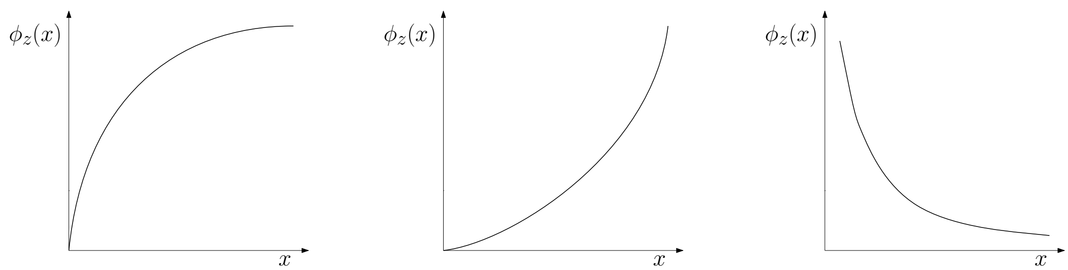


Figure 1: Graphical Interpretation of equation (5), for $z > 0.5$.



(a) $|z| < 0.5$, ϕ_z concave.

(b) $z < -0.5$, ϕ_z convex.

(c) $z > 0.5$, ϕ_z convex.

Figure 2: Plots of the functions ϕ_z, for different values of $z \in \mathbb{R} \setminus \{0.5, 0.5\}$.

3. *D^z can be extended by continuity at $z = 0.5$ and $z = -0.5$:*

$$\lim_{z \to 0.5} \frac{D^z(p,q)}{z - 0.5} = \sum_j p_j \log\left(\frac{p_j}{q_j}\right) =: D^{0.5}(p,q)$$

$$\lim_{z \to -0.5} \frac{D^z(p,q)}{-z - 0.5} = \sum_j q_j \log\left(\frac{q_j}{p_j}\right) =: D^{-0.5}(p,q).$$

The first point of the proposition highlights a symmetry in the family disagreement functions: for all z, the disagreement functions D^z is equivalent to D^{-z} after inverting p and q. The second point shows that D^z can be interpreted as the rescaled logarithm of a norm. This allows us to compare the distance between two beliefs as z changes, because $\|f\|_p^z \leq \|f\|_p^{z'}$ for all $z < z'$, and for all f. Finally, the third point shows that even though D^z are defined on $\mathbb{R} \setminus \{-0.5, 0.5\}$, they can be extended to the whole real line, once an appropriately rescaled limit is considered. Notice that $D^{0.5}$ corresponds to the Kullback-Leibler divergence, as $D^{0.5}(p,q) = D_{KL}(p\|q)$, whereas $D^{-0.5}(p,q) = D_{KL}(q\|p)$.

We stated Proposition 3 only for p, q in the interior of Δ_n. The following Lemma clarifies how the functions

D^z differ at the boundary of Δ_n.[7]

Lemma 1. *If* $|z| < 0.5$*, then* $D^z(p,q) = +\infty$ *if and only if* $Supp(p) \cap Supp(q) = \emptyset$*. If* $|z| \geq 0.5$ *then:*

- *if* $z \geq 0.5$ *then* $D^z(p,q) = +\infty$ *if and only if* $Supp(p) \setminus Supp(q) \neq \emptyset$*.*
- *if* $z \leq -0.5$ *then* $D^z(p,q) = +\infty$ *if and only if* $Supp(q) \setminus Supp(p) \neq \emptyset$*.*

This Lemma has some important implications regarding the continuity of D^z on $\Delta_n \times \Delta_n$. Recall that by assumption we have that all disagreement measures are smooth on $\Delta_n^\circ \times \Delta_n^\circ$, and hence continuous. This corollary shows when they are continuous also on the entire simplex Δ_n. For a fixed p, we denote by $D^z(p,\cdot) : \Delta_n \to \mathbb{R}$ the function $q \mapsto D^z(p,q)$.

Corollary 1.

1. *if* $z \geq 0.5$*,* $D^z(p,\cdot) : \Delta_n \to \mathbb{R}$ *is continuous for all fixed* $p \in \Delta_n$*. Furthermore* D^z *depends only on the states* $\theta \in Supp(p)$*:*
$$D^z(p,q) = \begin{cases} \log\left(\sum_{j \in Supp(p)} p_j \left(\frac{p_j}{q_j}\right)^{z-0.5}\right) & \text{if } z > 0.5 \\ \sum_{j \in Supp(p)} p_j \log\left(\frac{p_j}{q_j}\right) & \text{if } z = 0.5 \end{cases}.$$
On the other hand, $D^z(\cdot,q) : \Delta_n \to \mathbb{R}$ *is continuous if and only if* $q \in \Delta_n^\circ$*.*

2. *if* $z \leq -0.5$*,* $D^z(\cdot,q) : \Delta_n \to \mathbb{R}$ *is continuous for all fixed* $q \in \Delta_n$*. Furthermore* D^z *depends only on the states* $\theta \in Supp(q)$*:*
$$D^z(p,q) = \begin{cases} \log\left(\sum_{j \in Supp(q)} p_j \left(\frac{p_j}{q_j}\right)^{z-0.5}\right) & \text{if } z > 0.5 \\ \sum_{j \in Supp(q)} p_j \log\left(\frac{p_j}{q_j}\right) & \text{if } z = 0.5 \end{cases}.$$
On the other hand, $D^z(p,\cdot) : \Delta_n \to \mathbb{R}$ *is continuous if and only if* $p \in \Delta_n^\circ$*.*

3. *If* $|z| < 0.5$*, then* $D^z : \Delta_n \times \Delta_n \to \mathbb{R}$ *is continuous, so it is continuous on both variables separately, and:*
$$D^z(p,q) = -\log\left(\sum_{j \in Supp(p) \cap Supp(q)} p_j \left(\frac{p_j}{q_j}\right)^{z-0.5}\right),$$
with the convention that $\sum_{j \in \emptyset} = 0$*.*

Besides differing at the boundary of Δ_n, the disagreement functions D^z have very different properties also for $p, q \in \Delta_n^\circ$. The next proposition shows that for $|z| \geq 0.5$ the topology induced by D^z is very different from that induced by any norm on Δ_n. If $|z| < 0.5$, instead, the disagreement topology and the norm topology are equivalent. We state the proposition using the sup-norm on Δ_n, which is defined as $\|x\|_\infty := \sup_j |x_j|$. Since Δ_n is finite dimensional, the same result holds for any norm.[8]

Proposition 4. *Let* $|z| \geq 0.5$ *then for any* $\epsilon, \delta > 0$ *(arbitrary small) there exist* $\bar{p}, \bar{q}, \underline{p}, \underline{q} \in \Delta_n^\circ$ *such that:*

$$\|\bar{p} - \bar{q}\|_\infty > \max_{x,y \in \Delta_n} \|x - y\|_\infty - \delta, \qquad \|\underline{p} - \underline{q}\|_\infty < \epsilon,$$

[7] The boundary of Δ_n is defined by $\Delta_n \setminus \Delta_n^\circ$.
[8] All norms in finite dimensional spaces $\mathbb{R}^n$ are topologically equivalent, see Shores (2007).

and $D^z(\underline{p},\underline{q}) > D^z(\bar{p},\bar{q})$.

If $|z| < 0.5$*, instead,* $D^z(p,q)$ *is uniformly continuous with respect to the metric induced by* $\|\cdot\|_\infty$*. Namely, for all* ϵ *there exist a* δ *such that if* $\|p-q\|_\infty \le \delta$ *then* $D^z(p,q) < \epsilon$*.*

The first statement says that if we use disagreement measures D^z, with $|z| > 0.5$, we can find two pairs of beliefs such that the first pair is arbitrarily distant in the norm sense, and the second is arbitrarily close; and yet disagreement between the former is smaller than disagreement in the latter. On the other hand, if $|z| < 0.5$, the topology induced by D^z is equivalent to that induced by any norm.

The apparently counterintuitive property of measures D^z for $|z| > 0.5$ follows from the fact that our measures of disagreement are dispersions of the likelihood ratio (see equation (5)), unlike norms. To highlight this point consider the following example. Let $p^\epsilon = (1-\epsilon,\epsilon)$ and $q^\epsilon = (1-\epsilon^3,\epsilon^3)$. Plainly, as $\epsilon \to 0$ we have that $p^\epsilon, q^\epsilon \to (1,0)$ (in any norm), and so $\|p^\epsilon - q^\epsilon\| \to 0$. On the other hand:

$$D^z(p^\epsilon,q^\epsilon) = (1-\epsilon)\left(\frac{1-\epsilon}{1-\epsilon^3}\right)^z + \epsilon\left(\frac{\epsilon}{\epsilon^3}\right)^z \to +\infty, \quad \text{as } \epsilon \to 0,$$

for any $z > 0.5$. Therefore even though two beliefs converge to agreeing that the true distribution is $(1,0)$, their disagreement diverges. To rephrase this result in terms of volume of trade in financial markets, suppose that two agents with beliefs p^ϵ and q^ϵ trade an asset that pays 1 unit of good in state θ_2. Even as they converge to agreeing that state θ_2 will not be realized (i.e. that the asset is worthless), they will still be willing to trade the asset, and in fact the volume of trade will diverge, if one agent's speed of convergence is exponentially larger than the other's.

Proposition 4 might suggest that the disagreement functions D^z for $|z| < 0.5$ are preferable, in that they relate better to commonly used distances on $\mathbb{R}^n$. Nevertheless, the results we present in Section 4 will show that the case $|z| > 0.5$ satisfies other desirable properties of disagreement. For example, disagreement measures $D^z(p,q)$ with $z > 0.5$ decrease on average when agents observe the same piece of public information, and the ex-ante probability of a signal is taken to be p's (see Theorem 3).

2.3 Technical Literature Review

In this subsection, we will relate our disagreement measures to other divergence measures introduced in the information theory literature. Our measures overlap with several others in the literatures, so we cite the papers that introduced them and the ones that axiomatized them. The family that is most closely related to our disagreement measures is the Renyi's divergences. We defer a detailed description of Renyi's axiomatization and its differences from ours to Appendix B.

Name	Definition	Function D^z
Renyi's divergences	$R_\alpha(p,q)=\begin{cases}\frac{1}{\alpha-1}\log\left(\sum_j p_j\left(\frac{p_j}{q_j}\right)^{\alpha-1}\right) & \alpha\neq 1\\ \sum_j p_j\log\left(\frac{p_j}{q_j}\right) & \alpha=1\end{cases}$,	$R_\alpha(p,q)=\begin{cases}\frac{1}{\alpha-1}D^{\alpha-0.5}(p,q) & \text{if } \alpha\neq 0,1\\ D^{0.5}(p,q) & \text{if } \alpha=1\end{cases}$
Chernoff coefficients	$C_\beta(p,q)=\sum_j p_j^\beta q_j^{1-\beta}$, $\beta\in(0,1)$	$C_\beta(p,q)=\exp(-D^{\beta-0.5}(p,q))$
Kullback-Leibler divergence	$D_{KL}(p\|q)=\sum_j p_j\log\left(\frac{p_j}{q_j}\right)$	$D_{KL}(p\|q)=D^{0.5}(p,q)$
Symmetric divergence	$D_S(p,q)=\sum_j(p_j-q_j)\log\left(\frac{p_j}{q_j}\right)$	$D_S(p,q)=D^{0.5}(p,q)+D^{-0.5}(p,q)$
Bhattacharyya distance	$D_B(p,q)=-\log\left(\sum_j\sqrt{p_jq_j}\right)$	$D_B(p,q)=D^0(p,q)$

Renyi's divergences were introduced and axiomatized in Rényi (1961), as a generalization of Kullback-Leibler divergence. The Chernoff coefficients (Chernoff (1952)) were introduced as a bound for the asymptotic efficiency of a test, and axiomatized in Kannappan and Rathie (1972). The Kullback-Leibler divergence (Kullback and Leibler (1951)) was also obtained as a bound on the error when testing p versus q; it was axiomatized by several authors (Kullback and Khairat (1966), Campbell (1972), Kannappan and Rathie (1973)). The symmetric divergence was first studied by Jeffreys (1946) as a differential form invariant for all transformations of the distributions, and later axiomatized by Kannappan and Rathie (1988). The Bhattacharyya distance, Bhattacharyya (1946), was introduced to measure the divergence between multinomial samples, and it was never axiomatized (so far as we know). Csiszár (2008) reviews several different axiomatization and provides a guide through the different axioms assumed in the literature. Our paper differs from all the other axiomatizations both in terms of the axioms themselves, and in terms of the motivation.

Renyi's divergences differ from our disagreement measures in three ways: for $\alpha<0$ the Renyi divergences are negative (unlike our disagreement measures); Renyi's divergences do not include $D^{-0.5}(p,q)=\sum_j q_j\log\left(\frac{p_j}{q_j}\right)$; and Renyi's divergence do not include all the positive linear combinations of $D^{0.5}(p,q)$ and $D^{-0.5}(p,q)$. More importantly, the axioms of Rényi (1961) are very different from ours. First off, Renyi assumes that divergences satisfy a strengthening of our independence axiom, which implies that divergence of an event is the logarithm of the ratio. Furthermore, Renyi assumes that the divergence be a generalized mean, i.e. that there exists an increasing function g such that:

$$D(p,q)=g^{-1}\left(\sum_j p_j g(D(p_i,q_i))\right),$$

where $D(p_i,q_i)$ represents the divergence on state i. This assumption constraints the functional form of Renyi's divergences and therefore makes his axiomatization very different from ours. A more detailed comparison of the two approaches can be found in Appendix B.

3 Discussion of the Axioms

3.1 Zero Disagreement

Axiom 1 (Zero Disagreement)**.** *For all* Θ *and* $p, q \in \Delta(\Theta)$*:*

$$D_\Theta(p,q) = 0 \Leftrightarrow p = q.$$

Since we assume D to be weakly positive, this axiom says that agents with the same opinion have the minimum disagreement. Assuming that $D(p,q) > 0$ for $p \neq q$ amounts to *separate* different opinions.

3.2 Anonymity of the state space

Axiom 2 (Anonymity of the State Space)**.** *Consider two state spaces* Θ_1, Θ_2*. If* $\gamma : \Theta_1 \to \Theta_2$ *is a bijection, then for all* $p, q \in \Delta(\Theta_1)$*:*

$$D_{\Theta_2}(p \circ \gamma^{-1}, q \circ \gamma^{-1}) = D_{\Theta_1}(p,q),$$

where $p \circ \gamma^{-1}$ *is the distribution on* Θ_2 *defined by* $p \circ \gamma^{-1}(\theta_2) := p(\gamma^{-1}(\theta_2))$*.*

This axiom formalizes the idea that our disagreement functions are independent of the structure of the state space. In other words, our disagreement measures do not distinguish between any two state spaces with the same cardinality.

We impose this axiom because we are interested in disagreement functions that depend only on the relative probability that two agents assign to a particular state (without being concerned about what the state stands for, or how states are related). A lot of issues of economic relevance do involve a "structured" state spaces,[9] but we leave such analysis to a future project. The disagreement functions characterized in this paper can be used also to measure distance of opinions on spaces with a structure, but the structure of the space will not be captured by them. The next lemma formalizes the fact that any D_Θ satisfying Axiom 2 depends only on the cardinality of the state space, $n = |\Theta|$:

Lemma 2. *If a family of disagreement functions* D_Θ *for any state space* Θ *satisfies Axiom 2 then so does the family* $(\tilde{D}_n)_{n \geq 2, n \in \mathbb{N}}$*, where:*

$$\tilde{D}_n : \Delta(\Theta) \times \Delta(\Theta) \to \mathbb{R}^+ \cup \{+\infty\}, \qquad \tilde{D}_n(p,q) := D_\Theta(p,q),$$

for any Θ with $|\Theta| = n$.

Proof. All proofs of the results in this section are in Appendix A.2. □

Axiom 2 then allows to redefine the goal of the paper as defining a family of functions $(D_n)_{n \geq 2}$, where n is the cardinality of the state space. In the rest of the paper, we will drop the dependence of D_Θ on Θ.

[9] For example, disagreement over the price of a good tomorrow will depend on the labeling of the states.

3.3 In Betweenness

Suppose that there is a set $A \subset \Theta$ on whose probability two agents agree, $p(A) = q(A)$, and with $p(A) \in (0,1)$. Define $p(\cdot|A)$ as the conditional belief given A and $p(\cdot|A^c)$ the conditional belief given its complement. It is natural to assume that $D_n(p,q)$ be smaller than the maximum disagreement on the conditional beliefs (resp. $D_{|A|}(p(\cdot|A), q(\cdot|A))$ and $D_{|A^c|}(p(\cdot|A^c), q(\cdot|A^c))$). Formally:

$$D_n(p,q) \le \max\{D_{|A|}(p(\cdot|A), q(\cdot|A)), D_{|A^c|}(p(\cdot|A^c), q(\cdot|A^c))\}, \tag{6}$$

for all $A \subset \Theta$ with $p(A) = q(A) \in (0,1)$. This amounts to imposing that if we *agree* on the probability of a event A, and observing such event leads us to reduce our disagreement, then *not observing* A will have the effect of increasing disagreement.

Axiom 3 generalizes this property to any four beliefs p^1, p^2, q^1, q^2:[10]

Axiom 3 (In Betweenness)**.** *For all* $p^1, p^2, q^1, q^2 \in \Delta_n$*:*

$$D_n(\lambda p^1 + (1-\lambda)p^2, \lambda q^1 + (1-\lambda)q^2) \le \max\{D_n(p^1,q^1), D_n(p^2,q^2)\},$$

for all $\lambda \in [0,1]$.

Axiom 3 is equivalent to assuming $D_n : \Delta_n \times \Delta_n \to \mathbb{R} \cup \{+\infty\}$ quasi-convex. The next result shows that Axiom 3 implies three convexity properties of disagreement:

Lemma 3. *If D_n satisfies Axiom 3 and Axiom 1 then:*

1. *for all $p, q \in \Delta_n$ and for all $\lambda \in [0,1]$:*

$$D_n(p, \lambda p + (1-\lambda)q) \le D_n(p,q).$$

2. *For all $p, q, r \in \Delta_n$ and $\lambda \in [0,1]$:*

$$D_n(\lambda p + (1-\lambda)r, \lambda q + (1-\lambda)r) \le D_n(p,q).$$

3. *For all p, q^1, q^2 and $\lambda \in [0,1]$:*

$$D_n(p, \lambda q^1 + (1-\lambda)q^2) \le \max\{D_n(p,q^1), D_n(p,q^2)\}.$$

Also, this last property is equivalent to assuming that the balls:

$$B(p,\rho] := \{q \in \Delta_n \mid D_n(p,q) \le \rho\}$$

are convex for all D_n.

[10] Another noteworthy particular case is that in which instead of considering an event A, we consider a public signal s. Then, defining $p^1 = p(\cdot|s)$, $p^2 = p(\cdot|s^c)$, $q^1 = q(\cdot|s)$, $q^2 = q(\cdot|s^c)$, and $\lambda = \mathbb{P}_p(s) = \mathbb{P}_p(s)$, we find that if observing a public signal reduces disagreement ($D(p(s), q(s)) < D(p,q)$), then *not* observing it must increase it: $D(p,q) \le D(p(s^c), q(s^c))$ (if we agree on the ex-ante probability of the signal, $\mathbb{P}_p(s) = \mathbb{P}_p(s)$).

As a corollary we have that the maximum disagreement between two agents in any convex set $C \subset \Delta_n$ must be reached at the boundary of C.

Corollary 2. *For every $C \subset \Delta_n$ convex:*

$$\sup_{p,q \in C} D_n(p,q) = D_n(\bar{p},\bar{q}),$$

for some $\bar{p},\bar{q} \in \partial C$. This implies that for all $p,q \in \Delta_n$ we have that:

$$D_n(p,q) \leq D_n(e^1,e^2) \quad (= D_n(e^j,e^i) \quad \forall\, i,j),$$

where $e^1 = (1,0,0,\ldots,0)$ and $e^2 = (0,1,0,\ldots,0)$.

3.4 Coarsening

Consider a state space of cardinality n, and two beliefs $p,q \in \Delta_n$. If we merge states θ_1,θ_2 into a unique event $\{\theta_1,\theta_2\}$, we obtain two beliefs $p',q' \in \Delta_{n-1}$ defined by:

$$p \to p' = (p_1+p_2,p_3,\ldots,p_n) \quad \text{and} \quad q \to (q_1+q_2,q_3,\ldots,q_n).$$

This transformation has the effect of *eliminating* any disagreement on the states θ_1,θ_2 so we will impose that disagreement cannot increase if we coarsen the state space.

Axiom 4 (Coarsening)**.** *For all $p,q \in \Delta_n$:*

$$D_{n-1}((p_1+p_2,\ldots,p_n),(q_1+q_2,\ldots,q_n)) \leq D_n((p_1,p_2,\ldots,p_n),(q_1,q_2,\ldots,q_n)).$$

By induction, Axiom 4 implies that for any partition[11] $\mathscr{A} = (A_j)_j$ of Θ we have that:

$$D_{|\mathscr{A}|}(p_{\mathscr{A}},q_{\mathscr{A}}) \leq D_n(p,q),$$

where $p_{\mathscr{A}} \in \Delta(\mathscr{A})$, and $p_{\mathscr{A}}(A_j) := \sum_{\theta \in A_j} p(\theta_j)$, $\forall A_j \in \mathscr{A}$.

Notice that Axiom 4 allows to bound disagreement on a state space of cardinality n with disagreement on a state space of lower cardinality. This sets it apart from Axioms 1–3 which instead involved only D_n for a fixed n. The next axiom, Axiom 5, also allows us compare disagreement in different dimensions.

