Tales of Shifting Tails
Evidence from Weekly Options

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**Motivation**

Trading in Short-Dated Equity-Index Options (Weeklies) has Increased Dramatically

Growing Interest in Measuring and Interpreting (Risk-Neutral) Jump Tails

Left Jump Tail Risk Premium Prominent in Equity and Variance Risk Premium

Left Tail may Shift Independently of Volatility (AFT, JFE)

Identifying Features of Jump Process from Short-Dated Options “Easier”

Are Standard Models for Longer-Dated Options Consistent with *Weeklies*?

Are Jump Distributions Invariant over Time?
**Motivation**

Invariant Jump Distributions Critical for Standard Option Pricing Models

Jump Intensity an Affine Function of Volatility or Factors in Standard Models

Additional Latent Factor driving Jump Intensity Feasible (AFT, Ecma)

Separation of Jump Intensity and Jump Distribution Hard for Long-Dated Options

Weeklies Allow us to Revisit the Relevant Evidence regarding Jump Dynamics

What Constitutes a Good Parametric Approximation to the Jump Distribution

Does Variation in Jump Characteristics Explain Pricing Errors in Regular Models?

Weeklies Allow us to Obtain New Insights regarding the Jump Dynamics
Overview

Illustration of Economic Function and Properties of Weeklies

Stylized Facts about the Trading in Equity-Index Options

Do Regular Parametric Option Pricing Models Fit the Weeklies?

New Semi-Parametric Method for Approximate Pricing of Short-Dated Options

Extend to Unrestricted Time-Varying Volatility and Jump Intensity

Allow for Time-Varying Jump Distribution as well

Gaussian versus (Generalized) Tempered Stable Jump Distributions

Review of Implied Time-Variation in Jump Characteristics

Does Marked Jump-Variation Coincide with Errors in Standard Models?
Illustration: Left Tail Shift

Figure 1: Log Option Prices. September 20, 2012. Weeklies $\tau = 8$, Regular $\tau = 29$ Days; Moneyness: $\log(K/F)/(\sqrt{\tau}IV_{atm,\tau})$
. . . benchmarks managed to make good most of their losses, with the Dow finishing in the green as investors remained somewhat optimistic.

. . . economic data was dismal on the international front . . . The preliminary reading of the China HSBC manufacturing PMI was 47.8 for September . . . Separately, Markit reported that Euro-zone’s purchasing composite managers index dropped from 46.3 in August to 45.9 in September.

Economist at the Conference Board said: “The economy continues to be buffeted by strong headwinds domestically and internationally. As a result, the pace of growth is unlikely to change much in the coming months. Weak domestic demand continues to be a major drag on the economy”.
Illustration: Payoffs on Hedges for Short-Dated Tail Exposure

Assume Agent has Exposure Mimicking Short Put Writer; \( S_0 = 1,000 \) at \( t = 0 \);

**Exposure:** \(- (950 - S_7)^+ = (S_7 - 950) \cdot 1_{\{S_7 \leq 950\}}\)

Long Put, \( K = 950, \tau = 7 \); **Payoff** = − Option Price = \(- e^r \cdot P_{0,7} \) at \( t = 7 \).

Regular Tenor, \( \tau \in [10, 35] \) Days; **Δ Put Price** over \([0, 7]\): \( P_{7,\tau-7} - e^r \cdot P_{0,\tau} \)

Using Two-Factor SV Model with Co-Jumps in Price and Volatility (AFT, JFE, Ecma)

\( \mathbb{P} \) and \( \mathbb{Q} \) Parameters set to Estimates from Regular S&P 500 Option Sample

**Differential Payoff on Long vs Short OTM Put Hedge**

\[
EP(7, \tau) = P_{7,\tau-7} - (950 - S_7)^+ - e^r \cdot P(0, \tau) + e^r \cdot P(0, 7)
\]
Illustration: Differential Payoff on Long - Short Put Hedges

Figure 2: Excess P&L of Put Option Tail Hedges; 10 and 35 Days to Maturity.
S&P 500 Equity-Index Options

SPX Options Expire At Week’s End Around Middle of each Month

SPXQ Options Expire at Quarter’s End

SPXW Options Expire at Week’s End (unless SPX(Q) Expiry Scheduled)

