Pension Design in the Presence of Systemic Risk

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Abstract

I consider the possibility that individual agents’ savings and portfolio choices can have negative externalities on public finances, whenever retirement consumption drops below a minimum level. Within this framework, I discuss optimal pension design. I show the optimality of two policies. The first policy mandates that agents use part of their accumulated assets to purchase a claim providing a fixed income stream for the duration of their life. The second policy mandates the purchase of an appropriately structured portfolio insurance policy. Both policies are financed by an appropriate mandatory minimum savings requirement, while the agent is still a worker.

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1 Introduction

Mostly motivated by the aging of the population, several countries around the world have adopted new approaches in the ways their citizens prepare for retirement. Countries with substantially different economic structures and histories like Australia, Chile, Mexico and Sweden have partially replaced unfunded “pay as you go” retirement systems with funded retirement systems featuring private accounts.\(^1\) Even in many countries where such changes have not occurred (such as the US), private-sector defined contribution plans have increased in popularity and importance as opposed to defined benefit plans.

A common concern, acknowledged even by proponents of these trends, is that such changes increasingly expose retirement income to market risk. Such market risk could easily become “systemic” in the sense that it could result in negative externalities for public finances. For instance, a downturn in the stock market could create pressures to provide direct or indirect transfers to the affected retirees, increasing distortionary taxes.\(^2\) Indeed, the recent financial crisis is a reminder of how quickly a downturn in a major market (such as the housing market) can have adverse effects on public finances.

Because of these concerns, it is very common for countries to complement the shift towards private accounts and defined contribution plans with various measures to ensure a minimum standard of living in retirement. Such measures include minimum return guarantees, minimum retirement incomes, phased (as opposed to lump sum) withdrawals upon entering retirement, and mandates to use part of the accumulated balances to purchase a fixed annuity and ensure a minimum defined benefit. For instance, a recent Government Accountability Office report\(^3\) investigates such regulations in the UK, Switzerland and the Netherlands and documents that these countries use some combination of the above measures. The idea to use incentives or mandates, so that part of the accumulated balances in

\(^1\) Mitchell and Lachance (2003) report that more than 20 countries have established individual accounts.
\(^2\) For instance Shoffner et al. (2005) note in a Social Security Report that a common fear about individuals that have run out of assets is that “…Such individuals might then qualify for, and as a result place a greater burden on, means-tested antipoverty programs.”
\(^3\) Bovbjerg (2009)
defined contribution plans be taken in the form of a defined-benefit annuity, is also the topic of a current policy discussion in the US.\textsuperscript{4}

The pervasive use of measures to ensure a minimum standard of living in retirement has led to various studies that evaluate the costs and benefits of specific (and sometimes ad-hoc) policy interventions adopted in certain countries.\textsuperscript{5} Less emphasis has been placed on developing an integrated theoretical framework to derive the optimal government policies that would ensure a minimum standard of living in a fully funded retirement system. The present paper takes a first step in that direction by using methods developed in the last two decades in financial economics, and particularly in the strands of the literature analyzing portfolio insurance problems.

The proposed framework is in the tradition started by Ramsey (1927). A benevolent, rational government aims to maximize social welfare. Agents in the society maximize their individual welfare, which does not coincide with social welfare. The reason for the discrepancy is that a representative agent's consumption in retirement can have systemic, negative, external effects. This occurs when retirement consumption drops below a given minimum level and triggers redistributive pressures financed by distortionary taxes.\textsuperscript{6} To avoid such negative external effects, the government adopts policies to ensure that a retiree's consumption does not fall below the specified minimum level.

The allowed government policies are fully funded transfers from and to the agent. They can be chosen subject to two constraints:

The first constraint is informational. The government does not observe the agent's assets or consumption, so that it cannot condition taxes and transfers on these quantities. However,

\textsuperscript{4}See e.g. “The Obama administration is weighing how the government can encourage workers to turn their savings into guaranteed income streams following a collapse in retiree accounts when the stock market plunged.” by Theo Francis in http://www.bloomberg.com/apps/news?pid=20603037&sid=aHFCE999fWR0.

\textsuperscript{5}For some examples see e.g. Feldstein (2005b), Feldstein (2005a), Feldstein and Rangelova (2001), Fuster et al. (2008), Mitchell and Lachance (2003), Constantinides et al. (2002), and the numerous contributions in the special NBER volume edited by Campbell and Feldstein (2001).

\textsuperscript{6}For instance, such negative and external effects arise, when agents find it optimal to falsely claim that they experienced adverse idiosyncratic shocks, as a result overburdening the welfare system, which is financed by distortionary taxes on workers.
it does observe returns on financial markets. This informational constraint leads to a “hidden action” problem, analogous to the problems considered in the voluminous literature on moral hazard. The fact that the government cannot observe (and hence cannot dictate) the agent’s consumption, savings and portfolio choices implies that the government’s policies need to induce the agent to choose policies that are consistent with the goal of a minimum standard of living.

The second constraint is a full-financing constraint. The net present value of the transfers provided to the agent should be equal to the present value of mandatory savings accumulated by the agent. This second constraint is motivated by current policy discussions in the US, which focus on regulatory provisions for fully-funded private accounts and pensions. The broader issue of the advantages and disadvantages of full funding - as opposed to “pay as you go” - is outside the scope of this paper, and the reader is referred to the large literature that discusses this issue.\footnote{For some recent contributions, see e.g., Krueger and Kubler (2006) and Ball and Mankiw (2007) for two alternative views on the issue.} A practical implication of the full-financing constraint is that all the policies considered in the paper can be implemented by having the government simply specify the properties of optimal contracts and mandating that agents purchase the respective financial products by the private sector.

Besides their intuitive and practical appeal, the above two constraints also help to make the theoretical results of the paper most surprising for the following reason: Fundamental results in finance and macroeconomics imply that \textit{in the absence of frictions}, only the net present value of an agent’s resources guides her consumption choices. Hence, if agents can choose their consumption freely, and the transfers received by the agent are financed by herself during her work-years, then no government intervention can succeed in affecting the agent’s consumption choices. The agent will simply “undo” the effects of the transfers by altering her portfolio and her savings plans.\footnote{In the literature this insight is known as “Ricardian Equivalence”. Barro (1974) and Abel (1987) contain a modern treatment of this idea that is originally due to D. Ricardo.}

To overcome this hurdle, I assume a borrowing friction. Specifically, I assume that agents cannot borrow against future governmental transfers. (Such constraints can be easily
enforced in courts by forbidding securitization of such payments). Because of the resulting borrowing constraint, the government can affect the agent’s consumption choices and it becomes possible to discuss optimal mandatory savings and transfer processes.

I show that there can be multiple optimal forms of pension design. Within the context of the baseline model, one approach is to require new retirees to use part of their accumulated assets upon entering retirement to purchase a fixed income stream for the duration of their life, while leaving the rest of the assets at their disposal. The level of that fixed income stream is explicitly derived and shown to be a multiple of the minimum level of consumption that the government is aiming to enforce. Another optimal intervention implied by the model takes the form of “portfolio insurance”. The government (or some insurance company) sets certain incentive-compatible “guidelines” as to how the consumer is expected to consume, save and allocate her assets. Based on these guidelines, the government (or the insurance company) infers the agent’s asset evolution, and makes transfers once the value of the agent’s portfolio threatens to become zero. Both policies are optimally financed by mandatory savings that are accumulated during the latter years of an agent’s worklife, when borrowing constraints have ceased binding.

Methodologically the paper relates to the finance literature on optimal portfolio choice in the presence of constraints, and in particular to the literature that uses convex duality / dynamic Lagrange multiplier methods to solve such problems. (See, e.g., Basak and Cuoco (1998), Cuoco (1997), Cvitanic and Karatzas (1992), Dumas and Lysaoff (2010), Detemple and Serrat (2003), Gallmeyer and Hollifield (2008), He and Pages (1993), He and Pearson (1991)). The typical approach in this literature is to take the income process of the agent as given (see e.g. He and Pages (1993)) and derive the Lagrange multipliers associated with the borrowing constraint. The new methodological aspect of the present paper is that the convex duality approach is applied in reverse. The government first solves for the best possible Lagrange multiplier process that is associated with the borrowing constraint, and then determines a transfer process that is associated with these Lagrange multipliers. To the best of my knowledge this is the first paper to apply convex duality methods to solve a
problem involving hidden actions (consumption, portfolio choice) and a hidden state variable (wealth). This new methodology could prove useful in a variety of moral hazard setups in finance, where the principal designs compensation schemes that exploit constraints faced by the agent.

The paper also relates to the literature on portfolio insurance. (See, e.g., Basak (2002), Grossman and Zhou (1996)). In that literature some agents voluntarily place a requirement on the minimum level of their assets at some point in time. In the present paper agents do not place such a constraint on their assets voluntarily, but instead the government needs to design policies in order to induce agents to adopt consumption and asset accumulation plans that can safeguard a minimum standard of living in retirement.

Finally, the paper relates to the literature on “dynamic public finance”. (See, e.g., Cole and Kocherlakota (2001), Golosov and Tsyvinski (2006) amongst many others.) That literature considers optimal insurance and contract design problems predominantly in setups of “hidden information” and idiosyncratic shocks. The present paper differs from that literature in that it deals with a “hidden action” problem in the presence of aggregate shocks.

The paper is structured as follows. Section 2 sets up the model. Section 3 introduces a government with the task of keeping the agent’s consumption above a minimum level by usage of appropriate fully funded transfers. Section 4 considers the agent’s reaction to the presence of such intervention. Section 5 derives an upper bound to welfare no matter which set of admissible taxes/transfers is utilized. Section 6 illustrates two distinct ways of attaining that upper bound, which are hence optimal. Section 7 discusses pre-retirement implications and mandatory savings. Section 8 discusses the implications of closing the model in general equilibrium. Section 9 concludes. Appendix A provides further details on the assumed negative externality that arises when an agent’s consumption drops below the minimum level. All proofs are in appendix B.
2 The model

2.1 Agents, preferences, and endowments

The baseline model is very similar to the small open economy version of Blanchard (1985). Accordingly, the investment opportunity set (interest rate, equity premium etc.) is taken as given. Section 8 shows that the main conclusions of this baseline model remain valid in a closed, general-equilibrium economy.

All agents are identical. The typical agent faces a probability of death \( q \) per unit of time \( dt \). All agents have constant relative risk aversion \( \gamma \), and a constant discount rate \( \rho \). Accordingly, the maximization problem of an agent, who is born at time \( t^b \), is given by

\[
E_{t^b} \int_{t^b}^{\infty} e^{-(\rho + q)(t-t^b)} \frac{c^{1-\gamma}}{1-\gamma} dt.
\]

(1)

To expedite the exposition and shorten proofs, I concentrate on the empirically relevant case \( \gamma > 1 \).

Life has two phases. A “work” phase, which lasts for \( \tau \) years after birth, and is followed by a “retirement phase”. During the work phase agents receive a constant income stream equal to \( Y \) per unit of time. Once they retire, they receive no more labor income.

2.2 Investment opportunity set

Agents can invest in the money market, where they receive a constant strictly positive interest rate \( r > 0 \). In addition, they can invest in a risky security with a price per share that evolves as

\[
\frac{dP_t}{P_t} = \mu dt + \sigma dB_t,
\]

(2)

where \( \mu > r \) and \( \sigma > 0 \) are given constants and \( B_t \) is a one-dimensional Brownian motion.

\[\text{9With a few additional technical assumptions the results can be extended to } \gamma < 1, \text{ at the cost of lengthier proofs.}\]
on a complete probability space \((\Omega, F, P)\).\(^{10}\) The realization of this Brownian motion is the only source of uncertainty in this economy. The extension to multiple assets is straightforward and is left out.