[11] A partition is a set $(A_j)_j$ of subsets of Θ such that:

$$A_j \cap A_i = \emptyset, \qquad \bigcup_j A_j = \Theta.$$

3.5 Independence

Consider any two state spaces Θ_1, Θ_2 with $n = |\Theta_1|$ and $m = |\Theta_2|$ and the corresponding simplexes Δ_n, Δ_m. For any $p^{(1)} \in \Delta_n$ and $q^{(1)} \in \Delta_m$ we can define $r^{(1)} := p^{(1)} * q^{(1)} \in \Delta(\Theta_1 \times \Theta_2)$ by:

$$r^{(1)}(\theta_1, \theta_2) := p^{(1)}(\theta_1) q^{(1)}(\theta_2).$$

In words, $r^{(1)}$ is defined as the joint distribution with independent marginals $p^{(1)}$ and $q^{(1)}$. Whenever two agents both believe that issues on state spaces Θ_1 and Θ_2 are independent, we will assume that the disagreement can be summed across state spaces. Formally:

Axiom 5 (Independence)**.** *For all $n, m \in \mathbb{N}$, and all $p^{(1)}, p^{(2)} \in \Delta_n$ and $q^{(1)}, q^{(2)} \in \Delta_m$:*

$$D_{nm}(p^{(1)} * q^{(1)}, p^{(2)} * q^{(2)}) = D_n(p^{(1)}, p^{(2)}) + D_m(q^{(1)}, q^{(2)}).$$

Axiom 5 is equivalent to assuming that for all $p^1, q^1, p^2, q^2 \in \Delta_n$:

$$D_{nm}(p^2 * r, q^2 * s) - D_{nm}(p^1 * r, q^1 * s) = D_n(p^2, q^2) - D_n(p^1, q^1) \quad \forall r, s \in \Delta_m. \tag{7}$$

Let us illustrate why equation (7) is a desirable property of a disagreement function. Suppose an experimenter wants to measure the belief of two agents on Global warming, and takes a survey in two consecutive years, year 1 and 2. Let p^1 and p^2 (resp. q^1 and q^2) be the *beliefs on global warming* of agent P (resp. Q) in the two years. The experimenter's goal is to measure $D(p^2, q^2) - D(p^1, q^1)$, but other beliefs will be implicitly measured when a survey is taken. Say agent P and Q have different beliefs on the meaning of a word, and denote these beliefs by r and s respectively. Then the experimenter effectively measures (a statistic of) $p^1 * r$ and $q^1 * s$ in the first year, and $p^2 * r$ and $q^2 * s$ in the second year. Axiom 5 implies that if the experimenter can measure r and s, too, such disagreement on the meaning of words will not affect the change in beliefs from year 1 to 2.

Notice how Axiom 5 can be broken down into two properties: first, it requires disagreement to be separable in *independent issues*; second, it imposes an additive structure across independent state spaces. The separability is quite natural: if both agent believe the issues to be independent it is conceivable to assume away *complementarities* between state spaces. The *additive structure* instead sets it apart from the previous axioms, as Axioms 1, 2, 3, 4 are all ordinal.[12]

As it has been shown in Observation 1, the disagreement measures we identify are *ordinally* different. We show here that Axiom 5 can be relaxed to an ordinal axiom that yields a similar representation theorem.

Property 1 (Ordinal Independence)**.** *We say that D satisfies Ordinal Independence if both of the following conditions are satisfied:*

1. *Let $\underline{p}, \underline{q} \in \Delta_n$, $\bar{p}, \bar{q} \in \Delta_m$. If $D_n(\underline{p}, \underline{q}) \leq D_m(\bar{p}, \bar{q})$ then:*

$$D_{n \cdot n'}(\underline{p} * r, \underline{q} * s) \leq D_{m \cdot n'}(\bar{p} * r, \bar{q} * s),$$

[12] We say that an axiom is ordinal if whenever D satisfies it, so does any increasing transformation of D.

for all $r, s \in \Delta_{n'}$.

2. *Let* $p, q \in \Delta_n$ *and* $r \in \Delta_m$. *We assume that:*

$$D_n(p,q) = D_{n\cdot m}(p * r, q * r).$$

It is plain to see that a disagreement function satisfying Axiom 5 also satisfies Property 1. Besides being an ordinal property, Ordinal Independence also differs from Axiom 5 in that it is satisfied by constant disagreement functions. We will rule out such disagreement function by assuming that D is not locally constant:

Definition 1. We say that D is *not* locally constant if for all $p, q \in \Delta_n^\circ$ and for all U_q neighborhoods of q and U_p neighborhood of p there exists $p' \in U_p$ and $q' \in U_q$ such that:

$$D(p',q) \neq D(p,q) \neq D(p,q').$$

The following proposition shows that if a disagreement function satisfies Property 1 and Definition 1, then it is a strictly increasing transformation of a function that satisfies Axiom 5.

Proposition 5. *Let D be a smooth measure of disagreement satisfying Axioms 1–4. If D satisfies Property 1 and is not locally constant, there exists a strictly increasing function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that the disagreement function* $\tilde{D} := \phi(D)$ *satisfies Axiom 5.*

Therefore, assuming Axiom 5 instead of Property 1 amounts to choosing a convenient "cardinalization" of the orders represented by the disagreement functions of Theorem 1. For this reason, we will assume Axiom 5 instead of the weaker ordinal version. Let us derive the consequences of Axiom 5.

Lemma 4. *If* D_n *satisfies Axioms 1–5, then* $\forall p, q \in \Delta_n$*:*

$$D_n(p,q) = D_{n+1}((p_1,\ldots,p_n,0),(q_1,\ldots,q_n,0)).$$

This lemma implies that including states that both agents dim impossible has no effect on disagreement. A consequence of this Lemma (and Axioms 1–5) is that if two agents' beliefs have disjoint support, then their disagreement must be infinite.

Lemma 5. *Let* $p, q \in \Delta_n$ *and suppose that* $Supp(p) \cap Supp(q) = \emptyset$*, meaning* $p_j q_j = 0$*, for all* j*. Then if D satisfies Axioms 1–5,* $D(p,q) = +\infty$.

The next result shows how the statistic of interest when measuring disagreement on a state θ_j is given by the likelihood ratio. This is a key result in our characterization and it will simplify significantly the proof of Theorem 1. We break down the result in a proposition and a corollary. Proposition 6 shows that if the likelihood ratio on two states is the same, then the two states are undistinguishable in terms of disagreement; Corollary 3 states a direct consequence of the proposition.

Proposition 6. *Let* $p, q \in \Delta_n^\circ$ *be two beliefs, and suppose that* $\frac{p_1}{q_1} = \frac{p_2}{q_2}$*. Then:*

$$D_n(p,q) = D_{n-1}((p_1+p_2, p_3,\ldots,p_n),(q_1+q_2,q_3,\ldots,q_n)).$$

Corollary 3. *Fix a belief* $p \in \Delta_n^\circ$ *and consider the segment joining* $q^1 = (q_1 + q_2, 0, q_3, \dots, q_n)$ *and* $q^2 = (0, q_1 + q_2, q_3, \dots, q_n)$*:*

$$[q^1, q^2] = \{r \in \Delta_n \mid r = \lambda q^1 + (1-\lambda) q^2, \lambda \in [0,1]\}.$$

The minimum of the distance between p *and* $[q^1, q^2]$ *is reached at the* $q^* \in [q^1, q^2]$ *satisfying:*

$$\frac{q_1^*}{q_2^*} = \frac{p_1}{p_2},$$

And for all $r, r' \in [q^1, q^2]$ *we have that if* $r \in [r', q^*]$ *then*

$$D_n(p, r) \leq D_n(p, r').$$

Figure 3 shows graphically the result of Corollary 3. Given any two vectors p, q, the dashed line passing by q is the segment $[q^1, q^2]$ and the dashed line passing by p that represents the set of beliefs r such that $\frac{r_1}{r_2} = \frac{p_1}{p_2}$. The point at which they meet, q^*, represents the point on the segment with minimal distance to p, and for this reason we drew the ball of radius $D(p, q^*)$ around p tangent to $[q^1, q^2]$.

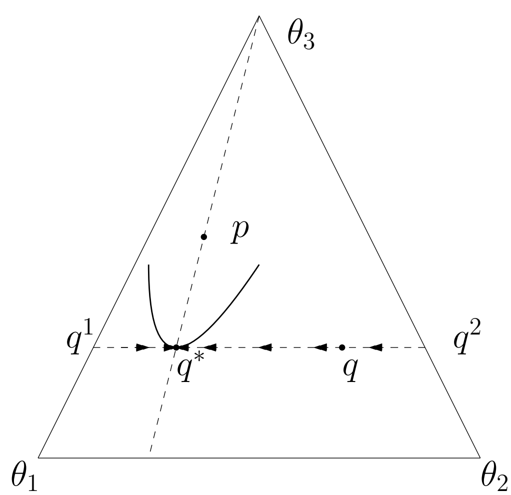


Figure 3: Graphical interpretation of Corollary 3 for $n = 3$. The curve represents the ball of radius $D(p, q^*)$ centered in p. The arrows imply that disagreement is decreasing as a belief approaches q^* on the segment $[q^1, q^2]$.

3.6 Marginal Rate of Substitution of Disagreement

This axiom constraints the local complementarities among states, by imposing a condition on the Marginal Rate of Substitution (MRS) of disagreement. In order to define a local condition on the disagreement functions, we define the derivatives along the simplex for $p, q \in \Delta_n^\circ$:

$$\partial_i D_n(p, q) := \lim_{\epsilon \to 0} \frac{D_n(p, (q_1, \dots, q_i - \epsilon, q_{i+1} + \epsilon, \dots, q_n)) - D_n(p, q)}{\epsilon} \quad \forall i = 1, \dots, n-1.$$

$\partial_i D_n(p,q)$ is the marginal change in D_n as q changes by reducing the likelihood of state i and increasing that on state $i+1$. For all (p,q) the derivatives $(\partial_i D_n(p,q))_{i=1}^{n-1}$ are a base for the differential of the function $D_n(p,\cdot): \Delta_n^\circ \to \mathbb{R}^+$, $q \mapsto D_n(p,q)$. In other words, the derivative in any two directions i,j,

$$\lim_{\epsilon \to 0} \frac{D_n(p,(q_1,\dots,q_i-\epsilon,\dots,q_j+\epsilon,\dots,q_n)) - D_n(p,q)}{\epsilon}$$

is a linear combinations of the derivatives $(\partial_i D(p,q))_{i=1}^{n-1}$. Our last axiom constraints the relative change in disagreement when beliefs change on states $i, i+1, j, j+1$:

Axiom 6 (MRS of Disagreement)**.** *Consider any pair of states $i,j \in \{1,\dots,n-1\}$. If $\partial_j D_n(p,q) \neq 0$, then:*

$$\frac{\partial_i D_n(p,q)}{\partial_j D_n(p,q)} = g(p_i, p_{i+1}, p_j, p_{j+1}; q_i, q_{i+1}, q_j, q_{j+1}), \tag{8}$$

that is, such ratio does not depend on the belief on any other state.

Axiom 6 means that the *relative change* in disagreement when beliefs change on states i and $i+1$ versus when they change on states j and $j+1$ locally depends only on the beliefs on those states. We interpret this axiom as limiting the complementarity among states. A stronger version of Axiom 6 is given by the following property:

Property 2 (Separability)**.** *For all n, and $\forall\, 1 \le j \le n-1$:*

- *if for some $\bar{p}, \bar{q}$:*

$$D((p_1,\dots,p_j,\bar{p}_{j+1},\dots,\bar{p}_n),(q_1,\dots,q_j,\bar{q}_{j+1},\dots,\bar{q}_n)) \\ \ge D((p'_1,\dots,p'_j,\bar{p}_{j+1},\dots,\bar{p}_n),(q'_1,\dots,q'_j,\bar{q}_{j+1},\dots,\bar{q}_n)),$$

- *then:*

$$D((p_1,\dots,p_j,\tilde{p}_{j+1},\dots,\tilde{p}_n),(q_1,\dots,q_j,\tilde{q}_{j+1},\dots,\tilde{q}_n)) \\ \ge D((p'_1,\dots,p'_j,\tilde{p}_{j+1},\dots,\tilde{p}_n),(q'_1,\dots,q'_j,\tilde{q}_{j+1},\dots,\tilde{q}_n)),$$

  for all $\tilde{p}, \tilde{q}$.

Separability implies Axiom 6. As a matter of fact, if for some $p,q \in \Delta_n$, we consider the function h implicitly defined by:

$$D_n(p,q) = D_n(p,(\dots,q_i+\epsilon, q_i-\epsilon,\dots,q_j+h(\epsilon), q_{j+1}-h(\epsilon))),$$

we have that Property 2 implies that h does not depend on states other than $i, i+1, j, j+1$. On the other hand, Axiom 6 only implies that $-\frac{\partial_j D(p,q)}{\partial_i D(p,q)} = h'(0)$ depends only on beliefs on $i, i+1, j, j+1$. In this sense, Axiom 6 can be interpreted as the local version of Property 2. As it follows from our representation theorem (Theorem 1), all the disagreement measures satisfy Property 2, so we could have assumed such property instead of Axiom 6 without affecting our main result.

4 Applications

In this section we analyze two applications of our measures of disagreement. Whenever referring to disagreement measures or functions, we mean functions satisfying Axioms 1–6.

The first application shows how disagreement among Bayesian agents changes when they observe a piece of public information. We first show that there exists, generically, a signal realization that increases disagreement. On the other hand, we show in Theorem 2 that averaging on signal realizations, a perfectly informed agent expects to disagree less with any other agent after observing public information (for any disagreement measure). Finally, we compare the expected disagreement between any two agents, finding that it is decreasing in the informativeness of the experiment if and only if the disagreement measure satisfies a continuity axiom, Axiom 7.

In the second application, we consider a model of trade of contingent assets. We show that if agents have Constant Relative Risk Aversion (CRRA) utility functions (with coefficient $1/z_j$) then: i) the equilibrium price of the contingent assets $(\Pi_i)_i$ is the belief that minimizes a weighted sum of disagreement, as measured by $D^{z_j+0.5}(\cdot,\cdot)$; ii) the volume of trade expected by an agent with beliefs p^j is proportional to $D^{z_j+0.5}(p^j,\Pi)+D^{0.5-z_j}(\Pi,p^j)$; iii) the expected gains from trade are an increasing function of $D^{z_j-0.5}(\Pi,p^j)$. We then show that similar results hold for any utility function, if disagreement is small.

4.1 Disagreement and Public Information

Rational agents update their beliefs upon observing signals that are correlated with the state of the world. Formally, let $\pi=(S,f(s|\theta))$ be an experiment given by a set of signals S (finite, for simplicity) and a family of conditional distributions $f(\cdot|\theta)\in\Delta(S)$. For all $\theta\in\Theta$, $f(s|\theta)$ represents the probability of observing signal s when the state of the world is θ. We assume that agents update their beliefs using Bayes rule, therefore, indicating by $p(s)=(p_1(s),\dots,p_n(s))$ the posterior beliefs, we have that:

$$p(\theta_j|s)=p_j(s)=\frac{f(s|\theta_j)p_j}{\sum_i f(s|\theta_i)p_i},$$

where the denominator represents the *ex-ante* probability of observing signal s, which we also denote by: $\mathbb{P}_p(s):=\sum_i f(s|\theta_i)p_i$.

The first proposition we state shows that in general there exist signal realizations that increase or decrease disagreement:[13]

Proposition 7. *For any $p,q\in\Delta_n^\circ$ with $p\neq q$ and for all D measures of disagreement there exists an experiment $\pi=(S,f(s|\theta))$ and a signal $s'\in S$ such that:*

$$D(p(s'),q(s'))<D(p,q).$$

[13]This result holds for a large class of distances on the space of beliefs, not only of our measures of disagreement. On the other hand, the positive result of this subsection (in particular, Theorem 2 and Theorem 3) do not generically hold for other metrics on the space of beliefs (norms, Euclidean distance, etc.).

For any D measure of disagreement, there exist $p,q \in \Delta_n$, an experiment $\pi = (S, f(s|\theta))$, and a signal $s \in S$ such that:

$$D(p(s), q(s)) > D(p,q).$$

This proposition involves single signal realizations, and does not take into account the fact that some signals are ex-ante more likely than others. In the rest of this subsection, we will analyze expected disagreement, that is: we will weight the disagreement in the posteriors by the ex-ante probability of the signal. Such ex-ante probability is defined only in terms of a certain belief on the state of the world, and we will take this belief to be p, the first of the two agents whose disagreement we are analyzing. Formally, for any experiment π, expected disagreement is defined as:

$$\mathbb{E}_p^{\pi}(D(p(s), q(s))) := \sum_s \mathbb{P}_p(s) D(p(s), q(s)).$$

Instead of comparing the expected posterior disagreement $\mathbb{E}_p^{\pi}(D(p(s), q(s)))$ with the disagreement in the priors $D(p,q)$, we will compare the expected posterior disagreement after observing two experiments ranked by Blackwell's notion of sufficiency. We will show under which conditions a more informative experiment induces lower expected disagreement, i.e. what assumptions imply that:

$$\mathbb{E}_p^{\pi}[D(p(s), q(s))] \leq \mathbb{E}_p^{\tilde{\pi}}[D(p(\tilde{s}), q(\tilde{s}))],$$

for all pairs of experiments such that π is sufficient for $\tilde{\pi}$. Statistical sufficiency is defined as follows:

Definition 2 (Sufficiency)**.** Let Θ be a state space, and $\pi = (S, f(s|\theta))$ and $\tilde{\pi} = (\tilde{S}, g(\tilde{s}|\theta))$ be two experiments. We say that π is sufficient for $\tilde{\pi}$ and write $\tilde{\pi} \preceq \pi$ if:

$$g(\tilde{s}|\theta) = \sum_s \lambda_{s,\tilde{s}} f(s|\theta),$$

for some set of positive $(\lambda_{s,\tilde{s}})_{s,\tilde{s}}$ such that $\sum_{\tilde{s}} \lambda_{s,\tilde{s}} = 1$.

In words, an experiment π is sufficient for $\tilde{\pi}$ if $\tilde{\pi}$ can be obtained by *garbling* the experiment π. For example, the letter grade of an exam is a garbling of the percentage grade, because the former can be obtained by adding noise to the latter.

In the next theorem, we will consider p to be the belief of a *perfectly informed agent*, an agent with degenerate belief on a state of the world θ, that we interpret as the *true state of the world*. The following theorem states that more information will make q disagree less with the correct state of the world. This result holds for all the measures of disagreement D.

Theorem 2. *Let p be a degenerate distribution on the true state of the world, and let $q \in \Delta_n$. Then for all measures of disagreement, and for all $\tilde{\pi} \preceq \pi$ we have that:*

$$\mathbb{E}_p^{\pi}[D(p(s), q(s))] \leq \mathbb{E}_p^{\tilde{\pi}}[D(p(\tilde{s}), q(\tilde{s}))].$$

This result says that, conditional on the true state of the world (i.e. belief p), more information will get any agent closer to the true state of the world, when we measure the distance via *any* of our disagreement measures. Notice how this result is *cardinal*, so Axiom 5, our only cardinal axiom, plays a key role in the proof of Theorem 2 (and similarly for the other results of this subsection). Francetich and Kreps (2014) analyze a property analogous to the result of Theorem 2, namely, they ask whether "Bayesian inference can lead us astray". They find a similar negative result, i.e. on average Bayesian inference lead us closer to the truth. Differently from us, though, they do not consider different measures of disagreement, and they do not compare different experiments.

If instead we consider any two agents with beliefs $p, q \in \Delta_n$, it is in general not true that more information decreases disagreement *for any measure of disagreement*, i.e. it will not be the case that:

$$\mathbb{E}_p^{\pi}[D(p(s), q(s))] \leq \mathbb{E}_p^{\tilde{\pi}}[D(p(\tilde{s}), q(\tilde{s}))], \tag{9}$$

for all D. A necessary and sufficient condition for this to happen is given by the following axiom:[14]

Axiom 7 (Absolute Continuity). *We say that D is absolutely continuous in the second variable if whenever $Supp(p) \setminus Supp(q) \neq \emptyset$, $D(p,q) = +\infty$.*

Let us illustrate with an example why Axiom 7 is necessary for information to decrease disagreement (for all p, q).

Example 1. *Suppose Axiom 7 is violated: let $p, q \in \Delta_2$ with $p = (p_1, 1-p_1)$, $q = (0,1)$, and $D(p,q) < \infty$. Consider a sequence of beliefs $q^n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ and notice that, for large n, $D(p, q^n) < D(p,q) < \infty$, because $q^n \in [q, p]$. Therefore $D(p, q^n)$ is finite and bounded uniformly in n. We can construct an experiment with signal s_0 such that $p(s_0)$ is arbitrary close to $(1,0)$ and such that $q^n(s_0) \approx (0,1)$, for n big enough. In other words, for all ϵ, we can pick an experiment and a prior q^n such that:*

$$D(p(s_0), q^n(s_0)) > D((1-\epsilon, \epsilon), (\epsilon, 1-\epsilon)).$$ [15]

It is easy to see that for all our measures of disagreement $D((1-\epsilon,\epsilon),(\epsilon,1-\epsilon)) \to \infty$, as $\epsilon \to 0$, and therefore we get that in this limit $\mathbb{E}_p^{\pi}(D(p(s), q^n(s))) \to +\infty$ while $D(p, q^n)$ is bounded, and hence:

$$D(p, q^n) < \mathbb{E}_p^{\pi}(D(p(s), q^n(s))).$$

The problem with disagreement functions that do not satisfy Axiom 7 is that, whenever $Supp(p) \setminus Supp(q) \neq \emptyset$, the disagreement between p and q can be made arbitrarily large by using the fact that $q(\theta|s) = 0$ for all $\theta \in Supp(p) \setminus Supp(q)$. Because of Bayesian updating, an agent who assigns zero probability to a state θ will assign zero probability after any signal. Therefore, unless $D(p,q) = \infty$, it is possible to find two priors and and an experiment such that $D(p,q) < \mathbb{E}_p^{\pi}[D(p(s), q(s))]$.

The next theorem, Theorem 3, shows that Axiom 7 is necessary and sufficient for more information to decrease expected disagreement. The proof uses the following characterization of the measures of disagreement that satisfy Axiom 7:

[14] Notice that this axiom is asymmetric in p, q (unlike all the previous ones) and this is related to the fact that equation (9) is also asymmetric in p and q.

[15] Choosing the right n and the right experiment precision requires some fine tuning, for all fixed ϵ. We leave these details to the proof of Theorem 3, in Appendix A.3.