From Jan 2014, Six Consecutive SPXW (weekly) Options Alive, String of Nearby Maturities

Trading of Short-Dated Weeklies has Grown Dramatically

We Construct Regular ($\tau > 9$) and Short-Dated ($\tau \leq 9$) Option Sample

Check for Quality: Trading Volume, Spread, Strike Coverage (Moneyness)

Defining Moneyness: $m = \frac{\ln(K/F_\tau)}{\sqrt{\tau} IV_{ATM,\tau}}$
Trading in Weeklies

Figure 3: **Weekly (SPXW) Options.** The Bar Plot (left y-axes) represents Average Daily Volume in Weekly Options. Light Gray Line (right y-axes) depicts Weekly Option Volume as Percentage of Total Volume in SPX (sum of SPX, SPXQ and SPXW)
Constructing Two Option Samples

We Apply the following Filter to All Options from OptionMetrics:

i. Front Maturity has Tenor $\tau \leq 9$ Calendar Days
ii. Each Tenor has at least Ten Quotes across Strike Range
iii. Maturity Less than One Year, $\tau \leq 365$ Calendar Days
iv. Moneyness is not Extreme: $-15 \leq m \leq 5$
v. Bid Quote is strictly Positive
vi. Ratio of Ask to Bid Price Less than Five: $\frac{Ask}{Bid} < 5$
vii. Front Maturity has Quotes Relatively Deep OTM: $m \leq -3.5$
viii. It is Not an Abbreviated Trading Day
ix. It is Not U.S. Holiday or Day Prior to U.S. Holiday

Split Remaining 973, 866 Option Contracts for 2011-2014 into two Samples:

\textbf{Short-Dated Options:} Maturity $\tau \leq 9$ Calendar Days
\textbf{Regular Options:} Maturity $9 < \tau \leq 365$ Calendar Days
Refining the Short-Dated Option Sample

Short-Dated Options Key to Semiparametric Inference for Jump Process

Short-Dated Options are Inexpensive:

Relatively Large (BSIV) Errors from Discreteness, Bid-Ask Spread

Apply Additional Filter for Short-Dated Options:

i. Retain only the most Liquid Front Maturity
ii. The moneyness is further restricted: \(-8 \leq m \leq 3\);
iii. Each Option Contract has Daily Trading Volume of at least Five Units
iv. Moneyness is not Extreme: \(-15 \leq m \leq 5\)
v. Remove Clear Violations of No-Arbitrage (Non-Convex Prices in \(m\))

41, 206 Short-Dated Options, 925 Days: Average of 54.5 Quotes/Day for Single Maturity

912, 110 Regular Options similar to Standard Samples in Tenor-Strike Composition
Bid-Ask Spread

Figure 4: **Bid Ask Spread.** Kernel regression of Relative (left axis) and Absolute (right axis) Bid-Ask Spread as Function of Moneyness $m$. 
Figure 5: **Moneyness Range.** A Left Panel: moneyness range for short maturity options between January 3rd 2011 and December 31st 2014. Right Panel: moneyness range for long maturity options between January 3rd 2011 and December 31st 2014. Moneyness is defined as $m = \frac{\ln(K/F)}{\sqrt{\tau} IV_{atm,\tau}}$ where $K$ is the strike price, $F$ the forward price of the underlying, $\tau$ is the time-to-maturity, and $IV_{atm,\tau}$ is the implied volatility of the front maturity option with strike price closest to $F$. 
Asset Price Dynamics

S&P 500 Index governed by following Risk-Neutral Dynamics,

\[
\frac{dX_t}{X_{t-}} = (r_t - \delta_t) \, dt + \sqrt{V_t} \, dW_t + \int_{\mathbb{R}^2} (e^x - 1) \tilde{\mu}(dt, dx, dy),
\]

\(r_t, \delta_t\) are Deterministic;

\(V\) is Diffusive Stochastic Variance;

\(\mu\) Counting Jump Measure, Compensator \(dt \otimes \nu_t(dx, dy)\);

\(\tilde{\mu}(dt, dx, dy) = \mu(dt, dx, dy) - dt \otimes \nu_t(dx, dy)\) associated MG Jump Measure;