As is well understood, dynamic trading in the stock and the bond leads to a dynamically complete market. (See e.g. Duffie (2001) or Karatzas and Shreve (1998)). As Karatzas and Shreve (1998) show, the assumptions of a constant interest rate and risk premium imply the existence of a unique stochastic discount factor \(H_t\), so that the time-\(t\) price of any claim that delivers dividends equal to \(D_u\), for \(u \geq t\) is given by\(^{11}\)

\[
E_t \int_t^\infty \frac{H_u}{H_t} D_u du,
\]

and \(H_t\) is given by

\[
\frac{dH_t}{H_t} = -rdt - \kappa dB_t, \quad \text{where } \kappa \equiv \frac{\mu - r}{\sigma}.
\] (3)

The agent can also enter into “annuity-style” contracts with a competitive life insurance company as in Blanchard (1985). Specifically, these contracts specify the following cash-flows: The insurance company offers an income stream of \(p\) per unit of time \(dt\), in exchange for receiving one dollar if the agent dies over the next interval \(dt\). Competition between insurance companies implies that \(p = q\). The presence of such annuities is inessential for the main arguments, but it simplifies some technical aspects of the analysis.

### 2.3 Portfolio and wealth processes

Throughout life, an agent chooses a portfolio process \(\pi_t\) and a consumption process \(c_t\). The portfolio process \(\pi_t\) is the dollar amount invested in the risky asset (the “stock market”) at time \(t\). The rest, \(W_t - \pi_t\), is invested in the money market. Since the key insights of the

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\(^{10}\)\(F = \{F_t\}\) denotes the \(P\)-augmentation of the filtration generated by \(B_t\).

\(^{11}\)From a macroeconomic perspective, one can also think of \(H_t\) as the marginal utility of consumption of the world-representative agent.
paper do not depend on the presence or absence of bequest motives, I simplify matters and assume that the agent has no bequest motives.\textsuperscript{12} As a result, the agent has an incentive to enter Blanchard-style annuity contracts for the full amount of her financial wealth. This results in an income stream of $qW_t$ per unit of time $dt$ while she is alive. In exchange, the entire remaining wealth of the agent gets transferred to the insurance company when the agent dies. Accordingly, the wealth process of a retired agent evolves as

$$dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt - c_t dt,$$

and the wealth process of a working agent is given by:

$$dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt + Y dt - c_t dt. \tag{5}$$

An additional requirement is that financial wealth must remain non-negative throughout:

$$W_t \geq 0 \text{ for all } t. \tag{6}$$

This constraint excludes un-collateralized borrowing.

2.4 Externalities when consumption falls below a minimum standard of living

As already mentioned in the introductory section, societies typically opt to introduce regulatory measures to ensure that retiree consumption does not fall below a minimum standard.

To capture the reasons for such interventions in a simple way, and expedite the presentation of the main results, it is easiest to start by assuming that if the representative retiree’s consumption were to fall below a level $\xi$, this drop would have a negative externality on social welfare. Existing literature attributes such negative social effects on the likely

\textsuperscript{12}Extending the model to include a homothetic bequest function is straightforward.
redistributive pressures that would arise in that event.\textsuperscript{13} Since such payments are financed through distortionary taxes, they lead to deadweight costs. Accordingly, the central planner wants to ensure that retirees’ consumption plans satisfy

\[ c_t \geq \xi \text{ for all } t > t^b + \tau, \]  

in order to avoid such deadweight costs. Appendix A studies one potential rationale behind (7) in more detail by presenting an extended model featuring a tax-financed welfare system, intended for agents experiencing unobserved catastrophic shocks in retirement. In the context of that model, I show that the constraint (7) arises \textit{endogenously} as a “truth telling” constraint, safeguarding that it is never worth-while for agents who have not experienced catastrophic shocks (and hence are not the intended recipients of welfare) to falsely claim that they have. A behavioral rationale leading to constraint (7) is also discussed.

To expedite the presentation of the main results, the body of the paper simply assumes that the central planner aims to impose constraint (7) on the consumption of retirees, in order to avoid the deadweight costs of redistributive pressures and welfare payments that might arise otherwise. For brevity, I will henceforth refer to the central planner as the government.

\section{Introducing a role for the government}

To achieve the goal of imposing constraint (7) on the agent’s choices, the government can use transfers to modify the agent’s behavior so that retirees’ consumption plans satisfy equation (7).

To make matters realistic, the government’s information set is limited. The government can observe an agent’s income and the realized returns on the stock market, but not the agent’s assets or her consumption.

Based on that information set, the government needs to structure fully funded transfers

\textsuperscript{13}See e.g., the various contributions in Campbell and Feldstein (2001).
to the individual so as to ensure that constraint (7) holds. To keep with the assumption that
the retirement system is fully funded, such transfers are financed by the agent upon entering
retirement.

To obtain these optimal transfers it is most useful to use backward induction and split
the problem into a “post-retirement” part (which is solved first) and a “pre-retirement”
part, which is solved subsequently. In the post-retirement part the government determines
the optimal transfer process that maximizes the agent’s retirement utility subject to (7),
assuming that these transfers are financed with an upfront payment upon entering retirement.
This is done in sections 3.1 - 6. The pre-retirement part is discussed in section 7.

3.1 The post-retirement problem

It is now possible to provide a mathematical formulation to the government’s post-retirement
problem. Because of the time-invariance of the problem, I henceforth simplify notation and
normalize the time of retirement $t^b + \tau$ to be equal to zero. For the analysis of the post-
retirement problem (sections 3.1 - 6), I also normalize the value of the stochastic discount
factor at retirement to be equal to $H_0 = 1$.\textsuperscript{14}

**Problem 1** The government’s objective is to determine an admissible cumulative non-decreasing
transfer process $G_t$ and an initial tax $D_0$ so as to maximize:

\[
\Omega (W_0) \equiv \max_{G_t, D_0} E_0 \int_0^\infty e^{-(\rho + q)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt
\]

subject to

\begin{align*}
    c_t &\geq \xi \text{ for all } t > 0, \\
    D_0 &= E_0 \int_0^\infty e^{-qt} H_t dG_t.
\end{align*}

\textsuperscript{14}This latter normalization is without loss of generality since all quantities of interest depend on the ratio
of the stochastic discount factor between two points in time, rather than its level.
and subject to the constraint that \( c_t \) solves the agent’s optimization problem given \( G_t \)

\[
c_t = \text{arg} \max_{\langle \xi_t, \pi_t \rangle} E_0 \int_0^\infty e^{-(\rho+q)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt
\]

s.t.

\[
dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt - c_t dt + dG_t
\]

\[
W_0^+ = W_0 - D_0
\]

\[
W_t \geq 0 \text{ for all } t > 0
\]

Equation (8) states that the government aims to maximize the agent’s welfare, subject to the additional requirement (9) that the agent’s consumption not fall below the minimum level \( \xi \).

Equations (10) and (13) state that the cost of providing the transfer process \( G_t \) to the consumer should be self-financed by an upfront payment \( D_0 \). Parenthetically, this “self-financing” requirement implies that the government is not required to implement the provision of transfers to consumers. It can simply specify the optimal process \( G_t \) that each consumer should purchase and leave it to competitive financial companies to price and provide these transfers.

Finally, equations (11)-(14) capture the “principal-agent” aspect of the problem. Equation (11) states that the optimal process \( c_t \) cannot be mandated by the government (since the government observes neither the consumption nor the assets of the agent). Instead, the optimal consumption process is chosen optimally by the agent, who is faced with the budget dynamics of equation (12). These dynamics are identical to the ones in equation (4), except for the presence of the transfers \( dG_t \), and the fact that the consumer needs to finance these transfers by paying the amount \( D_0 \) upon entering retirement (equation [13]). Accordingly, an instant after entering retirement, her wealth \( W_0^+ \) is equal to the funds she has accumulated in the pre-retirement phase \( (W_0) \) net of the lump sum payment \( D_0 \).

The final requirement that constrains a consumer’s choices is the borrowing constraint (14). This constraint plays a central role in the analysis. Without this constraint, it would
be impossible for the government to find any set of transfers that would induce the agent to choose a consumption path that satisfies (9). The reason is due to a well understood result in Public Finance, known as “Ricardian Equivalence”: Since the market is dynamically complete in the absence of the constraint (14), a consumer’s feasible consumption plans are constrained only by the requirement that the net present value of her consumption be equal to the wealth she has accumulated. Since the net present value of government transfers is equal to the lump sum payment $D_0$, the consumer’s intertemporal budget constraint is unaffected by the government intervention, no matter what process $G_t$ the government chooses. Accordingly, the transfers cannot affect the consumer’s plans. Agents can continue to consume as they would in the absence of government intervention and only modify their portfolios so as to undo the effects of the transfers.

The presence of a borrowing constraint such as (14), however, makes transfers non-neutral. The reason is that a borrowing constraint implies stronger restrictions than a simple intertemporal budget constraint on the agent’s feasible consumption choices. Hence, by a judicious choice of transfers, the government can affect the agent’s consumption. Importantly, the borrowing constraint (14) is realistic and easy to implement in practice. It suffices that the government instruct courts not to enforce agreements that would let lenders seize future government transfers as collateral for loans.

Because of the central role played by the borrowing constraint (14), the next section reviews some known results related to the implications of the constraint (14) for optimal consumption processes. Subsequent sections use these results to solve problem 1.

4 The agent’s consumption choices in the presence of government intervention and borrowing constraints

Suppose that at the time of retirement (time 0) the government collects an amount $D_0$ and then assumes the obligation to deliver an admissible cumulative transfer process $G_t$. It is natural to ask how the agent’s consumption choices will be affected by this intervention in
the presence of the borrowing constraint (14).

To gain some intuition, it is useful to start by assuming that there is no uncertainty \( (\sigma = 0) \), so that \( \mu = r \), the stochastic discount factor is deterministic \( (H_t = e^{-rt}) \), and the agent’s dynamic budget constraint is given by \( dW_t = (q + r)W_t dt - c_t dt + dG_t \). The deterministic dynamics of \( W_t, H_t \) imply that the constraint \( W_t \geq 0 \) amounts to the requirement

\[
\int_0^t e^{-qs} H_s \leq W_0 - D_0 + \int_0^t e^{-qs} H_s dG_s \quad \text{for all } t \geq 0. \tag{15}
\]

Applying the Lagrangian method, an agent’s problem can be converted into an unconstrained problem by attaching Lagrange multipliers \( \lambda, \zeta_t \geq 0 \) to obtain

\[
L = \int_0^\infty e^{-(\rho+q)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + \lambda \left[ W_0 - D_0 + \int_0^\infty e^{-qt} H_t (dG_t - c_t dt) \right] \tag{16}
\]

\[
+ \left\{ \int_0^\infty \zeta_t \left( W_0 - D_0 + \int_0^t e^{-qs} H_s (dG_s - c_s ds) \right) dt \right\}
\]

Applying integration by parts to the second line of (16) and imposing the transversality condition \( \lim_{t \to \infty} e^{-qt} H_t W_t = 0 \) gives

\[
L = \int_0^\infty e^{-(\rho+q)t} \left( \frac{c_t^{1-\gamma}}{1-\gamma} - \rho X_t c_t \right) dt + \lambda \int_0^\infty e^{-qs} H_s X_s dG_s + \lambda \left[ W_0 - D_0 \right], \tag{17}
\]

where \( X_t \equiv 1 - \int_0^t \frac{\zeta_s}{X} ds \). Maximizing \( L \) over \( c_t \) amounts to simply maximizing the expression inside round brackets in equation (17), which gives

\[
c_t = (\lambda e^{\rho t} H_t X_t)^{-\frac{1}{\gamma}}. \tag{18}
\]

If all \( \zeta_s = 0 \) (i.e., when the borrowing constraint \( W_t \geq 0 \) is not binding) then \( X_t = 1 \), and equation (18) amounts to the familiar result that an agent’s marginal utility of consumption \( (e^{-\rho t} c_t^{-\gamma}) \) be proportional to the stochastic discount factor \( H_t \).