Lemma 6. *D satisfies Axioms 1–7 if and only if $D(p,q) = aD^z(p,q)$ for some $z \geq 0.5$, and $a > 0$.*

Theorem 3. *Let D be a measure of disagreement. Then the following statements are equivalent:*

1. *D satisfies Axiom 7;*
2. *$D(p,q) = aD^z(p,q)$ for some $z \geq 0.5$ (and $a > 0$);*
3. *for all experiments $\pi \succeq \tilde{\pi}$ and priors $p, q \in \Delta_n$:*

$$\mathbb{E}_p^{\pi}[D(p(s), q(s))] \leq \mathbb{E}_p^{\tilde{\pi}}[D(p(\tilde{s}), q(\tilde{s}))].$$

4.2 Disagreement and Trade of contingent Assets

Consider an economy with incomplete information on a state space Θ (with $|\Theta| = n < +\infty$). Let J be a finite set of agents,[16] and with an abuse of notation suppose that $J \in \mathbb{N}$ denotes also the number of agents. Let there be one commodity $x \in \mathbb{R}^+$. Agent $j \in J$ has beliefs $p^j \in \Delta^\circ(\Theta)$ and von Neumann-Morgerstern (vNM) utility function $u_j : \mathbb{R}^+ \to \mathbb{R}$. For simplicity, we assume that for all j, u_j is strictly increasing, strictly concave and satisfies the Inada conditions $\lim_{x \to 0} u'(x) = \infty$, $\lim_{x \to \infty} u'(x) = 0$. These assumptions will imply that the equilibrium of our model exists, is unique, and is pinned down by the first order conditions.[17] The ex-ante utility of agent j is given by:

$$U_j(\mathbf{x}) := \mathbb{E}_{p^j}(u_j) := \sum_i p_i^j u_j(x_i^j),$$

where x_i^j is the amount of commodity consumed in state i, and we denote by $\mathbf{x} = (x_1^j, \ldots, x_n^j)$ the vector of contingent commodity to be consumed in states $\theta_1, \ldots, \theta_n$.

We assume that there is a market for Arrow Debreu (AD) securities for states $i \in I \subseteq \Theta$. I.e. there is a set of risky assets that pay 1 unit of the commodity x in state i and 0 in all the other states (for all $i \in I$). Let $I \geq 2$ be also the cardinality of the set I. The case of $I = \Theta$ corresponds to a complete market in which agents can insure themselves against any contingency. The prices of the Arrow Debreu securities will be denoted by $(\Pi_i)_{i \in I}$, and we normalize this vector to have $\sum_{i \in I} \Pi_i = 1$, as we will later interpret such price vector as the "market belief".

We assume that each agent is endowed with a unit of commodity in each state, and can trade it for units of commodity in other states, at prices Π_i. Equal endowments across agents and states implies that this is a purely speculative economy: agents trade if and only if they disagree with the market. Formally, an equilibrium of this economy is defined as follows:

Definition 3 (Equilibrium)**.** An equilibrium is a price vector $(\Pi_i)_{i \in I}$ and a set of allocations $(x_i^j)_{i \in I}^{j \in J}$ such that:

- for all $j \in J$ the vector $(x_i^j)_{i \in I}$ solves:

$$\max_{\mathbf{x} \in \mathbb{R}^n} \quad \sum_i p_i^j u_j(x_i^j) \qquad \text{s.t.} \sum_{i \in I} \Pi_i x_i \leq 1, \tag{10}$$

[16] Studying a model with countably many agents, or a continuum of them, would yield analogous results.

[17] In particular, notice that strict concavity implies that for all set of beliefs $(p^j)_j$ with $p^j \in \Delta^\circ(\Theta)$ the equilibrium amount of trade will be finite. This makes sure that infinite bets such as those described in Eliaz and Spiegler (2007) are never an equilibrium.

- market clears, i.e. for all states i:

$$\sum_{j \in J} x_i^j = J.$$

As noted by Sebenius and Geanakoplos (1983), if two agents with common prior and asymmetric information agree to trade, that information should induce both parties to update their beliefs on the state of the world. In particular, if agents have common knowledge of the information partitions (i.e. they agree to disagree, see Aumann (1976)) then discussion between the agents will reveal enough information to make the trade unappealing. In our model, we abstract from these considerations, as we do not model the source of heterogeneity in beliefs.

The quantities we will analyze are the expected volume of trade and gains from trade. The expected volume of trade of agent j is given by:

$$V_j(\mathbf{x}) := \sum_i p_i^j (x_i^j - 1),$$

i.e. the net amount of commodity that agent j expects to receive after uncertainty is resolved.[18] The gains from trade are defined as:

$$G_j(\mathbf{x}) := U_j(\mathbf{x}) - U_j(\mathbf{1}),$$

i.e. the difference of expected utility between the equilibrium of the model and the no-trade outcome, $\mathbf{1} = (1, \ldots, 1)$.

It is plain to see (from the first order conditions) that amount of asset x_i^j traded in state i is and increasing function of the likelihood ratio p_i^j/Π_i. Therefore, in order to measure the equilibrium vector $\mathbf{x}$ we expect to find a function of the vectors of likelihood ratios $(p_1^j/\Pi_i, \ldots, p_n^j/\Pi_n)$. This implies that norms, or metrics that are not based on likelihood ratios, are not a good proxy for the amount of trade. In other words, it is generically not the case that $G_j(\mathbf{x})$ is increasing in $\|p - \Pi\|$. The next subsection shows that, on the other hand, $G_j(\mathbf{x})$ is increasing in an appropriate function of disagreement $D(p, \Pi)$ whenever an agent has constant relative risk aversion (CRRA).

4.2.1 CRRA utility functions

Suppose that the utilities $(u_j)_{j \in J}$ exhibit Constant Relative Risk Aversion, i.e. utility functions parametrized by:

$$u_j(x) = \frac{x^{1-\frac{1}{z_j}}}{1 - \frac{1}{z_j}} \qquad z_j > 0, \quad x > 0.$$

The risk tolerance of an agent with utility u_j is $-\frac{u_j'(x)}{u_j''(x)} = z_j x$, and empirical estimation of CRRA utility functions typically yield $z_j \in (0,1)$.[19] For this reason, the next theorem will impose $z_j \in (0,1)$ and we defer to Remark 2 a discussion of the differences with the case $z_j > 1$.

[18] For each state i, agent j is endowed with 1 unit of commodity in state i and in equilibrium she will consume x_i^j, therefore the net amount of trade in state i is $x_i^j - 1$.

[19] See Table 1 in Neilson and Winter (2002). The values they find are positive and smaller than 1, with the exception of Hansen and Singleton (1983) who find $-\frac{xu''(x)}{u'(x)} = \frac{1}{z_j} \in [0.07, 0.62]$)).

Theorem 4. *Let $p^1, \dots, p^J \in \Delta_n^\circ$, and let $(u_j)_j$ be CRRA utility functions with parameters $z_j \in (0,1)$. The equilibrium of the economy exists and is unique, and it can be characterized as follows:*

- $(\Pi_i)_i$ *is the unique solution to the problem:*

$$\min_{q \in \Delta(I)} \sum_j D^{z_j+0.5}(q, p^j(\cdot|I)), \tag{11}$$

  i.e. the prices of the Arrow Debreu securities are the beliefs that minimize weighted disagreement with the agents;

- *the expected volume of trade is given by:*

$$V_j((x_i^j)_i) = \exp(D^{z_j+0.5}(p^j(\cdot|I), \Pi) + D^{-z_j+0.5}(\Pi, p^j(\cdot|I))) - 1 \quad \forall j \in J; \tag{12}$$

- *the gains from trade are given by:*

$$G_j((x_i^j)_i) = \frac{1}{\frac{1}{z_j} - 1}\left(1 - \exp\left(-\frac{D^{z_j-0.5}(\Pi, p^j(\cdot|I))}{z_j}\right)\right), \quad \forall j \in J. \tag{13}$$

The first point of the theorem says that the equilibrium price of the assets can be interpreted as the belief that minimizes the sum of disagreement among agents. Therefore, the market aggregates beliefs to a price that minimizes total disagreement. Secondly, the volume of trade and the gains from trade are both increasing in the disagreement between one agents' belief and the market belief.

Remark 2. We stated the theorem for $z_j \in (0,1)$ as this corresponds to the empirically relevant case. When $z_j > 1$ the formula for the gains from trade becomes:

$$G_j((x_i^j)_i) = \frac{1}{\frac{1}{z_j} - 1}\left(1 - \exp\left(\frac{D^{z_j-0.5}(\Pi, p^j(\cdot|I))}{z_j}\right)\right),$$

and therefore gains from trade are still increasing in disagreement $D^{z_j-0.5}$. On the other hand, the same does not hold for the volume of trade, as the equilibrium formula becomes:

$$V_j((x_i^j)_i) = \exp(D^{z_j+0.5}(p^j(\cdot|I), \Pi) - D^{-z_j+0.5}(\Pi, p^j(\cdot|I))) - 1.$$

This is due to the fact that if an agents' risk tolerance increases fast with wealth (i.e. $z_j \gg 0$) the shadow price of her income become increasing in disagreement with market belief. Therefore the equilibrium effect on gains from trade is not unambiguously increasing in disagreement.

4.2.2 Trade for Moderate Disagreement

We conclude this section by extending the results of Theorem 4 to agents with *generic* utility functions, under the additional assumption that agents' disagreement is small. Small disagreement, in our setting, means that the disagreement between agents is small compared to the rate at which their relative risk aversion changes, and then it can be approximated with a constant. Formally, we will model small disagreement as the limit model as agents' beliefs all converge to the same belief p^*.

This case is particularly interesting for two reasons: firstly, one rarely observes very large differences in beliefs in financial markets (with the exception of economic crisis or periods of political turmoil); secondly, most agents invest a small portion of their wealth so even when agents' preferences are not CRRA, approximating them locally with CRRA utility function provides a useful benchmark.

We formalize the idea of "moderate disagreement" by taking a limit of beliefs. Let the belief of agent j depend on an index $m \in \mathbb{N}$, let us denote them by $p^{j,(m)}$.

Definition 4 (Merging Beliefs)**.** We say that beliefs are merging if for some norm $\|\cdot\|$ on Δ_n:

$$\lim_{m\to+\infty}\left(\max_{j_1,j_2\in J}\|p^{j_1,(m)} - p^{j_2,(m)}\|\right) = 0. \tag{14}$$

In words, beliefs are merging if for all ϵ, we can find $\bar{m}$ such that all beliefs $p^{j,(m)}$ are in a ball of radius ϵ for all $j \in J$ and $m > \bar{m}$. Notice furthermore that Definition 4 implies that if beliefs are merging, then for all j, $(p^{j,(m)})_m$ has a limit in Δ_n as $m \to +\infty$, and such limit is the same for different agents j.[20] Since in general such limit could not belong to the interior of Δ_n we will assume that agents' beliefs are bounded away from the boundary of Δ_n, to avoid complications related to zero probability states.

Definition 5 (Uniformly Mixed Beliefs)**.** We say that the sequence of family of beliefs $(p^{j,(m)})^{j\in J}_{m\in\mathbb{N}}$ is uniformly mixed if there exists $\varepsilon > 0$ such that $p_i^{j,(m)} > \varepsilon > 0$ for all $i = 1,\dots,n$, $j \in J$, $m \in \mathbb{N}$.

In order to highlight the dependence of trade on beliefs, we will fix the preferences of the agents (i.e. they will not depend on m). For each j, let u_j be any smooth strictly increasing and concave utility function defined in a compact neighborhood of 1. As agents' beliefs merge, the allocation x_i^j converges to 1, the no trade outcome. Therefore, in the next theorem we approximate the utility functions u_j with the best CRRA approximation at 1. Without loss of generality we assume that $u_j'(1) = 1$ for all j, and then defining $z_j = -\frac{1}{u''(1)}$ we have that the function:

$$\tilde{u}_j(x) := \frac{x^{1-\frac{1}{z_j}}}{1-\frac{1}{z_j}}$$

is the only CRRA function such that $u'(1) = \tilde{u}_j'(1)$ and $u''(1) = \tilde{u}_j''(1)$. Theorem 5 implies that as agents' beliefs merge, the equilibrium of the model is asymptotic to the solution to the CRRA approximation. To simplify the notation, we assume that $I = \Theta$, i.e. the market for contingent securities is complete. It is easy to see that all the results extend to the case of $I \subset \Theta$.

Theorem 5. *Let $(u_j)_{j\in J}$ be a family of strictly concave, twice continuously differentiable utility functions defined in a compact neighborhood of* 1*. Suppose (without loss of generality) that $u_j'(1) = 1$, and assume that $-1/u_j''(1) =: z_j \in (0,1)$. Let $(p^{j,(m)})^{j\in J}_{m\in\mathbb{N}}$ be any family of merging and uniformly mixed beliefs.*

For all m the equilibrium exists and is unique, denote it by $((x_i^{j,(m)})^{j\in J}_{i=1,\dots,n}, (\Pi_i^m)_i)$. For all m, let $\tilde{\Pi}^m$ be the solution of the problem $\min_{q\in\Delta_n^\circ}\sum_j D^{z_j+0.5}(q, p^{j,(m)})$. We have that:

[20] This follows directly from the compactness of Δ_n.

- *for all j,* $\lim_{m\to\infty}\frac{D^{z_j+0.5}(p^{j,(m)},\Pi^m)}{D^{z_j+0.5}(p^{j,(m)},\tilde{\Pi}^m)}=1$*, so the disagreement between any belief p^j and the* approximate *equilibrium $\tilde{\Pi}^m$ is asymptotically equivalent to the disagreement with the market belief Π^m.*

- *The volume of trades is asymptotic to the volume of trades in the economy with CRRA utility functions:*

$$\lim_{m\to\infty}\frac{\sum_i p_i^{j,(m)}(x_i^{j,(m)}-1)}{D^{z_j+0.5}(p^{j,(m)},\tilde{\Pi}^m)+D^{-z_j+0.5}(\tilde{\Pi}^m,p^{j,(m)})}=1.$$

- *The gains from trade are asymptotic to the volume of trades in the economy with CRRA utility functions:*

$$\lim_{m\to\infty}\frac{\sum_i p_i^{j,(m)}u_i(x_i^{j,(m)})}{\frac{1}{1-z_j}D^{z_j-0.5}(\tilde{\Pi}^m,p^{j,(m)})}=1.$$

This theorem shows that whenever agents disagree moderately, their disagreement is a sufficient statistic for the volume and gains from trade, regardless of their utility functions. The parameter of the disagreement measures captures the local coefficient of relative risk aversion of the agents, around the no trade outcome.

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A Proofs

A.1 Proofs of Section 2

In this section we prove the main theorem, Theorem 1, and the results related to the analysis of the disagreement functions.

A.1.1 Proof of the main theorem

The proof proceeds by steps, that we state and prove separately for clarity. We summarize the steps here, pointing out at which point each Axiom we assumed plays a role. Axioms 1 and 2 are used at various points. The starting point of the Theorem is Axiom 6.

- Result 1 shows that the differential of $D(p,q)$ at $p \neq q$ is never identically 0, and it uses Axiom 5 and Axiom 1;

- Result 2 proves that $\partial_i D(p,q)$ can be written as the ratio of a function, h, that depends only on the beliefs in states i and $i+1$ and a function, α, that depends on the whole beliefs p,q, and is independent of the state i:

$$\partial_i D(p,q) = \frac{h(p_i, p_{i+1}, q_i, q_{i+1})}{\alpha(p,q)}.$$

- Result 3 uses the properties of the derivatives to show that $h^n_p(p_i, p_{i+1}, q_i, q_{i+1})$ can be written as a difference:

$$h(p_1, p_2, q_1, q_2) = i(p_1, q_1) - i(p_2, q_2),$$

  for some function $i : [0,1]^2 \to \mathbb{R}$.

- Result 4 then builds on Corollary 3 to show that $i(x,y)$ is a function of the ratio:

$$i(x,y) = G(x/y),$$

  for some $G : (0,+\infty) \to \mathbb{R}$ Since we use Corollary 3, this is the step of the proof where we use Axioms 3 and 4, together with Lemma 4, which was a consequence of those axioms and Axiom 5.

- Results 5 and 6 then apply Axiom 5 to show that the derivative of G must be a homogeneous function:

$$G'(x) = cx^{\alpha} \quad \exists \alpha, c \in \mathbb{R}.$$

  or $G'(x) = \frac{1}{x}\left(a + \frac{b}{x}\right)$. Result 6 is the main part of the proof.

- Result 7 then shows what the possible primitives of $G'(x)$ are, and Result 8 uses this to pin down the functional form of $\alpha(p,q)$.

- Finally we use the conclusions of Results 7 and 8 to wrap up the proof, after restating Theorem 1.

Result 1. *For all $p \neq q$ we have that there exists $j = 1, \ldots, n-1$ such that $\partial_j D(p,q) \neq 0$.*

Proof of Result 1. Let us prove that if $n = 2$, then for all $p \neq q$ we have that $\partial_1 D(p,q) \neq 0$. Suppose toward contradiction that instead $\partial_1 D(p,q) = 0$ for some p,q.

Now, for any $q' \in [p,q]$ we can define a function $\gamma(q,q') : \Delta_2 \times \Delta_2 \to \mathbb{R}$ such that:

$$D(p,q) = D(p,q') + D(p, p + \epsilon(q,q')z),$$

where $z = (1,-1)$. Therefore $\partial_1 D(p,q) = 0$ implies that:

$$\partial_1 D(p, p+\epsilon(q,q')z) = \partial_1 D(p, p+\epsilon(q,q')z)\frac{\partial \epsilon(q,q')}{\partial q} = 0, \tag{15}$$

and this holds for all $q' \in [p,q]$. Notice that as q' approaches q we have that $\epsilon(q,q') \to 0$, and we for a dense subset of $y \in (0,\delta) \subset \mathbb{R}$, $\partial_1 D(p, p+yz) \neq 0$.[21] This implies that $\frac{\partial \epsilon(q,q')}{\partial q} = 0$ for all q' in a neighborhood of q. Therefore for q' in a neighborhood of q we have that $\epsilon(q,q') = h(q')$ and since at $q = q'$ we have that $\epsilon(q,q') = 0$, then $h(q') = 0$. Therefore, we have that for $q' \in [p,q]$, and in a neighborhood of q, $D(p,q) = D(p,q')$.

In order to complete the proof, we show that $D(p,q)$ must be constant on the segment $(p,q]$, which implies, by continuity, that $0 = D(p,p) = D(p,q)$. Since this contradicts Axiom 1, proving that $D(p,q)$ is constant on $(p,q]$ suffices to conclude the proof.

Suppose not, i.e. suppose there exists $\bar{q} \in (p,q)$ such that $D(p,q') < D(p,\bar{q}) = D(p,q)$ for all $q'' \in [p,\bar{q})$. Plainly, $\partial_1 D(p,\bar{q}) = 0$, because $D(p,\cdot)$ is constant on $[\bar{q},q]$, and since it is differentiable its derivative must agree with the right derivative. Since $p \neq \bar{q}$ and $\partial_1 D(p,\bar{q}) = 0$, we can reiterate the argument used earlier for q to find for $q'' \in [p,\bar{q}]$ in a neighborhood of $\bar{q}$ we must have that $D(p,q'') = D(p,\bar{q})$, which contradicts the definition of $\bar{q}$. This concludes the proof for $n = 2$.

For general n, the proof is similar. If $\partial_j D(p,q) = 0$ for all j, then applying equation (15) for all $z_{i,j} = (0,\ldots,1,0,\ldots,-1,0,\ldots,0)$ we obtain that $D(p,q)$ must be locally constant for q' in a neighborhood of q. Consider then the intersection of this neighborhood with the segment $[p,q]$. Denote by $\bar{q}$ the belief in $[p,q]$ such that $D(p,\bar{q}) = D(p,q)$ and $D(p,q') < D(p,\bar{q})$ such that $q' \in [p,\bar{q}]$. Take the derivative of D in the direction of the segment $[p,q]$. Since $D(p,\cdot)$ is differentiable and it is constant for $q' \in [\bar{q},q]$, then we must have that $\partial_{q-p} D(p,\bar{q}) = 0$.[22] But then we can employ the argument used for $n = 2$ on the segment $[p,\bar{q}]$ obtaining the contradiction that $D(p,\cdot)$ must be constant in a neighborhood of $\bar{q}$. □

Result 2. *If $D_n(p,q)$ satisfies Axiom 6, then there exists a function $h : [0,1]^4 \to \mathbb{R}$,[23] and a function $\alpha(p,q) : \Delta_n \times \Delta_n \to \mathbb{R}$ such that:*

$$\partial_i D(p,q) = \frac{h(p_i, p_{i+1}, q_i, q_{i+1})}{\alpha(p,q)}, \tag{16}$$

for all $p,q \in \Delta_n$.

[21] If this was not the case, by continuity we have that $\partial_1 D(p,p+yz) = 0$ for all $y \in (0,\delta)$. But then $D(p,p+yz) = D(p,p) = 0$, contradicting Axiom 1.

[22] We denote by $\partial_{q-p} D(p,\bar{q})$ the derivative in the direction $q - p$, formally:

$$\partial_{q-p} D(p,\bar{q}) = \lim_{\epsilon \to 0} \frac{D(p,\bar{q} + \epsilon(q-p)) - D(p,\bar{q})}{\epsilon}.$$

[23] In practice the function h will be used only for vectors $x = (x_1,x_2,x_3,x_4)$ such that $x_1 + x_2 \leq 1$ and $x_3, x_4 \leq 1$. We did not clarify this in the domain of h to simplify the notation.

Furthermore, if (α, h) *and* $(\tilde{\alpha}, \tilde{h})$ *both satisfy equation* (16) *then:*

$$\alpha(p,q) = k\tilde{\alpha}(p,q) \quad \text{and} \quad h(p_i, p_{i+1}, q_i, q_{i+1}) = k\tilde{h}(p_i, p_{i+1}, q_i, q_{i+1}),$$

for some $k \neq 0$.

Proof of Result 2. Take any $p \neq q \in \Delta_n^\circ$. By Result 1 there exists a j such that $\partial_j D(p,q) \neq 0$. Apply then Axiom 6, and define:

$$h(p_i, p_{i+1}, q_i, q_{i+1}) := g(p_i, p_{i+1}, p_j, p_{j+1}, q_i, q_{i+1}, q_j, q_{j+1}),$$

where g the function defined in Axiom 6. Notice that in the definition of h, the values of $p_j, p_{j+1}, q_j, q_{j+1}$ are fixed. Also, notice that $h(p_i, p_{i+1}, q_i, q_{i+1}) = 0$ if and only if $\partial_i D_n(p,q) = 0$ (by Axiom 6) so for all i such that $\partial_i D(p,q)$ we have that:

$$\frac{\partial_i D(p,q)}{h(p_i, p_{i+1}, q_i, q_{i+1})} \quad \text{is independent of} \quad i,$$

because:

$$\frac{\partial_i D_n(p,q)}{h(p_i, p_{i+1}, q_i, q_{i+1})} = \partial_j D_n(p,p) = \frac{\partial_k D_n(p,q)}{h(p_k, p_{k+1}, q_k, q_{k+1})}$$

as it again follows from Axiom 6. Define $\alpha(p,q)$ to be such ratio:

$$\alpha(p,q) := \frac{h(p_i, p_{i+1}, q_i, q_{i+1})}{\partial_i D(p,q)},$$

we only need to show that if (α, h_p^n) and $(\tilde{\alpha}, \tilde{h}_p^n)$ both satisfy equation (16) then they are multiples by a constant *independent of* p, q.