Jump Specification has Two Components: \(x\) Price Jumps, \(y\) Volatility Jumps
Standard Parametric Option Pricing Models

Parametric Models Specify $V$ Dynamics, interaction with $W$, $\mu$

General Representation implemented in AFT(JFE, Ecma)

\[
dV_{1,t} = \kappa_1 (\bar{V}_1 - V_{1,t}) \, dt + \sigma_1 \sqrt{V_{1,t}} \, dB_{1,t} + \int_{\mathbb{R}^2} y \, \mu(dt, dx, dy),
\]
\[
dV_{2,t} = \kappa_2 (\bar{V}_2 - V_{2,t}) \, dt + \sigma_2 \sqrt{V_{2,t}} \, dB_{2,t},
\]
\[
\text{Corr} (dW_t, dB_{1,t}) = \rho_1 \sqrt{V_{1,t}}/V_t,
\]
\[
\text{Corr} (dW_t, dB_{2,t}) = \rho_2 \sqrt{V_{2,t}}/V_t.
\]
Alternative Jump Specifications

Gaussian Price – Exponential Volatility Co-Jumps

\[ \nu_t(dx, dy) = c(t) \frac{e^{-\frac{(x - \mu_x - \rho y)^2}{2\sigma^2}}}{\sqrt{2\pi} \sigma_x} \frac{e^{-\frac{y}{\mu_v}}}{\mu_v} 1\{y>0\} \]

\[ c(t) = c_0 + c_1 V_{1,t-} + c_2 V_{2,t-}. \]

Double-Exp Price - Volatility Co-Jumps; Asymmetric GARCH Feature

\[ \nu_t(dx, dy) = c^-(t) \cdot 1\{x<0, y=\mu_v x^2\} \cdot \lambda_- e^{-\lambda_- |x|} + c^+(t) \cdot 1\{x>0, y=0\} \cdot \lambda^+ e^{-\lambda^+ x} \]

\[ c^\pm(t) = c_0^\pm + c_1^\pm V_{1,t-} + c_2^\pm V_{2,t-}. \]
Standard Parametric Model Fit for Weeklies

State Vector $S_t = (V_{1,t}, V_{2,t})$, Log-Moneyness $k = \ln(K/F_{t,t+\tau})$

Quote Option Prices in BSIV: $\kappa_{k,\tau}(S_t, \theta)$,

Observed Option Prices in BSIV: $\kappa_{t,k,\tau}$

Fit to Regular Option Sample Using AFT – Accommodate Option Measurement Error

$$\left(\{\hat{S}_t\}_{t=1}^T, \hat{\theta}\right) = \arg\min_{\{S_t\}_{t=1}^T, \theta} \sum_{t=1}^T \left\{ \sum_{j: \tau_j > 9} \frac{(\kappa_{t,k_j,\tau_j} - \kappa_{k_j,\tau_j}(S_t, \theta))^2}{N_t} + \frac{k_n}{N_t} \left( \frac{\sqrt{\hat{V}_t^n} - \sqrt{V_{1,t} + V_{2,t}}}{\hat{V}_t^n/2} \right)^2 \right\}$$

Now, Fix Parameter Vector, Adjust State Vector to Sample of Weekly Options
Standard Parametric Model Fit for Weeklies

State Vector Re-Estimated Freely Day-by-Day to Adjust to Weeklies,

\[ \hat{S}_t = \arg\min_{S_t} \sum_{j: \tau_j \leq 9} \left( \kappa_{t,k_j,\tau_j} - \kappa_{k_j,\tau_j}(S_t, \hat{\theta}) \right)^2, \quad t = 1, \ldots, T. \]

Gaussian Price Jumps

RMSE = 1.52\%,  \quad \text{Mean Jump} = -21.15\%,  \quad \text{Avg Annual Jump Intensity}= 0.30

Double-Exponential Price Jumps

RMSE = 1.42\%,  \quad \text{Mean Jumps:}  -4.64\%, 2.05\%,  \quad \text{Avg Annual Jump Intensity}= 3.5
Diagnostic Test for Model Fit to Weeklies