15To derive this equation, note that in the deterministic case \( d(e^{-qt} H_t W_t) = e^{-qt} H_t (dG_t - c_t dt) \). Integrating the left and right hand side of this equation and imposing the requirement \( W_t \geq 0 \) leads to (15).
However, when the borrowing constraint is binding, then consumption is affected by the presence of the decreasing process $X_t$, which reflects the cumulative effect of the Lagrange multipliers associated with the borrowing constraint. By construction $X_t$ is a process that is non-increasing and starts at $X_0 = 1$.

To fully determine the solution to the consumer’s problem, one needs to determine the Lagrange multipliers $\lambda, \zeta$. He and Pages (1993) show that this amounts to first maximizing $L$ over $c_t$ (given arbitrary $\lambda, X_t$) and then minimizing the resulting expression over $\lambda, X_t$. Specifically, He and Pages (1993) show the following Proposition, which holds also in the presence of uncertainty:16

**Proposition 1** Let $D$ be the set of non-increasing, non-negative and progressively measurable processes that start at $X(0) = 1$. Then, the value function $V(W_0)$ of an agent can be expressed as:

$$V(W_0) = \min_{\lambda > 0, \lambda \in D} \left[ E \left( \int_0^\infty e^{-(\rho + q)s} \max_{c_s} \left( \frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-q s} H_s X_s dG_s \right) + \lambda (W_0 - D_0) \right]$$

(19)

Let $X^*_t, \lambda^*$ denote the process $X_t$ and the constant $\lambda$ that minimize the above expression. Then the optimal consumption process $c^*_t$ for a consumer faced with the borrowing constraint (14) is given by (18) evaluated at $\lambda = \lambda^*, X_t = X^*_t$. Moreover, the process $X^*_t$ decreases only when the associated wealth process $(W_t)$ falls to zero and is otherwise constant, i.e.:

$$\int_0^\infty W_t dX^*_t = 0$$

(20)

Finally, the resulting wealth process for any $t > 0$ is given by

$$W_t = \frac{E_t \left( \int_t^\infty e^{-q(s-t)} X^*_s H_s c^*_s ds \right)}{X_t^* H_t} - \frac{E_t \left( \int_t^\infty e^{-q(s-t)} X^*_s H_s dG_s \right)}{X_t^* H_t}$$

(21)

16Marcet and Marimon (1998) show a similar result in the context of recursive contracts.
5 Government transfers and their welfare effects: an upper bound

Proposition 1 gives an intuitive way to summarize the effects of the incentive compatibility requirement (equations [11]-[14]).

It asserts that every government transfer process $G_t$ will be associated with a constant $\lambda^*(G_t)$ and a Lagrange multiplier process $X_t^*(G_t)$. Given this correspondence between a choice of $G_t$ and the resulting pair $(\lambda^*, X_t^*)$, there is a straightforward way to obtain an upper bound to the value function of problem 1. In particular consider the following problem:

**Problem 2** Maximize:

$$J(W_0) \equiv \max_{c_t, X_t \in D, \lambda > 0} E_0 \int_0^\infty e^{-(\rho + q)s} \frac{1}{1 - \gamma} ds$$

subject to:

$$E_0 \left( \int_0^\infty e^{-qs} H_s c_s ds \right) \leq W_0$$  \hspace{1cm} (23)

$$c_t \geq \xi$$  \hspace{1cm} (24)

$$c_t = (\lambda e^{\rho \xi} H_t X_t)^{-\frac{1}{\gamma}}$$  \hspace{1cm} (25)

Problem 2 is the problem of a government that can choose directly the consumption of the agent, subject to the intertemporal budget constraint (23), the constraint on the minimum consumption level (equation [24]), and the additional requirement that any chosen consumption process should have a representation in the form of equation (25) for some $X_t$.

In effect, problem 2 allows the government to choose directly the Lagrange multipliers $(\lambda, X_t)$ without being concerned whether there exists any pair of taxes and transfers $(D_0, G_t)$ that would render these Lagrange multipliers as shadow values of the consumer’s optimization problem (11).

Figure 1 gives a graphical argument to show that the optimized value $J$ to Problem 2
Figure 1: An illustration of Lemma 1. The admissible choices of problem 1 map into a subset of the admissible choices of problem 2.

provides an upper bound to the value function of problem 1. Clearly, any admissible pair $G_t, D_0$ needs to induce a consumption process that satisfies (24). Additionally, because transfers are fully funded, they don’t alter the consumer’s intertemporal budget constraint, and hence any admissible consumption process of problem 1 needs to satisfy\textsuperscript{17} equation (23). Moreover, Proposition 1 asserts that there always exists some pair of $\lambda, X_t$ such that any admissible consumption process of problem 1 can be expressed in the form of equation (25). Therefore, any admissible $G_t, D_0$ maps into a subset of pairs $(X_t, \lambda)$ allowed by Problem 2, and the value function of problem 2 must therefore provide an upper bound to problem 1. The following Lemma provides a formal statement.

**Lemma 1** Let $\mathcal{G}$ be the class of all transfer processes $G_t$ that enforce (9) and satisfy (10). Furthermore, let $V(W_0)$ be given as in equation (19). Then the value functions of problems 1 and 2 are related by

$$\Omega(W_0) = \max_{D_0, G_t \in \mathcal{G}} V(W_0) \leq J(W_0)$$

\textsuperscript{17}The consumer’s dynamic budget constraint (12) implies the intertemporal budget constraint

$$W_0 - D_0 + \int_0^\infty e^{-qt} H_t(dG_t - c_t dt) \geq 0.$$ Combining the intertemporal budget constraint with condition (10) implies (23).
The remainder of this section derives an explicit solution to problem 2, while the next section shows that there exist transfer processes $G_t^*$ that are optimal, because they make equation (26) hold with equality.

As a first step towards solving problem 2, it is useful to ask whether constraints (23), (24), and (25) will bind at an optimum. The top panel of figure 2 gives an optimal consumption path for a random realization of $H_t$ assuming that one maximizes (22) subject only to the intertemporal budget constraint (23). The resulting solution is $c_t^{***} = (\lambda^{***} e^{\rho t} H_t)^{-\frac{1}{\gamma}}$ and it corresponds to what the consumer would choose, if left alone. Because $H_t$ is log-normal, so is $c_t$ and accordingly $c_t < \xi$ with positive probability. Imposing the constraint $c_t \geq \xi$ (but not
the constraint \([25]\)) leads to the optimal consumption path \(c_t^{**} = \max \left[ \xi, (\lambda^{**} e^{\rho H_t})^{-\frac{1}{\gamma}} \right].\)

The solution \(c_t^{**}\) is what the government would choose, if it could directly observe and mandate the agent’s consumption and portfolio choices.

However, the government cannot directly observe these choices. Instead, it needs to induce the agent to choose consumption paths that satisfy \(c_t \geq \xi\), by exploiting binding borrowing constraints. This is captured by equation (25). The bottom panel of Figure 2 shows that this incentive compatibility requirement is in general binding. Indeed, equation (25) implies that any admissible consumption process should satisfy the property that the ratio \(c_t/c_t^{***} = \left(\frac{\lambda^*}{\lambda^{***}}\right)^{-\frac{1}{\gamma}} X_t^{-\frac{1}{\gamma}}\) should be a non-decreasing process (since \(X_t\) is non-increasing). Clearly, the ratio \(c_t^*/c_t^{***}\) has decreasing sections and therefore \(c_t^{**}\) cannot satisfy (25). Therefore, \(J(W_0)\) (and accordingly the value function \(\Omega(W_0)\) in problem 1) will in general be lower than what the government could attain if it observed and mandated consumption.

The next proposition determines the solution of problem 2:

**Proposition 2** Let the constants \(\phi, K\) be defined as

\[
\phi \equiv -\left(\rho - r - \frac{\kappa^2}{2}\right) + \sqrt{(\rho - r - \frac{\kappa^2}{2})^2 + 2(\rho + q)\kappa^2} > 1, \tag{27}
\]

\[
K \equiv \frac{\gamma}{\gamma - 1} \frac{\kappa^2}{2} + \gamma (r + q) + (\rho - r), \tag{28}
\]

and assume that

\[
W_0 \geq W^{min} \equiv \frac{1}{\phi} + \frac{1}{\phi - 1} K\xi. \tag{29}
\]

---

\(^{18}\)Clearly, \((\lambda^{**})^{-\frac{1}{\gamma}} < (\lambda^{***})^{-\frac{1}{\gamma}}\), otherwise it would be impossible that both \(c_t^{***}\) and \(c_t^{**}\) satisfy (23).

\(^{19}\)To see why \(\phi > 1\), notice that \(\phi\) solves the quadratic equation

\[
\frac{\kappa^2}{2} \phi^2 + \left(\rho - r - \frac{\kappa^2}{2}\right) \phi - (\rho + q) = 0
\]

Evaluating the left hand side of this equation at \(\phi = 1\) gives \(-(r + q) < 0\). Hence the larger of the two roots of the quadratic equation is larger than 1.
Additionally, for any $\lambda > 0$, let the process $X^*_t$ be given by

$$X^*_t(\lambda) \equiv \min \left[ 1, \frac{\xi^{-\gamma}/\lambda}{\max_{0 \leq s \leq t} (e^{\rho s} H_s)} \right].$$

(30)

Then the value function of problem (2) is given by

$$J(W_0) = \min_{\lambda \geq 0} \left[ E \left( \int_0^\infty e^{-(\rho+q)s} \frac{(\lambda e^{\rho s} H_s X^*_s)^{1-\frac{1}{\gamma}}}{1 - \gamma} ds - \lambda \int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X^*_s)^{-\frac{1}{\gamma}} ds + \lambda W_0 \right) \right]$$

(31)

$$= \min_{\lambda \geq 0} \left[ -\frac{K \xi^{1-\gamma}}{\gamma \phi (\phi - 1)} \left( \frac{\lambda}{\xi^{-\gamma}} \right)^\phi + K \frac{\gamma}{1 - \gamma} \lambda^{1-\frac{1}{\gamma}} + \lambda W_0 \right].$$

(32)

Letting $\lambda^*$ be the scalar that minimizes (32), the optimal triplet that solves problem (2) is given by $\lambda^*$, $X^*_t = X_t(\lambda^*)$, and $c^*_t = (\lambda^* e^{\rho t} H_t X^*_t)^{-\frac{1}{\gamma}}$.

Proposition 2 provides an explicit expression for the value function of problem 2, assuming that the agent enters retirement with a level of assets that are no smaller than the lower bound of equation (29). Assumption (29) will be maintained henceforth and discussed in further detail in section 7.

6 Optimal Transfer Processes

This section illustrates two optimal distinct processes $G^*_t$ that attain the upper bound $V(W_0; G^*_t) = J(W_0)$.

6.1 A constant income stream

The simplest form of government transfer process is a constant income stream: The government collects a lump sum tax of $D_0 = \frac{y_0}{r+q}$ and in exchange it delivers a constant stream of $y_0$ until the agent dies. Surprisingly, this policy is optimal, as long as $y_0$ is chosen judiciously. The following proposition gives a closed form solution for $y_0$. 

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Proposition 3 Let $y_0$ be given by

$$y_0 \equiv (r + q) K \xi \left( \frac{1 + \phi - 1}{\phi - 1} \right),$$

(33)

where $K$ is given in (28) and $\phi$ is given in (27). The policy of collecting $D_0 = \frac{y_0}{r+q}$ and providing transfers equal to $y_0$ until the agent dies, attains the upper bound $V(W_0; G_t = y_0) = J(W_0)$ and is therefore optimal.

An interesting feature of the optimal policy in proposition 3 is contained in the following Lemma

Lemma 2 The optimal policy of proposition 3 has the property

$$\frac{y_0}{\xi} > 1.$$  

Lemma 2 shows that if the government wants to ensure a minimum consumption of one dollar, it needs to deliver more than one dollar in guaranteed income. This result is driven by the fact that agents cannot be excluded from markets, and the presence of a constant income guarantee incentivizes them to use some of the constant income to take risks in the stock market. Since part of the income is used for such investments, the government needs to set $y_0 > \xi$ in order to ensure that $c_t \geq \xi$.