To see this, denote for brevity $h(i) := h(p_i, p_{i+1}, q_i, q_{i+1})$, $\partial_i D := \partial_i D(p,q)$, and $\alpha := \alpha(p,q)$. Now suppose that (α, h) and $(\tilde{\alpha}, \tilde{h})$ satisfy (16). This means that:

$$\frac{h(j)}{\partial_j D} = \alpha \text{ and } \tilde{\alpha} = \frac{\tilde{h}(i)}{\partial_i D},$$

but then using that $\frac{\partial_j D}{\partial_i D} = \frac{h(j)}{h(i)}$, we obtain:

$$\frac{\tilde{\alpha}}{\alpha} = \frac{\tilde{h}(i)}{h(i)}.$$

Since the right hand side depends only on $p_i, q_i, p_{i+1}, q_{i+1}$, and the left hand side depends on the whole set of beliefs p, q (in general), we have that:

$$\frac{\tilde{\alpha}}{\alpha} = k = \frac{\tilde{h}(i)}{h(i)},$$

for some k independent of p, q. □

Result 3. *We have that for some function* $i : [0,1]^2 \to \mathbb{R}$*:*

$$h(p_1, p_2, q_1, q_2) = i(p_1, q_1) - i(p_2, q_2),$$

for all $p_1, p_2, q_1, q_2 \in [0,1]^4$.

Proof of Result 3. Denote by $\partial_{i,j}D(p,q)$ in the direction $z_{ij} = e_i - e_j$. Simple properties of the derivative imply that:

$$\partial_{i,i+1}D(p,q) + \partial_{i+1,i+2}D(p,q) = \partial_{i,i+2}D(p,q),$$

and rewriting this in terms of h we find:

$$\frac{h(p_i,p_{i+1},q_i,q_{i+1})}{\alpha(p,q)} + \frac{h(p_{i+1},p_{i+2},q_{i+1},q_{i+2})}{\alpha(p,q)} = \frac{h(p_i,p_{i+2},q_i,q_{i+2})}{\alpha(p,q)}.$$

Simplifying we obtain:

$$h(p_i,p_{i+1},q_i,q_{i+1}) + h(p_{i+1},p_{i+2},q_{i+1},q_{i+2}) = h(p_i,p_{i+2},q_i,q_{i+2}). \tag{17}$$

Since we have that h is differentiable (because D was assumed to be three times differentiable), we can take the derivative with respect to p_i in equation (17). So, defining h_j to be the derivative with respect to the j−th variable,[24] we get that:

$$h_1(p_i,p_{i+1},q_i,q_{i+1}) = h_1(p_i,p_{i+2},q_i,q_{i+2}),$$

and since $p_{i+1},q_{i+1},p_{i+2},q_{i+2}$ can take any value, we have that h_1 depends only on p_i and q_i. Analogously, we find that h_3 depends only on p_i and q_i. Hence h can be written as the sum of two functions $i^{(1)}$ and $i^{(2)}$ such that:

$$h(p_1,p_{i+1},q_i,q_{i+1}) = i^{(1)}(p_i,q_i) - i^{(2)}(p_{i+1},q_{i+1}). \tag{18}$$

The proof would be complete if we proved that the functions $i^{(1)}$ and $i^{(2)}$ are the same function.

To see this, consider equation (17) with

$$x := p_i = p_{i+1} = p_{i+2} \quad \text{and} \quad y := q_i = q_{i+1} = q_{i+2},$$

these values yield the equality:

$$2h(x,x,y,y) = h(x,x,y,y) \Rightarrow h(x,x,y,y) = 0, \quad \forall x,y \in [0,1]^2.$$

So rewriting this in terms of the functions $i^{(1)}$ and $i^{(2)}$ introduced in equation (18), we get:

$$i^{(1)}(x,y) - i^{(2)}(x,y) = 0, \quad \forall x,y \in [0,1]^2,$$

which means that $i^{(1)}(x,y) = i^{(2)}(x,y)$ so we call it i and find the thesis. □

Result 4. *We have that the function $i(x,y)$ introduced in Result 3 is a function of the ratio of x/y, i.e.:*

$$i(x,y) = G(x/y),$$

for a function $G : \mathbb{R}_+ \to \mathbb{R}$.

[24] Formally, $h_1(x,y,w,z) := \frac{\partial h}{\partial x}$, and similarly for h_2, h_3, h_4.

Proof of Result 4. Pick $p,q \in \Delta_n^\circ$. As in Corollary 3 define $q^1 = (q_1+q_2, 0, q_3, \dots, q_n)$ and $q^2 = (0, q_1+q_2, q_3, \dots, q_n)$ and consider the segment joining these two beliefs:

$$[q^1, q^2] = \{r \in \Delta_n \mid r = \lambda q^1 + (1-\lambda)q^2, \lambda \in [0,1]\}.$$

It was proved in the same Corollary that the minimum $\min_{r\in[q^1,q^2]} D(p,r)$ is achieved at the r for which $\frac{r_1}{r_2} = \frac{p_1}{p_2}$. Since D is differentiable, at such point the derivative in the direction 1,2 must be 0, i.e.:

$$\partial_1 D(p,r) = \frac{h(p_1,p_2,r_1,r_2)}{\alpha(p,r)} = 0,$$

but then using Result 3 we have that:

$$\frac{i(p_1,r_1) - i(p_2,r_2)}{\alpha(p,q)} = 0, \quad \text{if } \frac{p_1}{p_2} = \frac{r_1}{r_2},$$

or equivalently:

$$i(p_1,r_1) = i(p_2,r_2) \quad \text{if} \quad \frac{p_1}{r_1} = \frac{p_2}{r_2},$$

so i is a function of the ratio only:

$$i(p_1,r_1) = G\left(\frac{p_1}{r_1}\right).$$

□

In the following part of the proof, we will apply the Independence Axiom, Axiom 5, to obtain the functional form of G.

Consider $p = (p_1,\dots,p_n) \in \Delta_n^\circ$ and $q = (q_1,\dots q_n) \in \Delta_n^\circ$ and take $\lambda := (\lambda, 1-\lambda) \in \Delta_2^\circ$ and $\gamma = (\gamma, 1-\gamma) \in \Delta_2^\circ$.[25] We have that:

$$p * \lambda = (\lambda p_1, (1-\lambda)p_1, \dots, \lambda p_n, (1-\lambda)p_n),$$

$$q * \gamma = (\gamma q_1, (1-\gamma)q_1, \dots, \gamma q_n, (1-\gamma)q_n),$$

and by the independence Axiom, Axiom 5:

$$D(p*\lambda, q*\gamma) = D(p,q) + D(\lambda,\gamma),$$

so that perturbing p with ϵz_i we get the derivative on the right hand side is $\frac{\left(G\left(\frac{p_i}{q_i}\right) - G\left(\frac{p_{i+1}}{q_{i+1}}\right)\right)}{\alpha(p,q)}$. For brevity, call it:

$$RHS := \frac{\left(G\left(\frac{p_i}{q_i}\right) - G\left(\frac{p_{i+1}}{q_{i+1}}\right)\right)}{\alpha(p,q)}.$$

Similarly, the derivative of the left hand side will be defined as:

$$LHS := \frac{\lambda\left(G\left(\frac{\lambda p_i}{\gamma q_i}\right) - G\left(\frac{\lambda p_{i+1}}{\gamma q_{i+1}}\right)\right) + (1-\lambda)\left(G\left(\frac{(1-\lambda)p_i}{(1-\gamma)q_i}\right) - G\left(\frac{(1-\lambda)p_{i+1}}{(1-\gamma)q_{i+1}}\right)\right)}{\alpha(p*\lambda, q*\gamma)}$$

[25] Excuse the abuse of notation here.

Since we must have that $LHS = RHS$, then:

$$\frac{\alpha(p*\lambda, q*\gamma)}{\alpha(p,q)} = \frac{\lambda\left(G\left(\frac{\lambda p_i}{\gamma q_i}\right) - G\left(\frac{\lambda p_{i+1}}{\gamma q_{i+1}}\right)\right) + (1-\lambda)\left(G\left(\frac{(1-\lambda)p_i}{(1-\gamma)q_i}\right) - G\left(\frac{(1-\lambda)p_{i+1}}{(1-\gamma)q_{i+1}}\right)\right)}{\left(G\left(\frac{p_i}{q_i}\right) - G\left(\frac{p_{i+1}}{q_{i+1}}\right)\right)} \tag{19}$$

and for the sake of brevity define $r_i = \frac{p_i}{q_i}$, $R_1 = \frac{\lambda}{\gamma}$, and $R_2 = \frac{1-\lambda}{1-\gamma}$, so that we can rewrite:

$$\frac{\alpha(p*\lambda, q*\gamma)}{\alpha(p,q)} = \frac{\lambda(G(R_1 r_i) - G(R_1 r_{i+1})) + (1-\lambda)(G(R_2 r_i) - G(R_2 r_{i+1}))}{(G(r_i) - G(r_{i+1}))}$$

and notice that $\frac{\alpha(p*\lambda,q*\gamma)}{\alpha(p,q)}$ does not depend on r_i or r_{i+1} and then we obtain that for all $r'_i, r'_{i+1} \in \mathbb{R}^+$:

$$\frac{\lambda(G(R_1 r_i) - G(R_1 r_{i+1})) + (1-\lambda)(G(R_2 r_i) - G(R_2 r_{i+1}))}{(G(r_i) - G(r_{i+1}))}$$
$$= \frac{\lambda\left(G\left(R_1 r'_i\right) - G\left(R_1 r'_{i+1}\right)\right) + (1-\lambda)\left(G\left(R_2 r'_i\right) - G\left(R_2 r'_{i+1}\right)\right)}{\left(G\left(r'_i\right) - G\left(r'_{i+1}\right)\right)}$$

Therefore, we found that there exists a constant $K(\lambda,\gamma)$ such that:

$$\frac{\lambda(G(R_1 x) - G(R_1 y)) + (1-\lambda)(G(R_2 x) - G(R_2 y))}{(G(x) - G(y))} = K(\lambda,\gamma), \tag{20}$$

$\forall x, y \in [0, +\infty)$, where $R_1 := \frac{\lambda}{\gamma}$ and $R_2 := \frac{1-\lambda}{1-\gamma}$.

Next, we prove that equation (20) implies that $G(x)$ cannot be bounded, unless it is a constant. I.e. if G is not constant, then either $\lim_{x\to 0} G(x) = -\infty$ or $\lim_{x\to+\infty} G(x) = \infty$ (or both). Notice that $\lim_{x\to 0} G(x)$ and $\lim_{x\to+\infty} G(x)$ both exist because G is weakly monotone. As a matter of fact G is continuous and if for some x, x' we have that $G(x) = G(x')$ then we have that $G(y) = G(x)$ for all $y \in [x, x']$. This follows by Corollary 3, as disagreement is monotone on segment $[q^1, q^*]$ and segment $[q^*, q^2]$ (see also Figure 3).

Therefore the limits of $G(x)$ at 0 and ∞ exist, and furthermore:

$$\forall R > 0 \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} f(Rx), \tag{21}$$

as it is easy to check. We will invoke equation (21) repeatedly to prove the following result.

Result 5. *Either G is constant or G is unbounded, meaning:*

$$\sup_{x\in(0,+\infty)} |G(x)| = +\infty.$$

Proof. Suppose that G is bounded, $c_1 < G(x) < c_2$, for two $c_1, c_2 \in \mathbb{R}$. Then, take the limit for $x \to +\infty$, and define $I := \lim_{x\to\infty} G(x)$ (notice it is finite because we assumed G bounded). Using equation (21), we can rewrite (20) as:

$$\frac{\lambda(I - G(R_1 y)) + (1-\lambda)(I - G(R_2 y))}{(I - G(y))} = K(\lambda,\gamma).$$

Doing the same thing for the limit at $y \to 0$ (call $J := \lim_{y\to 0} G(y)$),

$$K(\lambda,\gamma) = \frac{\lambda(I-J)+(1-\lambda)(I-J)}{(I-J)} = 1.$$

But if $K(\lambda,\gamma) = 1$ for all λ and γ we get:

$$\lambda(G(R_1x) - G(R_1y)) + (1-\lambda)(G(R_2x) - G(R_2y)) = (G(x) - G(y)).$$

Now taking the limit for $\lambda \to 0$ we have that:

$$\lambda(G(R_1x) - G(R_1y)) \to 0,$$

because $(G(R_1x) - G(R_1y))$ is bounded. Then $R_2 \to \frac{1}{1-\gamma}$, and since G is continuous in $(0,+\infty)$ we get:

$$G\left(\frac{x}{1-\gamma}\right) - G\left(\frac{y}{1-\gamma}\right) = G(x) - G(y).$$

Then, considering again the limit at, say, $y \to +\infty$ and equation (21), we get that (for all $x > 0$ and $\gamma \in (0,1)$): $G\left(\frac{x}{1-\gamma}\right) = G(x)$, which implies that G is constant. □

Next, we will prove another important consequence of equation (20).

Result 6. *Either the derivative of G is homogeneous, i.e.:*

$$G'(x) = cx^{\alpha},$$

for some $\alpha \in \mathbb{R}$, $c \in \mathbb{R}$. *Or:*

$$G'(x) = \frac{a}{x^2} + \frac{b}{x},$$

for some $a, b \in \mathbb{R}$.

Proof. Clearly if G is constant, we have that $G'_p = 0$, and so the thesis of the Theorem holds true. Suppose that G is not constant. Then, by Result 5, we have that it is unbounded.

Clearly, since G is continuous on $(0,+\infty)$, if it is unbounded then either the limit for $x \to +\infty$ or $x \to 0$ (or both) have to be infinite. Suppose that $\lim_{x\to 0} G(x) = \infty$ (the case in which $\lim_{x\to 0} G(x) \in (-\infty,\infty)$ and $\lim_{x\to+\infty} G(x) = \pm\infty$ is analogous).

Recall that by (20):

$$\frac{\lambda(G(R_1x) - G(R_1y)) + (1-\lambda)(G(R_2x) - G(R_2y))}{(G(x) - G(y))} = K(\lambda,\gamma),$$

so the limit for $x \to 0$ must be the same value:

$$K(\lambda,\gamma) = \lim_{x\to 0} \frac{\lambda(G(R_1x) - G(R_1y)) + (1-\lambda)(G(R_2x) - G(R_2y))}{(G(x) - G(y))},$$

so, in particular, it must exist. Now, since $G(x) \to \infty$ as $x \to 0$, and y, R_1 and R_2 are fixed, then we have that:

$$\lim_{x\to 0} \frac{\lambda(G(R_1x)-G(R_1y))+(1-\lambda)(G(R_2x)-G(R_2y))}{(G(x)-G(y))} = \lim_{x\to 0} \frac{\lambda G(R_1x)+(1-\lambda)G(R_2x)}{G(x)},$$

so in particular the latter exists and is finite, for all λ and γ. This implies that

$$\lim_{x\to 0} \frac{G(Rx)}{G(x)} \quad \text{exists for all } R.$$

Notice that

$$\lim_{x\to 0} \frac{\lambda G(R_1x)+(1-\lambda)G(R_2x)}{G(x)} = \lim_{x\to 0} \frac{\lambda G(R_1x)+(1-\lambda)G(R_2x)}{G(Rx)} \cdot \frac{G(Rx)}{G(x)},$$

and the limit on the LHS exists, and so does $\lim_{x\to 0} \frac{\lambda G(R_1x)+(1-\lambda)G(R_2x)}{G(Rx)}$ (and it is generically not zero). In fact, we know that for all R, $\lim_{x\to 0} \frac{G(Rx)}{G(x)}$ must be finite because of equation (20). So we can define:

$$L(R) := \lim_{x\to 0} \frac{G(Rx)}{G(x)}, \tag{22}$$

and observe that $L(R) = R^\alpha$ for some $\alpha \in \mathbb{R}$. This is a classical result in the theory of *slowly varying function*, and it can be easily deduced by noticing that:

$$L(R) = \lim_{x\to 0} \frac{G(Rx)}{G(x)} = \lim_{x\to +\infty} \frac{G(Rx)}{G(R'x)} \frac{G(R'x)}{G(x)} = L\left(\frac{R}{R'}\right) L(R').$$

Using again equation (20), we then find that:

$$\lim_{x\to +\infty} \frac{\lambda(G(R_1x)-G(R_1y))+(1-\lambda)(G(R_2x)-G(R_2y))}{(G(x)-G(y))} = \lambda\left(\frac{\lambda}{\gamma}\right)^\alpha + (1-\lambda)\left(\frac{1-\lambda}{1-\gamma}\right)^\alpha.$$

But since by equation (20), the function

$$\frac{\lambda(G(R_1x)-G(R_1y))+(1-\lambda)(G(R_2x)-G(R_2y))}{(G(x)-G(y))}$$

does not depend on x, we have that *for all* $x \in (0, +\infty)$:

$$\frac{\lambda(G(R_1x)-G(R_1y))+(1-\lambda)(G(R_2x)-G(R_2y))}{(G(x)-G(y))} = \lambda\left(\frac{\lambda}{\gamma}\right)^\alpha + (1-\lambda)\left(\frac{1-\lambda}{1-\gamma}\right)^\alpha, \tag{23}$$

and this equation must hold for all x, y, γ, λ.

Moreover, having assumed that D is three times continuously differentiable, we can take the derivative with respect to x finding:

$$G'(x)\left(\lambda\left(\frac{\lambda}{\gamma}\right)^\alpha + (1-\lambda)\left(\frac{1-\lambda}{1-\gamma}\right)^\alpha\right) = \left(\lambda R_1 G'(R_1x) + (1-\lambda)R_2 G'(R_2x)\right), \tag{24}$$

for all $x \in [0, +\infty)$ and $\lambda, \gamma \in (0,1)$.

We now divide the study into the two cases of $\alpha = 0, -1$ and $\alpha \neq 0, -1$.

1. if $\alpha = 0$ or $\alpha = -1$ we get that equation (23) becomes:

$$\lambda(G(R_1x) - G(R_1y)) + (1-\lambda)(G(R_2x) - G(R_2y)) = (G(x) - G(y)),$$

and differentiating with respect to x we get:

$$G'(x) = \left(\lambda R_1 G'(R_1x) + (1-\lambda)R_2G'(R_2x)\right),$$

now multiply both sides by x and define $\phi(x) := xG'(x)$, which yields:

$$\phi(x) = \lambda\phi(R_1x) + (1-\lambda)\phi(R_2x),$$

and at $\lambda = \frac{1}{2}$ we obtain:

$$2\phi(x) = \phi\left(\frac{x}{2\gamma}\right) + \phi\left(\frac{x}{2(1-\gamma)}\right).$$

then differentiating with respect to γ we find:[26]

$$0 = -\frac{2x}{(2\gamma)^2}\phi'\left(\frac{x}{(2\gamma)}\right) + \frac{2x}{(2(1-\gamma))^2}\phi'\left(\frac{x}{(2(1-\gamma))}\right).$$

Now define $z = \frac{x}{2\gamma}$, and simplify so that we get:

$$\phi'\left(z\frac{\gamma}{1-\gamma}\right) = \left(\frac{\gamma}{1-\gamma}\right)^{-2}\phi'(z),$$

which implies that $\phi'(z) = \frac{c}{z^2}$ for some $c \in \mathbb{R}$ (since $\gamma \in (0,1)$ implies that $\gamma/(1-\gamma)$ spans $\mathbb{R}^+$, and z can be any positive number). But then $\phi(z) = \frac{a}{z} + b$ (by integrating $\phi'(z)$) and given the definition of $\phi(\cdot)$ we get that a further integration yields:

$$G'(x) = \frac{a}{x^2} + \frac{b}{x}, \tag{25}$$

as in the thesis.

2. If $\alpha \neq 0, -1$ let us consider equation (24) and once again define $\phi(x) := xG'(x)$ so that we get:

$$\phi(x)\left(\lambda\left(\frac{\lambda}{\gamma}\right)^{\alpha} + (1-\lambda)\left(\frac{1-\lambda}{1-\gamma}\right)^{\alpha}\right) = \lambda\phi(R_1x) + (1-\lambda)\phi(R_2x). \tag{26}$$

Observe that since the left hand side is continuous in $\lambda \in [0,1]$, so is the right hand side, and hence the limit for $\lambda \to 0$ is well-defined. Now, define the function:

$$c(r) := \lim_{\lambda \to 0} \lambda\phi(\lambda r). \tag{27}$$

[26]Here, in order to differentiate with respect to γ, we need to have that G' (which remember is the second derivative of D) is differentiable. This is the only step of the proof where we use D three times differentiable.

Notice that $c(r) = \frac{d}{r}$, for some $d \in \mathbb{R}$. As a matter of fact, for all $r' \neq r$ we have that:

$$c(r') = \lim_{\lambda\to 0} \lambda\phi\left(\lambda r'\right) = \frac{r}{r'} \lim_{\lambda r'\to 0} \lambda \frac{r'}{r} \phi\left(\lambda \frac{r'}{r} r\right) = \frac{r}{r'} c(r).$$

Now, suppose $\alpha > 0$, and take the limit for $\lambda \to 0$ in equation (26). We get that:

$$\phi(x)\frac{1}{(1-\gamma)^\alpha} = c\left(\frac{x}{\gamma}\right) + \phi\left(\frac{1}{(1-\gamma)}x\right), \tag{28}$$

where we used the continuity of ϕ in $(0, +\infty)$, and the definition of c given in (27). We will now prove that it must be that $c(r) = 0$. Rewriting (28) and substituting $c(r) = \frac{d}{r}$ we get that:

$$d = \frac{x\phi(x)}{\gamma}\frac{1}{(1-\gamma)^\alpha} - \frac{x}{\gamma}\phi\left(\frac{1}{(1-\gamma)}x\right),$$

and letting $x \to 0$ on the right hand side we get that $x\phi(x) \to d$ and $\frac{x}{\gamma}\phi\left(\frac{1}{(1-\gamma)}x\right) \to \frac{d(1-\gamma)}{\gamma}$, (because of equation (27)). Thus:

$$d = \lim_{x\to 0} \frac{x\phi(x)}{\gamma}\frac{1}{(1-\gamma)^\alpha} - \frac{x}{\gamma}\phi\left(\frac{1}{(1-\gamma)}x\right) = \frac{d}{\gamma(1-\gamma)^\alpha} - \frac{d(1-\gamma)}{\gamma},$$

and since this has to be true for all γ and $\alpha \neq -1, 0$, then the only possibility is $d = 0$, so the function $c(r)$ is constantly 0.

