A Tale of Two Option Markets: State-Price Densities Implied from S&P 500 and VIX Option Prices*

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Abstract

The S&P 500 and VIX option markets are closely connected as both options depend on the volatility dynamics. Capturing information in both option prices by nonparametric state-price densities (SPDs), we look into the dynamics of the index and its volatility, along with interactions between the two option markets. We find that SPDs of the index strongly depend on the current VIX level, and that such dependence is driven by information implied from VIX options beyond VIX time series, such as volatility of volatility and volatility skewness. In addition, SPDs of the VIX document three features of its risk-neutral dynamics, including positive skewness, mean-reversion, and high persistence. Moreover, the pricing kernel estimates exhibit a U-shape when the current VIX level is high and a decreasing pattern otherwise, both pre- and post-crisis. This pattern implies that stochastic volatility may be the missing key state variable for the preference of agents, which is responsible for the puzzling U-shape. Finally, we conduct nonparametric specification tests and find that the state-of-the-art stochastic volatility models in the literature cannot capture the S&P 500 and VIX option prices simultaneously. The identified asymmetric mean-reversion rate of volatility suggests non-affine specification as a necessary extension.

Key Words: Options, Pricing Kernel, State-Price Density, VIX

JEL classification: G12,G13

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1 Introduction

By reconciling evidence from the S&P 500 index and its options, empirical option pricing literature has documented stochastic volatility as an indispensable factor of the underlying dynamics of index returns. Nevertheless, determining the volatility dynamics is not straightforward because, as Aït-Sahalia et al. (2001) point out, volatility is a nontraded asset, and its risk-neutral behavior cannot be solely identified. Therefore, most studies in the literature either rely on strong beliefs in their favorite parametric models restricted within the affine class, such as Bakshi et al. (1997), Bates (2000), Pan (2002), Eraker (2004), Broadie et al. (2007), Duan and Yeh (2011), and Amengual and Xiu (2012), or ignore the unobserved volatility factor when conducting nonparametric analysis of the index dynamics, as in Aït-Sahalia and Lo (1998), Aït-Sahalia et al. (2001), Buraschi and Jackwerth (2001), Jackwerth and Rubinstein (1996), and Fan and Mancini (2009).

The lack of observable and tradable volatility has changed substantially since the Chicago Board of Options Exchange (CBOE) introduced the Volatility Index (VIX) in 1993,[1] and VIX derivatives such as futures and options in 2004 and 2006, respectively. The VIX provides investors with a direct measure of the volatility and VIX derivatives offer investors convenient instruments to trade the volatility of S&P 500 index.[2] The VIX, referred to as the fear gauge, is constantly exposed in the media spotlight. Also, VIX options achieve huge liquidity, becoming the third most active contracts at CBOE as of October 2011. Because the VIX is derived from S&P 500 options as the square root of the risk-neutral expectation of the return's average variance over the next 30 calendar days, it uncovers the unobservable volatility in a forward-looking way. In addition, VIX options are particularly informative about the distribution of the future VIX level. Consequently, two option markets exist simultaneously that both depend on the volatility dynamics.

Capturing information in both option prices by nonparametric state-price densities (SPDs), we investigate dynamics of the index and its volatility, as well as the interactions between the two option markets. These SPDs, also known as the prices of Arrow-Debreu securities, summarize all the information of the economic equilibrium in an uncertain environment.[3] They not only enable convenient pricing of vanilla options, but also allow consistent no-arbitrage pricing of exotic and less-liquid securities such as over-the-counter derivatives. Moreover, distinct from the parametric

[1] The VIX, from its inception, was calculated from S&P 500 index options by inverting the Black-Scholes formula. In 2003, the CBOE amended this approach and adopted a model-free method to calculate the VIX.

[2] Before the introduction of VIX derivatives, to trade volatility, investors have to take positions of option portfolios, such as straddles or strangles.

[3] SPDs give for each state x the price of a security paying one dollar if the state falls between x and $x + dx$. See Aït-Sahalia and Lo (1998) for detailed discussions.

studies in the literature, our nonparametric analysis of SPDs unveils potential misspecifications of the parametric forms, and may be advocated as a prerequisite to the construction of parsimonious models.

Nonparametric studies of SPDs were innovated by Aït-Sahalia and Lo (1998) and Jackwerth and Rubinstein (1996), disregarding the stochastic volatility factor.[4] Although it may be possible to estimate SPDs of the S&P 500 index conditional on an ex-post volatility proxy filtered from historical time series (see Li and Zhao (2009) for such a procedure with interest rate caps), SPDs of the volatility cannot be determined because options cannot be written on unobservable variables directly. In contrast, using information in both the S&P 500 and VIX option markets, we recover SPDs of both the S&P 500 index and VIX and hence indirectly spotlight the unobservable volatility dynamics without parametric assumptions. Our empirical study of the nonparametric SPDs documents several important findings about the dynamics of the index and its volatility and information content of the two option markets.

First, we find that SPDs of the S&P 500 index strongly depend on the current VIX level, not only in the short run but also for the long term, when VIX options become unavailable. To study the documented short-run dependence and illustrate how investors employ information in VIX options, we regress the future average volatility implied from S&P 500 densities on the VIX and moments of the VIX densities extracted from VIX option prices. We find that the short-run dependence is driven not only by the VIX, but also by information in its SPDs. This finding implies that the availability of VIX options indeed deliver incremental information for market participants to predict future index dynamics and price S&P 500 options accordingly. To understand the documented long-term dependence, we also implement a simple martingale procedure by which investors interpolate implied volatility of today's long-maturity S&P 500 options using yesterday's implied volatility surface. Regression results of implied volatilities of realized option prices on interpolated implied volatilities show that the martingale interpolation indeed agrees with the market practice and may offer a potential justification for the long-run dependence.

The nonparametric SPDs of Aït-Sahalia and Lo (1998), in the absence of the stochastic volatility factor, differ considerably from our estimates of SPDs conditional on the VIX when the current VIX level is either low or high, while only being approximately identical for the average VIX level. Therefore, with today's VIX level in the high and low ranges, our nonparametric SPD conditional on the VIX may provide more-accurate information about the future dynamics of the S&P 500 index.

Second, we document several empirical features of the risk-neutral dynamics of the VIX. In

[4]Many follow-up studies have been conducted based on these nonparametric SPDs, such as Aït-Sahalia and Lo (2000), Aït-Sahalia et al. (2001), Chabi-Yo et al. (2008), and Bakshi et al. (2010)

particular, we find that SPDs of the VIX are positively skewed across all maturities and current levels of the VIX. In addition, the risk-neutral dynamics of the VIX exhibit both mean-reversion and persistence. Although the volatility process under the physical measure is well documented to display a mean-reverting pattern using historical time series (Mencia and Sentana (2009)), its risk-neutral behavior is not crystal clear. Our finding confirms these patterns under the pricing measure without any parametric restrictions.

Third, we estimate pricing kernels conditional on the VIX, for the periods before and after the 2008 financial crisis. We find that the estimated pricing kernel exhibits a U-shape conditional on a high VIX level, while maintaing a decreasing pattern when the current VIX level is low for both pre- and post-crisis periods. The U-shape of the pricing kernel, described in Aït-Sahalia and Lo (2000), Jackwerth (2000), and Bakshi et al. (2010) as a puzzle, contradicts standard economic theory, in which the pricing kernel, equal to the scaled marginal rate of substitution, decreases in market index return. Our empirical result suggests that the shape of the pricing kernel is significantly affected by the stochastic volatility. The U-shape becomes visible when the market experiences a high volatility. This finding echoes the conclusions of equilibrium-based parametric models discussed in Jackwerth and Brown (2001), Chabi-Yo et al. (2008), Chabi-Yo (2011), and Christoffersen et al. (2010a), that the missing state variables in pricing kernels may result in the U-shape. Without restricting the form of pricing kernels, our result, however, implies that stochastic volatility is the key but missing state variable of pricing kernels.

Finally, our SPD estimates enable us to gauge the specifications of parametric models in pricing S&P 500 and VIX options simultaneously. Modeling both option prices jointly is important, as they both rely on the volatility factor: the volatility affects the distribution of the S&P 500 index, while determining the payoff function of a VIX option. Nevertheless, most studies in the option-pricing literature treat them separately; they either focus on pricing S&P 500 options (see e.g. Bates (2000), Pan (2002), Eraker (2004), and Broadie et al. (2007)) or pricing VIX options on top of a standalone process for VIX (e.g. Whaley (1993), Grunbichler and Longstaff (1996), Detemple and Osakwe (2000), and Mencia and Sentana (2009)). Although some parametric models have been developed to capture the evolution of S&P 500 index and VIX, along with joint pricing S&P 500 and VIX options (Amengual and Xiu (2012)), it remains unclear how well they capture empirical features of these option prices.

Combining SPDs of both the S&P 500 index and the VIX, we test the specification of state-of-the-art models that resemble those proposed by Pan (2002), Eraker (2004), and Broadie et al. (2007), but are more flexible in the specification of jumps in the return dynamics. Building upon the difference between parametric SPD estimates and nonparametric ones, our test statistics reject all these models under consideration. Closer scrutiny of the tests and SPD estimates of the VIX suggest an asymmetric mean-reversion rate of volatility, i.e., volatility reverts relatively faster when its current level is low, which further indicates non-affine models may be a promising start point to jointly pricing the two options. In addition, we also construct test statistics to check whether the VIX SPD depends on the current information set only through the VIX, for which we find no evidence of rejection. This finding justifies the VIX to be an effective volatility measure, and helps simplify our nonparametric analysis.

Our paper is also related to Boes et al. (2007) that tries to estimate the SPDs of returns, with realized volatility constructed from high-frequency data as a proxy for the unobservable spot volatility. In contrast, our estimation method draws on the observable VIX, and more importantly VIX options, which are particularly informative about volatility dynamics. Another relevant specification analysis includes Christoffersen et al. (2010b), who find that non-affine models outperform affine square root models using realized volatilities, S&P 500 returns, and an extensive panel of option prices. Our result pinpoints, nonparametrically, the exact inadequacy — constant rate of mean-reversion — of a variety of affine models. Engulatov et al. (2011), on the other hand, estimate the drift and volatility functions nonparametrically, and point out similar evidence of asymmetric mean-reversion.

The rest of the paper is organized as follows. In section 2, we discuss the relationship of stochastic volatility process and VIX, how to obtain SPDs from option prices, and construct our nonparametric estimators using a multivariate local linear approach. A specification test based on the estimated SPDs is also proposed, and parametric models for pricing S&P 500 options and VIX options consistently are introduced. Section 3 presents our empirical results, including nonparametric SPD estimates of the S&P 500 index and VIX, analysis of the dependence of S&P 500 index densities on the current VIX level, estimates of pricing kernels conditional on VIX, and nonparametric specification analysis of parametric pricing models. Section 4 concludes. The appendix provides technical assumptions, proofs, derivations of parametric SPDs, and tests for robustness checks.

2 State-Price Densities: Derivation, Estimation, and Test

2.1 Two Option Markets and State-Price Densities

In this section, we identify the underlying connection between S&P 500 and VIX options and introduce the state-price densities that span the two markets. To fix ideas, we denote the log price of the S&P 500 index as S_t, and the VIX as Z_t. Because the payoffs of S&P 500 and VIX options only depend on their own underlying indices S_T and Z_T, the option prices are determined by two SPDs $p(S_T|\mathcal{F}_t)$ and $p(Z_T|\mathcal{F}_t)$ of S_T and Z_T, respectively, where $\mathcal{F}_t$ summarizes the information up to time

t.

To infer these SPDs from option prices, we denote the instantaneous volatility of S_t as V_t, and write the price of the S&P 500 option with maturity T and strike x as

$$\begin{aligned} C(\tau, s_t, v_t, x) =& e^{-r\tau} E^Q \left[(e^{S_T} - x)^+ | S_t = s_t, V_t = v_t \right] \\ =& e^{-r\tau} \int_{\mathbb{R}} (e^{s_T} - x)^+ p(s_T | \tau, s_t, v_t) ds_T \end{aligned}$$

where $p(s_T|\tau, s_t, v_t)$ is the SPD of S_T, $\tau = T - t$ is the time-to-maturity, and r is the constant risk-free rate between t and T[5]. Building upon the insight of Breeden and Litzenberger (1978), the SPD of S_T can be recovered from the second order derivative of $C(\tau, s_t, v_t, x)$ with respect to x,

$$p(s_T|\tau, s_t, v_t) = e^{r\tau + s_T} \frac{\partial^2 C(\tau, s_t, v_t, x)}{\partial x^2}\Big|_{x=e^{s_T}}. \tag{1}$$

By the same reasoning and assumption, we write the VIX option price H as a function of s_t, v_t, time-to-maturity τ and strike y, i.e., $H(\tau, s_t, v_t, y)$, and obtain the SPD of Z_T as

$$p(z_T|\tau, s_t, v_t) = e^{r\tau} \frac{\partial^2 H(\tau, s_t, v_t, y)}{\partial y^2}\Big|_{y=z_T}. \tag{2}$$

Note that the derived SPDs of S and Z in (1) and (2) extract all the information in option prices that can be used to identify the risk neutral measure Q, and the dynamics of S and V.

Nevertheless, these two densities $p(s_T|\tau, s_t, v_t)$ and $p(z_T|\tau, s_t, v_t)$ are of little use in studying the two option markets, as the information regarding V_t is unobservable. Considering the functional relationship between Z_t and V_t,[6] we may rewrite the option prices as a function of z_t, i.e., $C(\tau, s_t, z_t, x)$ and $H(\tau, s_t, z_t, y)$,[7] and take second order derivatives to obtain

$$\begin{aligned} p(s_T|\tau, s_t, z_t) =& e^{r\tau + s_T} \frac{\partial^2 C(\tau, s_t, z_t, x)}{\partial x^2}\Big|_{x=e^{s_T}} \\ p(z_T|\tau, s_t, z_t) =& e^{r\tau} \frac{\partial^2 H(\tau, s_t, z_t, y)}{\partial y^2}\Big|_{y=z_T} \end{aligned}$$

[5]We treat r as a constant here following our empirical setup, which seems valid for the sample period we consider; see Section 3 for details.

[6]The CBOE constructs Z_t form a portfolio of options weighted by strikes according to the formula:

$$(Z_t/100)^2 = E^Q(QV_{t,T}|\mathcal{F}_t) = \frac{2e^{r\tau}}{\tau} \Big(\int_0^{F_t} \frac{P(\tau, x)}{x^2} dx + \int_{F_t}^{\infty} \frac{C(\tau, x)}{x^2} dx \Big) + \epsilon$$

where $QV_{t,T}$ denotes the quadratic variation of the log return process, $P(\tau, x)$ and $C(\tau, x)$ are put and call options with time-to-maturity τ and strike x, and $F_t = e^{r\tau + s_t}$ is the price of futures contracts, see e.g. Britten-Jones and Neuberger (2000) and Carr and Wu (2009).

[7]Strictly speaking, the function $C(\cdot)$ here is a composite function, which is different from the previous call option pricing function. We recycle it to simplify our notations.

with an information set that is fully observable. In fact, representing options in terms of S_t and Z_t amounts to conditioning on a coarser information set generated by S_t and Z_t, which is a subset of the original information set generated by S_t and V_t. Equating the two information sets assumes that Z_t, as a function of V_t and S_t, is invertible with respect to V_t.

Such an approximation is justifiable for a few reasons. First, VIX is always exposed in the media spotlight, and used by investors as a measure of market volatility. Second, a simple regression of the realized volatility (RV) against the lagged VIX and lagged RV indicates that the VIX has far more explanatory power compared with RV, implying the importance of the VIX in capturing dynamics of the unobserved volatility (see, e.g., Blair et al. (2001)). Third, although the VIX differs from unobserved volatility as being forward-looking because it is constructed using S&P 500 options with maturities up to 30 calendar days, the approximation errors are expected to become smaller and even negligible as time-to-maturity increases.[8] Finally, for specification analysis of parametric models, the approximation error is zero for most models in the literature as Z_t is a deterministic and invertible function of V_t; see Section 2.5 for details.

In summary, state-price densities $p(s_T|\tau, s_t, z_t)$ and $p(z_T|\tau, s_t, z_t)$ encapsulate all the price information in the two option markets. They complement each other to reveal an intact picture of the index and volatility dynamics and interactions of the two markets.