6.2 Portfolio Insurance

Providing agents with a constant income is not the unique optimal way to attain the upper bound in Proposition 2. The approach presented in this section also succeeds in attaining the same upper bound. An advantage of the approach presented here is that it allows one to always determine a general feasible policy that attains the upper bound of Proposition 2 for arbitrary stochastic discount factors. (Section 8 provides further discussion.)

To describe this approach, let $\lambda^*$ be the scalar that minimizes (32). Then define the
government’s transfer process as:

\[ dG_t = -\left(\frac{1}{\gamma} + \phi - 1\right) K \xi \frac{dX_t^*}{X_t^*} \]  (34)

where \( X_t^* (\lambda^*) \) is the process defined in (30).

This section shows the following two results:

a) The process (34) attains the upper bound of Proposition 2.

b) The process (34) has an intuitive economic interpretation as a type of minimum return guarantee (portfolio insurance) on the agent’s optimal portfolio of stocks and bonds.

The following proposition formalizes the first claim and provides results that are useful towards establishing the second claim.

**Proposition 4** Let \( \lambda^* \) be the scalar that minimizes (32) and \( X_t^* (\lambda^*) \) be the process that is given in (30). Consider an agent who anticipates transfers given by (34) and is faced with an initial tax of \( D_0 \), where \( D_0 \) satisfies (10). Then

a) her value function coincides with the upper bound given in (32).

b) Letting

\[ Z_t \equiv \lambda^* e^{\lambda t} H_t X_t^* \]  (35)

the agent invests

\[ \pi_t = \frac{\kappa}{\sigma} K \xi \left[ (\phi - 1) \left( \frac{Z_t}{\xi - \gamma} \right)^{\phi - 1} + \frac{1}{\gamma} \left( \frac{Z_t}{\xi - \gamma} \right)^{-\frac{1}{\gamma}} \right] \]  (36)

dollars in the stock market and consume

\[ c_t = Z_t^{\frac{1}{\gamma}} \]  (37)
while the agent’s optimal wealth process $W_t$ is given by

$$W_t = -K\xi \left( \frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} + KZ_t^{-\frac{1}{\phi}}.$$  \hspace{1cm} (38)

c) The initial payment $D_0$ associated with (34) is given by

$$D_0 = K\xi \frac{1}{\phi} + \phi - 1 \left( \frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi-1}.$$  \hspace{1cm} (39)

The portfolio policy (36) will aid in the interpretation of (34) as a form of portfolio insurance. To obtain some intuition on the nature of (34), consider first the following puzzling feature of the optimal portfolio policy: As $c_t \to \xi$, equation (37) implies that $Z_t \to \xi^{-\gamma}$ and (38) implies that $W_t \to 0$. However, the portfolio of the agent becomes

$$\lim_{Z_t \to \xi^{-\gamma}} \pi_t = \left( \frac{1}{\gamma} + \phi - 1 \right) K\xi \frac{\kappa}{\sigma} > 0.$$  \hspace{1cm} (40)

Because the agent’s financial wealth approaches zero as $Z_t \to \xi^{-\gamma}$, but her stock position doesn’t, a further negative return on the stock market would lead to a negative financial asset position in the absence of any transfers. To prevent such a negative asset position, the transfers given by (34) act as a minimum return guarantee, which ensures that the agent receives just enough funds to sustain her financial wealth at zero and keep her consumption at $\xi$.

It is useful here to clarify that these transfers do not require that the government actually observe the path of the agent’s assets or her consumption. By the definition of $X^*_t$ in equation (30), the government only needs to know the evolution of the stochastic discount factor $H_t$, which can be inferred from the path of the stock market$^{20,21}$.

A simple way of thinking about the transfer process $G_t$ in (34) is that the government

---

$^{20}$Note that $\log (H_t) - \log (H_0) = -(r + 0.5\kappa^2)t - \kappa (B_t - B_0) = -(r + 0.5\kappa^2)t - \frac{\xi}{\sigma} \sigma (B_t - B_0) = -(r + 0.5\kappa^2)t - \frac{\xi}{\sigma} \left[ \log P_t - \log P_0 - (\mu - 0.5\sigma^2) t \right] = \frac{\xi}{\sigma} (P_t - P_0) + \left( \frac{\xi}{\sigma} (\mu - 0.5\sigma^2) - (r + 0.5\kappa^2) \right) t$.

$^{21}$Section 7 also implies that the behavior of the stochastic discount factor allows the government to infer the agent’s initial assets at retirement.
and the agent have a joint understanding of how the consumer will consume and invest in
the presence of the transfers given by (34). Based on its (correct) understanding of the
consumer’s optimal policies, the government can infer the agent’s wealth and make just
enough transfers when needed, so as to keep the agent’s wealth above 0 and her optimal
consumption above $\xi$.

6.3 Comparing the two policies

Given that both policies attain the upper bound of equation (32), this means that they are
equivalent from a welfare perspective. The derivations in the appendix also show that they
imply exactly the same consumption process “path by path”.

However, the two policies do differ. They make transfers of different magnitudes in
different states of the world. The initial payments that they imply are also different. Indeed,
the initial payment associated with the constant income policy is:

$$D_0^{\text{const.}} = \frac{y_0}{r + q} = K\xi \left( \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right), \quad (41)$$

whereas by equation (39), the initial payment of the portfolio insurance policy is:

$$D_0^{p.i.} = K\xi \left( \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right) \left( \frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi - 1} \phi^{-1} \quad (42)$$

Since $c_0 \geq \xi$ and$^{22} c_0^{-\gamma} = \lambda^*$, it follows that $\lambda^*/\xi^{-\gamma} \leq 1$ and accordingly $\frac{D_0^{p.i.}}{D_0^{\text{const.}}} \leq 1$. Hence the “portfolio insurance” policy implies an initial payment that cannot be larger than the initial payment of the “constant income” policy. This is intuitive, since the constant income policy delivers the same transfers in all states of the world, including states of the world where the borrowing constraint doesn’t bind. By contrast, the “portfolio insurance” policy delivers payments only when the borrowing constraint binds.

However, when $c_0 = \xi$ (or alternatively $W_{0+} = 0$) the two policies imply the same initial

$^{22}$Recall that $H_0 = X_0^* = 1$. 

23
payment. Hence, the initial payment of the two policies differs only when the borrowing constraint is not binding, but is identical when the borrowing constraint does bind. This is the reason why the two policies imply different initial payments, but are identical from a welfare perspective. The additional resources delivered by the constant income policy are delivered in states of the world where the borrowing constraint is not binding and hence can be “undone” by agents’ portfolio choice.

The above discussion illustrates that simply comparing the costs of retirement benefit guarantees does not provide sufficient information for welfare comparisons.

7 Minimum level of assets and implications for pre-retirement savings

A maintained assumption of the analysis so far was that the agent’s assets upon entering retirement were above the minimum level of equation (29). As the next Proposition shows, this assumption is not only sufficient, but it is also necessary for the existence of any transfer processes that can induce a consumption process that satisfies $c_t \geq \xi$.

**Proposition 5** An admissible transfer process $G_t$ that can induce $c_t \geq \xi$ post-retirement exists if and only if (29) holds, i.e. if $W_0 \geq W_{min}$.

Proposition 5 has implications for the government’s pre-retirement problem. Specifically, the feasibility of enforcing the constraint $c_t \geq \xi$ post-retirement is equivalent to requiring that the agent arrive in retirement with assets that are at least as large as $W_{min}$.

Therefore, prior to retirement, the government needs to ensure that the agent saves an adequate fraction of labor income, so as to be able to finance the post-retirement optimal transfer processes $G_t$, which were described previously. Specifically, recalling that $t^b$ is an agent’s date of birth, $\tau$ the duration of work, and $t^b + \tau$ is the time of retirement (which is normalized to zero), the government can collect pre-retirement payments from the agent...
equal to $\int_{t^b}^{0} dS_t$, $dS_t \geq 0$, and rebate a lump-sum amount $L_0$ at retirement so that

$$W_0 \equiv W_{0^-} + L_0 \geq W_{min}, \quad (43)$$

and\(^{23}\)

$$E_t e^{-qt} \left( \frac{H_0}{H_{t^b}} \right) L_0 = E_t \int_{t^b}^{0} e^{-q(t-t^b)} \left( \frac{H_t}{H_{t^b}} \right) dS_t. \quad (44)$$

Equation (43) requires that the agent’s assets upon entering retirement ($W_0$) - which are comprised of the agent’s assets an instant before retirement ($W_{0^-}$) and the governmental lump-sum transfer ($L_0$) - be at least as large as $W_{min}$. Additionally, equation (44) is analogous to equation (10), since it requires that the present value of the lump sum transfer $L_0$ be equal to the present value of the pre-retirement payments $dS_t$. Because of equation (44), I refer to the transfers $dS_t$ as “mandatory savings” rather than “distortionary labor taxes”, since they are returned to the agent (compounded at a fair market return) in the form of a lump sum transfer at retirement. (A practical implication of this difference would arise in an extended model with endogenous, continuous labor supply, since taxes would distort the intra-temporal first order conditions for labor supply, whereas mandatory savings would not.)

An appealing feature of the post-retirement government policies of the previous sections is that they did not actually require the government to implement and administer them. The government could simply mandate that retirees purchase an “insurance” policy with payoffs $dG_t$, and then private competitive entities (insurance companies, pension funds, etc.) could provide this policy in exchange for an upfront fee $D_0$.

To ensure that the same “decentralization” through private entities is also feasible for the pre-retirement problem, I make an additional assumption on the allowable combinations of $S_t, L_0$. To motivate this assumption, suppose that the government determined a policy pair $< S_t, L_0 >$, and mandated that working agents make transfers to a pension fund equal

\(^{23}\)Clearly, for the purposes of this section $t < 0$, and $H_0$ is a random number, so that it cannot be normalized to 1 as in the post-retirement problem.
to \(dS_t\), in exchange for a transfer payment of \(L_0\) from the pension fund at retirement. Then, the financial assets of the pension fund are given by \(\tilde{W}_t\), with \(\tilde{W}_t\) defined as

\[
\tilde{W}_t \equiv E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) L_0 - E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) dS_u
\]  

Equation (45) follows from the fact that the financial assets of the pension fund \(\tilde{W}_t\) plus the remaining present value of transfers from the workers \(E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) dS_u\) must be equal to the present value of the lump sum transfer to be paid once the agents retire \(E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) L_0\). To prevent default in the spirit of Bulow and Rogoff (1989),\(^\text{24}\) I require that

\[
\tilde{W}_t \geq 0 \iff E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) L_0 - E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) dS_u \geq 0, \tag{46}
\]

for all \(t \in [t^b, 0]\). This non-negativity of financial assets for a pension fund is the direct analog of the non-negativity of financial assets for consumers (equation [6]).\(^\text{25}\) It is noteworthy that the requirement (46) is automatically satisfied for any deterministic policy pair \(dS_t, L_0\) that satisfies (44).

Finally, to ensure that there exist some feasible combination of \(S_t, L_0\) that satisfy (44), (46), and can enforce (43), I assume that the present value of \(W^{\text{min}}\) as of the birth of the agent is no larger than the respective present value of the agent’s income

\[
E_{t^b} e^{-q\tau} \left( \frac{H_0}{H_{t^b}} \right) W^{\text{min}} < E_{t^b} \int_{t^b}^0 e^{-q(t-t^b)} \left( \frac{H_t}{H_{t^b}} \right) Y dt. \tag{47}
\]

\(^\text{24}\)Specifically, Bulow and Rogoff show that whenever \(\tilde{W}_t < 0\) for any private entity, then there exist a profitable deviation whereby the private entity defaults on its lenders at \(t\), keeps receiving its income from \(t\) onwards (in the case of a pension fund \(dS_t\)), and finances its consumption (in the case of a pension fund the terminal payout \(L_0\)) without ever having to borrow from its lenders in the future.