But then going back to (28) and substituting $c(x/r) = 0$ we get that:

$$\phi\left(\frac{1}{(1-\gamma)}x\right) = \frac{1}{(1-\gamma)^\alpha}\phi(x).$$

For the generality of x and γ we find that $\phi(x)$ is homogeneous of degree α:

$$\phi(x) = cx^\alpha, \Rightarrow G'(x) = cx^{\alpha-1}, \tag{29}$$

and hence we get the thesis.

Observe that as α varies in $\mathbb{R}$ we find that $G'(x)$ can be *any homogeneous function*. Equation (29) covers all homogeneous functions except those of degree $-1, -2$ (since $\alpha \neq 0, -1$), but those are the cases covered in the first case, taking $a = 0$ or $b = 0$ (respectively) in equation (25). □

This implies that:

Result 7. *G takes either of these functional forms:*

$$G(x) = ax^\alpha + b \quad \exists\alpha \neq 0, -1$$

or

$$G(x) = a\log(x) + \frac{b}{x} + c,$$

where $a, b, c, z \in \mathbb{R}$.

Proof. In Result 6 we showed that $G'(x) = ax^{\alpha}$ for some $\alpha \in \mathbb{R}$ or $G'(x) = \frac{a}{x} + \frac{b}{x^2}$. This result gives a family of primitives for these functions $G'(x)$. □

The next part of the proof builds on Result 7 to pin down the function $\alpha(p,q)$:

Result 8. *If* $G(x) = a\log(x) + \frac{b}{x} + c$, *then* $\alpha(p,q) = K$.

If $G(x) = ax^z + b$, *then for all* $p, q \in \Delta_n^\circ$:

$$\alpha(p,q) = K\left(\sum_j p_j \left(\frac{p_j}{q_j}\right)^z\right).$$

Proof. Let us rewrite equation (19), for $\lambda, \gamma \in \Delta_m^\circ$ (for a general m, instead of $m = 2$ as for the proof of Result 6). We get:

$$\frac{\alpha(p * \lambda, q * \gamma)}{\alpha(p,q)} = \sum_{j=1}^{m} \lambda_j \left(\frac{\lambda_j}{\gamma_j}\right)^{\alpha},$$

The case of $\alpha = 0, -1$ is easier, and it yields $\frac{\alpha(p*\lambda, q*\gamma)}{\alpha(p,q)} = 1$, which implies that $\alpha(\cdot,\cdot)$ is constant, for the generality of p, q, λ, γ.

Now in the case $\alpha \neq 0, -1$ we can write:

$$\alpha(p * \lambda, q * \gamma) = \alpha(p,q) \sum_{j=1}^{m} \lambda_j \left(\frac{\lambda_j}{\gamma_j}\right)^{\alpha}.$$

Since $D(p * \lambda, q * \gamma) = D(\lambda * p, \gamma * q)$ (by Axiom 2) then its derivatives must also coincide and we get that flipping p, q with λ, γ:

$$\alpha(p * \lambda, q * \gamma) = \alpha(\lambda, \gamma) \sum_j p_j \left(\frac{p_j}{q_j}\right)^z,$$

and this gives that

$$\frac{\alpha(p,q)}{\sum_j p_j \left(\frac{p_j}{q_j}\right)^z} = \frac{\alpha(\lambda,\gamma)}{\sum_{j=1}^m \lambda_j \left(\frac{\lambda_j}{\gamma_j}\right)^z},$$

and then $\frac{\alpha(p,q)}{\sum_j p_j \left(\frac{p_j}{q_j}\right)^z}$ does not depend on p, q and so it must be constant. □

This final result, buys us the main theorem.

Theorem 1. *The only functions* D_Θ *that satisfy Axioms 1, 2, 3, 4, 5, 6 have the following functional form (for all* Θ *finite, and all* $p, q \in \Delta(\Theta)$*):*[27]

1. *either:*

$$D_\Theta(p,q) = a \sum_{\theta \in \Theta} p(\theta) \log\left(\frac{p(\theta)}{q(\theta)}\right) + b \sum_\theta q(\theta) \log\left(\frac{q(\theta)}{p(\theta)}\right), \tag{30}$$

[27] The expression is not well-defined for non fully-mixed beliefs. See Remark 1 after the Theorem for a clarification.

for some $a, b \geq 0$ (not both zero);

2. *or:*

$$D_\Theta(p,q) = a \log\left(\sum_\theta p(\theta)\left(\frac{p(\theta)}{q(\theta)}\right)^{z-0.5}\right) \quad \textit{where} \quad \begin{cases} a > 0 & \textit{if } |z| > 0.5 \\ a < 0 & \textit{if } |z| < 0.5 \end{cases} \tag{31}$$

for $z \in \mathbb{R} \setminus \{-0.5, 0.5\}$.

Proof of Theorem 1. Let us show first the first case, $z \neq 0$. We showed above that under Axioms 1–6 the derivatives:

$$\partial_i D_z(p,q) = \frac{h(p_i, p_{i+1}, q_i, q_{i+1})}{\alpha(p,q)} = a\frac{\left(\frac{p_i}{q_i}\right)^z - \left(\frac{p_{i+1}}{q_{i+1}}\right)^z}{\sum_j p_j \left(\frac{p_j}{q_j}\right)^z},$$

for some $a, z \in \mathbb{R}$.

Then integrating on the directions of $q + \epsilon z_{i,i+1}$ (for any fixed p) we get that all primitives are given by:

$$D(p,q) = a \log\left(\sum_j p_j \left(\frac{p_j}{q_j}\right)^z\right) + K,$$

for some K constant. Since by definition we assumed that $D(q,q) = 0$ we have that $K = 0$. Also, since $D(p,q) \geq 0$ for all $p, q \in \Delta_n$ it must be that:

- if $z \in (-1, 0)$, then $a < 0$; because for $z \in (-1,0)$ notice that x^{-z} is concave and then by Jensen inequality:

  $$\sum_j p_j \left(\frac{p_j}{q_j}\right)^z \leq \left(\sum_j q_j\right)^{-z} = 1,$$

  and so $a \log\left(\sum_j p_j \left(\frac{p_j}{q_j}\right)^z\right) \geq 0$ if and only if $a \leq 0$. Then because of Axiom 1 we must have that $a < 0$;

- if $z \in (-\infty, -1) \cup (0, +\infty)$, then $a > 0$. To see this notice that for $z \in (0, \infty)$ the function x^{-z} is convex, and then Jensen inequality implies that:

  $$\sum_j p_j \left(\frac{p_j}{q_j}\right)^z \geq \left(\sum_j q_j\right)^{-z} = 1,$$

  and then $D(p,q) \geq 0$ if and only if $a \geq 0$. If instead $z \in (-\infty, -1)$ then notice that:

  $$a \log\left(\sum_j p_j \left(\frac{p_j}{q_j}\right)^z\right) = a \log\left(\sum_j q_j \left(\frac{q_j}{p_j}\right)^{-z-1}\right),$$

  and since $-z-1 \in (0, +\infty)$ we have that $z+1 \in (-\infty, 0)$ and then by Jensen inequality:

  $$\sum_j q_j \left(\frac{q_j}{p_j}\right)^{-z-1} \geq \left(\sum_j p_j\right)^{z+1} = 1$$

Passing now to the case of $G(z) = a\log(x) + \frac{b}{x}$ we showed in Result 8 that $\alpha(p,q)$ is constant, which in turns implies that

$$\partial_i D(p,q) = a\left(\log\left(\frac{p_i}{q_i}\right) - \log\left(\frac{p_{i+1}}{q_{i+1}}\right)\right) + b\left(\frac{q_i}{p_i} - \frac{q_{i+1}}{p_{i+1}}\right), \tag{32}$$

for some constants $a, b \in \mathbb{R}$.

Now observe that for all p, the family of functions

$$a\sum_j q_j \log\left(\frac{q_j}{p_j}\right) + b\sum_j p_j \log\left(\frac{p_j}{q_j}\right) + K,$$

are a family of primitives, where a and b are the same constant of equation (32). Then the normalization $D(p,p) = 0$ implies that $K = 0$, and let us prove that a and b have to be positive in order for $D(p,q) \geq 0$.

If at least one of a and b is negative, say a, we have that taking $p = (\epsilon, 1-\epsilon, 0, \ldots, 0)$ and $q = (0.5, 0.5, 0, \ldots, 0)$ gives us:[28]

$$\lim_{\epsilon\to 0} b\sum_j p_j \log\left(\frac{p_j}{q_j}\right) = \lim_{\epsilon\to 0} b(\epsilon\log(2\epsilon) + (1-\epsilon)\log(2(1-\epsilon))) = b\log(2),$$

whereas:

$$\lim_{\epsilon\to 0} a\sum_j q_j \log\left(\frac{q_j}{p_j}\right) = -\infty, \quad \text{if } a < 0$$

so we have that $\lim_\epsilon D(p,q) = -\infty$, and we have that a, b have to be positive.

The proof above shows that if Axioms 1–6 are satisfied, then the disagreement functions must take the form of equations (30) and (31). We have not showed that those functions satisfy all the axioms (the *easiest* implication of the theorem). Let us prove it now:

1. Axiom 1 is trivial, just substitute $p = q$ and see that $D(p,p) = 0$;
2. Axiom 2 is also trivial, because changing the name of the labels the disagreement function does not change;
3. Axiom 3 is not trivial. Let us analyze first the functional form of equation (30). Notice that for all $x_1, x_2, y_1, y_2, \lambda \in [0,1]$ we have that:

$$(\lambda x_1 + (1-\lambda)x_2)\log\left(\frac{(\lambda x_1 + (1-\lambda)x_2)}{(\lambda y_1 + (1-\lambda)y_2)}\right) \leq \lambda x_1 \log\left(\frac{x_1}{y_1}\right) + (1-\lambda)x_2\log\left(\frac{x_2}{y_2}\right), \tag{33}$$

because this equation it is equivalent to:

$$-\log\left(\frac{\lambda x_1}{\lambda x_1 + (1-\lambda)x_2}\left(\frac{y_1}{x_1}\right) + \frac{(1-\lambda)x_2}{\lambda x_1 + (1-\lambda)x_2}\left(\frac{y_2}{x_2}\right)\right) \leq -\frac{\lambda x_1}{\lambda x_1 + (1-\lambda)x_2}\log\left(\frac{y_1}{x_1}\right) - \frac{(1-\lambda)x_2}{\lambda x_1 + (1-\lambda)x_2}\log\left(\frac{y_2}{x_2}\right), \tag{34}$$

[28] Here we use the continuity result that $\lim_x x\log(x) = 0$

which is true by the convexity of $-\log$. But then using equation (33) repeatedly we find that:

$$D_{\Theta}(\lambda p^1 + (1-\lambda)p^2, \lambda q^1 + (1-\lambda)q^2) \leq \lambda D(p^1, q^1) + (1-\lambda)D(p^2, q^2),$$

and since the RHS is smaller than the maximum we obtain Axiom 3. The proof for the functions of the form written in equation (31) is exactly analogous, after noting that $x^{z-0.5}$ is convex for $|z| > 0.5$ and concave for $|z| < 0.5$.

4. Axiom 4 also requires a small proof, and it resembles that just done for Axiom 3. Since we showed the proof of Axiom 6 for the functions of the form (30), let us show this axiom using the other, equation (31). Notice that proving that Axiom 4 is equivalent to proving that for all p_1, p_2, q_1, q_2 and for $|z| > 0.5$ we have that:

$$(p_1 + p_2)\left(\frac{p_1 + p_2}{q_1 + q_2}\right)^{z-0.5} \leq p_1\left(\frac{p_1}{q_1}\right)^{z-0.5} + p_2\left(\frac{p_2}{q_2}\right)^{z-0.5}.$$

To show this, again notice that it is equivalent to having:

$$\left(\frac{p_1}{p_1+p_2}\frac{q_1}{p_1} + \frac{p_2}{p_1+p_2}\frac{q_2}{p_2}\right)^{0.5-z} \leq \frac{p_1}{p_1+p_2}\left(\frac{q_1}{p_1}\right)^{0.5-z} + \frac{p_2}{p_1+p_2}\left(\frac{q_2}{p_2}\right)^{0.5-z},$$

and if $|z| > 0.5$ the function $x^{0.5-z}$ is convex so the result follows. If instead $|z| < 0.5$ we have to prove that the opposite inequality holds, i.e.

$$\left(\frac{p_1}{p_1+p_2}\left(\frac{q_1}{p_1}\right) + \frac{p_2}{p_1+p_2}\left(\frac{q_2}{p_2}\right)\right)^{0.5-z} \geq \frac{p_1}{p_1+p_2}\left(\frac{q_1}{p_1}\right)^{0.5-z} + \frac{p_2}{p_1+p_2}\left(\frac{q_2}{p_2}\right)^{0.5-z},$$

which is true because for $|z| < 0.5$, $0.5 - z \in (0,1)$ and then $x^{0.5-z}$ is concave.

5. Axiom 5 can be verified directly by using the fact that $\log(ab) = \log(a) + \log(b)$ and $\sum_{i,j}(p_i p_j)^z = \sum_i p_i^z \sum_j p_j^z$.

6. Axiom 6 can be proved by taking the derivatives explicitly, and does not require any proof.

□

A.1.2 Proofs on the Analysis of the disagreement functions

Proposition 1. *The only disagreement functions that satisfy Axioms 1–6 and are such that $D(p,q) = D(q,p)$ for all p,q, are proportional to:*

- *the symmetric divergence:*

$$D(p,q) = \sum_j (p_j - q_j)\log\left(\frac{p_j}{q_j}\right) = \sum_j p_j \log\left(\frac{p_j}{q_j}\right) + \sum_j q_j \log\left(\frac{q_j}{p_j}\right);$$

- *the Bhattacharyya distance:*

$$D(p,q) = -\log\left(\sum_j \sqrt{p_j q_j}\right).$$

Furthermore, the only symmetric disagreement function that is additively separable in the states[29] *is the symmetric divergence, and the only symmetric disagreement function such that* $D(p,q) < +\infty$ *if and only if* $Supp(p) \cap Supp(q) \neq \emptyset$ *is the Bhattacharyya distance.*

Proof of Proposition 1. Consider $p = (1,0)$ and $q = (q_1, 1-q_1)$ with $q_1 \neq 0.5$. Computing $D(p,q)$ for the disagreement functions of equation (3) (in Theorem 1) we find that $D(p,q) \neq D(q,p)$ unless $z = 0$. For the functions of equation (2), consider $p = (0.5, 0.5)$ and $q = (q_1, 1-q_1)$ and notice that $D(p,q) \neq D(q,p)$ if and only if $a \neq b$. The last statement of the theorem can also be obtained directly.

□

Proposition 2. *Let* $p,q,r \in \Delta_n^\circ$ *be beliefs ranked by Likelihood Ratio:*

$$p <_{LR} q <_{LR} r,$$

then for any disagreement function we have that:

$$D(p,q) \leq D(p,r).$$

Proof of Proposition 2. We prove this result by showing it holds for all the functions D characterized in Theorem 1.

- let $D(p,q) = \log\left(\sum_i p_i \left(\frac{p_i}{q_i}\right)^{z-0.5}\right)$ with $z > 0.5$. The statement is equivalent to proving that for all p,q,r with $p <_{LR} q <_{LR} r$,:
$$\sum_i p_i \left(\frac{p_i}{q_i}\right)^{z-0.5} \leq \sum_i p_i \left(\frac{p_i}{r_i}\right)^{z-0.5}.$$
To show this, let us prove something stronger, that is, for all $\epsilon \in (0,1)$ we have that $\partial \frac{\exp(D(p,\epsilon r+(1-\epsilon)q))}{\partial \epsilon} \geq 0$. Clearly, if this is true, then $D(p,r) \geq D(p,q)$. It is easy to find that:
$$\frac{\left(\partial \sum_i p_i \left(\frac{p_i}{\epsilon r_i + (1-\epsilon) q_i}\right)^{z-0.5}\right)}{\partial \epsilon} \geq 0 \Leftrightarrow \sum_i \left(\frac{p_i}{(1-\epsilon)q_i + \epsilon r_i}\right)^{z+0.5} r_i \leq \sum_i \left(\frac{p_i}{(1-\epsilon)q_i + \epsilon r_i}\right)^{z+0.5} q_i,$$
and we have that the latter holds because $\left(\frac{p_i}{(1-\epsilon)q_i+\epsilon r_i}\right)^{z+0.5}$ is decreasing (since $p <_{LR} \epsilon r + (1-\epsilon)q$) and $q <_{LR} r$.
- the case for $|z| < 0.5$ and $z < -0.5$ is analogous;
- the case $D(p,q) = \sum_i p_i \log\left(\frac{p_i}{q_i}\right)$ and $D(p,q) = \sum_i q_i \log\left(\frac{q_i}{p_i}\right)$ can be obtained as limit for $z \to 0.5, -0.5$ (see Proposition 3 below) so the same result must hold.

□

[29] A divergence metric is said to be additively separable in the states if $D(p,q) = \sum_j f_j(p_j, q_j)$, for some functions $f_j : [0,1] \times [0,1] \to \mathbb{R}$.

Observation 1. *For all $z \neq \tilde{z}$, the disagreement functions D^z and $D^{\tilde{z}}$ are* not *ordinally equivalent, i.e. there exist $p, q, r \in \Delta_n$ such that:*

$$D^z(p,q) < D^z(p,r) \quad \textit{and} \quad D^{\tilde{z}}(p,q) > D^{\tilde{z}}(p,r).$$

Proof of Observation 1. In order to prove this result, notice that by Axiom 6 we have that if $z \neq 0.5, -0.5$:

$$\frac{\partial_i D^z(p,q)}{\partial_j D^z(p,q)} = \frac{\left(\frac{p_i}{q_i}\right)^{z+1} - \left(\frac{p_{i+1}}{q_{i+1}}\right)^{z+1}}{\left(\frac{p_j}{q_j}\right)^{z+1} - \left(\frac{p_{j+1}}{q_{j+1}}\right)^{z+1}},$$

and observe that this is can be interpreted as (minus) the slope of implicit function defined by:

$$D(p,q) = K = D(p,(q_1,\ldots,q_i+\epsilon,q_{i+1}-\epsilon,\ldots,q_j+g_z(\epsilon),q_{j+1}-g_z(\epsilon),\ldots,q_n)),$$

plainly:

$$g_z'(0) = -\frac{\partial_i D^z(p,q)}{\partial_j D^z(p,q)} = -\frac{\left(\frac{p_i}{q_i}\right)^{z+1} - \left(\frac{p_{i+1}}{q_{i+1}}\right)^{z+1}}{\left(\frac{p_j}{q_j}\right)^{z+1} - \left(\frac{p_{j+1}}{q_{j+1}}\right)^{z+1}}. \tag{35}$$

Now for any $z \neq \tilde{z}$ we can find $p, q \in \Delta_n^\circ$ with:

$$g_z'(0) \neq g_{\tilde{z}}'(0).$$

This implies that locally around q the ball of radius $D(p,q)$ around p (in the plane spanned by z_i and z_j) are different.

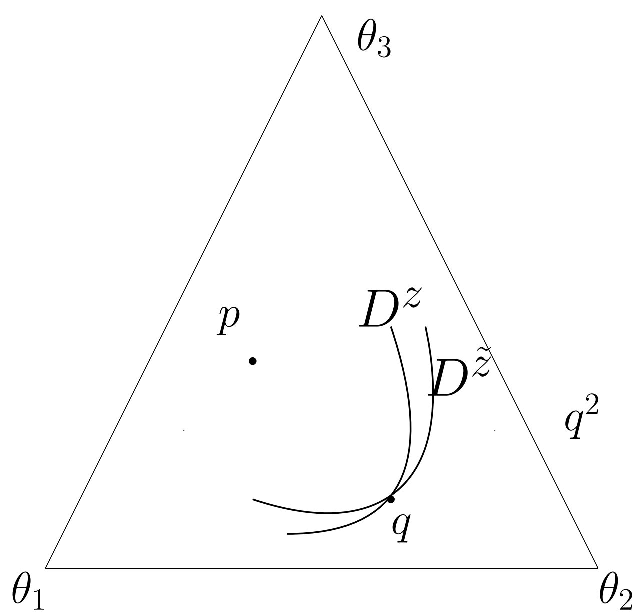

Figure 4: The two balls of radius $D^z(p,q)$ and $D^{\tilde{z}}(p,q)$. If $z \neq \tilde{z}$ then the tangent in p has different slope (generically), as it can be seen by equation (35).

This means we can find r arbitrary close to q such that:

$$D_z(p,q) < D_z(p,r) \quad \text{and} \quad D_{z'}(p,r) < D_{z'}(p,q).$$

If we consider disagreement functions such as those of equation (2), the result is analogous. □

Proposition 3. *For all $p,q \in \Delta_n^\circ$ and for all $z \neq 0.5, -0.5$:*

1. *$D^z(p,q) = D^{-z}(q,p)$;*
2. *for all $z \in \mathbb{R}$, $z \neq 0.5$:*
$$\frac{1}{z-0.5} D^z(p,q) = \log(\|p(\theta)/q(\theta)\|^p_{z-0.5}),$$
where the w-norm of a function $f: \Theta \to \mathbb{R}^+$, is defined by $\|f(\theta)\|^p_w := (\sum_i |f(\theta_i)|^w p(\theta_i))^{1/w}$;
3. *D^z can be extended by continuity at $z = 0.5$ and $z = -0.5$:*
$$\lim_{z\to 0.5} \frac{D^z(p,q)}{z-0.5} = \sum_j p_j \log\left(\frac{p_j}{q_j}\right) =: D^{0.5}(p,q)$$
$$\lim_{z\to -0.5} \frac{D^z(p,q)}{-z-0.5} = \sum_j q_j \log\left(\frac{q_j}{p_j}\right) =: D^{-0.5}(p,q).$$

Proof of Proposition 3. Points 1 and 2 follow simply by writing the definitions of D^z, the only point that requires a short proof is 3.