2.2 Multivariate Local Linear Estimators

Here we introduce our nonparametric estimation strategies to estimate the SPDs. To fix ideas, we assume the observed prices $\tilde{C}$ and $\tilde{H}$ are contaminated with observation errors, such that

$$\begin{aligned} C(\tau, s_t, z_t, x) &= E\Big(\tilde{C}\Big|\tilde{\tau} = \tau, S_t = s_t, Z_t = z_t, X = x\Big) \\ H(\tau, s_t, z_t, y) &= E\Big(\tilde{H}\Big|\tilde{\tau} = \tau, S_t = s_t, Z_t = z_t, Y = y\Big). \end{aligned}$$

We then construct nonparametric estimators of C and H, and take derivatives to estimate the SPDs. Different from the multivariate kernel regression approach adopted by Aït-Sahalia and Lo (1998), we use the multivariate local linear method, which enjoys better boundary performance and reduces the asymptotic bias without increasing asymptotic variance (Fan and Gijbels (1996)). Moreover, local linear regression provides a closed-form estimator for both the price function and its first order derivatives, so that estimators for SPDs can be obtained simply by a first-order differentiation with respect to the strike.

[8] Due to this forward-looking feature, we have used S&P 500 option prices with maturities up to 30 calendar days when changing the information set to that generated by S_t and Z_t. Therefore, caution is needed when employing $p(s_T|\tau, s_t, z_t)$ in practice with τ less than 30 calendar days.

To estimate the pricing function C as a function of $\mathbf{u}=(\tau,s,z,x)'$, we consider the following minimization problem,

$$\min_{\alpha,\boldsymbol{\beta}}\sum_{i=1}^{n}\left\{C_i-\alpha-\boldsymbol{\beta}'\left(\mathbf{U}_i-\mathbf{u}\right)\right\}^2 K_h\left(\mathbf{U}_i-\mathbf{u}\right)$$

where $\mathbf{U}_i=(\tau_i,S_{t_i},Z_{t_i},X_i)'$ and C_i are the characteristics and price of the $i-$th option in sample. K_h is a kernel function scaled by a bandwidth vector $\mathbf{h}=(h_\tau,h_s,h_z,h_x)$:

$$K_h(\mathbf{U}_i-\mathbf{u})=\frac{1}{h_\tau}k\left(\frac{\tau_i-\tau}{h_\tau}\right)\frac{1}{h_s}k\left(\frac{S_{t_i}-s}{h_s}\right)\frac{1}{h_z}k\left(\frac{Z_{t_i}-z}{h_z}\right)\frac{1}{h_x}k\left(\frac{X_i-x}{h_x}\right) \tag{3}$$

where $k(\cdot)$ is, for example, the density of standard normal distribution. The minimizer has a closed-form representation:

$$\begin{bmatrix}\widehat{\alpha}\\ \widehat{\boldsymbol{\beta}}\end{bmatrix}_{(1+4)\times 1}=\left(\boldsymbol{\Omega}'\mathbf{K}\boldsymbol{\Omega}\right)^{-1}\boldsymbol{\Omega}'\mathbf{K}\mathbf{C} \tag{4}$$

where

$$\boldsymbol{\Omega}=\begin{bmatrix}1 & (\mathbf{U}_1-\mathbf{u})'\\ \vdots & \vdots\\ 1 & (\mathbf{U}_n-\mathbf{u})'\end{bmatrix},\mathbf{C}=\begin{bmatrix}C_1\\ \vdots\\ C_n\end{bmatrix},\qquad \mathbf{K}=\begin{bmatrix}K_h\left(\mathbf{U}_1-\mathbf{u}\right) & & \\ & \ddots & \\ & & K_h\left(\mathbf{U}_n-\mathbf{u}\right)\end{bmatrix}.$$

The nonparametric local linear estimator for the option pricing function $C(\tau,s,z,x)$ is

$$\widehat{C}(\tau,s,z,x)=\widehat{\alpha}=\mathbf{e}_1'\left(\boldsymbol{\Omega}'\mathbf{K}\boldsymbol{\Omega}\right)^{-1}\boldsymbol{\Omega}'\mathbf{K}\mathbf{C},$$

with $\mathbf{e}_1=(1,0,0,0)'$ and the estimator $\widehat{p}(s'|\tau,s,z)$ for the SPD of S_t is

$$\widehat{p}(s'|\tau,s,z)=e^{r\tau+s'}\frac{\partial\widehat{\boldsymbol{\beta}}_4}{\partial x}\Big|_{x=e^{s'}}=e^{r\tau+s'}\frac{\partial\left(\mathbf{e}_4'\left(\boldsymbol{\Omega}'\mathbf{K}\boldsymbol{\Omega}\right)^{-1}\boldsymbol{\Omega}'\mathbf{K}\mathbf{C}\right)}{\partial x}\Big|_{x=e^{s'}}. \tag{5}$$

where $\mathbf{e}_4=(0,0,0,1)'$. The nonparametric estimator $\widehat{H}(\cdot)$ and $\widehat{p}(z'|\tau,s,z)$ can be constructed similarly.

In our empirical implementations, we choose the Gaussian kernel as $k(\cdot)$. Moreover, we set the bandwidth h_j ($j=\tau,s,z$, and x) as $h_j=c_j s\left(\mathbf{U}^j\right)n^{-1/6}$, where $s\left(\mathbf{U}^j\right)$ is the unconditional standard deviation of the regressor $\mathbf{U}^j$ ($j=\tau,s,z$, and x). This bandwidth choice ensures that the nonparametric pricing function achieves the optimal rate of convergence in the mean-squared sense among all possible nonparametric estimators for option prices. However, this rate leads to an asymptotic bias term for the estimator $\widehat{C}(\cdot)$ and $\widehat{p}(\cdot)$, as shown by Fan and Gijbels (1996). Hence, we follow Aït-Sahalia and Lo (1998) to select $c_j=c_{j0}/\ln(n)$, which results in a slightly slower rate of convergence, but centers the asymptotic distribution at zero. Similar to Li and Zhao (2009), the constant c_{j0} is chosen by minimizing the finite-sample mean-squared error of the estimator via simulations.

Suppose the sample sizes of the nonparametric SPD estimates of the S&P 500 index and VIX are n_C and n_H, respectively. Using the equivalent kernels introduced in Fan and Gijbels (1996) and following the derivation in Aït-Sahalia and Lo (1998), we obtain the asymptotic distributions of these estimators:

$$n_C^{1/2} h_x \left(h_\tau h_s h_z h_x\right)^{1/2} \left(\widehat{p}(s'|\tau, s, z) - p(s'|\tau, s, z)\right) \tag{6}$$
$$\xrightarrow{d} N\left(0, e^{2r\tau}\left[\int k^2(c)\, dc\right]^3 \left[\int \left(c\dot{k}(c) + k(c)\right)^2 dc\right] / \left[\int k(c)\, c^2 dc\right]^2 s_C^2(\tau, s, z, x) / \pi_C(\tau, s, z, x)\right),$$
$$n_H^{1/2} \overline{h}_y \left(\overline{h}_\tau \overline{h}_s \overline{h}_z \overline{h}_y\right)^{1/2} \left(\widehat{p}(z'|\tau, s, z) - p(z'|\tau, s, z)\right)$$
$$\xrightarrow{d} N\left(0, e^{2r\tau}\left[\int k^2(c)\, dc\right]^3 \left[\int \left(c\dot{k}(c) + k(c)\right)^2 dc\right] / \left[\int k(c)\, c^2 dc\right]^2 s_H^2(\tau, s, z, y) / \pi_H(\tau, s, z, y)\right),$$

where $s_C^2(\tau, s, z, x)$ and $s_H^2(\tau, s, z, y)$ are conditional variances for the local linear regressions of C and H on their state variables respectively, and $\pi_C(\tau, s, z, x)$ and $\pi_H(\tau, s, z, y)$ are marginal densities of these variables. The estimators $\widehat{s}_C^2(\cdot)$ and $\widehat{s}_H^2(\cdot)$ for $s_C^2(\cdot)$ and $s_H^2(\cdot)$ can be constructed using similar nonparametric regressions.

2.3 Dimension Reduction

One of the major issues of nonparametric estimation is the *curse of dimensionality.* The rate of convergence decreases rapidly as the dimension of state variables increases. In the most general forms, the pricing functions $C(\cdot)$ and $H(\cdot)$ depend not only on time-to-maturity, strike, VIX, and the S&P 500 index, but also on interest rates and dividends. Because interest rates and dividends do not vary much within the period of our empirical studies (see Section 3 for details), we take them as constants.

Furthermore, following many existing studies such as Li and Zhao (2009), we assume the S&P 500 option price is homogeneous of degree one in the current price level:

$$C(\tau, s, z, x) = e^s C(\tau, 0, z, x/e^s) = e^s \bar{C}(\tau, z, m)$$

where $m = x/e^s$ represents the moneyness of the S&P 500 option. Consequently, we obtain the estimate of $C(\tau, s, z, x)$ through multiplying the nonparametric estimate of $\bar{C}(\cdot)$ by e^s, and write the SPD of S_T as

$$p(s_T|\tau, s_t, z_t) = e^{r\tau + s_T - s_t} \frac{\partial^2 \bar{C}(\tau, z_t, m)}{\partial m^2}\Big|_{m = e^{s_T}/e^{s_t}}.$$

As for VIX options, we assume that the information about $Z_{t'}$ in S_t is fully incorporated into Z_t. In other words, conditional on Z_t, $Z_{t'}$ is independent of S_t, for any $t' > t$. This assumption further implies that the SPD of Z_T, obtained from the VIX option prices, depends on S_t only through Z_t, i.e., $p(z_T|\tau, s_t, z_t) = p(z_T|\tau, z_t)$. Thus, the number of state variables for the SPD of VIX is also decreased by one. We formally test these two assumptions in Section 3, and the results suggest that they are not rejected for the sample we consider.

2.4 Nonparametric Estimation of Pricing Kernels

In the equilibrium model of Aït-Sahalia and Lo (2000) with a representative agent, the ratio between risk-neutral and physical densities, i.e., the pricing kernel, is—up to a scaled factor—the marginal rate of substitution, which also reflects the market's aggregation of risk preferences implicit in option prices. In an economy with two state variables, the pricing kernel is supposed to be $\pi(s_T, z_T|\tau, s_t, z_t)$, which cannot be identified nonparametrically. Nevertheless, we may consider an integrated pricing kernel $\pi(s_T|\tau, s_t, z_t)$ as

$$\pi(s_T|\tau, s_t, z_t) = E_{S_T}(\pi(s_T, z_T|\tau, s_t, z_t)|s_t, z_t) = \frac{p(s_T|\tau, s_t, z_t)}{\widetilde{p}(s_T|\tau, s_t, z_t)}$$

where $\widetilde{p}(s_T|\tau, s_t, z_t)$ is the conditional physical density of S_T, $p(s_T|\tau, s_t, z_t)$ is the conditional risk neutral density, and $E_{S_T}(\cdot|s_t, z_t)$ denotes the projection onto S_T. Distinct from Aït-Sahalia and Lo (2000) and Jackwerth (2000), our pricing kernel $\pi(\cdot|\cdot)$ is conditional on the VIX so that it can reflect the risk preferences with respect to the volatility.

While estimating the risk neutral density $p(s_T|\tau, s_t, z_t)$ from option prices, we estimate $\widetilde{p}(s_T|\tau, s_t, z_t)$ using the time series of the S&P 500 index and VIX based on the kernel method. Suppose we have time series $\{S_{T_i}, S_{t_i}, Z_{t_i}\}_{i=1}^n$, with $T_i - t_i = T - t$ fixed. We construct the estimator of the density $\tilde{p}(s'|\tau, s, z)$ as:

$$\widehat{\widetilde{p}}(s'|\tau, s, z) = \frac{\sum_{i=1}^{n} \frac{1}{b_{s'}} k\left(\frac{S_{T_i} - s'}{b_{s'}}\right) \frac{1}{b_s} k\left(\frac{S_{t_i} - s}{b_s}\right) \frac{1}{b_z} k\left(\frac{Z_{t_i} - z}{b_z}\right)}{\sum_{i=1}^{n} \frac{1}{b_s} k\left(\frac{S_{t_i} - s}{b_s}\right) \frac{1}{b_z} k\left(\frac{Z_{t_i} - z}{b_z}\right)}$$

with bandwidths given by the following:

$$b_j = c_j \sigma_j n^{-1/(4+d)}, \ j = s', s, z$$

where σ_j is the unconditional standard deviation of the data, c_j is a constant, and d represents the dimension: $d = 3$ for those b_js that appear in the numerator, and $d = 2$ for those in the denominator.

Consequently, our pricing kernel can be estimated by

$$\widehat{\pi}(s'|\tau, s, z) = \frac{\widehat{p}(s'|\tau, s, z)}{\widehat{\widehat{p}}(s'|\tau, s, z)}.$$

As discussed in Aït-Sahalia and Lo (2000), the convergence rate is higher for $\widehat{\widehat{p}}(s'|\tau, s, z)$ than for $\widehat{p}(s'|\tau, s, z)$, and the asymptotic distribution of $\widehat{\pi}(s'|\tau, s, z)$ is identical to that of $\widehat{p}(s'|\tau, s, z)/\widetilde{p}(s'|\tau, s, z)$, so that the asymptotic variance can be easily estimated.

2.5 Specification Tests Based on State-Price Densities

The estimated SPDs enable us to evaluate the performance of the state-of-the-art parametric models under the unified framework of pricing S&P 500 and VIX options. We bring together and test a variety of stochastic volatility models, including those discussed in Pan (2002), Eraker (2004), and Broadie et al. (2007), which are shown to be successful in capturing empirical features of S&P 500 options such as volatility smile, smirk, and index option returns. It is important to check whether they can capture the empirical features of VIX options before employing them in practice, as the two option markets are highly integrated. However, few paper in the literature have discussed modeling them jointly. Our SPDs can uncover two distinct aspects— returns and volatility—of the parametric model dynamics, and hence may shed light on the inadequacy of the model. In particular, comparing our nonparametric SPD estimates with the parametric ones illustrates whether the rejection is mainly due to modeling S&P 500 options, VIX options, or both, and which characteristics we should focus on in order to improve the model specification.

The general model we consider is specified under the risk-neutral measure Q as

$$\begin{aligned} dS_t &= (r - d - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_t^Q + J_S^Q dN_t - \mu\lambda_t dt \\ dV_t &= \kappa(\xi - V_t)dt + \sigma\sqrt{V_t}dB_t^Q + J_V^Q dN_t \end{aligned} \tag{7}$$

where W_t^Q and B_t^Q are standard Brownian motions satisfying $E(dW_t^Q dB_t^Q) = \rho dt$, J_S^Q and J_V^Q are random jump sizes, dN_t is a pure-jump process with intensity $\lambda_t = \lambda_0 + \lambda_1 V_t$, and $\mu = E(e^{J_S^Q} - 1)$.

The benchmark model is the Heston stochastic volatility model (SV) without jumps in either returns or volatility, i.e., $\lambda_0 = \lambda_1 = 0$. To add jumps in returns, we consider two types of specification:

$$\text{SVJ1:} \quad J_S^Q \sim N(\mu_S, \sigma_S); \qquad \text{or SVJ2:} \quad J_S^Q \sim \begin{cases} exp(\beta_+) & \text{with probability } q \\ -\ exp(\beta_-) & \text{with probability } 1-q \end{cases}$$

where the jump intensity is constant λ_0, i.e. $\lambda_1 = 0$. The last two models incorporate the same exponentially distributed jumps in volatility with mean β_V:

$$J_V^Q \sim exp(\beta_V),$$

and we label them as SVCJ1 and SVCJ2, due to their differences in jump specification for returns. The intensity in these cases is linear in V_t. Compared with SVJ1 and SVCJ1, the jump specifications of return dynamics in model SVJ2 and SVCJ2 can incorporate heavy-tailed and skewed jumps.