\(^\text{25}\)It is useful to remark that in the post-retirement problem \(\tilde{W}_t = E_t \int_t^\infty e^{-q(u-t)} H_u dG_u \geq 0\), and the constraint (46) is automatically satisfied.
Using $E_{t^b} \left( \frac{H_{t^b}}{H_{t^b}} \right) = e^{-r(t-t^b)}$ inside (47), taking logarithms on both sides and rearranging leads to the following re-statement of (47), which is maintained throughout

$$
\tau > \frac{\log \left( 1 + (r + q) \frac{W_{	ext{min}}}{Y} \right)}{(r + q)}.
$$

(48)

The government’s pre-retirement problem can now be summarized in a manner analogous to the government’s post-retirement problem 1.

**Problem 3** Choose $S_t, L_0$ so as to maximize

$$
\Omega_{t^b} \equiv \max_{S_t, L_0} E_{t^b} \int_{t^b}^{0} e^{-\rho \gamma (t-t^b)} \left( c_t \right)^{1-\gamma} dt + E_{t^b} e^{-\rho \gamma \tau} J \left( W_0^- + L_{\tau} \right)
$$

subject to (43), (44), (46), and subject to the constraint that $W_t$ is the wealth process that results for the choices of $c_t, \pi_t$ that solve the agent’s pre-retirement optimization problem given $S_t, L_{\tau}$:

$$
\max_{<c_t, \pi_t>} E_{t^b} \int_{t^b}^{0} e^{-\rho \gamma (t-t^b)} \left( c_t \right)^{1-\gamma} dt + E_{t^b} e^{-\rho \gamma \tau} J \left( W_0^- + L_{\tau} \right)
$$

(50)

s.t.:

$$
dW_t = qW_t dt + \pi_t \left\{ \mu dt + \sigma dB_t \right\} + \left\{ W_t - \pi_t \right\} r dt + (Y - c_t) dt - dS_t
$$

$$
W_t \geq 0 \text{ for all } t \geq t^b.
$$

(51)

(52)

Equation (49) requires that the government choose mandatory saving policies $S_t$ and a lump sum transfer upon retirement so as to maximize the agent’s life-time expected utility at birth. The equations (50) - (52) require that the wealth process is the result of the agent’s optimal consumption and portfolio choices $c_t, \pi_t$ that result in the presence of the government policies $S_t, L_0$.

The discrepancy in the government’s and the agent’s objectives in the pre-retirement problem stems from the fact that the government wants to ensure that the agent arrives in retirement with a minimum amount of assets (in order satisfy the constraint $c_t \geq 0 \geq \xi$ post-
retirement), while the agent would not necessarily arrive in retirement with such a minimum level of assets, if left alone. Accordingly, the government policies $S_t, L_0$ need to induce her to choose pre-retirement consumption and portfolio policies that will result in a minimum level of retirement assets.

As a first step towards solving problem 3, it is useful to consider the solution of the following problem

**Problem 4** Choose $c_t, W_0$ so as to maximize

$$
J_t^b \equiv \max_{c_t, \pi_t} E_t^b \int_t^b e^{-(\rho+q)(t-t')} \left( \frac{c_t}{1-\gamma} \right)^{1-\gamma} 
\left[ dt + E_t^b e^{-(\rho+q)\tau} J(W_0) \right]
$$

subject to the dynamic budget constraint (5), the non-negativity of wealth constraint (6) and the additional constraint $W_0 \geq W_{min}$.

Problem 4 is the problem that would be solved by the agent, if she voluntarily imposed the constraint $W_0 \geq W_{min}$ on her own choices. As one might expect, the fact that the agent voluntarily imposes the constraint $W_0 \geq W_{min}$ on her own decisions implies that the value function of problem 4 is an upper bound to the solution of problem 3. This is formalized in the next Lemma

**Lemma 3** For $\Omega^b, J^b$ denoting the value functions of problems 3 and 4 respectively, $\Omega^b \leq J^b$.

Variants of problem 4 have been studied elsewhere, and especially in the literature on portfolio insurance. (See e.g., Basak (2002)). The new aspect of this paper is that the solution of Problem 4 acts as an upper bound to Problem 3 and hence can be used to check the optimality of various mandatory savings programs.

Solving Problem 4 in closed form is difficult, because of the presence of the borrowing constraint $W_t \geq 0$ and the extra state variable introduced by the agent’s distance to retirement. Fortunately, the exact solution of problem 4 is not required for the analysis that
follows. Instead, it is sufficient to establish the following property of any optimal solution to problem 4.

**Lemma 4** If \( W_t^* \) is the optimal wealth process associated with problem 4, then there exist a time \( \chi = -\frac{\log(1+(r+q)\frac{W_{\text{min}}}{r})}{(r+q)} \in (t^b, 0) \) such that \( W_t^* > 0 \) for all \( t \in (\chi, 0] \).

Lemma 4 asserts that there exists a time \( \chi \) prior to the retirement time zero, such that the borrowing constraint \( W_t^* \geq 0 \) is non-binding for any \( t \in [\chi, 0) \). The next proposition shows that a simple way to construct an optimal policy to the government’s problem 3 is to set \( dS_t > 0 \) only during \([\chi, 0)\).

**Proposition 6** Consider the government policy given by

\[
dS_t = \begin{cases} 
0 & \text{if } t \in [t^b, \chi) \\
Y dt & \text{if } t \in [\chi, 0]
\end{cases}
\]

and \( L_0 = W_{\text{min}} \). Then this policy is optimal, since the associated value function \( V_{t^b} \) of the agent satisfies \( V_{t^b} = J_{t^b} \).

Simply put, proposition 6 asserts that an optimal solution to the government’s problem is to make a transfer equal to \( W_{\text{min}} \) at retirement and finance this transfer by requiring mandatory savings only for a few years immediately prior to retirement. This “backloading” of savings is driven by Lemma 4 and in particular, the observation that the borrowing constraint stops binding for some time prior to retirement \( (t \in [\chi, 0]) \).

Intuitively, by the time \( \chi \), the agent has accumulated enough wealth that income can be channeled towards savings without distorting the optimal consumption decisions the agent would have made, if she was solving problem 4.

Summarizing, this section has shown that the feasibility of any fully funded post-retirement plan \( dG_t \) that can provide a minimum standard of living in retirement, is equivalent to the requirement that the agent arrive in retirement with assets at least as large as \( W_{\text{min}} \). In order to ensure that the agent has accumulated this amount of assets by retirement, the government needs to enforce a minimum amount of savings pre-retirement. Given the frictions
implied by the presence of borrowing constraints, the current framework suggests that these
mandatory savings should be done just prior to retirement, when the agent has accumulated
enough wealth to avoid consumption distortions.

Of course, Proposition 6 is mostly of theoretical interest: It helps highlight the fact that
borrowing constraints make it optimal to postpone mandatory savings to the years prior
to retirement. The stark difference in optimal mandatory savings before $\chi$ and after $\chi$ is
sensitive to the maintained assumption of an exogenous retirement time. It is reasonable
to conjecture that endogenizing the retirement time along the lines of Farhi and Panageas
(2007) would lead to a smoother age-dependent mandatory savings profile. This extension
would be lengthy and would require a separate paper. However, it is reasonable to conjecture
that the presence of borrowing constraints would still tend to backload mandatory savings,
albeit not completely.

8 Arbitrary stochastic discount factors and general equi-
librium

The assumption of a small open economy facilitated the analysis by rendering the stochastic
discount factor exogenous to the model. Another simplifying assumption is that everything
is driven by a single shock. Neither of these assumptions is restrictive. Even if the stochastic
discount factor were endogenous and driven by multiple sources of uncertainty, most of the
results of the paper would survive.

Specifically, the fact that equation (31) provides an upper bound to problem 1 remains
valid for any continuous stochastic discount factor $H_t$. It is also straightforward to show that
an appropriate variant of the portfolio insurance policy would attain the upper bound of
proposition 2 for any stochastic discount factor, and accordingly for the general equilibrium
version of the present model.

However, the result that seems to depend on the constant nature of the investment
opportunity set is the optimality of the constant income policy. Nevertheless, since the upper
bound of equation (31) is attainable for any stochastic discount factor (and in particular for the one obtained when the model is closed in general equilibrium), one can use equation (31) to evaluate the magnitude of potential welfare losses, and balance these losses against the simplicity of a constant income policy.

In summary, the qualitative findings of the model would survive even in a closed, general equilibrium economy.\textsuperscript{26} Even though the stochastic discount factor and the price of all guarantees would change, most of the key results of the paper, namely the nature of the upper bound of equation (31), and the optimality of the portfolio insurance policy would remain unchanged.

9 Conclusion

By exploiting borrowing restrictions of agents, this paper proposed a framework to discuss optimal transfer processes that can ensure a minimum standard of living in retirement.

Within the framework of the baseline life-cycle model, two policies were shown to be optimal: According to the first policy, retirees use part of their accumulated assets to purchase a fixed annuity that pays off a constant income stream. The second policy is an appropriate form of portfolio insurance that ensures retirees against further negative returns, once their assets approach zero. Optimal transfers are financed by mandating pre-retirement savings, which optimally take place in the years leading up to retirement, and not at the beginning of the life-cycle.

Several issues are unexplored by the present paper. A first issue concerns unobserved preference heterogeneity. If agents have different risk aversions, or subjective discount factors, then the government needs to offer menus of contracts in the spirit of discriminatory pricing. An open question is whether the need to enforce sorting into different types of contracts would affect the qualitative features of the guarantees. A further extension of the

\textsuperscript{26}Of course in general equilibrium care should be taken to make sure that aggregate consumption stays above the level $\xi$ multiplied by the mass of retirees. In an endowment economy this could be done by boundedness assumptions on the aggregate endowment. Alternatively one could introduce production in the spirit of Cox et al. (1985) and relax the bounds on the fundamental shocks.
present model would be to allow agents to choose their retirement time endogenously and examine the implications for pre-retirement savings. Studying these two questions is left for future research.
Appendix

A Justifications for the constraint $c_t \geq \xi$

An underlying assumption behind problem 1 is that there is a wedge between the government’s objective and the individual agent’s objective. The wedge stems from the assumed negative external effects that arise, when an agent’s consumption falls below the level $\xi$. This section revisits the reasons behind the assumed wedge between the government’s and the agent’s objectives.

A.1 The constraint $c_t \geq \xi$ in the presence of a welfare system

One potential reason for safeguarding that agents can self-finance a minimum standard of living in retirement is to deter them from over-burdening the (distortionary-tax financed) welfare system by claiming welfare benefits, when they are not the intended recipients of such benefits. In that sense agents’ financial decisions can be a source of systemic risk, in the sense that they can constitute a negative externality for the economic system.

To substantiate this claim, I enrich the model and introduce a simple reason for the existence of a welfare system, along with a stylized model of the welfare system. Specifically, assume that until retirement agents are identical in every respect. Upon entering retirement, however, a small fraction $\theta$ of agents experiences an unobservable and idiosyncratic shock that results in a negative income stream of $Y$ for the rest of their lives. (The remaining $1 - \theta$ fraction of the agents remain identical to the agents described in the paper sofar). The idiosyncratic shock is catastrophic, in the sense that no agent could self-insure against that shock by accumulating savings

$$\int_{t^b}^{0} e^{-(q+r)(t-t^b)}Y dt < \int_{0}^{\infty} e^{-(q+r)(t-t^b)}Y dt$$

Equation (53) states that even if an agent saved all her labor income, the resulting present value would still be smaller than the present value of the negative shock $Y$.