Avg Pricing Error over part of Strike Region \( \mathcal{K} \) on Day \( t \),

\[
Z_{\mathcal{K},t} = \sum_{j: k_j \in \mathcal{K}, \tau_j < 9} \left( \kappa_{t,k_j,\tau_j} - \hat{\kappa}_{k_j,\tau_j}(\hat{S}_t, \hat{\theta}) \right)
\]

Discrepancy of Model-Implied and Nonparametric Spot Vol Estimate on Day \( t \),

\[
\hat{V}_t - \hat{V}_n^t = \left( \hat{V}_{1,t} + \hat{V}_{2,t} \right) - \hat{V}_n^t
\]

For Correct Model Specification, \textbf{Z-Scores Asymptotically Standard Normal},

\[
\hat{T}_{\mathcal{K},t}^{of} = Z_{\mathcal{K},t} / \sqrt{\text{Avar}(Z_{\mathcal{K},t})}
\]

\[
\hat{T}_{t}^{vf} = (\hat{V}_n^t - \hat{V}_t) / \sqrt{\text{Avar}(\hat{V}_n^t - \hat{V}_t)}
\]
Standard Model Fit to Weekly Options

Figure 6: Two-Factor Double Exponential Full Sample.
Weekly Option Implied Model Fit to Nonparametric Spot Volatility

Figure 7: Two-Factor Double Exponential Full Sample.
Extracting Information from Short-Maturity Option Prices

No Restriction on Volatility Dynamics, Time-Invariant Parameter Vector $\theta$
Risk-Neutral Jump Intensity: $dt \otimes \nu(dx; J_t, \theta)$
$\nu(dx; J_t, \theta)$ Parametric Jump Distribution, Time-Varying State Vector $J_t$

For Deep OTM Short-Dated Options (Carr & Wu (2003), Bollerslev & Todorov (2013)):
\[
\frac{O_{t,\tau,k}}{\tau X_{t-}} \rightarrow \begin{cases} 
\int_{\mathbb{R}} (e^x - e^k)^+ \nu(dx; J_t, \theta), & \text{if } k > 0 \\
\int_{\mathbb{R}} (e^k - e^x)^+ \nu(dx; J_t, \theta), & \text{if } k < 0 
\end{cases}, \quad \text{as } \tau \downarrow 0.
\]

For ATM Short-Dated Options (Durrleman (2008)):
\[
\kappa_{t,0,\tau} \rightarrow \sqrt{V_t}, \quad \text{as } \tau \downarrow 0.
\]

For Options with Non-Trivial Tenor, Approximation works only for Few Strikes
**New Semiparametric Approximation to Short-Maturity Option Prices**

Approximate Option Price for \( X \), with Constant Variance, Jump Intensity, \( \tilde{X}_t = X_t \).

\[
\frac{d\tilde{X}_s}{\tilde{X}_{s-}} = (r_t - \delta_t) ds + \sqrt{V_t} dW_s + \int_{\mathbb{R}} (e^x - 1)(\mu(ds, dx) - ds \otimes \nu(dx; J_t, \theta)), \quad s \geq t
\]

For Short Interval, \((\mathbb{Q})\) Volatility and Jump Intensity Not Expected to Vary Much

\( \tilde{X} \) Approximates \( X \) for \( s > t \), **Freezing Drift, Volatility, Jump Intensity at \( t \).**

Conditional on \( \mathcal{F}_t \), \( \tilde{X} \) is a Lévy Process \( \rightarrow \) **Pricing Options Easy**

Theoretical Option Price, in BSIV, at \( t \): \( \tilde{\kappa}_{k,\tau}(S_t, \theta), \quad S_t = (V_t, J_t) \)
Semiparametric Approximation Models, Fixed Jump Distribution

Vol, Jump Intensity Free, Parametric on Jump Distribution, \( \nu(dx; J_t, \theta) \)

**Gaussian Jumps**, (Arbitrary) Time-Varying Intensity, \( c_t \)

\[
\nu(dx; J_t, \theta) = c_t \frac{1}{\sqrt{2\pi \sigma_x}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} dx .
\]

Model has \( \theta = (\mu_x, \sigma_x^2) \) and \( J_t = c_t \)