For each model, the theoretical value of the VIX Z_t can be calculated, as shown by Britten-Jones and Neuberger (2000) and Carr and Wu (2009):

$$Z_t^2 = 10^4 \cdot \frac{1}{\tau} E_t^Q \Big(\int_t^{t+\tau} V_s ds + \sum_{N_s \geq 0,\ t \leq s \leq t+\tau} (J_S^Q)^2 \Big) = aV_t + b \tag{8}$$

where a and b are constants determined by the specific model; see Appendix B for the formulae of a and b and details of derivation. On top of (8), we derive VIX option prices in addition to S&P 500 option prices.[9] These formulae regarding the VIX and its option prices can greatly simplify the parametric inference, as the unobservable volatility factor can be replaced by a function of the VIX.[10]

The dynamics of (S_t, V_t), combined with (8), imply the dynamics of (S_t, Z_t) for which we perform our nonparametric specification analysis of the model (7). In particular, model (7) specifies parametric forms of the SPDs of S&P 500 index and VIX, i.e., $p(s_T|\tau, s_t, z_t;\theta)$ and $p(z_T|\tau, z_t;\theta)$, respectively (see Appendix B for details of derivation). The idea of our nonparametric analysis is to compare these parametric SPDs with nonparametric estimates $\widehat{p}(s_T|\tau, s_t, z_t)$ and $\widehat{p}(z_T|\tau, z_t)$ and check whether the differences are statistically significant.

Recall that we employ two dimension-reduction techniques in obtaining $p(s_T|\tau, s_t, z_t)$ and $p(z_T|\tau, z_t)$, i.e., homogeneous of degree 1 in S_t for S&P 500 options and the dependence of VIX densities on S_t only through Z_t, both of which are satisfied for the parametric model (7). Based on these simplifications, we propose to test the joint null hypothesis $\mathbb{H}_0$,

$$\mathbb{H}_0: \ \Pr\left\{p(m_T|\widetilde{\tau}, Z_t; \boldsymbol{\theta}_0) = p(m_T|\widetilde{\tau}, Z_t),\ p(Z_T|\widetilde{\tau}, Z_t; \boldsymbol{\theta}_0) = p(Z_T|\widetilde{\tau}, Z_t)\right\} = 1$$

against the alternative hypothesis $\mathbb{H}_A$,

$$\mathbb{H}_A: \ \Pr\left\{p(m_T|\widetilde{\tau}, Z_t; \boldsymbol{\theta}_0) = p(m_T|\widetilde{\tau}, Z_t),\ p(Z_T|\widetilde{\tau}, Z_t; \boldsymbol{\theta}_0) = p(Z_T|\widetilde{\tau}, Z_t)\right\} < 1,$$

where $m_T = e^{S_T}/e^{S_t}$. If the null hypothesis $\mathbb{H}_0$ is rejected, we test the following two separate specification hypotheses,

$$\mathbb{H}_{0,\text{S\&P}}: \ \Pr\left\{p(m_T|\widetilde{\tau}, Z_t; \boldsymbol{\theta}_0) = p(m_T|\widetilde{\tau}, Z_t)\right\} = 1,$$

[9] The closed-form pricing formulae have been derived by Amengual and Xiu (2012) for more-general models, see also Sepp (2008) for derivation under the Heston model.

[10] The CBOE VIX contains numerical errors due to jumps and discretization and hence may not equal to the theoretical VIX in (8) exactly. However, these approximation errors are negligible under commonly used models and model parameters, as shown by Jiang and Tian (2005) and Carr and Wu (2009).

$$\mathbb{H}_{0,\mathrm{VIX}}: \ \Pr\left\{p(Z_T|\widetilde{\tau}, Z_t; \boldsymbol{\theta}_0) = p(Z_T|\widetilde{\tau}, Z_t)\right\} = 1,$$

i.e., whether the model is consistent with S&P 500 option prices and VIX option prices separately. The test statistics based on the differences between parametric and nonparametric SPDs are constructed and converge to convenient standard normal distributions under the null hypothesis; see Appendix A for details.

3 Empirical Results

In this section, we estimate nonparametric SPDs implicit in the S&P 500 and VIX options, along with recovering their interactions. Before delving into the details, we introduce the dataset.

3.1 Data

We collect close prices of S&P 500 Index, VIX, and their options from the OptionMetrics group for the year 2009. The best bid and offer prices for options are quoted between 3:59 p.m. and 4:00 p.m. EST. Figure 1 plots the joint time series of the S&P 500 index and the VIX, Table 1 provides their summary statistics, and Table 3 presents summary statistics of the option prices. We follow the data-cleaning routine commonly used in literature, see, e.g., Aït-Sahalia and Lo (1998). First, observations with bid or ask prices smaller than 0.05 are eliminated to mitigate the effect of price discreteness. For each option, we take the midquote as the option price. Due to liquidity concerns, we eliminate any options with zero open interests or trading volumes, as well as options with time-to-maturity of less than 10 days. Also, it is clear from Table 2, that in-the-money S&P 500 options are less liquid than out-of-the-money options. Therefore, we delete in-the-money options, and use the put-call parity to construct in-the-money call options from out-of-the-money put options. There is no such pattern for VIX options and hence we only consider VIX call options. The last step is to eliminate option contracts that violate no-arbitrage conditions. The resulting sample covers a broad cross section of options, including 128,883 S&P 500 call options, and 14,539 VIX call options.

3.2 Monte Carlo Simulations

We provide simulation evidence for our local linear estimators. The Monte Carlo experiments are designed to match our empirical studies. First, we select the same option characteristics as those traded on CBOE in 2009. Second, we fix a sample path generated by the SVCJ2 model. We then calculate option prices, according to the closed-form formulae given in the Appendix. Finally, we pollute the prices with multiplicative measurement error following log-normal distribution with 1% standard deviation. Option prices below 0.05 are eliminated to mimic the reality.

Based on the generated sample, we evaluate our nonparametric estimators of index option pricing functions on the grid of time-to-maturity and current index level, with Z_t and X fixed at their sample averages. We also calculate the index densities on the grid of τ and S_T, with S_t fixed at the sample mean, to evaluate our density estimators. The nonparametric estimators of VIX option prices and densities are evaluated similarly. All of these quantities and their percentage errors are reported in Figures 2 and 3, averaged over 500 replications. These figures show that the nonparametric estimates are within 2% and 5% of their theoretical values for S&P 500 and VIX options, respectively. The errors for densities are slightly larger, as also shown in both Figures 2 and 3, due to the fact that derivatives are estimated with slower rates of convergence, the so-called *curse of differentiation.*

3.3 State-Price Densities of S&P 500 Index and VIX

3.3.1 Empirical Characteristics of State-Price Densities

Figure 4 presents nonparametric SPD estimates of the S&P 500 index for both low and high levels of the VIX with time-to-maturity equal to 21, 42, and 84 days. The low and high levels of the VIX are obtained as the 20% and 80% quantiles, respectively, of the VIX in our sample. The 95% confidence intervals are obtained through the asymptotic theory given in (6). As expected, Figure 4 shows that the S&P 500 index densities are negatively skewed across all maturities and VIX values. More importantly, the index densities strongly depend on the VIX Z_t. Take $\tau = 21$ as an example. When Z_t is low, the index density $p(s_T|\tau, s_t, z_t)$ has pronounced spikes; as Z_t rises to the high level, the density becomes more dispersed and more negatively skewed. This pattern becomes stronger for longer maturities.

In Figure 5, we benchmark the nonparametric SPD estimates proposed by Aït-Sahalia and Lo (1998) (AL) that neglect the volatility variable, and compare it with our density estimates conditional on different VIX levels. It is evident that the AL density estimate departs from ours substantially when Z_t stays at high or low levels: our estimate is more compact with higher spikes for a low Z_t but more dispersed with heavier tails for a high Z_t. In contrast, the two densities are very close when Z_t is at the medium level, indicating that the AL density estimator actually approximates our density estimator for medium VIX levels. Hence, our nonparametric SPD estimator may be more informative about the future dynamics of the S&P 500 index than the AL density.

Figure 6 provides nonparametric estimates of the VIX SPDs for different levels of Z_t with τ equal to 21, 42, and 84 days. We find three important features of the risk-neutral densities of the VIX.

First, the VIX densities are positively skewed across all maturities and current VIX levels. Second, the risk-neutral dynamics of the VIX exhibit a mean-reverting pattern, which is most apparent when Z_t is at the high level. Specifically, the left tail of the VIX SPD becomes higher as maturity increases, showing that when Z_t is high, the probability of Z_T mean-reverts to the medium level increases with maturity. This mean-reverting behavior also shows up when Z_t is at the low level, although less strongly, as evidenced by the lower peaks at the left end of the VIX densities for longer maturities. Third, the risk-neutral dynamics of the VIX exhibit high persistence. Conditional on a low Z_t, the shape of the VIX density changes slowly as maturity increases. This change implies that the probability of Z_T to stay at low levels conditional on a low Z_t is high. Even conditional on a high Z_t level when the mean reversion effect is strong, Z_T can maintain at high levels with large probability as maturity increases from 21 to 42 days. Moreover, for any fixed maturity, the VIX density is more compact (dispersed), conditional on a low (high) level of the current VIX.

3.3.2 Information from VIX Options about S&P 500

As shown in the last section, SPDs of the S&P 500 index strongly depend on Z_t at maturities of 21, 42, and 84 days, for which VIX options are available. Such short-run dependence can be driven through two channels. First, investors may employ historical time series of the VIX up to time t to predict the volatility of the future index S_T. Second, investors may use VIX option prices to obtain information about the distribution of the future VIX Z_T and form expectations of the volatility of S_T. We now study whether VIX options contain information about market future volatility beyond VIX time series for the future volatility of index dynamics.

We compute the average volatility over some future 30-calendar-day interval $[T, T+21]$ by,

$$\mathrm{V}_{t,T}^{S} = \sqrt{\mathrm{Var}_t\left[S_{T+21}\right] - \mathrm{Var}_t\left[S_T\right]}$$

where $\mathrm{Var}_t\left[S_T\right]$ is the conditional variance of S_T implied from the density estimate $p(s_T|\tau, s_t, z_t)$. We then regress $\mathrm{V}_{t,T}^{S}$ on both Z_t and characteristics of the VIX density, corresponding to information in VIX time series and VIX options, respectively. The characteristics of VIX densities we employ are the conditional standard deviation $\mathrm{Std}_t\left[Z_T\right]$ and conditional skewness $\mathrm{Skew}_t\left[Z_T\right]$ of $p(z_T|\tau, z_t)$, which proxy the volatility of volatility and volatility skewness implicit in VIX options. All of the quantities are annualized and estimated using nonparametric SPDs.

For $\tau = 21, 42, 63$ and 84, we run the following regression

$$\mathrm{V}_{t,t+\tau}^{S} = \alpha + \beta_1 Z_t + \beta_2 \mathrm{Std}_t\left[Z_{t+\tau}\right] + \beta_3 \mathrm{Skew}_t\left[Z_{t+\tau}\right] + \varepsilon_t, t = 1, 2, \ldots, 252 \tag{9}$$

and the results are presented in Table 3. We find that the VIX level Z_t has predictability for the S&P 500 option-implied future volatility at $\tau = 21$ and 84, but is less significant with $\tau = 42$ and 63 days.

Most importantly, both the volatility of volatility $\text{Std}_t[Z_{t+\tau}]$ and volatility skewness $\text{Skew}_t[Z_{t+\tau}]$ are significant in explaining $V^S_{t,t+\tau}$, with the former more pronounced for maturities of 63 and 84 days and the latter more evident for all maturities except for 84 days. For example, Panels C and D show that adding the volatility of volatility $\text{Std}_t[Z_{t+\tau}]$ to the regression with single explanatory variable Z_t increases the adjusted R^2 from less than 1% to over 12%. Moreover, Panels A, B, and C show that adding volatility skewness $\text{Skew}_t[Z_{t+\tau}]$ to the regression against Z_t and $\text{Std}_t[Z_{t+\tau}]$ leads to increases in the adjusted R^2, by 20% for maturities equal to 21 and 63 days and 300% for 42 days.

In summary, we find that the dependence of index densities on the VIX is driven not only by the VIX time series, but also by the volatility of volatility and volatility skewness that can only be extracted from VIX option prices. That is to say, investors consider both the VIX level Z_t and the whole distribution of volatility when forming expectations of future volatility and pricing S&P 500 options. Hence, the availability of VIX options indeed delivers incremental information for market participants beyond the VIX time series.

3.3.3 In-Sample and Out-of-Sample Forecasts

We benchmark our nonparametric estimator (SX) against two alternative methods discussed in Aït-Sahalia and Lo (1998): the nonparametric approach without volatility state (AL) for both densities and option prices, and the martingale approach (MKT) for option prices, which interpolates tomorrow's implied volatility using today's implied volatility surface.

Intuitively, one of the potential advantages of the SX and AL approaches lies in their inclusion of historical prices of options with similar characteristics into the weighted prediction of option prices. As opposed to the MKT approach that replies exclusively on the cross section of options at each date, both SX and AL tend to maintain more-stable pricing structures over time, and hence they are expected to outperform the MKT approach in out-of-sample forecasting, although they may fit the cross section of option prices poorly on certain days, whereas perfectly for some other days in-sample. In terms of the comparison between SX and AL, the SX approach further explores the predictability from the VIX, hence the SX is expected to outperform the AL in- and out-of-sample for both option prices and densities.

Panel A of Table 4 reports the forecasting performance of the SX and AL density, whereas Panel B presents the price prediction comparisons including, additionally, the MKT approach. For each date t, we adopt a preceding six-month window, within which options are selected to predict the prices of traded options with strikes equal to 900, 925, 950, and 975 (the four strikes around the median of the sample) and maturities closest to four months, along with the SPDs, on day $t+\gamma$, for $\gamma = 0$ (in-sample), and $\gamma = 7, 14, 21, 28, 35$, and 42 days (out-of-sample) progressively. We repeat the procedure for each day t from the second half of the year, and aggregate across t to obtain the root-mean-squared percentage difference between the predictions and the values realized on day $t+\gamma$. The realized density on day $t+\gamma$ is proxied by either the AL or the SX estimator, whereas the observed market prices are taken to be the realized option prices.

Regarding density forecasts, the AL densities produce a better fit in-sample and over short horizons (for γ up to 14 days), but only when the realized density is represented by the AL method (the left part of Panel A). In contrast, the SX estimates are uniformly better over every horizon in predicting the future density when the realized density is represented by the SX density (the right part of Panel A), but also when it is proxied by the AL density for γ longer than 21 days. For market prices, the AL estimates are uniformly outperformed by our estimators at every horizon except at $\tau = 14$ days. Moreover, the simple martingale interpolation leads to a better fit in-sample and for less than a month ($\tau = 0, 7, 14$, and 21 days), whereas over longer forecasting horizons, it is outperformed by the SX densities.

For forecasting option prices, Panel B shows that the MKT approach outperforms the AL approach uniformly in forecasting prices for the sample period we consider, which is in contrast with findings of Aït-Sahalia and Lo (1998), at least for longer horizons. However, this finding is not puzzling as the AL estimator behaves as an estimator conditional on the average volatility level and performs better when the volatility stays around the mean level. In our sample period, the VIX decreases monotonically and stays at lower levels for the entire second half of the year. Therefore, the uniform better performance of the MKT over the AL method may be caused by the decreasing volatility path in 2009, as the assumption of stability across time deteriorates the performance. This performance actually highlights, indirectly, the importance of including volatility in the prediction from the fact that the SX outperforms the MKT in the long run, as shown by Panel B.

3.4 Pricing Kernels Conditional on VIX

In addition to SPDs under the risk-neutral measure, we also estimate the physical densities of the S&P 500 index, and hence obtain estimates of the pricing kernels conditional on VIX by combining the SPDs under both measures.

To highlight the potential change of average risk preference due to recent financial crisis, we compare the empirical characteristics of the pricing kernels over two different periods: the pre-crisis[11] period from June 1, 2006 to November 30, 2007, and the post-crisis period from June 1, 2009 to November 30, 2010. Moreover, we focus on a 2-month ($\tau = 42$ days) horizon. Figure 7 reports the

[11]We follow the NBER Business Cycle Committee and regard the 18-month duration of 2008 financial crisis as from Nov 1, 2007 to May 31, 2009. Then we chose 18-month periods before and after the crisis, respectively.

pricing kernel estimates, for both pre-crisis (in the upper panel) and post-crisis (in the lower panel) at different VIX levels. The low and high levels of Z_t before crisis, are chosen as 11.56 and 20.03, whereas the low and high VIX levels post-crisis are taken as 20.03 and 35.455, respectively. Note that the low level of VIX post-crisis is the same as the high level pre-crisis, as the VIX stays in a high regime after crisis.