Because of these catastrophic and unobservable idiosyncratic shocks, the government can raise the welfare of the time $t^b-$cohort of agents by creating a “welfare” system, which aims to ensure agents against catastrophic idiosyncratic shocks. Clearly, the fact that shocks are unobservable requires some way to ensure that agents truthfully declare whether they
suffered a shock. In reality, non-pecuniary costs (such as standing in queues, filing paper-work etc.) can help “separate” different types of agents. To formalize this notion, suppose that the welfare system works as follows: any agent who enters retirement can obtain transfers \((dN_t \geq 0)\) from the government, by performing an activity associated with non-pecuniary, time-related costs \(\xi^{-\gamma}dN_t\), such as standing in a queue.\(^{27}\) (An alternative interpretation is that the agent gets hired and paid a wage \(dN_t\) for performing a task of zero productivity, but with a time-related disutility to herself given by \(\xi^{-\gamma}dN_t\).) Accordingly, the agent’s objective is to maximize

\[ E_0 \left( \int_0^\infty e^{-(\rho+q)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt - \int_0^\infty e^{-(\rho+q)t} \xi^{-\gamma}dN_t \right). \] (54)

The assumption of a constant cost of time \((\xi^{-\gamma})\) per unit of transfer simplifies the analysis, but is not key. As will become clear shortly, the crucial feature of (54) is that the non-pecuniary costs act as a useful screening device so as to separate the agents who have experienced idiosyncratic shocks from those who haven’t.

A final assumption is that the transfers \(dN_t\) are financed by distortionary labor taxes when the cohort of agents is still working. Specifically, the government collects a labor tax equal to \(\bar{\omega}Y\), during the work years of the agents, so as to finance any welfare transfers \(dN_t\) later on. An important difference between the mandatory savings discussed in section 7 and the labor tax \(\bar{\omega}Y\) is that this tax is withheld by the government and redistributed only to agents who experience idiosyncratic shocks, so that there is no direct linkage between the taxes paid and the benefits received. As is well understood in the literature (see e.g., Barro (1979), Lucas and Stokey (1983)), this decoupling leads to a distortion of the labor-leisure tradeoff and results in deadweight costs. Even though it is straightforward to model such distortions explicitly, as in Panageas (2010), for the purposes of this paper it suffices to treat such deadweight costs in a simple reduced-form way and assume that only a fraction \((1-\delta)\bar{\omega}Y\) of an agent’s taxes reaches the government. (The literature sometimes refers to such a simple modeling of deadweight costs as “iceberg” costs). The constant \(\delta\) captures the fraction of income that gets “wasted” due to work-disincentives. The resulting budget

\(^{27}\)The idea that queues can act as devices to elicit hidden information is well established in the public finance literature. See e.g. Stiglitz (1992).
constraint of the government is given by
\[
\int_{t^b}^{0} e^{-(\rho+q)(t-t^b)} (1-\delta) \varpi Y = \theta \int_{0}^{\infty} e^{-q(t-t^b)} \left( \frac{H_t}{H_t^b} \right) dN_t,
\]
assuming that only the \( \theta \)-fraction of agents who suffer the idiosyncratic shock ever request transfers.

Letting \( i = \mathcal{I} \) if agent \( i \) has experienced an idiosyncratic shock and \( i = \mathcal{N}\mathcal{I} \) otherwise, the “truth-telling” requirement\(^{28}\) can be expressed as
\[
\int_{0}^{\infty} dN_t^i = 0, \text{ whenever } i = \mathcal{N}\mathcal{I}.
\]

Using the shorthand notation \( \theta^\mathcal{I} \equiv \theta \), \( \theta^\mathcal{N}\mathcal{I} \equiv 1 - \theta \) and \( Y^\mathcal{I} \equiv \overline{Y}, Y^\mathcal{N}\mathcal{I} \equiv 0 \), the full government’s post-retirement problem can be expressed as

**Problem 5** Choose \( G_t, D_0 \) to maximize
\[
\max_{G_t, D_0} E_0 \sum_{i=\mathcal{I}, \mathcal{N}\mathcal{I}} \theta^i \left( \int_{0}^{\infty} e^{-(\rho+q)t} \frac{(c_t^i)^{1-\gamma}}{1-\gamma} \big) dt - \int_{0}^{\infty} e^{-(\rho+q)t} \xi^{-\gamma} dN_t^i \right) \tag{56}
\]
subject to (10), (55), and subject to the constraint that \( c_t^i, dN_t^i \) solve the agent’s optimization problem given \( G_t \)

\[
c_t^i = \arg \max_{c_t^i, \pi_t^i, dN_t^i} E_0 \int_{0}^{\infty} e^{-(\rho+q)t} \frac{(c_t^i)^{1-\gamma}}{1-\gamma} \big) dt - \int_{0}^{\infty} e^{-(\rho+q)t} \xi^{-\gamma} dN_t^i \tag{57}
\]
s.t.:
\[
dW_t^i = qW_t^i dt + \pi_t^i \{ \mu dt + \sigma dB_t \} + \{ W_t^i - \pi_t^i \} r dt
\]
\[
- c_t^i dt + dG_t + dN_t^i - Y^i dt \tag{58}
\]
\[
W_0^i = W_0^i - D_0 \tag{59}
\]
\[
W_t^i \geq 0 \text{ for all } t > 0 \tag{60}
\]

Problem 5 is almost identical to problem 1, with the main exception that the constraint \( c_t \geq \xi \) is replaced by the separation requirement (55). Once again, the government introduces

\(^{28}\)Because of the distortions associated with labor taxation (and the deadweight costs associated with screening), separation of types is optimal, in the sense that only agents who experience an idiosyncratic shock should be receiving welfare transfers \( dN_t \).
a fully funded transfer process, but not in an attempt to keep consumption above a minimum standard, but instead with the goal to induce agents who have not experienced idiosyncratic shocks to refrain from using the welfare system. However, the setup does not prohibit agents who have experienced idiosyncratic shocks to use the welfare system, and in general they will.

The link between the two problems is given by the following Lemma.

**Lemma 5** \( dN_t^{NI} = 0 \) whenever \( c_t^{NI} \geq \xi \).

Lemma 5 shows that the constraint (7) is a standard “truth-telling” constraint, which ensures that agents with asset dynamics given by equation (12) (i.e. agents who have not experienced idiosyncratic shocks) do not find it optimal to access the welfare system. Because of this correspondence, problem 1 can be viewed as a limiting case of problem 5 as \( \theta \) becomes sufficiently small.

**Proposition 7** Let \( \Omega^*(W_0) \) denote the value function of problem 5 and let \( \Omega(W_0; G_t^*, D_0^*) \) denote the value of the objective function of problem 5 assuming that the government follows any of the optimal policies \( G_t^*, D_0^* \) for problem 1. Then \( \lim_{\theta \to 0} \Omega(W_0; G_t^*, D_0^*) = \Omega^*(W_0) \).

### A.2 Behavioral justifications

Problem 1 is also consistent with a behavioral interpretation. Several authors in behavioral economics model the inability of an agent to commit as a principal-agent problem. The principal is taken as the “prudent”, time-zero “self”, who has a different objective than the subsequent “reckless selves” who are making decisions.\(^{29}\) For instance, if one were to interpret \( \xi \) as inelastic retirement expenditures associated with aging (say medical costs), then the “prudent” self would like to impose the constraint \( c_t \geq \xi \) on the choices made by the subsequent “reckless” selves, who will simply ignore this constraint. In such a behavioral interpretation of the problem, the government’s choice of post-retirement transfers maximizes the welfare of the prudent “self”, by exploiting borrowing constraints on the “reckless” selves.

\(^{29}\)For economic applications of the concept of “multiple selves”, see e.g. Amador et al. (2006).
B Proofs

Proof of proposition 1. Subject to minor modifications, the proof of this proposition is identical to the first theorem of He and Pages (1993) and the reader is referred to that paper for a proof.

Proof of Lemma 1. The proof of this lemma is contained in the proof of proposition 2 (Particularly Lemma 7).

Proof of Proposition 2. The proof of this proposition is established in steps. The following Lemma contains a useful first result.

Lemma 6 Take any \( \lambda \in (0, \xi^{-\gamma}] \) and any process \( G_t \) and define

\[
\hat{X}_t \equiv \arg \min_{X_s \in \mathcal{D}} E_0 \left( \int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left( \frac{c_s^{1-\gamma} - \lambda e^{\rho s} X_s c_s}{1 - \gamma} \right) ds + \lambda \int_0^\infty e^{-qs} H_s (X_s - 1) dG_s \right).
\]  

(62)

Then:

\[
\lambda E_0 \left( \int_0^\infty e^{-qs} H_s (\hat{X}_s - 1) dG_s \right) = E_0 \int_0^\infty e^{-(\rho+q)s} \left( e^{\rho s} \lambda H_s \hat{X}_s \right)^{1-\frac{1}{\gamma}} \left( 1 - \frac{1}{\hat{X}_s} \right) ds. \]  

(63)

Proof of Lemma 6. Let \( \Lambda_t \equiv 1 - \frac{1}{X_t} \). Applying Ito’s Lemma to \( \Lambda_t \), one obtains

\[
d\Lambda_t \equiv d\hat{X}_t \hat{X}_t \]

Hence \( \Lambda_t \) changes when and only \( \hat{X}_t \) changes. By Theorem 1 of He and Pages (1993):

\[
\int_0^\infty \left[ E_t \left( \int_t^\infty \hat{X}_s e^{-qs} H_s dG_s \right) - E_t \left( \int_t^\infty \hat{X}_s e^{-qs} H_s c_s ds \right) \right] d\hat{X}_t = 0, \]  

(64)

where \( c_s \) is given explicitly by (25). Plugging (25) into (64), and observing that \( \Lambda_t \) changes when and only when \( \hat{X}_t \) changes implies that

\[
\int_0^\infty \left( E_t \int_t^\infty \hat{X}_s e^{-qs} H_s dG_s - E_t \int_t^\infty \hat{X}_s e^{-qs} H_s \left( e^{\rho s} \lambda H_s \hat{X}_s \right)^{\frac{1}{\gamma}} ds \right) d\Lambda_t = 0.
\]

Then, for any admissible \( G_t \) and \( \hat{X}_t \) given by (62)

\[
\lambda E_0 \left( \int_0^\infty e^{-qs} H_s (\hat{X}_s - 1) dG_s \right) = \]

\[
\lambda E_0 \left[ \int_0^\infty e^{-qs} H_s (\hat{X}_s - 1) dG_s - \int_0^\infty \left( E_t \int_t^\infty \hat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t \right] \]

\[
+ \lambda E_0 \left\{ \int_0^\infty E_t \left[ \int_t^\infty \hat{X}_s e^{-qs} H_s \left( e^{\rho s} \lambda H_s \hat{X}_s \right)^{-\frac{1}{\gamma}} ds \right] d\Lambda_t \right\}. \]  

(65)
Next consider the martingale
\[ M_t \equiv E_t \int_0^\infty \tilde{X}_s e^{-qs} H_s dG_s = \int_0^t \tilde{X}_s e^{-qs} H_s dG_s + E_t \int_0^\infty \tilde{X}_s e^{-qs} H_s dG_s. \] (66)

According to the martingale representation theorem, there exists a square integrable \( \tilde{\psi}_s \) such that
\[ M_t = M_0 + \int_0^t \tilde{\psi}_s dB_s. \] (67)
Combining (66) and (67) gives
\[ d \left( E_t \int_0^\infty \tilde{X}_s e^{-qs} H_s dG_s \right) = dM_t - \tilde{X}_t e^{-qt} H_t dG_t = \tilde{\psi}_t dB_t - \tilde{X}_t e^{-qt} H_t dG_t. \]

Now, fixing an arbitrary \( \varepsilon > 0 \), letting \( \tau_\varepsilon \) be the first time \( t \) such that \( |\Lambda_t| \geq \frac{1}{\varepsilon} \), applying integration by parts and using the fact that \( \Lambda_0 = 0 \), gives
\[ -E_0 \int_0^{T \wedge \tau_\varepsilon} \left( E_t \int_0^\infty \tilde{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t = -E_0 \int_0^{T \wedge \tau_\varepsilon} \Lambda_s \tilde{X}_s e^{-qs} H_s dG_s + E_0 \int_0^{T \wedge \tau_\varepsilon} \Lambda_s \tilde{\psi}_s dB_s \]
\[ -E_0 \left[ \Lambda_{T \wedge \tau_\varepsilon} \left( E_{T \wedge \tau_\varepsilon} \int_0^\infty \tilde{X}_s e^{-qs} H_s dG_s \right) \right]. \]