**Generalized Tempered Stable Jumps**, \( \alpha \) Fixed, (Arbitrary) Time-Varying Intensity \( c_t \)

\[
\nu(dx; J_t, \theta) = c_t \left\{ \frac{e^{-\lambda^-|x|}}{|x|^{1+\alpha}} 1_{\{x<0\}} + \frac{e^{-\lambda^+|x|}}{|x|^{1+\alpha}} 1_{\{x>0\}} \right\} dx, \quad \alpha < 2
\]

Model has \( \theta = (\lambda^-, \lambda^+) \) and \( J_t = c_t \)

\( \alpha = -1 \) **Double-Exponential**, \( \alpha = 0 \) **Variance Gamma**, CGMY (2002, 2003)
Semiparametric Approximation Models, Fixed Jump Distribution

Estimation for Weekly Options, Using AFT Procedure,

\[
\left( \{\hat{S}_t\}_{t=1}^T, \hat{\theta} \right) = \arg\min_{\{S_t\}_{t=1}^T, \theta} \sum_{t=1}^T \left\{ \sum_{j: \tau_j \leq 9} \frac{\left( \kappa_{t,k_j,\tau_j} - \tilde{\kappa}_{k_j,\tau_j}(S_t, \theta) \right)^2}{M_t} + \frac{k_n}{M_t} \left( \sqrt{\hat{V}_{t^n}} - \sqrt{V_t} \right)^2 \right\}
\]

Gaussian Price Jumps

RMSE = 1.19\%, Mean Jump = −5\%, Avg Annual Jump Intensity= 2.5

Double-Exponential Price Jumps

RMSE = 1.02\%, Mean Jumps: −3.8\%, 1.6\%

\( c_t \) Varies Freely, but Similar Form for \( c_t^- \) and \( c_t^+ \) (Right Tail Info Lacking!)
Semiparametric, Fixed Jump Distribution Fit to Weekly Options

Figure 8: Double Exponential Constant Tail, Weekly Sample.
Semiparametric, Fixed Jump Distribution Fit to Spot Volatility

Figure 9: Double Exponential Constant Tail, Weekly Sample.
Semiparametric Approximation, Time-Varying Jump Distribution

No Restrictions on Volatility, Jump Intensity, **Time-Varying Jump Parameters**

**Gaussian Jumps**, Time-Varying Jump Intensity and Parameters

\[
\nu(dx; J_t, \theta) = c_t \frac{1}{\sqrt{2\pi \sigma_{x,t}}} e^{-\frac{(x-\mu_{x,t})^2}{2\sigma_{x,t}^2}} \, dx.
\]

Model has \( \theta = \emptyset \); \( S_t = (V_t, J_t) \); \( J_t = (c_t, \mu_{x,t}, \sigma_{x,t}) \)

**Generalized Tempered Stable Jumps**, \( \alpha \) Fixed, Varying Intensity & Jump Parameters

\[
\nu(dx; J_t, \theta) = c_t \left\{ \frac{e^{-\lambda^+_t|x|}}{|x|^{1+\alpha}} 1\{x<0\} + \frac{e^{-\lambda^-_t|x|}}{|x|^{1+\alpha}} 1\{x>0\} \right\} \, dx, \quad \alpha < 2
\]

Model has \( \theta = \emptyset \); \( S_t = (V_t, J_t) \); \( J_t = (c_t, \lambda^+_t, \lambda^-_t) \)
Semiparametric Approximation, Time-Varying Jump Distribution

AFT Procedure for Weekly Options: Sequence of Separate Optimization Problems:
Spot Volatility, Jump Intensity & Parameters Estimated Independently each Day

Gaussian Price Jumps
RMSE = 0.85%, Mean Jump = −4.7%, Avg Annual Jump Intensity= 5.7

Double-Exponential Price Jumps
RMSE = 0.68%, Mean Jumps: −3.4%, 1.0%

Tempered Stable Price Jumps, $\alpha = 0.5$
RMSE = 0.57%
Semiparametric, Varying Jump Distribution, Option Fit for Weeklies

Figure 10: Gaussian, Varying Jump Parameters, Weeklies
Semiparametric, Varying Jump Distribution, Vol Fit for Weeklies