We observe that the pre-crisis pricing kernels, along with the post-crisis pricing kernel conditional on a low level, $Z_t = 20.03$, decrease as market index return increases. This decrease is consistent with standard economic theory that the pricing kernel decreases when wealth rises, as implied by a decreasing risk aversion. In contrast, the pricing kernel conditional on a high VIX level, Z_t=35.455, in the post-crisis period exhibits a pronounced U-shape. Therefore, our empirical result suggests that the shape of the pricing kernel is influenced by the stochastic volatility. In particular, the two pricing kernels conditional on the same VIX level, Z_t=20.03 (the top right and low left panels), although separately estimated using non-overlapping data before and after the crisis, exhibit a similar decreasing pattern.[12]

Overall, consistent with Jackwerth and Brown (2001), Chabi-Yo et al. (2008), Chabi-Yo (2011), and Christoffersen et al. (2010a), our nonparametric analysis reveals that the documented U-shape of pricing kernels could be due to the missing stochastic volatility variable.

3.5 Test Results

Here we report the nonparametric test results of parametric models (7) using S&P 500 and VIX options. For each parametric model, we estimate the model parameter θ by minimizing the mean squared percentage pricing errors:

$$\widehat{\boldsymbol{\theta}} = \min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left\{ \sum_{i=1}^{n_C} \left[\log\left(\frac{\widehat{C}(\tau_i, s_{t_i}, z_{t_i}, x_i)}{C(\tau_i, s_{t_i}, z_{t_i}, x_i; \boldsymbol{\theta})} \right) \right]^2 + \sum_{i=1}^{n_H} \left[\log\left(\frac{\widehat{H}(\tau_i, z_{t_i}, x_i)}{H(\tau_i, z_{t_i}, x_i; \boldsymbol{\theta})} \right) \right]^2 \right\}.$$

It is worth pointing out that our estimation strategy avoids replacing the unobserved volatility process by some VIX proxy as in Aït-Sahalia and Kimmel (2007), or high frequency volatility measures, e.g., Wu (2011). Instead, we derive the exact relationship between the instantaneous volatility and the VIX, and estimate the model using both VIX time series and VIX option prices, without which the parameters in the volatility process cannot be identified. This estimation leads to a secondary contribution of this paper—a new parametric estimation strategy.

We follow the common practice in the literature to select a subsample of S&P 500 and VIX options for parametric fitting, in order to accelerate the speed of calculation and ensure the quality

[12]The difference in the supports of these two pricing kernels over the wealth axis is caused by different levels of S&P 500 index pre- and post-crisis.

of the fitting. For each day in 2009, we first choose three shortest, yet, different maturities, and for each maturity, we randomly select three options with different moneyness. The resulting subsample includes a wide cross section of options. Parametric estimation results are shown in Table 5. Recall that the SVCJ1 and SVCJ2 models are augmented with volatility jumps compared with the SVJ1 and SVJ2 models. The difference in fitting results indicates that adding jumps into volatility dynamics, regardless of jump specifications, indeed reduces the fitting errors from 60% to 40% and significantly decreases the volatility of volatility (σ). Moreover, the insignificant λ_0 estimates for both SVCJ1 and SVCJ2 suggest that a state-dependent jump intensity without constant terms seems adequate. Finally, the estimates for the average jump sizes in S&P 500 returns are negative, as shown by $\mu_s = -0.119$ for SVCJ1, and $\beta_- > \beta_+$, with the probability of downward jump equal to $1 - q = 0.722$ for SVCJ2.

Using both parametric and nonparametric estimates, the test statistics discussed in Section 2.5 are computed and reported in Table 6. We find that all of the models are rejected with p-values close to zero, implying that they cannot capture S&P 500 and VIX option prices simultaneously. To gauge the performance in fitting each type of option prices, Table 6 reports test statistics of separate hypotheses. The result shows that for SVCJ1 and SVCJ2, the rejection for S&P 500 options is more serious compared with VIX options, which have smaller values of test statistics. This result is not surprising though, as adding VIX options in fitting amounts to increasing the weight of volatility dynamics.

Our SPD estimates can provide more-detailed information about the mis-specification of the model and pave the way for potential improvement. We report in Figures 8 and 9 the nonparametric and parametric SPDs of S_t and Z_t at both high and low levels of the VIX across maturities. It is obvious from the plot that for both index returns and the VIX, parametric estimates of SPDs are closer to nonparametric ones when the current VIX level is high, whereas they tend to underestimate the dispersion when volatility is low. This asymmetric pattern suggests an asymmetric mean-reverting rate of volatility—the speed is high at low levels of volatility and low otherwise—because the affine models under consideration have a constant mean-reverting rate, κ. Therefore, the parametric models under consideration are not flexible enough to capture the volatility dynamics implied by S&P 500 and VIX options simultaneously, indicating that non-affine models may be better choices.[13]

[13] In a related study, Engulatov et al. (2011) nonparametrically estimate the drift function of the volatility process and document a similar nonlinear pattern.

3.6 When VIX Options are Not Available

As shown in Table 1, the longest maturity for VIX option contracts in 2009 is 132 days whereas S&P 500 options with maturities from 133 days to a year are still very liquid. As a result, investors who would like to buy S&P 500 options with maturities longer than 132 days have no information available from VIX options about the volatility of the future index. In this case, do investors predict the long-term future volatility using current VIX Z_t and price the S&P 500 options accordingly? Or do they just price the long-maturity S&P 500 options by taking the future volatility level as a constant, which may equal the long-run mean? Answering these questions will shed light on how investors employ historical time series of the VIX up to time t to predict the volatility of the future S&P 500 index S_T.

To answer these questions, we direct our nonparametric estimation specifically to the long-term SPDs of S&P 500 index with maturities longer than 132 days. The remaining sample size is as large as 19,876 data points. Figure 10 provides nonparametric SPD estimates with maturities equal to 168, 231, 294, and 357 days for both high and low levels of the VIX. Across the long maturities, we find that the SPDs of the index conditional on high VIX levels differ significantly from the SPDs conditional on low current VIX levels, i.e., the former density is more dispersed than the latter. The significance indicates that investors may predict the future volatility through certain mechanism and price S&P 500 options accordingly, even when VIX options are not available.

One possible mechanism for such long-run dependence is the simple martingale procedure, which is shown to work well even in the short run when VIX options are available. Specifically, investors can first compute implied volatility of yesterday's S&P 500 option prices, interpolate the implied volatility of long-term options using today's corresponding characteristics, and then treat them as the implied volatility of today's long-term options. This martingale mechanism may result in the observed long-run dependence as short-term options may carry information from Z_t. To study whether this mechanism works in practice, we run the following regression

$$IV_t^\tau = \alpha^\tau + \beta^\tau IIV_t^\tau + \varepsilon_t^\tau$$

of the implied volatility IV_t^τ on the interpolated implied volatility IIV_t^τ for the maturity $\tau = 168, 231, 294,$ and 357 days, corresponding to the documented long-term dependence in Figure 10. The results in Table 7 show that the β^τ coefficients are significant and close to one across all long maturities with adjusted R^2 larger than 72%. This result suggests that the martingale procedure agrees with the market practice and may offer a potential explanation for the long-run dependence of the future S&P 500 index on current VIX levels.

3.7 Robustness Checks

Here we verify, for the sample period, the two dimension-reduction techniques employed in our nonparametric analysis, including homogeneous of degree 1 in S_t for S&P 500 options, and the dependence of VIX option prices on S_t only through Z_t.

Suppose $\widehat{C}(\tau, s_t, z_t, x)$ is the nonparametric estimate of the S&P 500 call option price $C(\tau, s_t, z_t, x)$ without the homogeneous of degree 1 assumption, and $\widehat{C}(\tau, z_t, m_t)$ is the estimate we have been using. For VIX options, $\widehat{p}(z_T|\tau, z_t)$ and $\widehat{p}(z_T|\tau, s_t, z_t)$ are nonparametric estimates of the SPDs with and without the assumption. In Figure 11, we plot $\widehat{C}(\tau, z_t, m_t)$ and $\widehat{p}(z_T|\tau, z_t)$ on the left panel along with their percentage errors relative to $\widehat{C}(\tau, s_t, z_t, x)$ and $\widehat{p}(z_T|\tau, s_t, z_t)$ on the right panel. Close scrutiny of the plots shows that the percentage errors are mostly close to zeros in spite of reaching 20% and 10% around the boundaries where the estimates incur larger biases. Overall, our dimension-reduction assumptions seem valid for the sample period in the empirical study.

We also propose rigorous statistical testing to check whether these assumptions hold. The null hypotheses are given below:

$$\mathbb{H}_{1,\text{S\&P}}: \ \Pr\left\{C(\tau, S_t, Z_t, X) = C(\tau, Z_t, M_t)\right\} = 1$$

for S&P 500 options where $M_t = X/e^{S_t}$ and

$$\mathbb{H}_{1,\text{VIX}}: \ \Pr\left\{p(Z_T|\tau, Z_t, S_t) = p(Z_T|\tau, Z_t)\right\} = 1$$

for VIX options. We construct test statistics $\widehat{M}_{S\&P}$ and $\widehat{M}_{VIX}$ based on the differences between the two estimators for the same quantity. Under certain regularity conditions, both statistics converge to standard normal distributions under the null hypotheses; see Appendix C for details. Table 6 reports values of the test statistics and the corresponding p-values. There is no statistical evidence for rejection with p-values around 0.5.

4 Conclusion

In this paper, we proposed SPD estimates of the index and volatility and document several important empirical facts relevant to the understanding of the interactions between the two option markets. In particular, the dependence of SPDs of the index on the VIX suggests pervasive volatility information in the VIX and its option market. Our results also highlight the missing volatility variable that may be responsible for the pricing kernel puzzle. In addition, our testing results call the affine models into question, and suggest that the inadequacy of these models lies in the inability to adjust mean-reverting speeds for different volatility regimes.

In fact, our nonparametric pricing method carries over the strategies introduced in Aït-Sahalia and Lo (1998) to the case with stochastic volatility, by exploring additional information from VIX and its options. By nature, our method enjoys several beneficial features, such as being model-free, robust to misspecification of models and pricing measures, and computationally efficient. In contrast, parametric stochastic volatility models face an unfortunate compromise between misspecification and computation burden. For example, the convenient affine model with closed-form option pricing formulae are usually subject to misspecification errors. Moreover, even with affine models, the computation often involves implementation of numerical algorithms, such as inverse Fourier transforms and partial differential equations. Our closed-form regression formulae, however, alleviate the dilemma to a great extent. Of course, efforts are needed to choose suitable bandwidths for our approach.

Appendix

A Specification Test

We measure the differences between the parametric and nonparametric SPDs by the sum of squared deviations:

$$\begin{aligned} M\left(\widehat{p}\left(\cdot\right),p\left(\cdot\right)\right) =& E\left[\left(\widehat{p}(m_T|\tau,Z_t)-p(m_T|\tau,Z_t,\boldsymbol{\theta}_0)\right)^2 a_C\left(\tau,Z_t,m_T\right)\right.\\ &\left.+\left(\widehat{p}(Z_T|\tau,Z_t)-p(Z_T|\tau,Z_t;\boldsymbol{\theta}_0)\right)^2 a_H\left(\tau,Z_t,Z_T\right)\right] \end{aligned}$$

where $a_C\left(\cdot\right):\mathbb{R}^3\to\mathbb{R}^+$ and $a_H\left(\cdot\right):\mathbb{R}^3\to\mathbb{R}^+$ are weighting functions. The idea behind our test is simple: If the null $\mathbb{H}_0$ holds, then SPDs of both S&P 500 index and VIX should be close to their nonparametric estimates. Suppose we have a consistent estimator $\widehat{\boldsymbol{\theta}}$ for the parameter θ. Then we can estimate $M\left(\widehat{p}\left(\cdot\right),p\left(\cdot\right)\right)$ by its sample analog

$$\begin{aligned} \widehat{M}\left(\widehat{p}\left(\cdot\right),p\left(\cdot\right)\right) =& \frac{1}{n_C}\sum_{i=1}^{n_C}\left[\widehat{p}(m_{T_i}|\tau_i,Z_{t_i})-p(m_{T_i}|\tau_i,Z_{t_i};\widehat{\boldsymbol{\theta}})\right]^2 a_C\left(\tau_i,Z_{t_i},m_{T_i}\right)\\ &+\frac{1}{n_H}\sum_{i=1}^{n_H}\left[\widehat{p}(Z_{T_i}|\tau_i,Z_{t_i})-p(Z_{T_i}|\tau_i,Z_{t_i};\widehat{\boldsymbol{\theta}})\right]^2 a_H\left(\tau_i,Z_{t_i},Z_{T_i}\right) \end{aligned} \tag{A.1}$$

The use of weighting functions $a_C\left(\cdot\right)$ and $a_H\left(\cdot\right)$ are not uncommon in the literature and often used to remove extreme observations. As noted by Aït-Sahalia et al. (2001), appropriate choices of weighting functions can reduce the influences of unreliable estimates and make the tests focus on a particular empirical question of interest.

Denote the bandwidth for the state variable m_T as h_m. Our test statistic for $\mathbb{H}_0$ against $\mathbb{H}_A$ is an appropriately standardized version of (A.1):

$$\begin{aligned} \widehat{M} =& \left\{h_m^2\left(h_\tau h_z h_m\right)^{1/2}\sum_{i=1}^{n_C}\left[\widehat{p}(m_{T_i}|\tau_i,Z_{t_i})-p(m_{T_i}|\tau_i,Z_{t_i};\widehat{\boldsymbol{\theta}})\right]^2\times a_C\left(\tau_i,Z_{t_i},m_{T_i}\right)-\widehat{C}_1\right\}/\sqrt{\sqrt{2}\widehat{D}_1}\\ &+\left\{\overline{h}_y^2\left(\overline{h}_\tau\overline{h}_z\overline{h}_\tau\right)^{1/2}\sum_{i=1}^{n_H}\left[\widehat{p}(Z_{T_i}|\tau_i,Z_{t_i})-p(Z_{T_i}|\tau_i,Z_{t_i};\widehat{\boldsymbol{\theta}})\right]^2 a_H\left(\tau_i,Z_{t_i},Z_{T_i}\right)-\widehat{C}_2\right\}/\sqrt{\sqrt{2}\widehat{D}_2} \end{aligned}$$

where

$$\begin{aligned} \widehat{C}_1 =& \left(h_\tau h_z h_m\right)^{-1/2}\left[\int k^2\left(c\right)dc\right]^2\left[\int\left(c\dot{k}\left(c\right)+k(c)\right)^2 dc\right]/\left[\int k\left(c\right)c^2 dc\right]^2\\ &\times e^{2r\tau}\int\widehat{s}_C^2\left(\tau,z,m\right)a_C\left(\tau,z,m\right)d\tau dz dm\\ \widehat{C}_2 =& \left(\overline{h}_\tau\overline{h}_z\overline{h}_y\right)^{-1/2}\left[\int k^2\left(c\right)dc\right]^2\left[\int\left(c\dot{k}\left(c\right)+k(c)\right)^2 dc\right]/\left[\int k\left(c\right)c^2 dc\right]^2 \end{aligned}$$

$$
\begin{aligned}
&\times e^{2r\tau}\int \widehat{s}_H^2\left(\tau,z,y\right)a_H\left(\tau,z,y\right)d\tau dzdy \\
\widehat{D}_1 =&2\left\{\int\left[\int\left(c_1\dot{k}\left(c_1\right)+k(c_1)\right)\left((c_1+c_2)\dot{k}\left(c_1+c_2\right)+k(c_1+c_2)\right)dc_1\right]^2dc_2\right\} \\
&\times\left\{\int\left[\int k\left(c_1\right)k\left(c_1+c_2\right)dc_1\right]^2dc_2\right\}^2/\left[\int k\left(c\right)c^2dc\right]^4 \\
&\times e^{4r\tau}\int \widehat{s}_C^4\left(\tau,z,m\right)a_C\left(\tau,z,m\right)d\tau dzdm \\
\widehat{D}_2 =&2\left\{\int\left[\int\left(c_1\dot{k}\left(c_1\right)+k(c_1)\right)\left((c_1+c_2)\dot{k}\left(c_1+c_2\right)+k(c_1+c_2)\right)dc_1\right]^2dc_2\right\} \\
&\times\left\{\int\left[\int k\left(c_1\right)k\left(c_1+c_2\right)dc_1\right]^2dc_2\right\}^2/\left[\int k\left(c\right)c^2dc\right]^4 \\
&\times e^{4r\tau}\int \widehat{s}_H^4\left(\tau,z,y\right)a_H\left(\tau,z,y\right)d\tau dzdy
\end{aligned}
$$

where $s_C\left(\tau,z,m\right)$ is the square-root of the conditional variance for the local linear regression of C_i on $\mathbf{U}_i$ and $s_H\left(\tau,z,y\right)$ is defined similarly. The bandwidths are chosen to satisfy $\frac{\ln n_C}{n_C h_m^4 h_\tau h_z}\to 0$, $h_m^2 h_\tau h_z \ln n_C \to 0$, $\frac{\ln n_H}{n h_y^4 h_\tau h_z}\to 0$, and $h_y^2 h_\tau h_z \ln n_H \to 0$ as $n_C, n_H\to\infty$, with n_C/n_H converging to a non-zero constant. Under certain regularity conditions, we have

$$
\widehat{M}\overset{\mathrm{d}}{\longrightarrow}N\left(0,1\right)\ \text{ under } \mathbb{H}_0 \text{ as } n_C, n_H\to\infty
$$

When the null hypothesis $\mathbb{H}_0$ is rejected, we follow similar procedures as above to construct test statistics $\widehat{M}_{\text{S\&P}}$ and $\widehat{M}_{\text{VIX}}$ to test separate specification hypotheses $\mathbb{H}_{0,\text{S\&P}}$ and $\mathbb{H}_{0,\text{VIX}}$, i.e., whether the model is consistent with S&P 500 option prices and VIX option prices separately.