To show this limit recall that for all f positive we have that:

$$\lim_{w\to 0} \|f\|^p_w = \exp\left(\sum_j p_j \log\left(\frac{p_j}{q_j}\right)\right),$$

so that applying this to the function $p(\theta)/q(\theta)$ and taking the logarithm we obtain the first limit result of point 3. To get the second result notice that thanks to point 1 we get:

$$\lim_{z\to -0.5} \frac{D^z(p,q)}{-z-0.5} = \lim_{z\to -0.5} \frac{D^{-z}(q,p)}{-z-0.5} \overset{w=-z}{=} \lim_{w\to 0.5} \frac{D^w(q,p)}{w-0.5},$$

so the result follows from the first limit. □

Lemma 1. *If $|z| < 0.5$, then $D^z(p,q) = +\infty$ if and only if $Supp(p) \cap Supp(q) = \emptyset$. If $|z| \geq 0.5$ then:*

- *if $z \geq 0.5$ then $D^z(p,q) = +\infty$ if and only if $Supp(p) \setminus Supp(q) \neq \emptyset$.*
- *if $z \leq -0.5$ then $D^z(p,q) = +\infty$ if and only if $Supp(q) \setminus Supp(p) \neq \emptyset$.*

Proof of Lemma 1. Recall that $D^z(p,q)$ is defined, for $|z| < 0.5$

$$D^z(p,q) = -\log\left(\sum_j p_j^{z+0.5} q_j^{0.5-z}\right),$$

hence if $|z| < 0.5$, then $0.5 + z > 0 < 0.5 - z$. Then, we have that the argument of the logarithm is finite, for all $p, q \in \Delta_n$. Therefore we have that $D^z(p,q) = +\infty$ if and only if $\sum_j p_j^{z+0.5} q_j^{0.5-z} = 0$. But since this is a sum of positive terms, it is 0 only if each term is 0. On the other hand, for $z > 0.5$, notice that the exponent of q_j is negative, so if $\theta_j \in Supp(p) \setminus Supp(q)$ we have that the sum in the argument of D^z is infinite, because of the j−th term alone. For the opposite implication, note that if $Supp(p) \subseteq Supp(q)$ then whenever $q_j = 0$, so is p_j, and then $D(p,q) < +\infty$.

□

Corollary 1.

1. *if $z \geq 0.5$, $D^z(p,\cdot) : \Delta_n \to \mathbb{R}$ is continuous for all fixed $p \in \Delta_n$. Furthermore D^z depends only on the states $\theta \in Supp(p)$:*

$$D^z(p,q) = \begin{cases} \log\left(\sum_{j \in Supp(p)} p_j \left(\frac{p_j}{q_j}\right)^{z-0.5}\right) & \text{if } z > 0.5 \\ \sum_{j \in Supp(p)} p_j \log\left(\frac{p_j}{q_j}\right) & \text{if } z = 0.5 \end{cases}.$$

   On the other hand, $D^z(\cdot, q) : \Delta_n \to \mathbb{R}$ is continuous if and only if $q \in \Delta_n^\circ$.

2. *if $z \leq -0.5$, $D^z(\cdot, q) : \Delta_n \to \mathbb{R}$ is continuous for all fixed $q \in \Delta_n$. Furthermore D^z depends only on the states $\theta \in Supp(q)$:*

$$D^z(p,q) = \begin{cases} \log\left(\sum_{j \in Supp(q)} p_j \left(\frac{p_j}{q_j}\right)^{z-0.5}\right) & \text{if } z > 0.5 \\ \sum_{j \in Supp(q)} p_j \log\left(\frac{p_j}{q_j}\right) & \text{if } z = 0.5 \end{cases}.$$

   On the other hand, $D^z(p, \cdot) : \Delta_n \to \mathbb{R}$ is continuous if and only if $p \in \Delta_n^\circ$.

3. *If $|z| < 0.5$, then $D^z : \Delta_n \times \Delta_n \to \mathbb{R}$ is continuous, so it is continuous on both variables separately, and:*

$$D^z(p,q) = -\log\left(\sum_{j \in Supp(p) \cap Supp(q)} p_j \left(\frac{p_j}{q_j}\right)^{z-0.5}\right),$$

   with the convention that $\sum_{j \in \emptyset} = 0$.

Proof of Corollary 1. 1. ($z \geq 0.5$) Consider $z > 0.5$ first. The fact that $D^z(p,\cdot)$ is continuous on Δ_n° is trivial, because we assumed that D^z be three times continuously differentiable in both variables. To show that it is continuous also on the boundary of Δ_n, consider a sequence of $(q^{(n)})_n$ converging to a point in the boundary, say $q_j^{(n)} \to 0$. If $p_j = 0$ the statement is trivial, because $D^z(p, q^{(n)})$ does not depend on the state j. If instead $p_j > 0$, we have that:

$$D^z(p, q^{(n)}) \geq \log\left(p_j^{z+0.5} (q_j^{(n)})^{0.5-z}\right) \geq (0.5 - z)\log(q_j^{(n)}) \to +\infty,$$

since $q_j^{(n)} \to 0$ and $(0.5 - z) < 0$. Therefore $D^z(p, q^{(n)}) \to +\infty$, and by our conventions (Remark 1) $D(p,r) = +\infty$ anytime $p_j > 0$ and $r_j = 0$. Hence $D^z(p,\cdot)$ is continuous on Δ_n. Given this result we have that:

$$D^z(p,q) = \log\left(\sum_{j \in Supp(p)} p_j \left(\frac{p_j}{q_j}\right)^{z-0.5}\right).$$

Let us prove that $D^z(\cdot,q):\Delta_n\to\mathbb{R}$ is continuous if and only if $q\in\Delta_n^\circ$. If $q\in\Delta_n^\circ$, then $D^z(\cdot,q)$ is continuous, because it is finite on Δ_n. On the other hand, let us show that if $q\notin\Delta_n^\circ$ then $D^z(\cdot,q)$ is not continuous. Without loss of generality, consider q with $q_1=0$, and $q_j>0$ for all $j=2,\dots,n$. Let us prove that if $z>0.5$, $D^z(p,q)$ is discontinuous in p. Consider a sequence $p^{(m)}$ defined by:

$$p_j^{(m)}:=\begin{cases}\frac{1}{m} & \text{if } j=1\\ \frac{(m-1)}{m(n-1)} & \text{if } >1\end{cases}$$

it is easy to see that $D_w(p^{(m)},q)=+\infty$ for all m, because $Supp(p)\setminus Supp(q)\neq\emptyset$, and

$$p^m\to p^*=\left(0,\frac{1}{n-1},\dots,\frac{1}{n-1}\right),$$

so $D^z(p^*,q)<+\infty$ because $Supp(q)\setminus Supp(p)=\emptyset$.

The case of $z=0.5$ is similar.

2. this case is analogous to the first one, changing p with q and using the first point of Proposition 3;

3. finally if $|z|<0.5$ notice that both $z+0.5$ and $z-0.5$ are positive, hence there are no discontinuities at the boundary and trivially:

$$D^z(p,q)=-\log\left(\sum_{j\in Supp(p)\cap Supp(q)} p_j^{z+0.5}q_j^{z-0.5}\right).$$

□

Proposition 4. *Let $|z|\geq 0.5$ then for any $\epsilon,\delta>0$ (arbitrary small) there exist $\bar p,\bar q,\underline{p},\underline{q}\in\Delta_n^\circ$ such that:*

$$\|\bar p-\bar q\|_\infty>\max_{x,y\in\Delta_n}\|x-y\|_\infty-\delta,\qquad \|\underline{p}-\underline{q}\|_\infty<\epsilon,$$

and $D^z(\underline{p},\underline{q})>D^z(\bar p,\bar q)$.

If $|z|<0.5$, instead, $D^z(p,q)$ is uniformly continuous with respect to the metric induced by $\|\cdot\|_\infty$. Namely, for all ϵ there exist a δ such that if $\|p-q\|_\infty\leq\delta$ then $D^z(p,q)<\epsilon$.

Proof of Proposition 4. For the case $|z|>0.5$ it is enough to prove the statement for $n=2$. Then the result for a general n will follow by considering vectors of the form $(p,1-p,0,\dots,0)$.

Notice that $\max_{x,y\in\Delta_2}\|x-y\|_\infty=1$. Fix ϵ and δ small. Pick $\bar p,\bar q\in\Delta_n^\circ$ with $\|\bar p-\bar q\|>1-\delta$. Since $\bar p,\bar q\in\Delta_n^\circ$ we have that $D(\bar p,\bar q)<+\infty$. Let us show that we can find two sequences $(p^m)_m,(q^m)_\in\Delta_n^\circ$ such that $\|p^m-q^m\|\leq\epsilon$ and $D(p^m,q^m)\to+\infty$ as $m\to+\infty$. Once we prove this, the thesis will follow.

Pick $q^m:=(1/m,1-1/m)$ and $p^m=(1/m+\epsilon,1-1/m-\epsilon)$. Clearly, $\|p^m-q^m\|\leq\epsilon$. If $z>0.5$:

$$D^z(p,q)\geq\log\left(p_1^m\left(\frac{p_1^m}{q_m^1}\right)^{z-0.5}\right)=\log\left[\left(\frac{1}{m}+\epsilon\right)\left(\frac{\frac{1}{m}+\epsilon}{\frac{1}{m}}\right)^{z-0.5}\right],$$

and as $m \to +\infty$ the right hand side clearly tends to ∞, so we get our result. If $z = 0.5$ we can use the same sequence, that yields:

$$D^{0.5}(p,q) \geq p_1^m \log\left(\frac{p_1^m}{q_1^m}\right) = \left(\frac{1}{m} + \epsilon\right) \log\left(\frac{\frac{1}{m} + \epsilon}{\frac{1}{m}}\right),$$

which also diverges as $m \to +\infty$, for any $\epsilon > 0$. The case $z < -0.5$ is analogous, after inverting p with q.

To get that instead D^z is uniformly continuous for $|z| < 0.5$, let us show it first for $n = 2$ and then explain how the same argument holds for $n \geq 2$ too. Consider all the beliefs of the form $p = (x, 1-x)$ and $q = (x+\epsilon, 1-x-\epsilon)$, as $x \in [0, 1-\epsilon]$. It is easy to see that (let $\alpha := z + 0.5$ and notice that $\alpha \in (0,1)$, if $|z| < 0.5$):

$$D^z(p,q) = -\log\left(x^\alpha(x+\epsilon)^{1-\alpha} + (1-x)^\alpha(1-x-\epsilon)^{1-\alpha}\right),$$

and one can check that both $x^\alpha(x+\epsilon)^{1-\alpha}$ and $(1-x)^\alpha(1-x-\epsilon)^{1-\alpha}$ are concave. Therefore $x^\alpha(x+\epsilon)^{1-\alpha} + (1-x)^\alpha(1-x-\epsilon)^{1-\alpha}$ achieves its minimum at $x = 0$ or $x = 1-\epsilon$, and we have that:

$$D^z(p,q) \leq \max\{-\log(\epsilon^\alpha + (1-\epsilon)^{1-\alpha}); -\log((1-\epsilon)^\alpha + \epsilon^{1-\alpha})\}.$$

For $\epsilon \to 0$ we have that both $-\log(\epsilon^\alpha + (1-\epsilon)^{1-\alpha})$ and $-\log((1-\epsilon)^\alpha + \epsilon^{1-\alpha})$ are positive and converge to 0, thus simply take

$$\delta := \max\{-\log(\epsilon^\alpha + (1-\epsilon)^{1-\alpha}); -\log((1-\epsilon)^\alpha + \epsilon^{1-\alpha})\},$$

and we have the thesis for $n = 2$.

For a general n, notice that for any p, q if $p_j = 0$ for some j (say $p_1 = 0$) then $D^z(p,q) \leq D^z(p, (0, q_1 + q_2, \ldots, q_n))$, so we can reduce ourselves to a case of dimension $n-1$ and the argument follows by induction. Suppose instead that $p \in \Delta_n^\circ$. Pick q such that $\|p - q\| \leq \epsilon$, this implies that $|p_j - q_j| \leq \epsilon$ for all j. Now suppose without loss that $p_1 < q_1$ and consider the beliefs:[30]

$$p' = (0, p_1 + p_2, p_3, \ldots, p_n), \quad q' = (q_1 - p_1, q_2 + p_1, q_3, \ldots, q_n).$$

An argument identical to that done for $n = 2$ shows that:

$$D^z(p,q) \leq D^z(p', q'),$$

but now we have that $p' \notin \Delta_n^\circ$, so we can reduce ourselves to a simplex of dimension $n-1$ and we are done. □

A.2 Proofs of Section 3

Lemma 2. *If a family of disagreement functions D_Θ for any state space Θ satisfies Axiom 2 then so does the family $(\tilde{D}_n)_{n \geq 2, n \in \mathbb{N}}$, where:*

$$\tilde{D}_n : \Delta(\Theta) \times \Delta(\Theta) \to \mathbb{R}^+ \cup \{+\infty\}, \qquad \tilde{D}_n(p,q) := D_\Theta(p,q),$$

[30] The notation makes sense if $q_2 + p_1 < 1$, notice that if this were not the case we can find another j for which $q_j + p_1 < 1$. I assume such j is 2.

for any Θ with $|\Theta| = n$.

Proof of Lemma 2. Trivial. □

Lemma 3. *If D_n satisfies Axiom 3 and Axiom 1 then:*

1. *for all $p, q \in \Delta_n$ and for all $\lambda \in [0,1]$:*

$$D_n(p, \lambda p + (1-\lambda)q) \le D_n(p,q).$$

2. *For all $p, q, r \in \Delta_n$ and $\lambda \in [0,1]$:*

$$D_n(\lambda p + (1-\lambda)r, \lambda q + (1-\lambda)r) \le D_n(p,q).$$

3. *For all p, q^1, q^2 and $\lambda \in [0,1]$:*

$$D_n(p, \lambda q^1 + (1-\lambda)q^2) \le \max\{D_n(p,q^1), D_n(p,q^2)\}.$$

Also, this last property is equivalent to assuming that the balls:

$$B(p,\rho] := \{q \in \Delta_n \mid D_n(p,q) \le \rho\}$$

are convex for all D_n.

Proof of Lemma 3. Notice that the first statement follows directly by taking $p^1 = p^2 = p$ and $q^1 = p$ in Axiom 3. the second statement follows taking $p^1 = p$, $p^2 = r = q^2$, and $q^1 = q$. Finally the third statement follows by taking $p^1 = p^2 = p$. The fact that balls in the metric induced by D_n are convex follows directly from equation 3. □

Corollary 2. *For every $C \subset \Delta_n$ convex:*

$$\sup_{p,q \in C} D_n(p,q) = D_n(\bar{p}, \bar{q}),$$

for some $\bar{p}, \bar{q} \in \partial C$. This implies that for all $p, q \in \Delta_n$ we have that:

$$D_n(p,q) \le D_n(e^1, e^2) \quad (= D_n(e^j, e^i) \quad \forall\, i,j),$$

where $e^1 = (1,0,0,\dots,0)$ and $e^2 = (0,1,0,\dots,0)$.

Proof of Corollary 2. Let $s := \sup_{p,q \in C} D_n(p,q)$ (in general $s \in \mathbb{R}^+ \cup \{+\infty\}$). Consider a sequence of $(p_m, q_m) \in C$ such that $D_n(p_m, q_m) \to s$. Now consider the sequence $(\tilde{p}_m, \tilde{q}_m)$ such that $\tilde{p}_m, \tilde{q}_m \in \partial C$, and[31] $[p_m, q_m] \subseteq$

[31] For any two $a, b \in \Delta_n$, the set $[a,b]$ denotes the segment joining them:

$$[a,b] := \{q \in \Delta_n \mid q = \lambda a + (1-\lambda)b, \quad \exists \lambda \in [0,1]\}.$$

$[\tilde{p}_m, \tilde{q}_m]$. By Lemma 3 we have that $D_n(\tilde{p}_m, \tilde{q}_m) \geq D_n(p_m, q_m)$. So we have that $\lim_m D_n(\tilde{p}_m, \tilde{q}_m) \geq s$, which implies that $\lim_m D_n(\tilde{p}_m, \tilde{q}_m) = s$, because s is the sup on C.

The set ∂C is compact (it is closed and bounded, and we are in a normed space), so $(\tilde{p}_m, \tilde{q}_m)$ admits a converging subsequence, with limit in ∂C. Call such limit (p^*, q^*), it must be that $D_n(p^*, q^*) = s$ (by continuity of D), hence the thesis. □

Proposition 5. *Let D be a smooth measure of disagreement satisfying Axioms 1–4. If D satisfies Property 1 and is not locally constant, there exists a strictly increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that the disagreement function $\tilde{D} := \phi(D)$ satisfies Axiom 5.*

Proof of Proposition 5. Recall by Corollary 2 we have that $D_n(e^1, e^2) \geq D_n(p, q)$ for all $p, q \in \Delta_n$, where $e^1 = (1, 0, \dots, 0)$ and $e^2 = (0, 1, 0, \dots, 0)$. Also, notice that for all $p, q \in \Delta_n$ we have that:

$$D_n(p, q) = D_{n+1}((p_1, \dots, p_n, 0), (q_1, \dots, q_n, 0)),$$

by Lemma 4.

Let $n = 2$ and notice that there are two mutually exclusive and collectively exhaustive cases:

1. $D_2(e_1, e_2) > D_2(e_1, (x, 1-x))$ for all $x \in (0, 1)$;
2. $D_2(e_1, e_2) = D_2(e_1, (x, 1-x))$ for some $x \in (0, 1)$;

Consider the first case. Let us prove that for all $p, q \in \Delta_n$ there must exist an $x \in (0, 1)$ such that:

$$D_n(p, q) = D_2(e^1, (x, 1-x)),$$

and for brevity define the function $\phi(x) = D_2(e^1, (x, 1-x))$. This follows directly from the fact that $\phi(1) = 0$ and $\phi(x) \to \infty$ as $x \to 0$, therefore for all $p, q \in \Delta_n$ there exists x such that $D_n(p, q) = D_2(e^1, (x, 1-x))$. Using then that D_2 cannot be locally constant implies that such x is unique.

Let us now show that for all D_n, we can find ψ such that $\psi(D_n(p * q, r * s)) = \psi(D_n(p, r)) + \psi(D_n(q, s))$. Fix a measure D_n, and suppose that for some x, y, z:

$$D_4(e^1 * e^1, (x, 1-x) * (y, 1-y)) = D_2(e^1, (z, 1-z)).$$

We define ψ so that if this is the case, then:

$$\psi(D_2(e^1, (x, 1-x))) + \psi(D_2(e^1, (y, 1-y))) = \psi(D_2(e^1, (z, 1-z)))$$

To show that this is a well-defined function, we need to show that if for some x' and y' we have that:

$$D_4(e^1 * e^1, (x, 1-x) * (y, 1-y)) = D_4(e^1 * e^1, (x', 1-x') * (y', 1-y')), \tag{36}$$

then $\psi(\phi(x)) + \psi(\phi(y)) = \psi(\phi(x')) + \psi(\phi(y'))$. To prove this last point, suppose, without loss, that $\phi(x') > \phi(x)$.

Then there exists w such that:

$$D_4(e^1 * e^1, (w, 1-w) * (x, 1-x)) = D_2(e^1, (x', 1-x')),$$

and then by Property 1 we have that:

$$D_8(e^1 * e^1 * e^1, (w, 1-w) * (x, 1-x) * (y, 1-y)) = D(e^1 * e^1, (x', 1-x') * (y, 1-y)),$$

and by (36) we get that:

$$D_8(e^1 * e^1 * e^1, (w, 1-w) * (x, 1-x) * (y, 1-y)) = D_8(e^1 * e^1 * e^1, (w, 1-w) * (x', 1-x') * (y', 1-y')),$$

so combining these last two equations (and eliminating the x' term) we get that:

$$D_4(e^1 * e^1, (w, 1-w) * (y', 1-y')) = D_2(e^1, (y, 1-y)).$$

Now since $\phi(x') > \phi(x)$, it must be that $\phi(y') < \phi(y)$, so there exists w' such that:

$$D_4(e^1 * e^1, (w', 1-w') * (y', 1-y')) = D_2(e^1, (y, 1-y)),$$

and repeating the same procedure just carried out for w we obtain that:

$$D_2(e^1, (w, 1-w)) = D_2(e^1, (w', 1-w')),$$

which implies that $\psi(\phi(y')) - \psi(\phi(y)) = -(\psi(\phi(x')) - \psi(\phi(x)))$ and then yields that $\psi(\phi(x)) + \psi(\phi(y)) = \psi(\phi(x')) + \psi(\phi(y'))$.

Let us now prove that for all p, q, r, s we have that:

$$\psi(D_{nm}(p * q, r * s)) = \psi(D_n(p, r)) + \psi(D_m(q, s)).$$

Consider $x_{p,r} \in (0,1)$ and $x_{q,s}$ to be defined as follows:

$$D_n(p, r) = D_2(e^1, (x_{p,r}, 1-x_{p,r})) \quad \text{and} \quad D_m(q, s) = D_2(e^1, (x_{q,s}, 1-x_{q,s})). \tag{37}$$

As proved earlier such $x_{p,r}$ and $x_{q,s}$ are unique. Let us now show that:

$$D_{nm}(p * q, r * s) = D_{nm}(e^1 * e^1, (x_{p,r}, 1-x_{p,r}) * (x_{q,s}, 1-x_{q,s})). \tag{38}$$

This follows directly from Property 1, as its first condition can be used for first with $D_n(p, r) \le D_2(e^1, (x_{p,r}, 1-x_{p,r}))$, and then with $D_n(p, r) \ge D_2(e^1, (x_{p,r}, 1-x_{p,r}))$. This shows that:

$$D_{nm}(p * q, r * s) = D_{nm}(e^1 * q, (x_{p,r}, 1-x_{p,r}) * s),$$

and applying the same argument to q, s we find equation (38).