Since \( \psi_s \) is square integrable and \( |\Lambda_s| \) is bounded in \( [0, \frac{1}{\varepsilon}] \) the second term on the right hand side of the above expression is 0. Also note that
\[ -E_0 \left[ \Lambda_{T \wedge \tau_\varepsilon} \left( E_{T \wedge \tau_\varepsilon} \int_0^\infty \tilde{X}_s e^{-qs} H_s dG_s \right) \right] = -E_0 \left[ \tilde{X}_{T \wedge \tau_\varepsilon} \Lambda_{T \wedge \tau_\varepsilon} J \right], \] (68)
where
\[ J \equiv \left( E_{T \wedge \tau_\varepsilon} \int_0^\infty \frac{\tilde{X}_s}{X_{T \wedge \tau_\varepsilon}} e^{-qs} H_s dG_s \right) \leq E_{T \wedge \tau_\varepsilon} \int_0^\infty e^{-qs} H_s dG_s, \] (69)
since \( \tilde{X}_t \) is non-increasing. Combining (69) with (68) and noting that \( 0 < \tilde{X}_t \leq 1 \),
\[ -E_0 \left[ \tilde{X}_{T \wedge \tau_\varepsilon} \Lambda_{T \wedge \tau_\varepsilon} J \right] = E_0 \left[ (1 - \tilde{X}_{T \wedge \tau_\varepsilon}) J \right] \leq E_{T \wedge \tau_\varepsilon} \int_0^\infty e^{-qs} H_s dG_s. \] (70)
Given that \( E \int_0^\infty e^{-qs} H_s dG_s < \infty \) it follows that
\[ E_{T \wedge \tau_\varepsilon} \int_0^\infty e^{-qs} H_s dG_s \to 0, \] (71)
as $\varepsilon \to 0, T \to \infty$. This leads to the inequalities:

$$-E_0 \int_0^\infty \left( E_t \int_t^\infty \hat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t \geq -E_0 \int_0^{T \wedge \tau} \left( E_t \int_t^\infty \hat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t \geq -E_0 \int_0^{T \wedge \tau} \Lambda_s \hat{X}_s e^{-qs} H_s dG_s.$$  

Letting $\varepsilon \to 0, T \to \infty$, using the monotone convergence theorem, and using (70) and (71), gives

$$- \int_0^\infty \left( E_t \int_t^\infty \hat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t = -E_0 \int_0^\infty \Lambda_s \hat{X}_s e^{-qs} H_s dG_s.$$  

(72)

Using (72) and the definition of $\Lambda_t$ gives

$$\lambda E_0 \left[ \int_0^\infty e^{-qs} H_s \left( \hat{X}_s - 1 \right) dG_s - \int_0^\infty \left( E_t \int_t^\infty \hat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t \right] =$$

$$= E_0 \left[ \lambda \int_0^\infty e^{-qs} H_s \left( \hat{X}_s - 1 \right) dG_s - \lambda \int_0^\infty e^{-qs} H_s \hat{X}_s dG_s \right] = 0.$$

Returning now to (65) and using the above equation yields

$$\lambda E_0 \left( \int_0^\infty e^{-qs} H_s \left( \hat{X}_s - 1 \right) dG_s \right) = \lambda E_0 \left\{ \int_0^\infty E_t \left[ \int_t^\infty \hat{X}_s e^{-qs} H_s \left( e^{qs} \lambda H_s \hat{X}_s \right)^{-\frac{1}{\gamma}} ds \right] d\Lambda_t \right\}$$

(73)

$$= E_0 \left[ \int_0^\infty e^{-(p+q)t} \left( e^{qt} \lambda H_t \hat{X}_s \right)^{1-\frac{1}{\gamma}} \Lambda_t dt \right],$$

(74)

where (74) follows from a similar integration by parts argument as the one in equations (66)-(72).

The next Lemma uses Lemma 6 to prove (26).

**Lemma 7** For all admissible processes $G_t \in G$:

$$\max_{G_t \in G} V(W_0) \leq \min_{\lambda \in (0, \xi^{-\gamma})} \left[ E \left( \int_0^\infty e^{-(p+q)s} \left( \lambda e^{qs} H_s \hat{X}_s \right)^{1-\frac{1}{\gamma}} ds \right) - \lambda \int_0^\infty e^{-qs} H_s \left( e^{qs} \lambda e^{qs} H_s \hat{X}_s \right)^{1-\frac{1}{\gamma}} ds + \lambda W_0 \right]$$

(75)

**Proof of Lemma 7.** Proposition 1 along with Lemma 6 implies that for any admissible process
Given $G_t$, there exists a $\lambda^G > 0$ and a decreasing process $X_t^G \in \mathcal{D}$ that minimizes (19) such that

$$V(W_0) = E \left( \int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left( \frac{1}{1-\gamma} - \lambda^G e^{\rho_s} H_s X_s^G c_s \right) ds + \lambda^G \int_0^\infty e^{-qs} H_s (X_s^G - 1) dG_s \right) + \lambda^G W_0$$

$$= E \int_0^\infty e^{-(\rho+q)s} \left( \frac{(e^{\rho_s} \lambda^G H_s X_s^G)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda^G e^{\rho_s} H_s (e^{\rho_s} \lambda^G H_s X_s^G)^{-\frac{1}{\gamma}} \right) ds + \lambda^G W_0.$$  \hspace{1cm} (76)

Moreover, since the process $G_t$ enforces $c_t \geq \xi$, equation (18) implies that $\lambda^G \leq \xi^{-\gamma}$. Next take an arbitrary $\lambda > 0$. Since $c_t = (e^{\delta t} \lambda^G H_t X_t^G)^{-\frac{1}{\gamma}}$ is an optimal consumption process, it exhausts the “budget constraint” of the consumer so that

$$E \int_0^\infty e^{-(\rho+q)s} e^{\rho_s} H_s (e^{\rho_s} \lambda^G H_s X_s^G)^{-\frac{1}{\gamma}} ds = W_0 - D_0 + E \int_0^\infty e^{-qs} H_s dG_s.$$  \hspace{1cm} (77)

Using (10), this implies that $E \int_0^\infty e^{-(\rho+q)s} e^{\rho_s} H_s (e^{\rho_s} \lambda^G H_s X_s^G)^{-\frac{1}{\gamma}} = W_0$. This furthermore implies that (76) can be rewritten as

$$V(W_0) = E \int_0^\infty e^{-(\rho+q)s} \left( \frac{(e^{\rho_s} \lambda^G H_s X_s^G)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda^G e^{\rho_s} H_s (e^{\rho_s} \lambda^G H_s X_s^G)^{-\frac{1}{\gamma}} \right) ds + \lambda W_0.$$  \hspace{1cm} (78)

Next define $X_t^\gamma$ as in equation (30), and let the process $N_t$ be given as $N_t \equiv \frac{\lambda^G X_t^{G^*}}{X_t^\gamma}$. Using $N_t$, one can rewrite equation (77) as

$$V(W_0) = E \int_0^\infty e^{-(\rho+q)s} \left( \frac{(e^{\rho_s} \lambda H_s X_s^\gamma N_s)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho_s} H_s (e^{\rho_s} \lambda H_s X_s^\gamma N_s)^{-\frac{1}{\gamma}} \right) ds + \lambda W_0.$$  \hspace{1cm} (79)

Since $\lambda^G X_t^{G^*}$ is a decreasing process that starts at $\lambda^G$ and always stays below $\xi^{-\gamma}$, the Skorohod equation\(^{30}\) implies that there exists another decreasing process $\lambda^G X_t^G$ that also starts at $\lambda^G$ and stays below $\xi^{-\gamma}$, with the property

$$\lambda^G X_t^G \leq \lambda^G X_t^{G^*}.$$  \hspace{1cm} (80)

This process is given by $X_t^{G^*} = \min \left[ 1, \frac{\xi^{-\gamma}/\lambda^G}{\max_{0 \leq s \leq t}(e^{\rho_s} H_s)} \right]$. Note that $X_t^{G^*}$ is identical to $X_t^*$ with the exception that $\lambda$ replaces $\lambda^G$. Using (79) and the definition of $N_t$ yields

$$N_t = \frac{\lambda^G X_t^G}{\lambda X_t^*} \leq \frac{\lambda^G X_t^{G^*}}{\lambda X_t^*}.$$  \hspace{1cm} (80)

Using (80) and (78) leads to
\[ V(W_0) \leq E \int_0^\infty e^{-(\rho+q)s} A(s)ds + \lambda W_0, \tag{81} \]
where
\[ A(s) \equiv \max_{N_s \leq Q_s} \left( \bar{A}(s) \right), \tag{82} \]
and \( \bar{A}(s) \) is defined as
\[ \bar{A}(s) \equiv \frac{\left(e^{\rho s} \lambda H_s X_s^* N_s\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s \left(e^{\rho s} \lambda H_s X_s^* N_s\right)^{-\frac{1}{\gamma}}, \]
while \( Q_s \equiv \max \left[ 1, \frac{\lambda G X_s^G}{X_s^*} \right] \).

To study the maximization problem of equation (82) it is useful to compute the derivative of \( \bar{A}_s \) with respect to \( N_s \).

Performing this computation and combining terms gives
\[ \frac{\partial \bar{A}_s}{\partial N_s} = \frac{1}{\gamma} \left(e^{\rho s} \lambda H_s X_s^* N_s\right)^{1-\frac{1}{\gamma}} N_s^{-1} \left(1 - \frac{1}{N_s X_s^*}\right). \tag{83} \]

At this stage it is useful to consider two cases separately. The first case is \( \lambda > \lambda^G \). In this case, it is straightforward to show that \( Q_s = 1 \). Hence in maximizing \( \bar{A}(s) \), one can constrain attention to values of \( N_s \leq 1 \). An examination of (83) reveals that \( \frac{\partial \bar{A}_s}{\partial N_s} \geq 0 \) for all \( N_s \leq 1 \) and all \( X_s^* \), since \( X_s^* \leq 1 \). Hence the solution to (82) is \( N_s = 1 \) when \( \lambda > \lambda^G \).

In the case where \( \lambda < \lambda^G \) it is also true that the optimal \( N_s \) in (82) is equal to one. To see this, observe that
\[ Q_s = \begin{cases} \frac{\lambda^G X_s^G}{X_s^*} & \text{when } X_s^* = 1 \\ 1 & \text{when } X_s^* < 1 \end{cases} \]

Using this observation in (83) reveals that the optimal choice for \( N_s \) is always equal to 1.\(^{31}\)

The above reasoning shows that the optimal solution of (82) is given by \( N_s = 1 \). Returning to (81), this implies that
\[ V(W_0) \leq E \int_0^\infty e^{-(\rho+q)s} \left(\frac{\left(e^{\rho s} \lambda H_s X_s^*\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s \left(e^{\rho s} \lambda H_s X_s^*\right)^{-\frac{1}{\gamma}}\right)ds \] \[ + \lambda W_0. \]

Since this bound holds for arbitrary \( \lambda \in (0, \xi^-] \) and arbitrary \( G_t \in G \), it also holds for the \( \lambda \in (0, \xi^-] \) that minimizes the right hand side of the above equation and the \( G_t \in G \) that maximizes the right hand side. Hence (75) follows. \( \blacksquare \)

The next part of the proof of Proposition 2 is to show that equation (31) holds. A first step is to show that (31) provides an upper bound to \( J(W_0) \).