B Parametric SPDs and Option Pricing

We derive the theoretical value of the VIX Z_t for Models SVCJ1 and SVCJ2, since the other models are nested. In fact, we have

$$
Z_t^2 = 10^4\cdot\frac{1}{\tau}E_t^Q\Big(\int_t^{t+\tau}V_s ds+\sum_{N_s\geq 0,\ t\leq s\leq t+\tau}(J_S^Q)^2\Big)=aV_t+b
$$

where

$$
\begin{aligned}
a =&\frac{1-e^{-(\kappa-\beta_V\lambda_1)\tau}}{(\kappa-\beta_V\lambda_1)\tau}(1+\lambda_1\chi)\cdot 10^4 \\
b =&\Big(\frac{\kappa\xi+\beta_V\lambda_0}{\kappa-\beta_V\lambda_1}\Big(1-\frac{1-e^{-(\kappa-\beta_V\lambda_1)\tau}}{(\kappa-\beta_V\lambda_1)\tau}\Big)(1+\lambda_1\chi)+\lambda_0\chi\Big)\cdot 10^4
\end{aligned}
$$

$$\chi = \begin{cases} \mu_S^2 + \sigma_S^2 & \text{for Model SVCJ1,} \\ 2(q\beta_+^2 + (1-q)\beta_-^2) & \text{for Model SVCJ2.} \end{cases}$$

In the following, we derive the conditional densities $p(s_T|\tau, s_t, z_t;\theta)$ and $p(z_T|\tau, s_t, z_t;\theta)$. Due to the fact that $Z_t = \sqrt{aV_t + b}$, we have by change of variable that

$$p(z_T|\tau, s_t, z_t; \boldsymbol{\theta}) = \frac{2z_T}{a} p(v_T|\tau, s_t, v_t; \boldsymbol{\theta})\Big|_{\left\{v_T = \frac{z_T^2 - b}{a}, v_t = \frac{z_t^2 - b}{a}\right\}}$$

Therefore, it is sufficient to calculate the marginal density of V. Now let $\Psi(\tau, \mathbf{u}, \mathbf{w}, s, v)$ be the time-t generalized conditional characteristic function of (S_T, V_T), that is,

$$\Psi(\tau, (u_s, u_v), (w_s, w_v), s, v; \boldsymbol{\theta}) = E_t^Q(e^{(u_s + iw_s)S_T + (u_v + iw_v)V_T}).$$

where $\tau = T - t$, $\mathbf{u} = (u_s, u_v) \in \mathbb{R}^2$ and $\mathbf{w} = (w_s, w_v) \in \mathbb{R}^2$. The conditional density $p(s_T, v_T|s_t, v_t;\theta)$ is given by

$$p(s_T, v_T|s_t, v_t; \boldsymbol{\theta}) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-iw_s s_T - iw_v v_T} \Psi(\tau, \mathbf{0}, \mathbf{w}, s_t, v_t; \boldsymbol{\theta}) d\mathbf{w}$$

Moreover, the marginal densities $p(s_T|\tau, s_t, v_t;\theta)$ and $p(v_T|\tau, s_t, v_t;\theta)$ are:

$$\begin{aligned} p(s_T, \tau|s_t, v_t; \boldsymbol{\theta}) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iw_s s_T} \Psi(\tau, \mathbf{0}, (w_s, 0), s_t, v_t; \boldsymbol{\theta}) dw_s \\ &= \frac{1}{\pi} \int_0^\infty \mathrm{Re}\Big(e^{-iw_s s_T} \Psi(\tau, \mathbf{0}, (w_s, 0), s_t, v_t; \boldsymbol{\theta})\Big] dw_s \\ p(v_T, \tau|s_t, v_t; \boldsymbol{\theta}) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iw_v v_T} \Psi(\tau, \mathbf{0}, (0, w_v), s_t, v_t; \boldsymbol{\theta}) dw_v \\ &= \frac{1}{\pi} \int_0^\infty \mathrm{Re}\Big[e^{-iw_v v_T} \Psi(\tau, \mathbf{0}, (0, w_v), s_t, v_t; \boldsymbol{\theta})\Big] dw_v \end{aligned}$$

As the model is affine, $\Psi(\tau, \mathbf{u}, \mathbf{w}, s, v;\theta)$ has a closed form formula. In fact, $\Psi(\tau, \mathbf{u}, \mathbf{w}, s, v;\theta)$ satisfies the following PDE:

$$\begin{aligned} 0 = &-\frac{\partial \Psi}{\partial \tau} + (r - d - \frac{1}{2}v - \mu(\lambda_0 + \lambda_1 v))\frac{\partial \Psi}{\partial s} + \kappa(\xi - v)\frac{\partial \Psi}{\partial v} + \frac{1}{2}v\frac{\partial^2 \Psi}{\partial s^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 \Psi}{\partial v^2} \\ &+ \rho\sigma v \frac{\partial^2 \Psi}{\partial s \partial v} + (\lambda_0 + \lambda_1 v) \iint_{\mathbb{R}^2} \Big(\Psi(\tau, \mathbf{u}, \mathbf{w}, s + \eta, v + \zeta) - \Psi(\tau, \mathbf{u}, \mathbf{w}, s, v)\Big)\nu_S(\eta)\nu_V(\zeta) d\eta d\zeta \end{aligned}$$

with the initial condition:

$$\Psi(0, \mathbf{u}, \mathbf{w}, s, v; \boldsymbol{\theta}) = e^{(u_s + iw_s)s + (u_v + iw_v)v}$$

Since the model is affine in v, we can obtain a closed-form formula for Ψ:

$$\Psi(\tau, \mathbf{u}, \mathbf{w}, s, v; \boldsymbol{\theta}) = e^{(u_s + iw_s)s + A(\tau;\boldsymbol{\theta}) + B(\tau;\boldsymbol{\theta})v}$$

where A and B satisfy the system of ordinary differential equations below:

$$0 = -\dot{B} + \frac{1}{2}\sigma^2 B^2 + ((u_s + iw_s)\rho\sigma - \kappa)B - (\frac{1}{2} + \mu\lambda_1)(u_s + iw_s) + \frac{1}{2}(u_s + iw_s)^2 + \lambda_1(l(B) - 1)$$

$$0 = -\dot{A} + \kappa\xi B + (r - d - \mu\lambda_0)(u_s + iw_s) + \lambda_0(l(B) - 1)$$

where $A(0;\theta) = 0$, $B(0;\theta) = u_v + iw_v$, and

$$\begin{aligned} l(a) &= \iint_{\mathbb{R}^2} e^{(u_s+iw_s)\eta+\zeta a}\nu_S(\eta)\nu_V(\zeta)d\eta d\zeta \\ &= \begin{cases} (1-\beta_V a)^{-1} e^{\mu_s(u_s+iw_s)+\frac{1}{2}\sigma_s^2(u_s+iw_s)^2} & \text{for Model SVCJ1,} \\ (1-\beta_V a)^{-1}\Big(q(1-(u_s+iw_s)\beta_+)^{-1} + (1-q)(1+(u_s+iw_s)\beta_-)^{-1}\Big) & \text{for Model SVCJ2.} \end{cases} \end{aligned}$$

The price of the S&P 500 Call option with strike X and maturity τ is given by:

$$\begin{aligned} C(\tau, s_t, v_t, X; \boldsymbol{\theta}) =& e^{-r\tau} E_t^Q\Big((e^{S_T} - X)^+\Big) \\ =& \frac{e^{-r\tau}}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{\Psi((u_s,0),(w_s,0),\tau,s_t,v_t;\boldsymbol{\theta})Xe^{-(u_s+iw_s)\log X}}{(u_s+iw_s)(u_s+iw_s-1)}\Big]du_s \end{aligned}$$

for $u_s > 1$. Finally, the VIX option price with maturity T and strike price Y is give by:

$$\begin{aligned} H(\tau, v_t, Y; \boldsymbol{\theta}) =& e^{-r\tau} E_t^Q\Big((\sqrt{aV_T + b} - Y)^+\Big) \\ =& e^{-r\tau}\sqrt{\frac{a}{\pi}}\int_0^\infty \mathrm{Re}\Big[\frac{\Psi((0,u_v),(0,w_v),\tau,s_t,v_t;\boldsymbol{\theta})e^{\frac{b}{a}(u_v+iw_v)}\mathrm{erfc}\Big(Y(\frac{u_v+iw_v}{a})^{\frac{1}{2}}\Big)}{2(u_v+iw_v)^{\frac{3}{2}}}\Big]dw_v \end{aligned}$$

where $u_v > 0$, and $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$ is the complementary error function. Using the fact that $z_t = \sqrt{av_t + b}$, we can rewrite functions C and H in terms of z_t, i.e., the VIX, which is observable from the market.

In order to calculate standard errors for parametric estimation, we need the derivatives of option prices with respect to $\boldsymbol{\theta}$. This is straightforward for S&P 500 options, since

$$\frac{\partial C}{\partial \boldsymbol{\theta}} = \frac{e^{-r\tau}}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{\partial\Psi((u_s,0),(w_s,0),\tau,s_t,v_t;\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\frac{Xe^{-(u_s+iw_s)\log X}}{(u_s+iw_s)(u_s+iw_s-1)}\Big]du_s$$

where

$$\frac{\partial\Psi(\mathbf{u},\mathbf{w},\tau,s_t,v_t;\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} = \Big(\frac{\partial A(\tau;\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} + \frac{\partial B(\tau;\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}v\Big)\Psi(\mathbf{u},\mathbf{w},\tau,s_t,v_t;\boldsymbol{\theta})$$

and $\frac{\partial A(\tau;\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}$, $\frac{\partial B(\tau;\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}$ can be calculated by augmenting the ODE systems with more unknowns. Regarding VIX options, the formula becomes more complicated, since a and b also depend on $\boldsymbol{\theta}$. Therefore, we have

$$\begin{aligned} \frac{\partial H}{\partial\boldsymbol{\theta}} =& e^{-r\tau}\sqrt{\frac{1}{\pi}}\int_0^\infty \mathrm{Re}\Big[\frac{\partial\Psi((0,u_v),(0,w_v),\tau,s_t,v_t;\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\frac{\sqrt{a}e^{\frac{b}{a}(u_v+iw_v)}\mathrm{erfc}\Big(Y(\frac{u_v+iw_v}{a})^{\frac{1}{2}}\Big)}{2(u_v+iw_v)^{\frac{3}{2}}}\Big]dw_v \\ &+ e^{-r\tau}\sqrt{\frac{1}{\pi}}\int_0^\infty \mathrm{Re}\Big[\frac{\Psi((0,u_v),(0,w_v),\tau,s_t,v_t;\boldsymbol{\theta})}{2(u_v+iw_v)^{\frac{3}{2}}}\frac{\partial\Big(\sqrt{a}e^{\frac{b}{a}(u_v+iw_v)}\mathrm{erfc}\Big(Y(\frac{u_v+iw_v}{a})^{\frac{1}{2}}\Big)\Big)}{\partial\boldsymbol{\theta}}\Big]dw_v \end{aligned}$$

and then follow similar calculations as for S&P 500 options.

C Tests for Robustness Checks

We measure the differences between $\widehat{C}(\tau, S_t, Z_t, X)$ and $\widehat{C}(\tau, Z_t, M_t)$, and between $\widehat{p}(z_T|\tau, z_t)$ and $\widehat{p}(z_T|\tau, s_t, z_t)$ by the sum of squared deviations. The test statistics are appropriately standardized versions. For the S&P 500 options, we have

$$\begin{aligned}&\widehat{M}_{\text{S\&P}}\\&=\left\{(h_\tau h_z h_s h_x)^{1/2}\sum_{i=1}^{n_C}\left[\widehat{C}(\tau, S_t, Z_t, X)-\widehat{C}(\tau, Z_t, M_t)\right]^2 a_C\left(\tau_i, S_{t_i}, Z_{t_i}, X_i\right)-\widehat{C}_{\text{S\&P}}\right\}/\sqrt{\widehat{D}_{\text{S\&P}}}\end{aligned}$$

where

$$\begin{aligned}\widehat{C}_{\text{S\&P}} =&(h_\tau h_z h_s h_x)^{-1/2}\left(\int k^2(c)\,dc\right)\\&\int\left[\widehat{s}_C^2(\tau,z,s,x)+\widehat{s}_C^2(\tau,z,m)\right]a_C(\tau,z,s,x)\,d\tau dz ds dx,\\\widehat{D}_{\text{S\&P}} =&2\left\{\int\left[\int k(c_1)\,k(c_1+c_2)\,dc_1\right]^2 dc_2\right\}^4\\&\int\left[\widehat{s}_C^2(\tau,z,s,x)+\widehat{s}_C^2(\tau,z,m)\right]^2 a_C^2(\tau,z,s,x)\,d\tau dz ds dx,\end{aligned}$$

and $\widehat{s}_C^2(\tau,z,s,x)$ and $\widehat{s}_C^2(\tau,z,m)$ are estimators of conditional variances of regressing option prices C_i on $(\tau_i, S_{t_i}, Z_{t_i}, X_i)$ without and with assuming the homogenous of degree 1.

For the VIX density, we have

$$\begin{aligned}&\widehat{M}_{\text{VIX}}\\&=\left\{\overline{h}_y^2\left(\overline{h}_\tau\overline{h}_z\overline{h}_s\overline{h}_y\right)^{1/2}\sum_{i=1}^{n_H}\left[\widehat{p}(Z_{T_i}|\tau_i,S_{t_i},Z_{t_i})-\widehat{p}(Z_{T_i}|\tau_i,Z_{t_i})\right]^2 a_H\left(\tau_i,S_{t_i},Z_{t_i},Z_{T_i}\right)-\widehat{C}_{\text{VIX}}\right\}/\sqrt{\widehat{D}_{\text{VIX}}}\end{aligned}$$

where

$$\begin{aligned}\widehat{C}_{\text{VIX}} =&\left(\overline{h}_\tau\overline{h}_z\overline{h}_s\overline{h}_y\right)^{-1/2}\left[\int k^2(c)\,dc\right]^3\left[\int\left(c\dot{k}(c)+k(c)\right)^2 dc\right]/\left[\int c^2 k(c)\,dc\right]^2\\&\times e^{2r\tau}\int\left[\widehat{s}_H^2\left(\tau,z,s,z'\right)+\widehat{s}_H^2\left(\tau,z,z'\right)\right]a_H\left(\tau,z,s,z'\right)d\tau dz ds dz',\\\widehat{D}_{\text{VIX}} =&2\left\{\int\left[\int\left(c_1\dot{k}(c_1)+k(c_1)\right)\left((c_1+c_2)\dot{k}(c_1+c_2)+k(c_1+c_2)\right)dc_1\right]^2 dc_2\right\}\\&\times\left\{\int\left[\int k(c_1)\,k(c_1+c_2)\,dc_1\right]^2 dc_2\right\}^3/\left[\int c^2 k(c)\,dc\right]^4\\&\times e^{4r\tau}\int\left[\widehat{s}_H^2\left(\tau,z,s,z'\right)+\widehat{s}_H^2\left(\tau,z,z'\right)\right]^2 a_H^2\left(\tau,z,s,z'\right)d\tau dz ds dz',\end{aligned}$$

and $\widehat{s}^2_H(\tau, z, s, z')$ and $\widehat{s}^2_H(\tau, z, z')$ are estimators of conditional variances of regressing option prices H_i on $(\tau_i, S_{t_i}, Z_{t_i}, Y_i)$ without and with assuming that H does not depend on S_t. By similar regularity conditions and bandwidth choices to those for the specification test $\widehat{M}$, we have

$$\widehat{M}_{\text{S\&P}} \xrightarrow{\text{d}} N(0,1) \text{ under } \mathbb{H}_{1,\text{S\&P}} \text{ as } n_C \to \infty,$$
$$\widehat{M}_{\text{VIX}} \xrightarrow{\text{d}} N(0,1) \text{ under } \mathbb{H}_{1,\text{VIX}} \text{ as } n_H \to \infty.$$

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Table 1: Summary Statistics of the S&P 500 Index and VIX

	Mean	Std	Skew	Kurt	Median	25%	75%
S&P 500 Index	948.0464	115.5682	-0.1954	2.0342	939.145	855.86	1057.3
S&P 500 Return	0.0007	0.0171	-0.0625	4.9126	0.0018	-0.0081	0.0097
VIX	31.4786	9.0777	0.6825	2.1979	28.57	24.275	39.425

Note: This table reports the summary statistics of the time series of S&P 500 index, return, and VIX from January 2 to December 31, 2009.