But on pair of beliefs of the form $(e^1 * e^1, (x_{p,r}, 1 - x_{p,r}) * (x_{q,s}, 1 - x_{q,s}))$ we proved earlier that:

$$\psi(D_4(e^1 * e^1, (x_{p,r}, 1 - x_{p,r}) * (x_{q,s}, 1 - x_{q,s}))) = \psi(D_2(e^1, (x_{p,r}, 1 - x_{p,r}))) + \psi(D_2(e^1, (x_{p,r}, 1 - x_{p,r}))).$$

Now using Property 1 we have that:

$$D_4(e^1 * e^1, (x_{p,r}, 1 - x_{p,r}) * (x_{q,s}, 1 - x_{q,s})) = D_{nm}(p * q, r * s),$$

and using equation (37) we obtain:

$$D_{nm}(p * q, r * s) = D_n(p, r) + D_m(q, s).$$

This concludes the first case, in which:

$$D_2(e_1, e_2) > D_2(e_1, (x, 1 - x)), \quad \forall x \in (0, 1).$$

If instead we have that $D_2(e_1, e_2) = D_2(e_1, (x, 1 - x))$ for some $x \in (0, 1)$, then we can consider a similar construction in which we define $\phi(z) := D_2((z, 1 - z), (x, 1 - x))$, for all $z \in (0, 1)$. A similar function ψ can then be defined so that $\psi(D(\cdot, \cdot))$ is additive on all vectors $p, q, r, s \in \Delta_2$. And finally one can show that it must be additive on Δ_n, analogously to what was done in the first case (as the only difference between these two cases is the references point – earlier it was e^1 and now it is x). □

Lemma 4. *If D_n satisfies Axioms 1–5, then $\forall p, q \in \Delta_n$:*

$$D_n(p, q) = D_{n+1}((p_1, \ldots, p_n, 0), (q_1, \ldots, q_n, 0)).$$

Proof of Lemma 4. By the coarsening axiom, Axiom 4, we have that for all $m \geq n$:

$$D_{m+1}(\bar{p}^{m+1}, \bar{q}^{m+1}) \geq D_m(\bar{p}^m, \bar{q}^m), \tag{39}$$

so the sequence $D_m(\bar{p}^m, \bar{q}^m)$ is non-decreasing in m.

By Axiom 5, we can construct the beliefs $p * (1, 0)$:

$$p * (1, 0) := (p_1, \ldots, p_n, 0, \ldots, 0) = \bar{p}^{2n}$$

and $q * (1, 0) = \bar{q}^{2n}$. By the independence axiom, Axiom 5, we have that:

$$D_{2n}(\bar{p}^{2n}, \bar{q}^{2n}) = D_{2n}(p * (1, 0), q * (1, 0)) = D_n(p, q) + D_2((1, 0), (1, 0)),$$

but $D_2((1, 0), (1, 0)) = 0$ by Axiom 1, so that we get $D_{2n}(\bar{p}^{2n}, \bar{q}^{2n}) = D_n(p, q)$, and by induction one can prove that:

$$D_{mn}(\bar{p}^{mn}, \bar{q}^{mn}) = D_n(p, q), \tag{40}$$

for all $m \in \mathbb{N}$. Hence we have that the sequence of numbers $D_m(\bar{p}^m, \bar{q}^m)$ is weakly increasing in m (by equation

(39)) and periodic (by equation (40)), but then it must be that it is a constant sequence, and hence we get the thesis. □

Lemma 5. *Let $p, q \in \Delta_n$ and suppose that $Supp(p) \cap Supp(q) = \emptyset$, meaning $p_j q_j = 0$, for all j. Then if D satisfies Axioms 1–5, $D(p,q) = +\infty$.*

Proof of Lemma 5. Notice that by Axiom 4 we have that for all $p, q \in \Delta_n$ with disjoint support:

$$D(p,q) \geq D((1,0),(0,1)),$$

simply by taking the partition of Θ consisting in $A_1 := Supp(p)$, $A_2 = \Theta \setminus Supp(p)$. Hence if we prove that $D((1,0),(0,1)) = +\infty$, the general statement follows.

Assume that $D((1,0),(0,1)) < \infty$, and consider the joint distributions:

$$(1,0) * (1,0) = (1,0,0,0) \quad \text{and} \quad (0,1) * (0,1) = (0,0,0,1).$$

By Lemma 4 we have that $D((1,0,0,0),(0,0,0,1)) = D((1,0),(0,1))$, and by Axiom 5, $D((1,0)*(1,0),(0,1)*(0,1)) = 2D((1,0),(0,1))$.

Hence:

$$D((1,0),(0,1)) = 2D((1,0),(0,1)),$$

then assuming $D((1,0),(0,1)) < \infty$ we obtain that $D((1,0),(0,1)) = 0$ - which contradicts the fact that $D(p,q) > 0$ for all $p \neq q$, so $D((1,0),(0,1)) = \infty$. □

Proposition 6. *Let $p, q \in \Delta_n^\circ$ be two beliefs, and suppose that $\frac{p_1}{q_1} = \frac{p_2}{q_2}$. Then:*

$$D_n(p,q) = D_{n-1}((p_1+p_2, p_3, \ldots, p_n),(q_1+q_2, q_3, \ldots, q_n)).$$

Proof of Proposition 6. Let's prove this by showing that both inequalities hold. Namely:

1. $D_n(p,q) \geq D_{n-1}((p_1+p_2, p_3, \ldots, p_n),(q_1+q_2, q_3, \ldots, q_n))$;
2. $D_n(p,q) \leq D_{n-1}((p_1+p_2, p_3, \ldots, p_n),(q_1+q_2, q_3, \ldots, q_n))$;

(1) follows directly from the coarsening axiom, Axiom 4:

$$D(p,q) \geq D((p_1+p_2, p_3, \ldots, p_n),(q_1+q_2, q_3, \ldots, q_n)).$$

In order to prove (2) notice that by the expandability Lemma (Lemma 4) we have that:

$$D((p_1+p_2, p_3, \ldots, p_n),(q_1+q_2, q_3, \ldots, q_n)) = D((0, p_1+p_2, p_3, \ldots, p_n),(0, q_1+q_2, q_3, \ldots, q_n))$$

and:

$$D((p_1+p_2, p_3, \ldots, p_n),(q_1+q_2, q_3, \ldots, q_n)) = D((p_1+p_2, 0, p_3, \ldots, p_n),(q_1+q_2, 0, q_3, \ldots, q_n)).$$

Notice also that:

$$p = \lambda(p_1 + p_2, 0, p_3, \dots, p_n) + (1-\lambda)(0, p_1 + p_2, p_3, \dots, p_n),$$

with $\lambda = \frac{p_1}{p_1+p_2}$, and similarly:

$$q = \lambda'(q_1 + q_2, 0, q_3, \dots, q_n) + (1-\lambda')(0, q_1 + q_2, q_3, \dots, q_n),$$

with $\lambda' = \frac{q_1}{q_1+q_2}$. But if $\frac{p_1}{p^2} = \frac{q^1}{q^2}$ then $\lambda = \lambda'$ and then we can use Axiom 3 to obtain that:

$$\begin{aligned} D(p,q) \leq \max\{&D((0, p_1 + p_2, p_3, \dots, p_n), (0, q_1 + q_2, q_3, \dots, q_n)),\\ &D((p_1 + p_2, 0, p_3, \dots, p_n), (q_1 + q_2, 0, q_3, \dots, q_n))\} =\\ &D((p_1 + p_2, p_3, \dots, p_n), (q_1 + q_2, q_3, \dots, q_n)), \end{aligned} \tag{41}$$

where the last equality follows from Lemma 4. This proves (2) and gives the thesis. □

Corollary 3. *Fix a belief $p \in \Delta_n^\circ$ and consider the segment joining $q^1 = (q_1 + q_2, 0, q_3, \dots, q_n)$ and $q^2 = (0, q_1 + q_2, q_3, \dots, q_n)$:*

$$[q^1, q^2] = \{r \in \Delta_n \mid r = \lambda q^1 + (1-\lambda) q^2, \lambda \in [0,1]\}.$$

The minimum of the distance between p and $[q^1, q^2]$ is reached at the $q^* \in [q^1, q^2]$ satisfying:

$$\frac{q_1^*}{q_2^*} = \frac{p_1}{p_2},$$

And for all $r, r' \in [q^1, q^2]$ we have that if $r \in [r', q^*]$ then

$$D_n(p, r) \leq D_n(p, r').$$

Proof of Corollary 3. Thanks to Proposition 6 we have that if $\frac{p_1}{q_1^*} = \frac{p_2}{q_2^*}$ then:

$$D(p, q^*) = D((p_1 + p_2, \dots, p_n), (q_1 + q_2, \dots, q_n)),$$

and since $\forall r \in [q^1, q^2]$ we have that:

$$D(p, r) \geq D((p_1 + p_2, \dots, p_n), (q_1 + q_2, \dots, q_n)),$$

by Axiom 4, then we obtain that for all $r \in [q^1, q^2]$:

$$D(p, r) \geq D(p, q^*), \quad \forall r \in [q^1, q^2].$$

The latter part of the result is then easily obtained by the quasi-convexity of the function $f(q) := D(p, q)$. Namely, pick r and r' with $r \in [r', q^*]$, then by Axiom 3 we have that:

$$D(p, r) \leq \max\{D(p, r'), D(p, q^*)\},$$

but since $D(p,q^*) \leq D(p,r')$ we have that $D(p,r) \leq D(p,r')$. □

A.3 Proofs of Section 4

Proposition 7. *For any $p,q \in \Delta_n^\circ$ with $p \neq q$ and for all D measures of disagreement there exists an experiment $\pi = (S, f(s|\theta))$ and a signal $s' \in S$ such that:*

$$D(p(s'), q(s')) < D(p,q).$$

For any D measure of disagreement, there exist $p,q \in \Delta_n$, an experiment $\pi = (S, f(s|\theta))$, and a signal $s \in S$ such that:

$$D(p(s), q(s)) > D(p,q).$$

Proof of Proposition 7. The first statement is trivial, as we can pick s' that fully reveals state θ_1, $f(s'|\theta_1) = 1$ and $f(s'|\theta_j) = 0$ for all $j = 2, \ldots, n$. Since p,q are fully mixed we have that $p(s') = q(s') = (1,0,\ldots,0)$, and hence:

$$D(p(s'), q(s')) = 0 < D(p,q),$$

because $p \neq q$.

To prove the second statement, notice that for any $p' \neq q'$ we can find a signal s' such that $D(p'(s'), q(s')) < D(p', q')$. But then notice that we can define a signal s by:

$$f(s|\theta) = \frac{k}{f(s'|\theta)}, \quad \forall \theta,$$

where k is a constant that makes sure $f(s|\theta) < 1$. Now, defining $p := p'(s')$, $q := q'(s')$ we have that $p(s) = p'$ and $q(s) = q'$ and therefore:

$$D(p,q) = D(p'(s'), q(s')) < D(p', q') = D(p(s), q(s)),$$

which yields the thesis.

□

Theorem 2. *Let p be a degenerate distribution on the true state of the world, and let $q \in \Delta_n$. Then for all measures of disagreement, and for all $\tilde{\pi} \preceq \pi$ we have that:*

$$\mathbb{E}_p^{\pi}[D(p(s), q(s))] \leq \mathbb{E}_p^{\tilde{\pi}}[D(p(\tilde{s}), q(\tilde{s}))].$$

Proof of Theorem 2. Let us consider the characterization of the disagreement functions given by equation (4). If $z \geq 0.5$ the results follows from the proof of Theorem 3 below, so we do not repeat it here (notice how the statement of this Theorem is a particular case of point 3 in Theorem 3).

Now, let $z \leq -0.5$ and let $p = (1,0,\ldots,0)$ without loss. Let S be the signals of the experiment π such that $f(s|\theta_1) > 0$ and $\tilde{S}$ those of $\tilde{\pi}$ with the same properties. If for all $s \in S$ we have that $f(s|\theta_j) = 0$ for all $j \geq 2$, then we have that:

$$\mathbb{E}_p^{\pi}[D(p(s),q(s))] = 0,$$

so the statement of the theorem is true. If instead there exists a signal s such that $f(s|\theta_j) > 0$ for some $\theta_j \geq 2$, then there exists a $\tilde{s} \in \tilde{S}$ such that $g(\tilde{s}|\theta_j) > 0$ because:

$$g(\tilde{s}|\theta_j) = \sum_s \lambda_{s,\tilde{s}} f(s|\theta_j), \tag{42}$$

and $\sum_{\tilde{s}} \lambda_{s,\tilde{s}} = 1$. But then observe that upon observing such signal $\tilde{s}$ (which has positive probability according to p) we have that $q_j(\tilde{s}) > 0 = p_j(\tilde{s})$, and then $D^z(p(\tilde{s}),q(\tilde{s})) = \infty$ and therefore we obtain:

$$\mathbb{E}_p^{\tilde{\pi}}[D(p(\tilde{s}),q(\tilde{s}))] = \infty,$$

so the statement of the Theorem is trivially true.

The only case that we are left to analyze is $|z| < 0.5$, and this is the only non-trivial part of the proof. Notice that, for all $s \in S$, $p_1(s) = 1$ and then $D^z(p(s),q(s)) = -\log\left(q_1(s)^{-z+0.5}\right)$, and similarly for $\tilde{s}$, therefore we have to prove that:

$$-(0.5-z)\sum_s f(s|\theta_1)\log(q_1(s)) \leq -(0.5-z)\sum_{\tilde{s}} g(\tilde{s}|\theta_1)\log(q_1(\tilde{s})),$$

and since $-(0.5-z) < 0$ for $|z| < 0.5$ this is equivalent to

$$\sum_{\tilde{s}} g(\tilde{s}|\theta_1)\log(q_1(\tilde{s})) \leq \sum_s f(s|\theta_1)\log(q_1(s))$$

Rewrite the RHS using the fact that, for all s, $\sum_{\tilde{s}} \lambda_{s,\tilde{s}} = 1$, that gives:

$$\sum_s f(s|\theta_1)\log(q_1(s)) = \sum_s \sum_{\tilde{s}} \lambda_{s,\tilde{s}} f(s|\theta_1)\log(q_1(s)) = \sum_{\tilde{s}} \sum_s \lambda_{s,\tilde{s}} f(s|\theta_1)\log(q_1(s)),$$

and now multiply and divide by $g(\tilde{s}|\theta_1)$ finding:

$$\sum_s f(s|\theta_1)\log(q_1(s)) = \sum_{\tilde{s}} \sum_s \lambda_{s,\tilde{s}} f(s|\theta_1)\log(q_1(s)) = \sum_{\tilde{s}} g(\tilde{s}|\theta_1) \sum_s \frac{\lambda_{s,\tilde{s}} f(s|\theta_1)}{g(\tilde{s}|\theta_1)} \log(q_1(s)), \tag{43}$$

now notice that the terms $\frac{\lambda_{s,\tilde{s}} f(s|\theta_1)}{g(\tilde{s}|\theta_1)}$ are probability distributions as $\sum_s \frac{\lambda_{s,\tilde{s}} f(s|\theta_1)}{g(\tilde{s}|\theta_1)} = 1$, and then using the fact that $-\log(\cdot)$ is convex we get that:

$$\sum_s \frac{\lambda_{s,\tilde{s}} f(s|\theta_1)}{g(\tilde{s}|\theta_1)}(-\log(q_1(s)^{-1})) \geq -\log\left(\sum_s \frac{\lambda_{s,\tilde{s}} f(s|\theta_1)}{g(\tilde{s}|\theta_1)} \frac{f(s;q)}{q_1 f(s|\theta_1)}\right) = -\log\left(\frac{\sum_s \lambda_{s,\tilde{s}} f(s;q)}{q_1 g(\tilde{s}|\theta_1)}\right),$$

and since $\sum_s \lambda_{s,\tilde{s}} f(s;q) = f(\tilde{s};q)$, we have that:

$$-\log\left(\frac{\sum_s \lambda_{s,\tilde{s}} f(s;q)}{q_1 g(\tilde{s}|\theta_1)}\right) = \log(q_1(\tilde{s})),$$

and plugging this into (43) we get:

$$\sum_s f(s|\theta_1)\log(q_1(s)) \geq \sum_{\tilde{s}} g(\tilde{s}|\theta_1)\log(q_1(\tilde{s})),$$

which is the thesis.

□

Lemma 6. *D satisfies Axioms 1–7 if and only if* $D(p,q) = aD^z(p,q)$ *for some* $z \geq 0.5$*, and* $a > 0$.

Proof of Lemma 6. Notice that for all $z \geq 0.5$, D^z satisfies Axioms 7, so it is enough to show that for $z < 0.5$ there exists p,q with $q_j = 0 < p_j$ such that $D(p,q) < +\infty$. To this extent, just consider $q = (1,0)$ and $p = (\alpha, 1-\alpha)$ and the result follows.

□

Theorem 3. *Let D be a measure of disagreement. Then the following statements are equivalent:*

1. *D satisfies Axiom 7;*
2. $D(p,q) = aD^z(p,q)$ *for some* $z \geq 0.5$ *(and* $a > 0$*);*
3. *for all experiments* $\pi \succeq \tilde{\pi}$ *and priors* $p,q \in \Delta_n$*:*

$$\mathbb{E}_p^{\pi}[D(p(s),q(s))] \leq \mathbb{E}_p^{\tilde{\pi}}[D(p(\tilde{s}),q(\tilde{s}))].$$

Proof of Theorem 3. The equivalence of 1 and 2 was established in Lemma 6 above. To prove that 2⇔ 3 we will prove separately that 2 implies 3 and vice versa:

(2 ⇒ 3) First off, notice that we can assume without loss of generality that $p \in \Delta_n^\circ$, and if $q \notin \Delta_n^\circ$ then we have that on both sides disagreement will be infinite, so the statement is trivial. For the rest of the proof, then, assume $p,q \in \Delta_n^\circ$.

We will use one of the characterizations of sufficiency proved by Blackwell and Girshick (1954):

Result 9. *We have that* $\tilde{\pi} \preceq \pi$ *if and only if for all* $\phi : \Delta_n \to \mathbb{R}$ *concave and all priors* $p \in \Delta_n$*:*

$$\sum_{s\in S} \mathbb{P}_p(s)\phi(p(s)) \leq \sum_{\tilde{s}\in\tilde{S}} \mathbb{P}_p(\tilde{s})\phi(p(\tilde{s})).$$

Therefore we are done if we prove that the functions $D_z(p(s),q(s))$ can be written as a concave function of $p(s)$ – for all $z \geq 0.5$. To do this, we apply the formula proved by Alonso and Camara (2016) that allows to write the posterior $q(s)$ as a function of $p(s)$ (and of both priors, p and q).

$$q_j(s) = \frac{\frac{q_j}{p_j}p_j(s)}{\sum_i \frac{q_i}{p_i}p_i(s)} = \frac{r_j p_j(s)}{p(s)\cdot r}, \tag{44}$$

where we defined the likelihood ratio $r_j := \frac{q_j}{p_j}$ and denoted by $\cdot$ the scalar product in $\mathbb{R}^n$. Using this formula we get:

$$D_z(p(s), q(s)) = \log\left(\sum_j p_j(s)\left(\frac{p_j(s)}{q_j(s)}\right)^{z-0.5}\right)$$
$$= \log\left(\sum_j p_j(s)\left(\frac{p(s)\cdot r}{r_j}\right)^{z-0.5}\right)$$
$$= (z-0.5)\log(p(s)\cdot r) + \log(p(s)\cdot r^z), \quad (45)$$

where we defined $r_j^z := \frac{1}{r_j^{z-0.5}}$.

If $z > 0.5$, then, we have that $D_z(p(s), q(s))$ is a concave function of $p(s)$ because r^z is strictly positive (so $\log(p(s)\cdot r^z)$ is concave), and $(z-0.5)\log(p(s)\cdot r)$ is also concave because $r \gg 0$ and $z - 0.5 > 0$.

On the other hand, if $z = 0.5$, we get:

$$D_{0.5}(p(s), q(s)) = \sum_j p_j(s)\log\left(\frac{p_j(s)}{q_j(s)}\right)$$
$$= \sum_j p_j(s)(\log(p(s)\cdot r) - \log(r_j)) = -p(s)\cdot\log(r) + \log(p(s)\cdot r), \quad (46)$$

where we defined the vector $\log(r) := (\log(r_1), \ldots, \log(r_n))$.

Then we have that $D_{0.5}(p(s), q(s))$ is the sum of a linear and a concave function of $p(s)$, and then it is a concave function of $p(s)$.

(3 ⇒ 2) To prove this implication fix a $z < 0.5$. Pick Then let $p \in \Delta_2^\circ$ and $q^n = (1/n, 1-1/n)$. Notice that for all n, $D^z(p,q) < D^z(p,(0,1)) < +\infty$, let us show that we can find a sequence of experiments $\pi(\epsilon)$ such that as $\epsilon \to 0$, $\mathbb{E}_p^{\pi(\epsilon)}(D(p(s), q^n(s))) \to +\infty$ (provided we pick n big enough).

This will imply that 3 is violated (with $\tilde{\pi}$ being the null experiment). Pick $\pi(\epsilon)$ to be an experiment with two signals, and conditional probabilities defined as follows:

$$f(s_0|\theta_1) = 1-\epsilon, \quad f(s_1|\theta_1) = \epsilon,$$

$$f(s_1|\theta_2) = \epsilon \quad f(s_0|\theta_2) = 1-\epsilon.$$

Consider the posteriors after observing signal s_1:

$$p_2(s_0) = \frac{\epsilon p_2}{\epsilon p_2 + (1-\epsilon)p_1}; \quad q_2(s_0) = \frac{\left(1-\frac{1}{n}\right)\epsilon}{\left(1-\frac{1}{n}\right)\epsilon + \frac{1}{n}(1-\epsilon)}.$$

Notice that if we take $\epsilon = n^{-0.5}$, as $n \to +\infty$ we have that $p_2(s_0) \to 0$ and $q_2(s_0) \to 1$. Therefore as we let $n \to \infty$ we have that:

$$\mathbb{E}_p^{\pi(n^{-0.5})}(D(p(s), q(s))) \geq \mathbb{P}_p(s_0) D^z((1-\delta,\delta),(\delta,1-\delta)), \quad (47)$$

for all $\delta \in (0,1)$. Since for all n we have $\mathbb{P}_p(s_0) > 0$ and $D^z(p,q^n) < D^z(p,(0,1)) < \infty$ we obtain that there exists an n such that:

$$\mathbb{E}_p^{\pi(n^{-0.5})}\left(D^z(p(s),q(s))\right) > D^z(p,q^n),$$

because we know that for all z, $D^z((1-\delta,\delta),(\delta,1-\delta)) \to \infty$ as $\delta \to 0$ and hence we can use equation (47) to conclude.