Figure 11: Gaussian, Varying Jump Parameters, Weeklies
Semiparametric, Varying Jump Distribution, Option Fit for Weeklies

Figure 12: Tempered Stable, Varying Jump Parameters, Weeklies
Semiparametric, Varying Jump Distribution, Vol Fit for Weeklies

Figure 13: Tempered Stable, Varying Jump Parameters, Weeklies
Illustration: Option Fit for Weeklies, Varying Jump Parameters

Figure 14: Log Option Prices, October 15, 2014. Left: Fitted Log Option Prices in Gaussian Model. Right: Fitted Log Option Prices in Generalized Tempered Stable Model with $\alpha = 0.5$. Dots Represent Observed Option Prices, Light Gray Line is Model-Implied Prices.
Implications of Time-Varying Jump Distributions

Great Flexibility in Fitting Option Skew Day-by-Day

Raises Inevitable Issues about potential Over-Fitting

Are Isolated Tail Shifts Truly Genuine Phenomena?

Does the Time-Variation Look Sensible?

How Precisely Are Different Model Features Estimated?

Are the Weeklies Trading in Segmented Market?

Can we Link Them to other Economic or Pricing Events

If Jump Features Vary Significantly, Standard Models Misspecified

Structural Models must be Cautious in Extracting Information from Options
Time-Varying Parameters, Smoothed

Figure 15: Parameter Estimates, Gen Temp Stable, $\alpha = 0.5$, MA-20
Example of Left Tail Shift – 2011

Figure 16: **Log Option Prices.** January 7, 2011. Weeklies $\tau = 7$, Regular $\tau = 42$ Days; Moneyness: $\log(K/F)/\left(\sqrt{\tau}IV_{atm,\tau}\right)$
Example of Left Tail Shift – 2012

Figure 17: Log Option Prices. September 20, 2012. Weeklies $\tau = 8$, Regular $\tau = 29$ Days; Moneyness: $\log(K/F)/(\sqrt{\tau} IV_{atm,\tau})$
Example of Left Tail Shift – 2013

Figure 18: **Log Option Prices.** December 20, 2013. Weeklies $\tau = 7$, Regular $\tau = 28$ Days; Moneyness: $\log(K/F) / (\sqrt{\tau} IV_{atm,\tau})$
Example of Left Tail Shift – 2014

Figure 19: Log Option Prices. October 15, 2014. Weeklies $\tau = 7$, Regular $\tau = 42$ Days; Moneyness: $\log(K/F)/(\sqrt{\tau IV_{atm,\tau}})$
**Weeklies’ Tail Variation vs. Regular Model Misspecification**

**Figure 20:** **Jump Variation** \( m \leq -3 \)

**Top:** Jump Variation, 2-F Double-Exp, Regular Options, Vs. Gen. Tempered Stable, \( \alpha = 0.5 \), Weeklies; 
\( m_t \) Negative Log-Return \( = 3 \times \) Daily ATM IV

**Bottom:** Option Z-Scores for Double-Exponential, Regular Options – Reporting Z-Scores > 2.5
Correlation of Shaded Areas: 0.54. All Series are Five-Day MA.
Figure 21: **Estimation Precision.** The Gray Areas represent ±2 Standard Error Bands Computed from Delta-Method, Spot Quarticity obtained via Nearest Neighbor Truncation Estimator
Figure 22: **Estimation Precision**. The Gray Areas represent ±2 Standard Error Bands Computed from Delta-Method, Spot Quarticity obtained via Nearest Neighbor Truncation Estimator
Conclusion

Weekly Options Offer Unprecedented Opportunity to Study Tail Behavior

Left Jump Tail Can be Estimated Well Day-by-Day

Daily Semiparametric Volatility, Jump Intensity and Distribution

Strong Evidence for Tempered Stable Type Tail Decay

Jump Intensities and Jump Distributions Vary

(Imperfect) Correlation between Volatility and Jump Intensity

Lots of Unrelated Variation in Left Tail Jump Distribution

Affine Mapping of Volatility cannot Capture Tail Dynamics

Tail Variation Predict Poor Fit for Regular Option Pricing Models