Table 2: Summary Statistics of S&P 500 and VIX Options

		S&P 500 Call		S&P 500 Put		VIX Call		VIX Put	
Moneyness		OTM	ITM	OTM	ITM	OTM	ITM	OTM	ITM
# of Contracts		54,123	21,599	82,658	20,530	11,571	4,083	3,909	4,243
Trading Volume	$\times 10^6$	72.2	29.8	135.5	16.9	19.4	1.5	7.0	5.4
	Min	0.025	6.75	0.025	5.55	0.025	0.325	0.025	0.275
	25%	1.00	47.40	1.25	50.40	0.350	5.450	0.175	3.150
Option Price	50%	6.70	71.15	6.60	83.90	1.000	7.600	0.675	5.950
	75%	22.20	111.10	22.75	151.55	2.275	11.350	1.750	11.688
	Max	184.60	622.05	197.95	1583.10	7.450	45.10	16.80	73.00
	Min	680	300	50	680	20.00	10.00	10.00	20.00
	20%	940	800	650	895	35.00	20.00	22.50	27.50
Strike Price	50%	1025	875	775	970	42.50	25.00	27.50	35.00
	75%	1125	965	875	1090	55.00	32.50	35.00	45.00
	Max	2500	1125	1125	2500	100.00	55.00	55.00	100.00
	Min	3	3	3	3	2	2	2	2
	25%	21	17	20	19	26	20	22	17
Time-to-Maturity	50%	39	33	38	37	49	40	41	34
	75%	80	69	78	86	80	72	67	54
	Max	783	780	783	780	132	130	130	131
	Min	0.136	0.153	0.153	0.105				
	25%	0.216	0.262	0.319	0.237				
Implied Volatility	50%	0.267	0.317	0.406	0.312				
	75%	0.336	0.395	0.518	0.377				
	Max	1.552	2.996	2.103	2.933				

Note: This table reports the summary statistics (minimum, quantiles, and maximum) for all traded S&P 500 and VIX options in 2009, including the option price, strike price, time-to-maturity, implied volatility, and trading volume. In total, there are 178,910 trading records for S&P 500 options with valid implied volatility and trading volumes, 23,806 records for VIX options with trading volumes.

Table 3: Dependence of S&P 500 Index SPD on the VIX SPD

Intercept		Z_t		$\text{Std}_t\,[Z_{t+\tau}]$		$\text{Skew}_t[Z_{t+\tau}]$		Adj.R^2
Panel A: Regression Results for $\tau = 21$								
48.097	(1.7106)	-0.2385	(0.0487)					0.1063
80.409	(27.299)	-0.0495	(0.0166)	-4.3725	(3.6869)			0.1082
86.670	(27.251)	-0.1899	(0.0179)	-3.8173	(3.6671)	-3.6036	(1.7711)	0.1226
Panel B: Regression Results for $\tau = 42$								
39.289	(1.7079)	-0.0633	(0.0487)					0.0036
74.546	(27.981)	-0.0677	(0.1146)	-4.2847	(3.3942)			0.0066
51.251	(30.215)	0.1735	(0.1676)	-0.0180	(4.0120)	-5.2299	(2.6299)	0.0212
Panel C: Regression Results for $\tau = 63$								
-29.282	(1.8518)	0.0202	(0.0528)					0.0008
-85.347	(21.036)	0.1122	(0.0520)	11.402	(2.0853)			0.1270
-64.527	(23.110)	0.0969	(0.0520)	9.9069	(2.1873)	-4.3522	(2.0823)	0.1421
Panel D: Regression Results for $\tau = 84$								
-24.893	(2.5586)	0.1286	(0.0729)					0.0108
-74.216	(13.050)	0.3859	(0.0720)	8.7851	(1.1396)			0.2416
-69.434	(21.840)	0.3831	(0.0729)	8.4237	(1.7469)	-0.8486	(3.1035)	0.2379

Note: This table reports regression results of the future average volatility $V^S_{t,T}$ of the S&P 500 index on the square root of the conditional second moment, conditional skewness, and conditional kurtosis of the VIX densities, with time-to-maturity equal to 21, 42, 63, and 84 days. The numbers in parentheses are standard errors.

Table 4: In-Sample and Out-of-Sample Forecasts

Panel A: Density Forecast Error				
	Forecast of AL-Density		Forecast of SX-Density	
γ	AL	SX	AL	SX
0	0.0000	1.5582	1.6719	0.0000
7	0.5422	1.4679	1.7012	0.2091
14	0.9451	1.1815	1.7362	0.3322
21	1.2332	1.0107	1.9444	0.4428
28	1.5370	0.7206	2.3313	0.6040
35	1.8695	0.7399	2.5789	0.8338
42	2.0787	0.8326	2.9918	1.1627

Panel B: Market-Price Forecast Error			
τ	AL	SX	MKT
0	9.36	8.54	0.00
7	8.79	7.76	4.65
14	6.46	6.97	5.63
21	5.68	4.68	4.62
28	5.79	2.41	3.28
35	4.90	1.74	2.95
42	5.09	1.27	2.11

Note: Panel A reports average forecast errors of the densities produced by the Aït-Sahalia and Lo (1998) estimator (AL) and our SPD estimator (SX) conditional on the VIX, while Panel B reports those of prices by the AL, SX, and a martinglae interpolation (MKT) method. The nonparametric option prices and their corresponding SPDs are estimated with daily data from January 2 to September 30, 2009 and out-of-sample forecasts are generated for various forecast horizons t on a daily rolling basis from October 1 to December 31, 2009.

Table 5: Estimates of Parametric Models

Param	SV		SVJ1		SVJ2		SVCJ1		SVCJ2	
κ	3.672	(0.094)	2.323	(0.086)	3.619	(0.102)	4.362	(0.162)	3.109	(0.159)
ξ	0.145	(0.002)	0.183	(0.004)	0.147	(0.002)	0.028	(0.002)	0.032	(0.002)
σ	1.033	(0.009)	0.921	(0.008)	1.023	(0.010)	0.370	(0.015)	0.410	(0.011)
ρ	-0.708	(0.005)	-0.953	(0.011)	-0.786	(0.011)	-0.652	(0.039)	-0.611	(0.021)
β_+					0.047	(0.001)			0.002	(0.005)
β_-					0.032	(0.006)			0.138	(0.019)
q					0.128	(0.028)			0.316	(0.100)
β_V							0.067	(0.003)	0.075	(0.004)
μ_s			0.021	(0.013)			-0.119	(0.009)		
σ_s			0.022	(0.006)			0.154	(0.006)		
λ_0			6.768	(2.864)	5.182	(1.371)	0.000	(0.039)	0.001	(0.048)
λ_1							33.276	(2.526)	26.991	(2.096)
RMSE	0.602		0.583		0.585		0.348		0.376	

Note: This table reports the parametric estimates for a variety of models. Standard errors are given in parentheses.

Table 6: Nonparametric Specification Tests of Parametric Models

Tests	SV	SVJ1	SVJ2	SVCJ1	SVCJ2
S&P	25.6055	14.5384	11.8035	13.0357	9.8441
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
VIX	21.9234	11.0747	15.5492	8.2278	6.9282
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Joint	33.6080	16.1774	19.3413	12.7787	11.8598
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Note: This table reports the nonparametric specification tests for a variety of models. P values are given in parentheses.

Table 7: Regression Analysis for Long-Term Dependence

Time-to-Maturity	168	231	294	357
Intercept	-0.0275	-0.0082	-0.0366	-0.0112
	(0.0035)	(0.0028)	(0.0056)	(0.0046)
IIV	1.1165	1.0459	1.1403	1.0476
	(0.0108)	(0.0090)	(0.0173)	(0.0142)
Adj.R^2	0.7290	0.7705	0.7967	0.8317

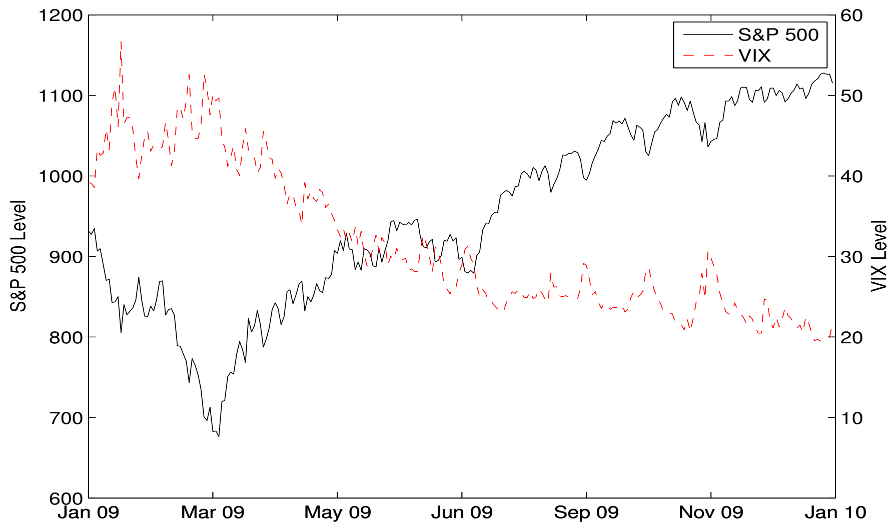
Note: This table reports regression results of implied volatility of observed option prices on the interpolated implied volatilities for long maturities. Standard errors are given in parentheses.

Table 8: Robustness Tests

	S&P 500	VIX
Test Statistics	0.0439	0.0154
p-value	0.5136	0.5025

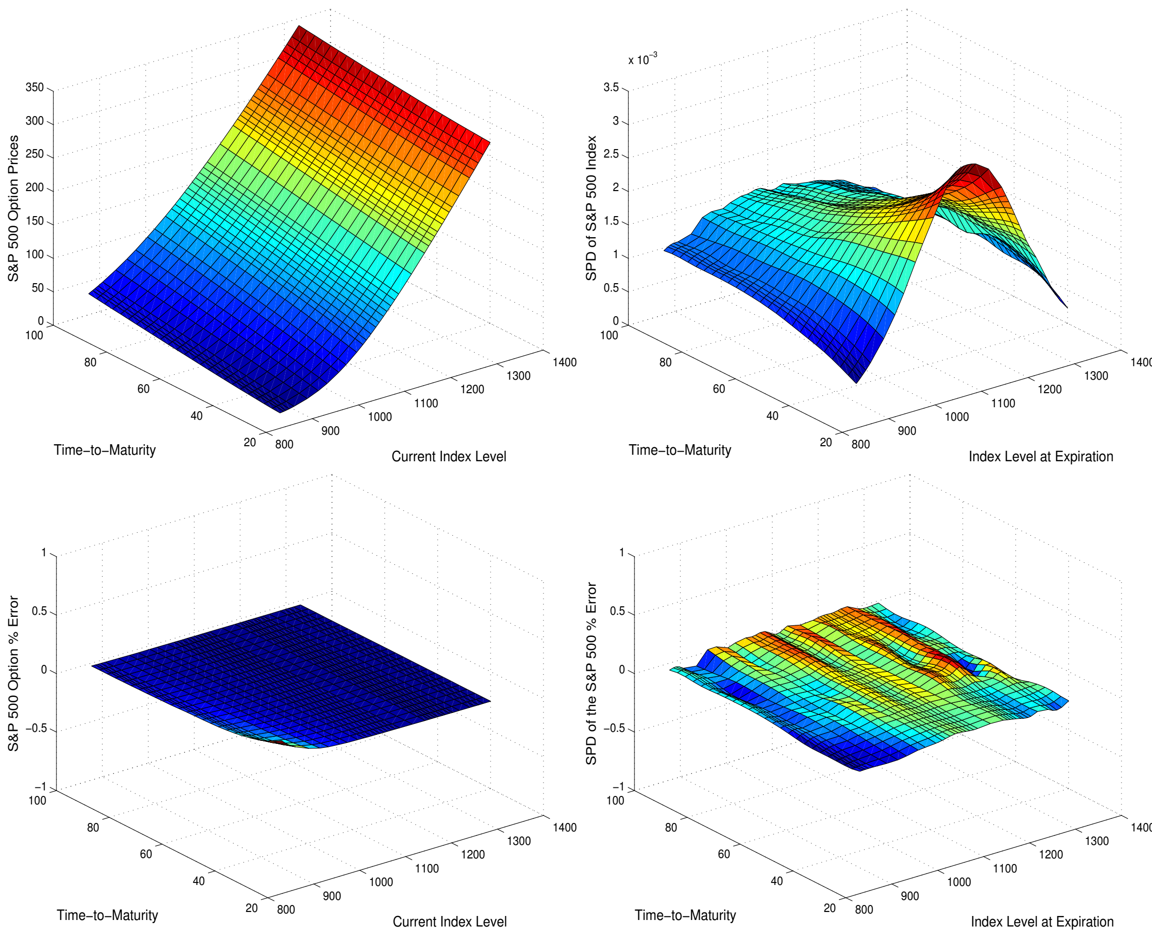
Note: This table reports the nonparametric tests for the two dimension reduction assumptions: homogeneous of degree 1 for S&P 500 options and dependence of VIX option prices on S_t only through Z_t.

Figure 1: Time Series of the S&P 500 Index and the VIX



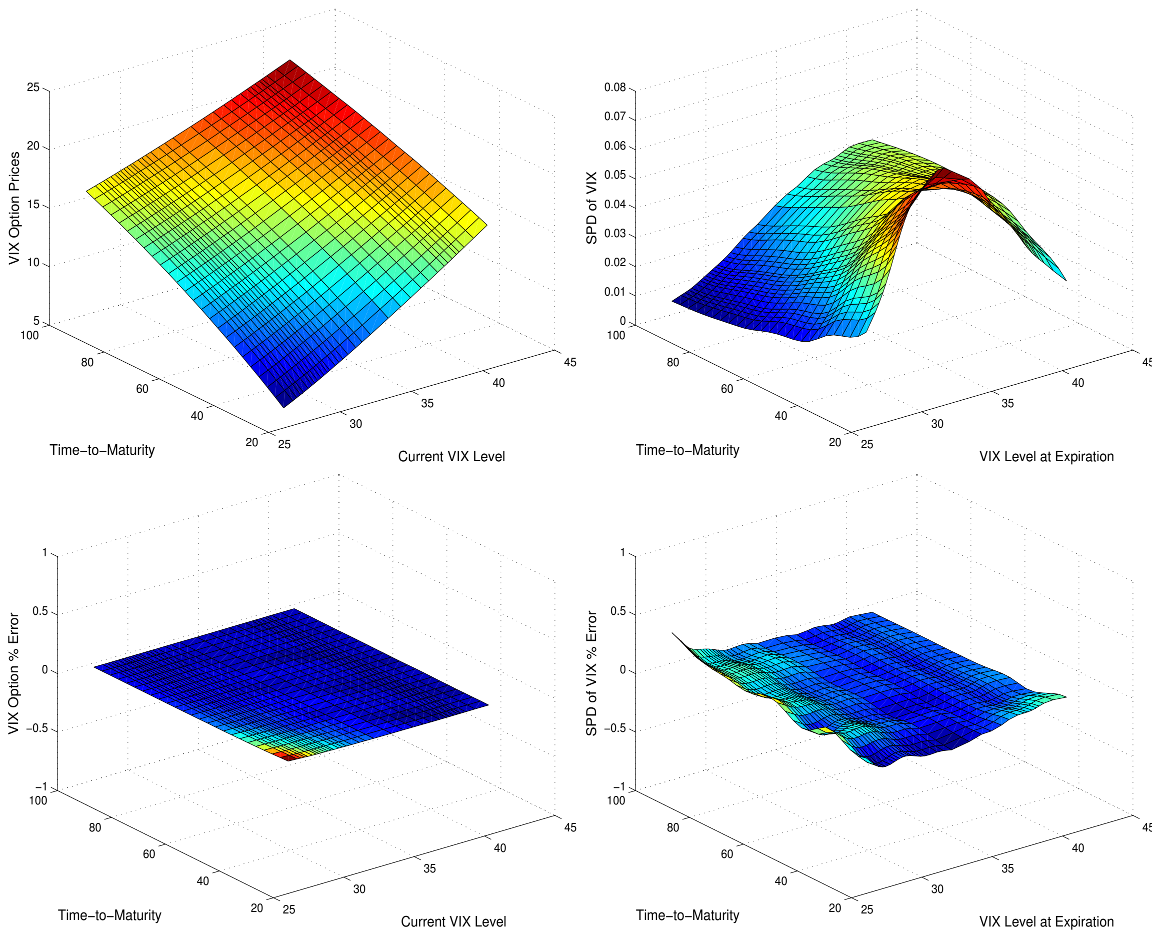
Note: This figure plots the time series of S&P 500 index and VIX from January 2 to December 31, 2009.