□

Theorem 4. *Let $p^1,\ldots,p^J \in \Delta_n^\circ$, and let $(u_j)_j$ be CRRA utility functions with parameters $z_j \in (0,1)$. The equilibrium of the economy exists and is unique, and it can be characterized as follows:*

- $(\Pi_i)_i$ *is the unique solution to the problem:*

$$\min_{q\in\Delta(I)} \sum_j D^{z_j+0.5}(q, p^j(\cdot|I)), \tag{48}$$

  i.e. the prices of the Arrow Debreu securities are the beliefs that minimize weighted disagreement with the agents;

- *the expected volume of trade is given by:*

$$V_j((x_i^j)_i) = \exp(D^{z_j+0.5}(p^j(\cdot|I),\Pi) + D^{-z_j+0.5}(\Pi, p^j(\cdot|I))) - 1 \quad \forall j \in J; \tag{49}$$

- *the gains from trade are given by:*

$$G_j((x_i^j)_i) = \frac{1}{\frac{1}{z_j}-1}\left(1-\exp\left(-\frac{D^{z_j-0.5}(\Pi,p^j(\cdot|I))}{z_j}\right)\right), \quad \forall j \in J. \tag{50}$$

Proof of Theorem 4. To prove the first point, let us solve the maximization problem of agent j:

$$\max_{(x_i^j)_i} \sum_i p_i^j \frac{(x_i^j)^{1-\frac{1}{z_j}}}{1-\frac{1}{z_j}} \quad \text{subject to} \quad \sum_i \Pi_i x_i \le 0,$$

this gives first order conditions (for the amount of good x_i^j):

$$p_i^j (x_i^j)^{-1/z_j} - \lambda \Pi_i = 0,$$

where λ is the Lagrainge multiplier. Isolating x_i^j we get:

$$x_i^j = \left(\frac{p_i^j}{\lambda \Pi_i}\right)^{z_j}, \tag{51}$$

and in equilibrium the budget constraint binds, $\sum_i \Pi_i x_i^j = 1$, so that:

$$\lambda^{z_j} = \sum_i \Pi_i \left(\frac{p_i^j}{\Pi_i}\right)^{z_j},$$

so plugging this into (51) we get:

$$x_i^j = \frac{\left(\frac{p_i^j}{\Pi_i}\right)^{z_j}}{\sum_k \Pi_k \left(\frac{\Pi_k}{p_k^j}\right)^{-z_j}}, \tag{52}$$

so summing across agents j we get:

$$1 = \sum_j x_{i,j} = \sum_j \frac{\left(\frac{p_i^j}{\Pi_i}\right)^{z_j}}{\sum_k \Pi_k \left(\frac{\Pi_k}{p_k^j}\right)^{-z_j}},$$

which in particular implies that the RHS does not depend on i and we obtain (picking i and $i+1$):

$$\sum_j \frac{\left(\frac{p_i^j}{\Pi_i}\right)^{z_j} - \left(\frac{p_{i+1}^j}{\Pi_{i+1}}\right)^{z_j}}{\sum_k \Pi_k \left(\frac{\Pi_k}{p_k^j}\right)^{-z_j}} = 0,$$

and notice that this can be rewritten as:

$$\sum_j \partial_i^1 D^{z_j+0.5}(\Pi, p^j(\cdot|I)),$$

where $\partial_i^1 D$ stands for the derivative of D on the first argument (Π, in this case) along the direction $z_{i,i+1}$. Thanks to the quasi-convexity of D (Axiom 3) we have the first result of our theorem.

To obtain the second result, simply notice that using equation (52) we find:

$$V_j((x_i^j)_i) = \sum_i p_i^j x_i^j - 1 = \frac{\sum_i p_i^j \left(\frac{\Pi_i}{p_i^j}\right)^{-z_j}}{\sum_k \Pi_k \left(\frac{\Pi_k}{p_k^j}\right)^{-z_j}} - 1 = \exp(D^{z_j+0.5}(p^j(\cdot|I),\Pi) + D^{-z_j+0.5}(\Pi, p^j(\cdot|I))) - 1.$$

Finally, to get the formula for the gains from trade, notice that $u_j(1) = \frac{1}{1-\frac{1}{z_j}}$, and:

$$u_j(x_i^j) = \frac{1}{1-\frac{1}{z_j}} \left(\frac{\left(\frac{\Pi_i}{p_i^j}\right)^{-z_j}}{\sum_k \Pi_k \left(\frac{\Pi_k}{p_k^j}\right)^{-z_j}} \right)^{1-\frac{1}{z_j}} \Rightarrow \sum_i p_i^j u(x_i^j) = \frac{1}{1-\frac{1}{z_j}} \left(\sum_i p_i^j \left(\frac{\Pi_i}{p_i^j}\right)^{1-z_j} \right) \cdot \left(\sum_k \Pi_k \left(\frac{\Pi_k}{p_k^j}\right)^{-z_j} \right)^{\frac{1}{z_j}-1},$$

so the gains from trade are:

$$G_j((x_i^j)_i) = \frac{1}{\frac{1}{z_j}-1} \left(1 - \exp\left(-\frac{D^{z_j-0.5}(\Pi, p^j(\cdot|I))}{z_j}\right)\right), \quad \forall j \in J.$$

□

Theorem 5. *Let $(u_j)_{j\in J}$ be a family of strictly concave, twice continuously differentiable utility functions de-*

fined in a compact neighborhood of 1*. Suppose (without loss of generality) that* $u_j'(1) = 1$*, and assume that* $-1/u_j''(1) =: z_j \in (0,1)$*. Let* $(p^{j,(m)})_{m\in\mathbb{N}}^{j\in J}$ *be any family of merging and uniformly mixed beliefs.*

For all m *the equilibrium exists and is unique, denote it by* $((x_i^{j,(m)})_{i=1,\dots,n}^{j\in J}, (\Pi_i^m)_i)$*. For all* m*, let* $\tilde{\Pi}^m$ *be the solution of the problem* $\min_{q\in\Delta_n^\circ} \sum_j D^{z_j+0.5}(q, p^{j,(m)})$*. We have that:*

- *for all* j*,* $\lim_{m\to\infty} \frac{D^{z_j+0.5}(p^{j,(m)},\Pi^m)}{D^{z_j+0.5}(p^{j,(m)},\tilde{\Pi}^m)} = 1$*, so the disagreement between any belief* p^j *and the* approximate *equilibrium* $\tilde{\Pi}^m$ *is asymptotically equivalent to the disagreement with the market belief* Π^m*.*
- *The volume of trades is asymptotic to the volume of trades in the economy with CRRA utility functions:*

$$\lim_{m\to\infty} \frac{\sum_i p_i^{j,(m)}(x_i^{j,(m)} - 1)}{D^{z_j+0.5}(p^{j,(m)}, \tilde{\Pi}^m) + D^{-z_j+0.5}(\tilde{\Pi}^m, p^{j,(m)})} = 1. \tag{53}$$

- *The gains from trade are asymptotic to the volume of trades in the economy with CRRA utility functions:*

$$\lim_{m\to\infty} \frac{\sum_i p_i^{j,(m)} u_i(x_i^{j,(m)})}{\frac{1}{1-z_j} D^{z_j-0.5}(\tilde{\Pi}^m, p^{j,(m)})} = 1. \tag{54}$$

Proof of Theorem 5. The existence and uniqueness of the equilibrium is granted by the strictly concavity of the utility functions, and by the compactness of the domain. Notice also that for m large enough such solution must be in the interior of the domain, because as $p^{j,(m)}$ merge, the equilibrium $x_i^{j,(m)} \to 1$, which is in the interior of the domain of u_j (by hypothesis). In the rest of the proof we assume that first order conditions hold with equality, even though in general this will hold only eventually.

Let $\tilde{u}_j$ to be the CRRA approximation of u_j at 1, that is:

$$\tilde{u}_j(x) := \frac{x^{1-\frac{1}{z_j}}}{1-\frac{1}{z_j}},$$

where $z_j = -\frac{1}{u''(1)}$.

For a given J-tuple of beliefs $\mathbf{p} := ((p_i^1)_i, \dots, (p_i^J)_i)$ denote by $((x_i^j(\mathbf{p}))_{i,j}, (\Pi_i(\mathbf{p}))_i, (\lambda_j(\mathbf{p}))_j)$ to be the solution of the model with beliefs $\mathbf{p}$ and utility functions u_j, where x_i^j are the allocations, Π_i are the prices, and λ_j are the Lagrainge multipliers. Analogously, let $((\tilde{x}_i^j(\mathbf{p}))_{i,j}, (\tilde{\Pi}_i(\mathbf{p}))_i, (\tilde{\lambda}_j(\mathbf{p}))_j)$ be the solution to the problem with utility functions $\tilde{u}_j$.

It is a well-known result of smooth economies that the functions $\mathbf{p} \mapsto x_i^j(\mathbf{p})$ are local diffeomorphism (for all j, i) and so are $\mathbf{p} \mapsto \Pi_i(\mathbf{p})$ (for all i) and $\mathbf{p} \mapsto \lambda_j(\mathbf{p})$ (for all j). Furthermore, denoting by $\mathbf{p}^* = (p^*, \dots, p^*)$ the vector of agreement beliefs we have that:

$$\partial_{\mathbf{v}} x_i^j(\mathbf{p}^*) = \partial_{\mathbf{v}} \tilde{x}_i^j(\mathbf{p}^*), \tag{55}$$

where we denote by ∂_v the directional derivative in the direction $\mathbf{v} = (v^1, \dots, v^J)$, where for all $j \in J$: $\sum_i v_i^j = 0$, so that $p^j + \epsilon v^j \in \Delta_n^\circ$ for ϵ small. The equalities in (55) follow from the fact that $((x_i^j(\mathbf{p}))_{i,j}, (\Pi_i(\mathbf{p}))_i, (\lambda_j(\mathbf{p}))_j)$ and

$((\tilde{x}_i^j(\mathbf{p}))_{i,j}, (\tilde{\Pi}_i(\mathbf{p}))_i, (\tilde{\lambda}_j(\mathbf{p}))_j)$ are the solutions to a system of equations given by first order conditions (FOC), budget constraints (BC), and market clearing conditions (MC). Equations BC and MC are the same in the general economy (utility functions u_j) and in the CRRA approximation (utility functions $\tilde{u}_j$). On the other hand, FOC depend on the utility functions for general vectors of beliefs $\mathbf{p}$. Nonetheless, the FOC:

$$p_i^{j,(m)} u_j'(x_i^{j,(m)}) - \lambda_j^m \Pi^m = 0, \quad \text{and} \quad p_i^{j,(m)} \tilde{u}_j'(\tilde{x}_i^{j,(m)}) - \tilde{\lambda}_j^m \tilde{\Pi}^m = 0,$$

agree for $\mathbf{p}$ close $\mathbf{p}^*$, since we picked $\tilde{u}_j$ to satisfy $\tilde{u}_j'(1) = u_j'(1)$ and $\tilde{u}_j''(1) = u_j''(1)$, and $x_i^j = 1$ is the only equilibrium at $\mathbf{p} = \mathbf{p}^*$. Therefore the Implicit function theorem implies that:

$$\partial_{\mathbf{v}} x_i^j(\mathbf{p}^*) = \partial_{\mathbf{v}} \tilde{x}_i^j(\mathbf{p}^*), \tag{56}$$

and similarly:

$$\partial_{\mathbf{v}} \Pi_i(\mathbf{p}^*) = \partial_{\mathbf{v}} \tilde{\Pi}_i(\mathbf{p}^*) \quad \text{and} \quad \partial_{\mathbf{v}} \lambda_j(\mathbf{p}^*) = \partial_{\mathbf{v}} \tilde{\lambda}_j(\mathbf{p}^*). \tag{57}$$

Now, for any sequence of beliefs $\mathbf{p}$ convergent to $\mathbf{p}^*$, let us prove that (for all i, j):

$$\lim_{\mathbf{p} \to \mathbf{p}^*} \frac{x_i^j(\mathbf{p}) - 1}{\tilde{x}_i^j(\mathbf{p}) - 1} = 1, \tag{58}$$

where the limit is taken in the usual norm on the product space $\Delta_n^\circ \times \cdots \times \Delta_n^\circ$. This result implies the statement of the theorem for any sequence of merging beliefs.

Notice that because of the FOC $x_i^j = (u')^{-1}\left(\frac{\lambda_j(\mathbf{p})\Pi_i(\mathbf{p})}{p_{i,j}}\right)$, and $\tilde{x}_i^j(\mathbf{p}) = \left(\frac{\tilde{\lambda}_j(\mathbf{p})\tilde{\Pi}_i(\mathbf{p})}{p_{i,j}}\right)^{-z_j}$, where in the latter we used the fact that $\tilde{u}_j$ is CRRA. Denote by $\phi(\mathbf{p}) := \frac{\lambda_j(\mathbf{p})\Pi_i(\mathbf{p})}{p_{i,j}}$, and similarly define $\tilde{\phi}(\mathbf{p})$. Thanks to the local results at $\mathbf{p}^*$ (equations (57) and (56)) we have that:

$$\partial_{\mathbf{v}} \phi(\mathbf{p}^*) = \partial_{\mathbf{v}} \tilde{\phi}(\mathbf{p}^*), \quad \forall \mathbf{v}.$$

The limit in equation (58) is of the form 0/0 and because of the smoothness assumptions we can apply the multidimensional de l'Hopital rule. In other words, if we prove that for all $\mathbf{v}$:

$$\lim_{\mathbf{p} \to \mathbf{p}^*} \frac{\partial_{\mathbf{v}}(x_i^j(\mathbf{p}) - 1)}{\partial_{\mathbf{v}}(\tilde{x}_i^j(\mathbf{p}) - 1)} = 1, \tag{59}$$

then the limit in equation (58) follows. Rewriting the limit (59) we get:

$$\lim_{\mathbf{p} \to \mathbf{p}^*} \frac{-\frac{1}{u''((u')^{-1}(\phi(\mathbf{p})))} \partial_{\mathbf{v}} \phi(\mathbf{p})}{-z_j \tilde{\phi}(\mathbf{p}) \partial_{\mathbf{v}} \tilde{\phi}(\mathbf{p})}, \tag{60}$$

and $\phi(\mathbf{p}) \to 1$, $\tilde{\phi}(\mathbf{p}) \to 1$, $\frac{1}{u''((u')^{-1}(\phi(\mathbf{p})))} = -z_j$, and $\partial_{\mathbf{v}} \phi(\mathbf{p}), \partial_{\mathbf{v}} \tilde{\phi}(\mathbf{p}) \to \partial_{\mathbf{v}} \phi(\mathbf{p}^*) \neq 0$.[32] This proves equation (58).

[32]The fact that these differentials are not null follows from the fact that ϕ is a local diffeomorphism at $\mathbf{p}^*$.

Summing on states and weighting by p_i^j we get:

$$\lim_{\mathbf{p}\to\mathbf{p}^*} \frac{\sum_i x_i^j(\mathbf{p}) p_i^j}{\sum_i \tilde{x}_i^j(\mathbf{p}) p_i^j} = 1,$$

and the denominator is asymptotic to $D^{z_j+0.5}(p^j, \tilde{\Pi}(\mathbf{p})) + D^{-z_j+0.5}(\tilde{\Pi}(\mathbf{p}), p^j)$ as $\mathbf{p}\to\mathbf{p}^*$. Therefore equation (54) follows and (53), is obtained analogously (thanks to the continuity of u_j and $\tilde{u}_j$).

To obtain equation disagreement between the prices Π and the approximated prices $\tilde{\Pi}$ notice that:

$$\frac{p_i^j}{\Pi_i^j(\mathbf{p})} = \lambda_j(\mathbf{p}) u'(x_i^j(\mathbf{p})) \quad \text{and} \quad \frac{p_i^j}{\tilde{\Pi}_i^j(\mathbf{p})} = \tilde{\lambda}_j(\mathbf{p}) \tilde{u}'(\tilde{x}_i^j(\mathbf{p})),$$

so equation (58) implies that for all i, j:

$$\lim_{\mathbf{p}\to\mathbf{p}^*} \frac{\frac{p_i^j}{\Pi_i^j(\mathbf{p})}}{\frac{p_i^j}{\tilde{\Pi}_i^j(\mathbf{p})}} = 1,$$

and then the first statement of the theorem follows from the continuity of the logarithm and the sum. □

B Renyi's Divergence Axiomatization

The first main difference between Renyi's approach and ours is that Renyi's postulates involve *generalized distributions*, that is finite dimensional vectors $(x_i)_i$ satisfying:

$$x_i \geq 0, \quad \text{and} \quad \sum_i x_i \leq 1.$$

We will denote the set of generalized distributions in dimension n as G_n. Even though this appears as a small difference between Renyi's and our axiomatization, it will have important implications, because Renyi's axioms will apply to a much larger set than our (all vectors in G_n, not only on the vectors in Δ_n). Let us introduce Renyi's postulates (with the numbering of Rényi (1961)):

(P6) $I(x|y)$ is unchanged if the elements of x and y are rearranged;

(P7) for all $n \in N$ and for all $x, y \in G_n$, $I(x, y) \geq 0$ if $x_i \geq y_i$ for all i; and $I(x|y) \leq 0$ if $x_i \leq y_i$ for all i;

(P8) $I(1, 1/2) = 1$;

(P9) for all $n, m \in \mathbb{N}$ and for any $x^1, y^1 \in G_n$ and $x^2, y^2 \in G_m$: $I(x^1 * x^2 | y^1 * y^2) = I(x^1|y^1) + I(x^2|y^2)$;

(P10) for all $x \in G_n$ and $y \in G_m$, denote by $x \cup y := (x_1, \ldots, x_n, y_1, \ldots, y_m) \in G_{n+m}$. There exists strictly increasing function $g:\mathbb{R}\to\mathbb{R}$ such that if x^1, x^2, y^1, y^2, $x^1 \cup x^2$ and $y^1 \cup y^2$ are generalized distributions:[33]

$$I(x^1 \cup x^2 | y^1 \cup y_2) = g^{-1}\left(\frac{\left(\sum_i x_i^1\right) g(I(x^1|x^2)) + \left(\sum_j x_j^2\right) g(I(y^1|y^2))}{\sum_i x_i^1 + \sum_j x_j^2}\right)$$

The first postulate, $(P6)$, is equivalent to our Axiom 2 (this axiom is assumed by all the axiomatization mentioned in this literature review). $(P7)$ has few implications on our analysis since we consider probability distributions. It only implies our Axiom 1, because for proper probability distributions x, y, $x_i \geq y_i$ for all i if and only if $x = y$, and then $(P7)$ implies that $I(p|p) = 0$. $P(8)$ is a normalization. $(P9)$ is very similar to our Axiom 5, but it is much stronger because it applies for all generalized distributions. In particular, considering $x^1, y^1, x^2, y^2 \in G_1 = [0,1]$ it implies that:

$$I(x^1 x^2 | y^1 y^2) = I(x^1|y^1) + I(x^2|y^2).$$

This equation (and a smoothness condition on I) imply that for $x, y \in [0,1]$, $I(x|y) = \beta \log(x/y)$, for some $\beta \in \mathbb{R}$. Therefore $(P9)$ alone implies that I is the logarithm on generalized distributions in dimension 1 (i.e. real numbers $x, y \in [0,1]$). Our Axiom 5, instead, apply only to actual distributions so does not have the implications of $(P9)$. Finally, $(P10)$ imposes a functional form on the divergence measures $I(x|y)$, and since for $z, z' \in [0,1]$, $I(z|z') = \beta \log(z/z')$, $(P10)$ implies that for all $x, y \in G_n$:

$$I(x|y) = g^{-1}\left(\sum_j \frac{x_j}{\sum_i x_i} g(\alpha \log(x_j/y_j))\right),$$

by induction. The final part of Renyi's proof shows that the only increasing functions g that make sure I satisfies $(P9)$ are parametrized by $g(x) := \gamma e^{\alpha x}$. Finally, the normalization $(P8)$ imply that:

$$I(x|y) = \frac{1}{\alpha - 1} \log\left(\frac{\sum_j x_j \left(\frac{x_j}{y_j}\right)^{\alpha-1}}{\sum_j x_j}\right),$$

which boils down to the expression $R_\alpha(p,q)$ when applied to probability distributions:

$$R_\alpha(p,q) = \frac{1}{\alpha - 1} \log\left(\sum_j p_j \left(\frac{p_j}{q_j}\right)^{\alpha-1}\right).$$

Summarizing, the main differences between Renyi's axiomatization and this paper are the following: Rényi (1961) models generalized distributions, and this (with additivity) buys that the divergence of two events is given by the logarithm of the ratio of the probabilities.[34] In our model, we also find that disagreement measures involve the dispersion of the likelihood ratio, but we derive such functional form from other axioms (that

[33] Observe that for any $x \in G_n$ and $y \in G_m$, $x \cup y$ is a generalized distribution if and only if:

$$\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j \leq 1.$$

[34] Aczel and Daroczy (1975) provided an axiomatization of Renyi's divergences that does not use any generalized distributions, but Aczel and Daroczy (1975) constraints the divergence to be a weighted mean of $\log(p_i/q_i)$.

are related to properties of disagreement). Secondly. Renyi's ($P10$) constraints the divergences to have the functional form of a generalized mean,[35] whereas we impose only local conditions the disagreement measures (Axiom 6). Thirdly, Renyi's postulate ($P10$) is not symmetric in x and y, meaning: if $(x,y) \mapsto I(x|y)$ satisfies it, $(x,y) \mapsto I(y|x)$ does not. In our case, instead if $(x,y) \mapsto D(x,y)$ satisfies our set of axioms, then so does $(x,y) \mapsto D(y,x)$.[36] This implies that the function $D(p,q) = \sum_j q_j \log\left(\frac{q_j}{p_j}\right)$ satisfies our axioms and is *not* a Renyi divergence. Finally, our characterization is based on basic principles that intuitively describe disagreement, whereas Renyi's motivation is generalizing the entropy and relative entropy.

[35]The generalized mean of n real numbers $x_1, \ldots, x_n \in \mathbb{R}$ is defined as:

$$M_{\phi}^{p}(x_1, \ldots, x_n) = \phi^{-1}\left(\frac{\sum_j p_j \phi(x_j)}{n}\right),$$

for some $\phi(\cdot)$ increasing.

[36]This result is obvious for Axioms 1–5, while Axiom 6 is apparently asymmetric in p and q. Nonetheless it is clear from our characterization that Axiom 6 holds for both p and q.