Figure 2: Nonparametric Option Prices and State-Price Densities of S&P



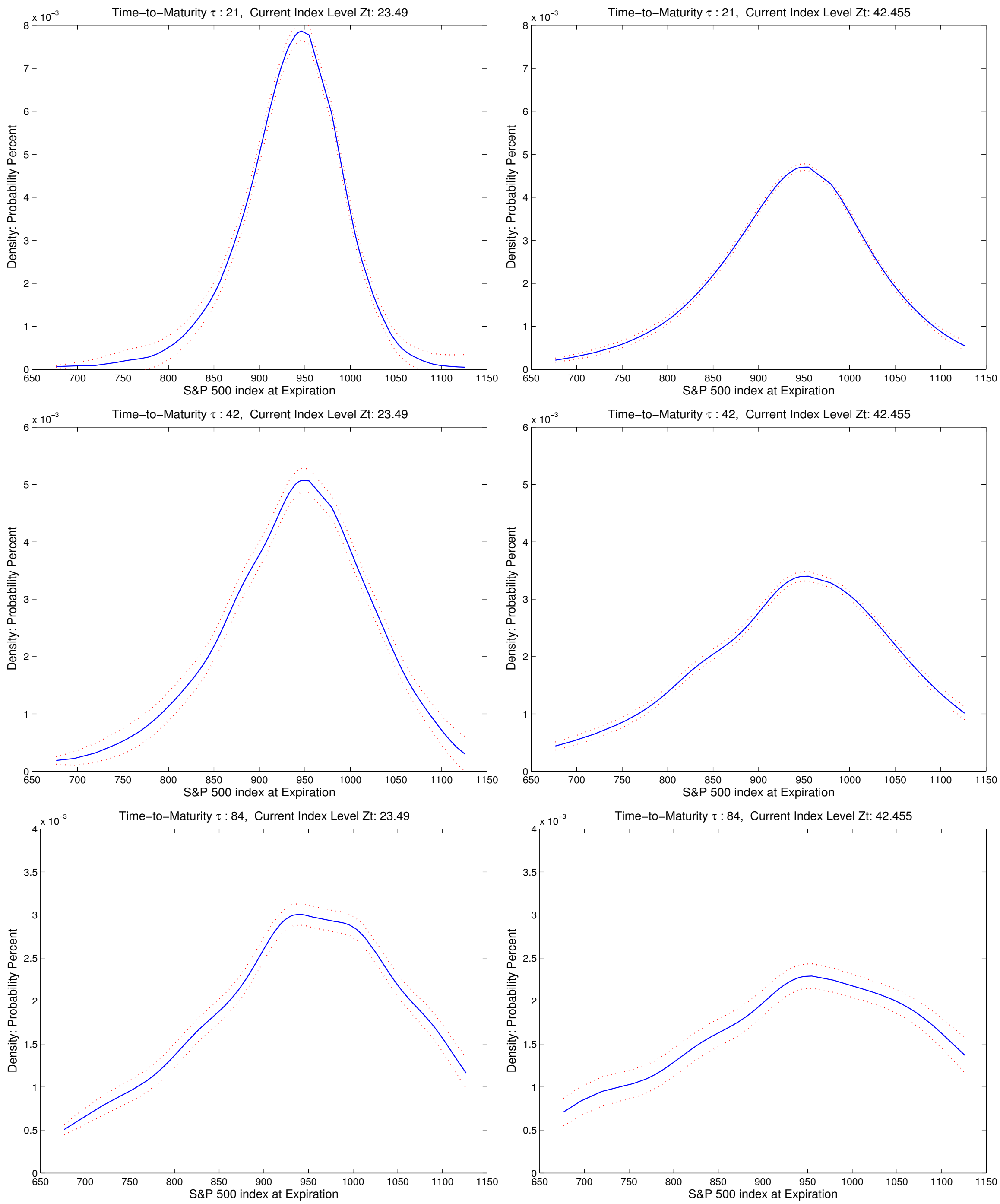
Note: In the upper panel, we plot the nonparametric estimates of S&P 500 option prices and the SPD of the index, fixing the current volatility level. The percentage estimation errors, averaged by 500 replications, are plotted in the lower panel. The data are simulated from the model SVCJ2, with empirical parameter estimates given in Section 3.

Figure 3: Nonparametric Option Prices and the State-Price Density of the VIX



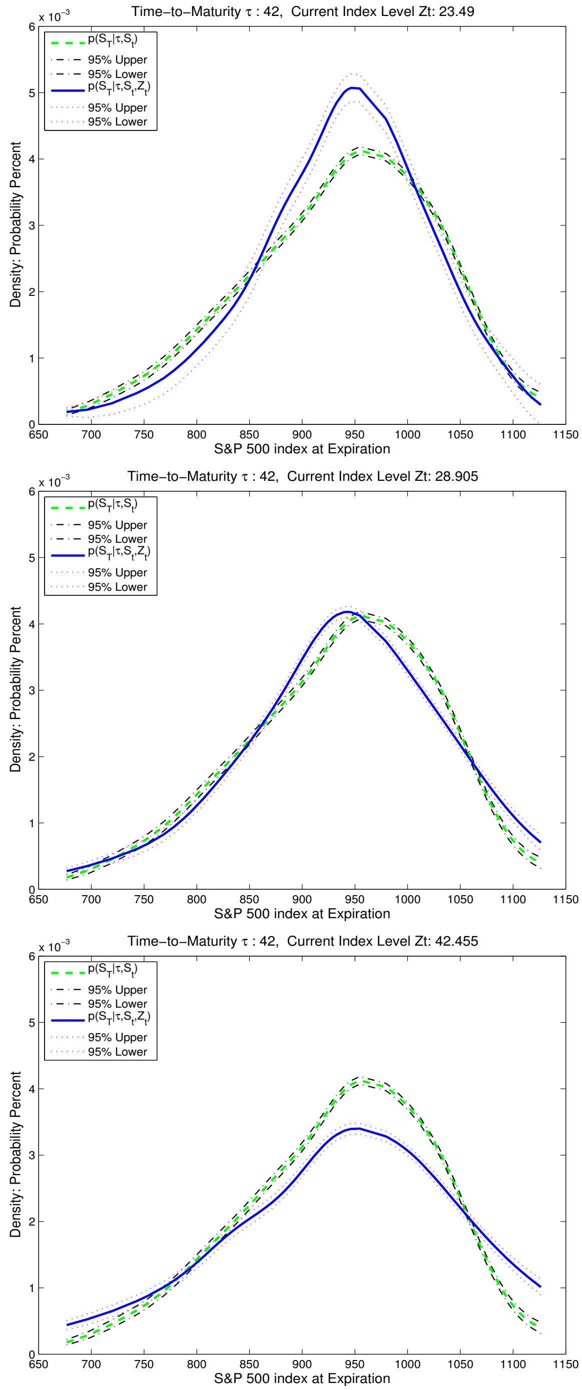
Note: In the upper panel, we plot the nonparametric estimates of VIX option prices and the SPD of the VIX, fixing current volatility level at medium. The percentage estimation errors, averaged by 500 replications, are plotted in the lower panel. The data are simulated from the model SVCJ2, with empirical parameter estimates given in Section 4.

Figure 4: Estimates of the State-Price Density of the S&P 500



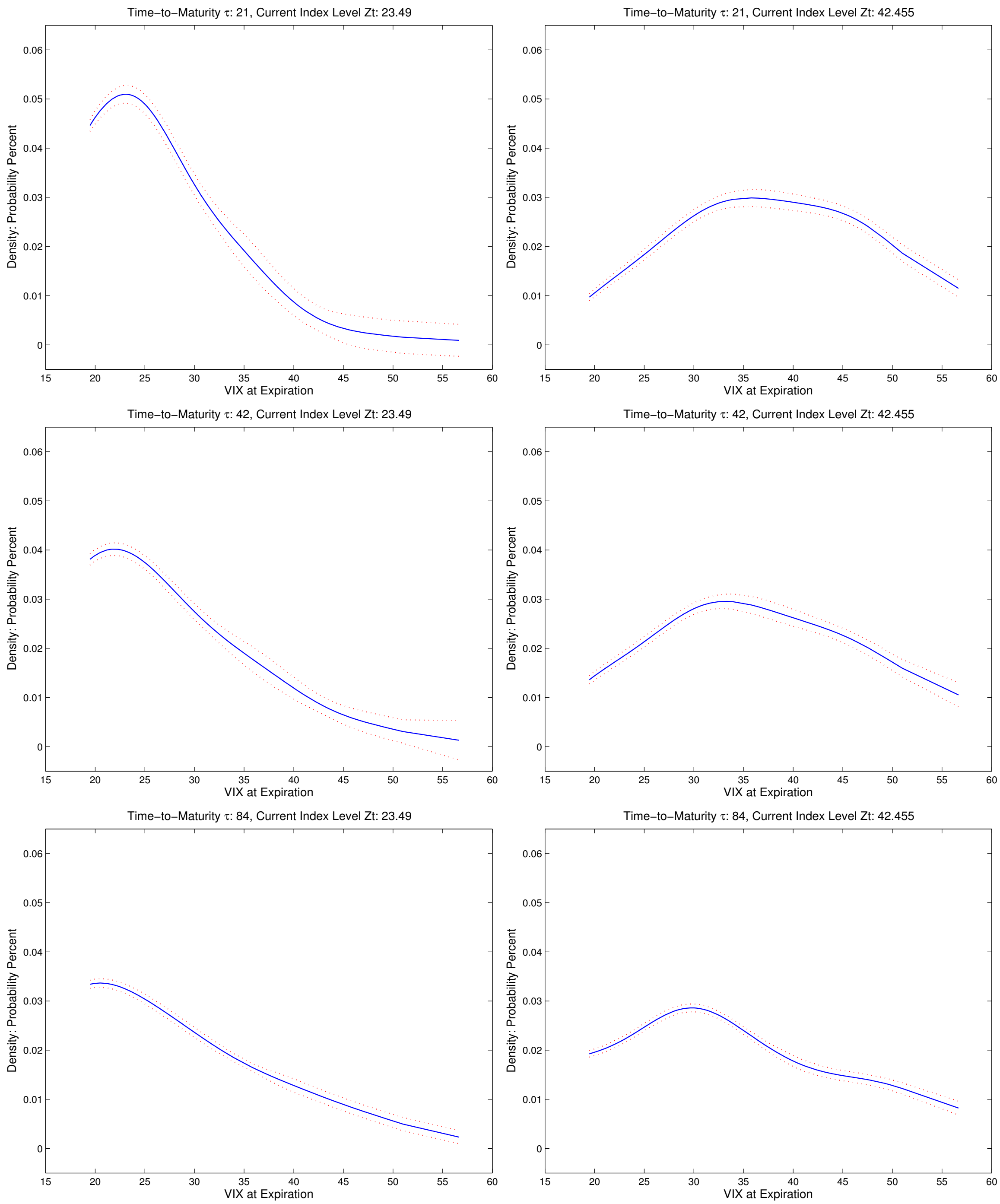
Note: This figure plots nonparametric SPD estimates (solid lines) of the S&P 500 index at maturities of 21, 42, and 84 days, and for two current VIX levels of 23.490 and 42.455. Dotted lines around each SPD estimate are 95 percent confidence intervals constructed from the asymptotic distribution theory in (6). The two levels of the current VIX index are obtained by the 20 and 80 percent quantiles of the VIX time series of 2009 and correspond to low and high levels of the VIX, respectively.

Figure 5: SPDs of S&P 500 Conditional on VIX vs Estimates that Ignore VIX



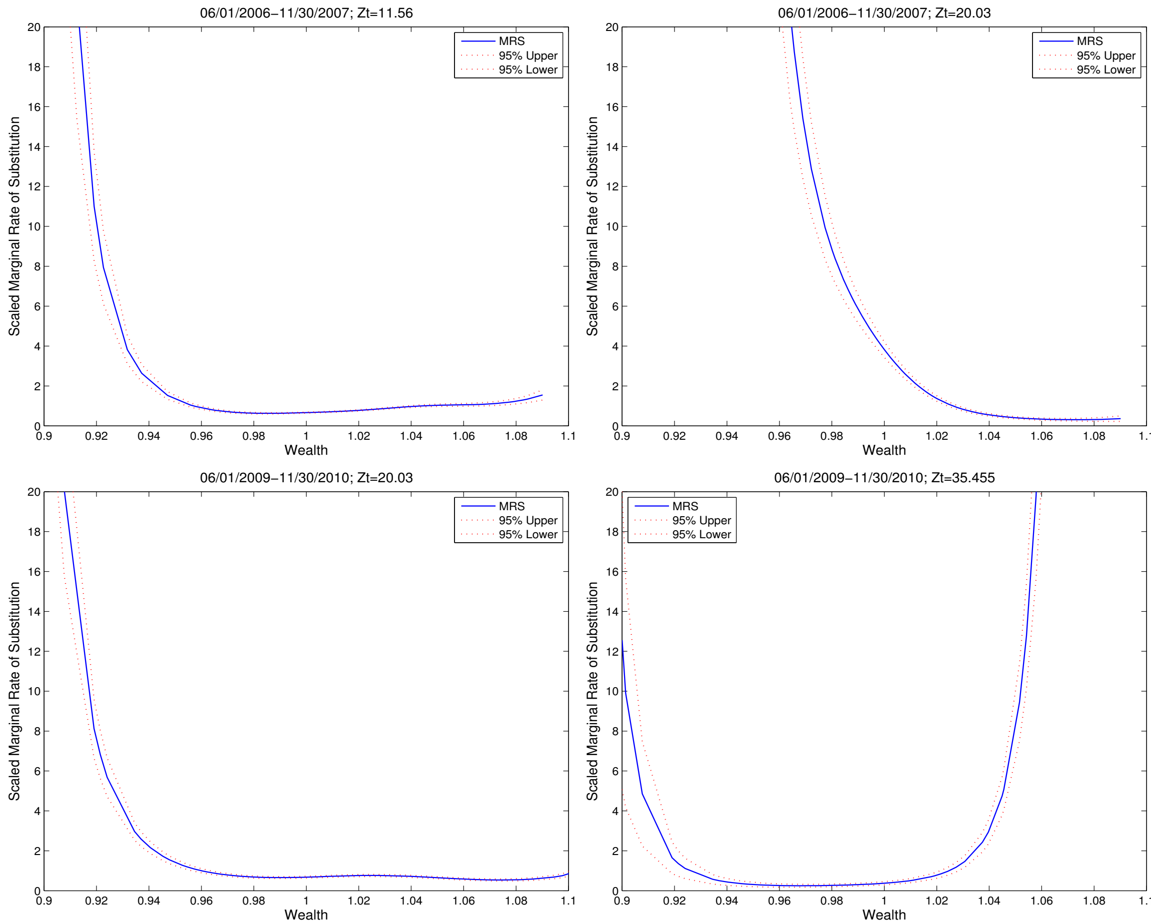
Note: This figure overlays nonparametric SPD estimates (solid lines) of the S&P 500 index conditional on the current VIX level with the nonparametric estimates that ignore the VIX variable (dashed lines). Dotted and dash-dotted lines are 95 percent confidence intervals constructed from the asymptotic distribution theory in (6). The current values of the VIX are fixed at 20%, 50% or 80% quantile of the VIX time series in 2009. The time-to-maturity is fixed at 42 days.

Figure 6: Estimates of the State-Price Density of the VIX



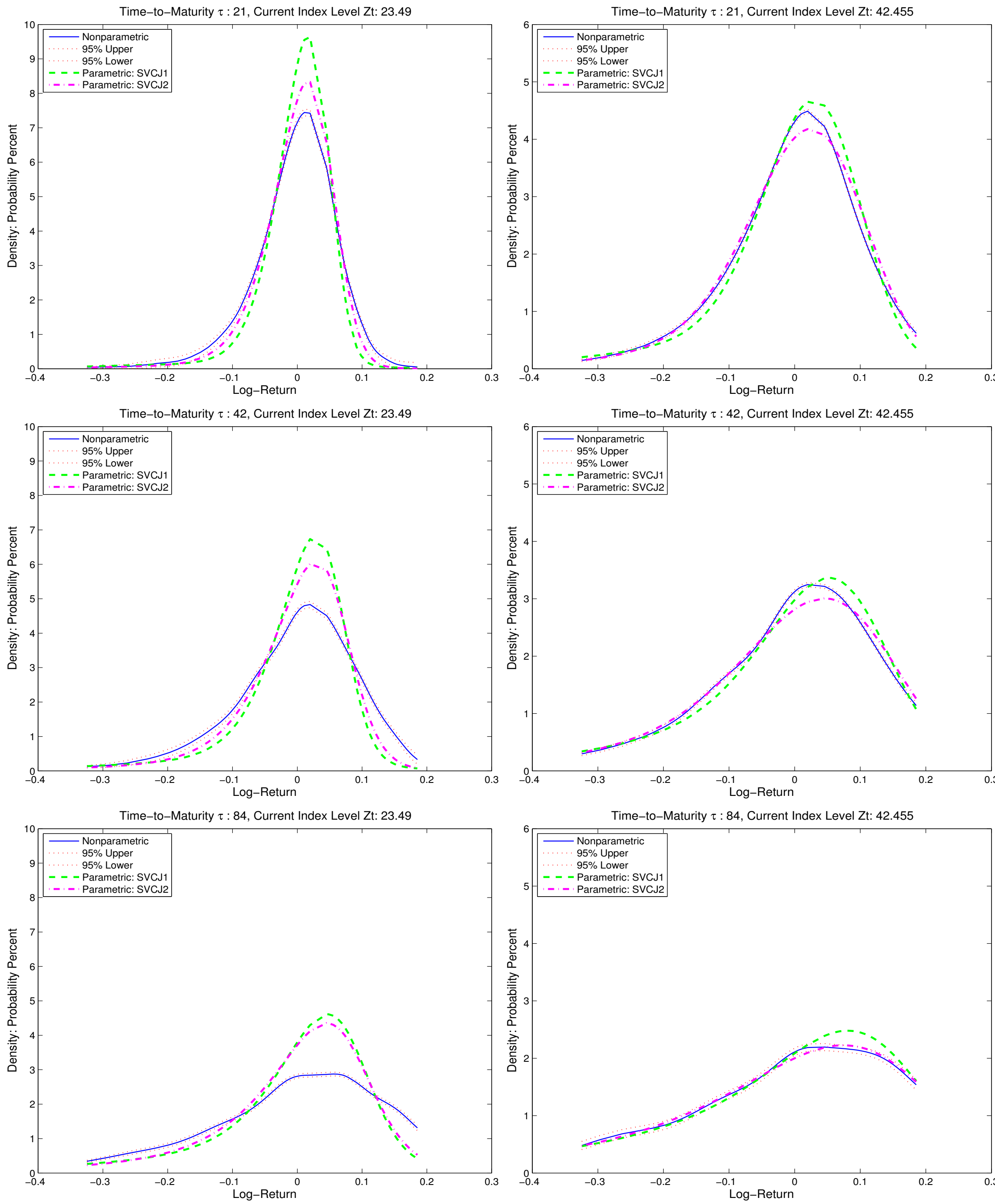
Note: This figure plots nonparametric SPD estimates (solid lines) of the VIX at maturities of 21, 42, and 84 days, and for two current VIX levels at 23.490 and 42.455. Dotted lines around each SPD estimate are 95 percent confidence intervals constructed from the asymptotic distribution theory in (6). The two levels of the current VIX index are the 20th and 80th percent quantiles of the VIX time series of 2009.

Figure 7: Nonparametric Estimates of Pricing Kernels



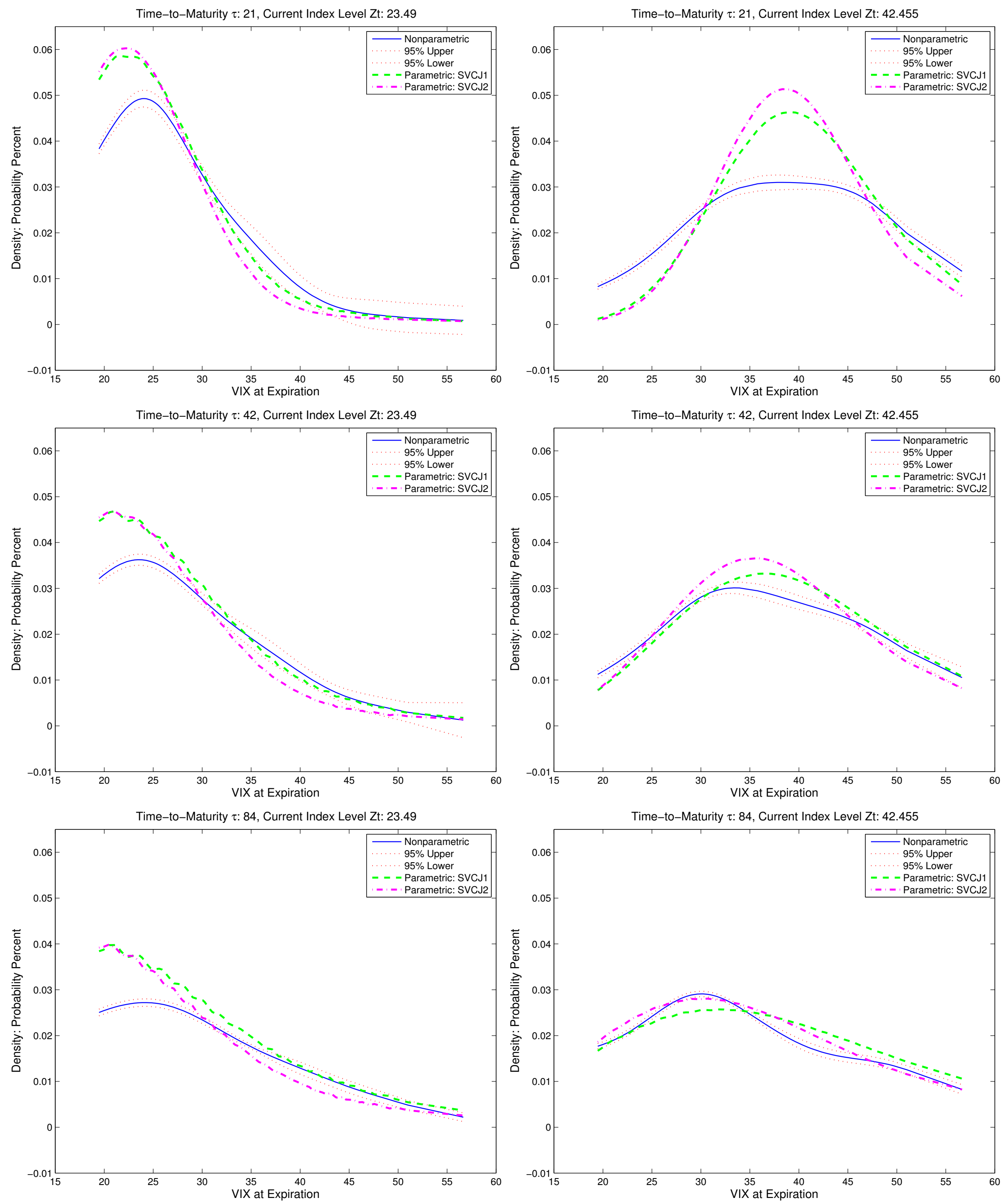
Note: In the upper panel, we plot nonparametric estimates of the pricing kernels for the 18-month sample period before the 2008 financial crisis. Correspondingly, the post-crisis estimates for the 18-month sample period from June 1, 2009 to December 30, 2010 are reported in the lower panel. Dotted lines around each pricing kernel are 95 percent confidence intervals. The two levels of Z_t, 11.56 and 20.03, represent low and high levels of VIX pre-crisis, whereas the low and high levels of VIX after the crisis are chosen to be 20.03 and 35.455, respectively. In addition, we choose time-to-maturity τ to be 42 days when reporting the estimates.

Figure 8: Nonparametric Versus Parametric Estimates of the SPDs of the S&P



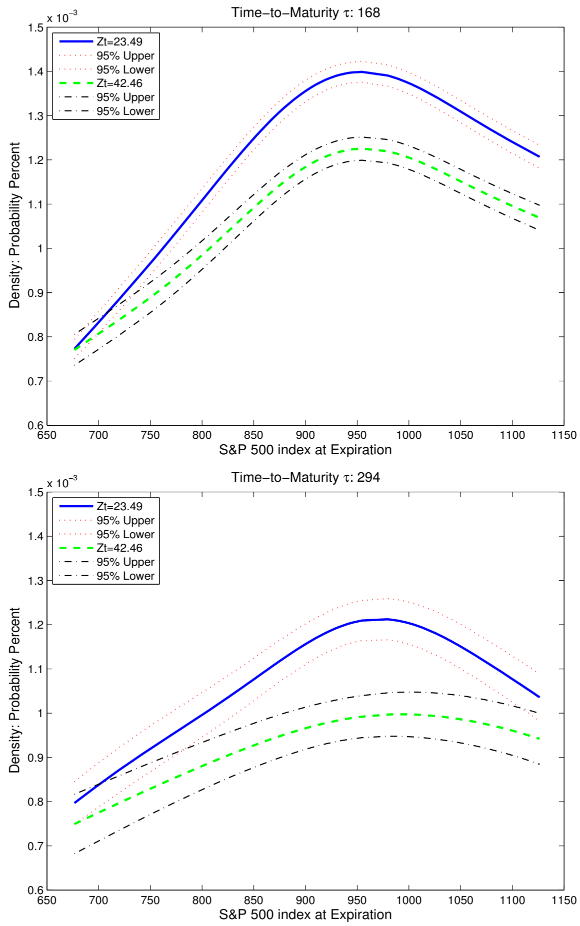
Note: This figure overlays nonparametric SPD estimates (solid lines) of the S&P 500 index with the corresponding parametric SPDs (dashed lines) with time-to-maturity equal to 21, 42, and 84 days, and for two current VIX index levels of 23.490 and 42.455. Dotted lines are 95 percent confidence intervals constructed from the asymptotic distribution theory in (6). The parametric SPDs are obtained using formulas in Appendix B. The two levels of the current VIX are the 20th and 80th percent quantiles of the VIX time series of 2009.

Figure 9: Nonparametric Versus Parametric Estimates of the SPDs of VIX



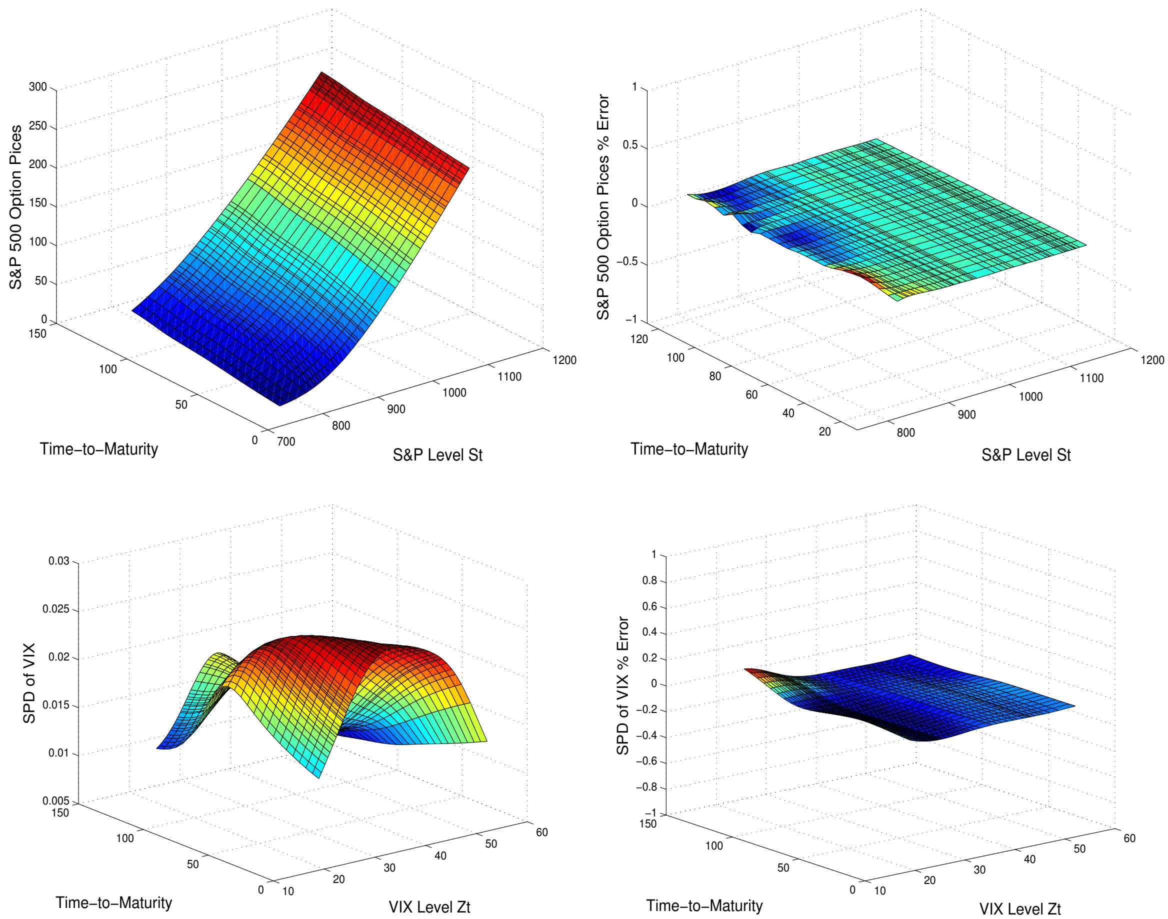
Note: This figure overlays nonparametric SPD estimates (solid lines) of the VIX with the corresponding parametric SPDs (dashed lines) with time-to-maturity equal to 21, 42, and 84 days, and for two current VIX index levels of 23.490 and 42.455. Dotted lines are 95 percent confidence intervals constructed from the asymptotic distribution theory in (6). The parametric SPDs are obtained using formulas in Appendix B. The two levels of the current VIX are the 20th and 80th percent quantiles of the VIX time series of 2009.

Figure 10: Estimates of S&P 500 Index SPDs When VIX Options are Unavailable



Note: This figure plots nonparametric SPD estimates of the S&P 500 index at long maturities of 168, 231, 294, and 357 days, and for two current VIX levels of 23.490 (solid lines) and 42.455 (dashed lines), respectively. Dotted and dash-dotted lines are 95 percent confidence intervals constructed from the asymptotic distribution theory in (7). The two levels of the current VIX index are obtained by the 20 and 80 percent quantiles of the VIX time series of 2009 and correspond to low and high levels of the VIX, respectively.

Figure 11: Robustness Checks



Note: In the left panel, we plot nonparametric estimates of the S&P 500 option prices and of the state-price density of the VIX using the two dimension reduction techniques. The right panel reports the percentage errors relative to the estimates without using the two dimension reduction techniques.