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\(^{31}\) To see this distinguish cases. When \( X_s^* = 1 \), then solving \( \frac{\partial \bar{A}_s}{\partial N_s} = 0 \) gives \( N_s = 1 \leq Q_s \). Hence \( N_s \) is the unique interior solution. When \( X_s^* < 1 \), then \( \frac{\partial \bar{A}_s}{\partial N_s} > 0 \) for all \( N_s \leq Q_s = 1 \). Hence the solution is given by the corner \( N_s = Q_s = 1 \).
Lemma 8. The value function of problem 2 is bounded above by

\[ J(W_0) \leq \min_{\lambda \in (0, \xi - \gamma]} \left[ E \left( \int_0^\infty e^{-(\rho + q)s} \frac{\lambda e^{\rho s} H_s X^*_s}{1 - \gamma} ds - \lambda \int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X^*_s)^{-\gamma} ds + \lambda W_0 \right) \right]. \]  

(84)

Proof of Lemma 8. The proof of this Lemma follows identical steps to the proof of the previous Lemma. To see this, take an arbitrary triplet \( \hat{\lambda}, X_t, c_t \) that satisfies equations (23)-(25) of Problem 2. Then for any \( \lambda > 0 \), one obtains

\[ J(W_0) \leq E \left( \int_0^\infty e^{-(\rho + q)s} \frac{\lambda e^{\rho s} H_s X^*_s}{1 - \gamma} ds - \lambda \int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X^*_s)^{-\gamma} ds + \lambda W_0 \right) \]

Notice that this equation is identical to equation (77), with the exception that \( \lambda \) is replaced by \( \hat{\lambda} \) and \( X^*_t \) is replaced by \( X_t \). Since the equations following (77) hold for any \( \lambda \), \( X^*_t \) they also hold for \( \hat{\lambda}, X_t \). Accordingly, by repeating the same steps, one can arrive at (84).

The next step in the proof of the proposition is to show that the inequality in (84) holds with equality for the optimal policy. The following Lemma presents a step in this direction.

Lemma 9. Let \( F(\lambda) \) be given by

\[ F(\lambda) = E \left( \int_0^\infty e^{-(\rho + q)s} \frac{\lambda e^{\rho s} H_s X^*_s}{1 - \gamma} ds - \lambda \int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X^*_s)^{-\gamma} ds \right) \]

(85)

Then

\[ F(\lambda) = -\frac{K}{\gamma \phi (\phi - 1)} \left( \frac{\lambda}{\xi - \gamma} \right)^\phi + K^{\frac{\gamma}{\phi - 1}} \lambda^{\frac{1}{\gamma}} \]

(86)

Assume moreover that (29) is met. Then

\[ \min_{\lambda \in (0, \xi - \gamma]} [F(\lambda) + \lambda W_0] = \min_{\lambda > 0} [F(\lambda) + \lambda W_0] \]

(87)

and (84) can be rewritten as \( J(W_0) \leq \min_{\lambda > 0} [F(\lambda) + \lambda W_0] \). Moreover, letting \( \lambda^* \) be given as \( \lambda^* \equiv \arg \min_{\lambda > 0} [F(\lambda) + \lambda W_0] \) implies that \( E_0 \left[ \int_0^\infty e^{-qs} H_s (\lambda^* e^{\rho s} H_s X^*_s)^{-\gamma} ds \right] = W_0 \), and accordingly

\[ c^*_s = (\lambda^* e^{\rho s} H_s X^*_s)^{-\gamma} \]

is a feasible consumption process for problem 2.

Proof of Lemma 9. To save notation, let

\[ Z_t \equiv \lambda e^{\rho t} H_t X^*_t, \]

(88)

and note that \( Z_0 = \lambda \), and that \( Z_t \in (0, \xi - \gamma] \) by the definition of \( X^*_t \) in equation (30). Equation
(85) can now be rewritten as
\[ F(\lambda) = E \left[ \int_0^\infty e^{-(\rho+q)s} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds - \int_0^\infty e^{-(\rho+q)s} \frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^\prime} ds \right]. \] (89)

It will be convenient to compute the two terms inside equation (89) separately. Define first
\[ G(Z_t) \equiv E \left[ \int_t^\infty e^{-(\rho+q)(s-t)} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds | Z_t \right]. \] (90)

To compute \( G(Z_t) \), it is easiest to let \( \tau^\varepsilon \) be the first hitting time of \( Z_t \) to the level \( \varepsilon > 0 \), namely \( \tau^\varepsilon = \inf_{s \geq t} \{ Z_s = \varepsilon \} \), and then compute the expression:
\[ G^\varepsilon (Z_t) = E \left[ \int_t^{\tau^\varepsilon} e^{-(\rho+q)s} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds | Z_t \right]. \] (91)

To compute (91), apply first Ito’s Lemma to (88) to obtain \( dZ_t = (\rho - r) dt - \kappa dB_t + \frac{d\gamma}{X_t^2} \). Next, construct a function \( G^\varepsilon (Z) \) that satisfies the ODE
\[ \frac{\kappa^2}{2} G^\varepsilon_{ZZ} Z^2 + G^\varepsilon_Z (\rho - r) - (\rho + q) G^\varepsilon + \frac{1}{1-\gamma} (Z)^{1-\frac{1}{\gamma}} = 0, \] (92)
subject to the boundary conditions \( G^\varepsilon_Z (\xi^{-}\gamma) = 0, G^\varepsilon (\varepsilon) = 0 \).

Equation (92) is a linear ordinary differential equation with general solution
\[ G^\varepsilon (Z) = C_1 Z^{\phi^-} + C_2 Z^\phi + K \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}}, \]
where \( C_1, C_2 \) are arbitrary constants, \( K \) is given in equation (28), \( \phi > 0 \) in (27), and \( \phi^- \) is given by
\[ \phi^- = - \left( \rho - r - \frac{\kappa^2}{2} \right) - \sqrt{\left( \rho - r - \frac{\kappa^2}{2} \right)^2 + 2(\rho + q) \kappa^2} \frac{\kappa^2}{\kappa^2} < 0 \] (93)

To satisfy the two boundary conditions \( G^\varepsilon_Z (\xi^{-}\gamma) = 0, G^\varepsilon (\varepsilon) = 0 \), the constants \( C_1 \) and \( C_2 \) must be chosen so that
\[ \phi^- C_1 (\xi^{-}\gamma)^{\phi^-} + \phi C_2 (\xi^{-}\gamma)^{\phi} - \frac{1}{\gamma} K (\xi^{-}\gamma)^{1-\frac{1}{\gamma}} = 0, \]
\[ C_1 e^{\phi^-} + C_2 e^\phi + K \frac{1}{1-\gamma} e^{1-\frac{1}{\gamma}} = 0. \]

Solving this system yields:
\[ C_2 = \frac{K \left[ 1 \phi^- (\xi^{-}\gamma)^{1-\frac{1}{\gamma}} - e^{\phi^-} + 1 \frac{1}{1-\gamma} e^{1-\frac{1}{\gamma}} \right]}{\phi^- (\xi^{-}\gamma)^{\phi^-} - e^{\phi^-}}, \]
\[ C_1 = -C_2 e^{\phi^-} - K \frac{1}{1-\gamma} e^{1-\frac{1}{\gamma}} - \phi^- . \]
It remains now to verify that \( G^\varepsilon (Z_t) \) satisfies (91). To this end, apply Ito’s Lemma to 
\[ e^{-(\rho+q)t}G^\varepsilon (Z_t) \]
to obtain for any time \( T \land \tau^\varepsilon \)
\[
e^{-(\rho+q)T}G^\varepsilon (Z_{T\land\tau}) - e^{-(\rho+q)t}G^\varepsilon (Z_t) = \int_t^{T\land\tau^\varepsilon} \left( \frac{\kappa^2}{2} G^\varepsilon_{ZZ} Z_s^2 + G^\varepsilon_Z Z_s (\rho - r) - (\rho + q) G^\varepsilon \right) e^{-(\rho+q)s} ds \\
- \int_t^{T\land\tau^\varepsilon} e^{-(\rho+q)s} \kappa G^\varepsilon_Z Z_s dB_s + \int_t^{T\land\tau^\varepsilon} e^{-(\rho+q)s} G^\varepsilon_{ZZ} (\xi^{-\gamma}) \xi^{-\gamma} dX_s^* / X_s^*.
\]

Using (92) inside the first term on the right hand side of the above equation along with 
\( G^\varepsilon_Z (\xi^{-\gamma}) = 0 \) inside the third term, letting \( T \to \infty \) along with \( G^\varepsilon (\varepsilon) = 0 \), and using the monotone convergence theorem gives
\[
G^\varepsilon (Z_t) = E_t \left[ \int_t^{\tau^\varepsilon} e^{-(\rho+q)(s-t)} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds + \int_t^{\tau^\varepsilon} e^{-(\rho+q)(s-t)} \kappa G^\varepsilon_Z Z_s dB_s \right]. 
\](94)

Since \( G^\varepsilon_Z Z \) is bounded between \( t \) and \( \tau^\varepsilon \), the second term in the above expression is a martingale and hence (113) follows. Next, letting \( \varepsilon \to 0 \), it is straightforward to show that
\[
C_2 = K \left[ \frac{1}{\gamma \phi^\varepsilon} (\xi^{-\gamma})^{1-\frac{1}{\gamma}} - \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}} \phi^{-\varepsilon} \right] \\
\to K \frac{1}{\gamma \phi} (\xi^{-\gamma})^{1-\frac{1}{\gamma}} - \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}},
\]
since \( \varepsilon^{\phi-\phi^{-}} \to 0 \) and \( \varepsilon^{1-\frac{1}{\gamma}-\phi^{-}} \to 0 \). By a similar argument it is easy to show that \( C_1 \to 0 \) and hence:
\[
\lim_{\varepsilon \to 0} G^\varepsilon (Z) = G(Z) = \frac{1}{\phi} K \xi^{1-\gamma} \left( \frac{Z}{\xi^{-\gamma}} \right)^{\phi} + K \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}}.
\](95)

Equation (90) follows as a consequence of the monotone convergence theorem.

It remains to compute the expression
\[
N (Z_t, X_t^*) = E_t \left[ \int_t^{\infty} e^{-(\rho+q)(s-t)} Z_s^{1-\frac{1}{\gamma}} X_s^* ds \right]. 
\](96)

Following similar steps as for \( G(Z_t) \), \( N(Z, X^*) \) is given by
\[
N(Z, X^*) = \frac{1}{(\phi - 1) \gamma} K \xi^{1-\gamma} \left( \frac{Z}{\xi^{-\gamma}} \right)^{\phi} + K \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}} / X^*.
\](97)

It is now possible to compute \( F(\lambda) \) which is given by
\[
F(\lambda) = G(\lambda) - N(\lambda, 1) = - \frac{K \xi^{1-\gamma}}{\gamma \phi (\phi - 1)} \left( \frac{\lambda}{\xi^{-\gamma}} \right)^{\phi} + K \frac{\gamma}{1-\gamma} \lambda^{1-\frac{1}{\gamma}}.
\](98)
To show the second part of the proposition, observe that (96), (88) and (97) imply that
\[
\frac{N(\lambda, 1)}{\lambda} = \frac{1}{\lambda} E_0 \left( \int_0^\infty e^{-(\rho+q)s} Z_s \frac{1}{X_s} ds \right) = \int_0^\infty e^{-qs} H_s (\lambda e^{qs} X_s)^{-\frac{1}{\gamma}} ds =
\]
\[
\frac{K \xi^{1-\gamma}}{(\phi - 1) \gamma} \frac{1}{\lambda} \left( \frac{\lambda}{\xi - \gamma} \right)^{\phi - 1} \gamma + K \lambda^{-\frac{1}{\gamma}}.
\]
(99)

Moreover, computing \( F'(\lambda) \) in (98) yields
\[
F'(\lambda) = \frac{-K \xi^{1-\gamma}}{(\phi - 1) \gamma} \frac{1}{\lambda} \left( \frac{\lambda}{\xi - \gamma} \right)^{\phi - 1} \gamma - K \lambda^{-\frac{1}{\gamma}}.
\]
(100)

Combining (99) and (100) leads to
\[
F'(\lambda) = -\frac{N(\lambda, 1)}{\lambda} = -E_0 \left( \int_0^\infty e^{-qs} H_s (\lambda e^{qs} X_s)^{-\frac{1}{\gamma}} ds \right).
\]
(101)

Using the formula for \( F(\lambda) \), equation (84) can be expressed as \( \min_{\lambda \in [0, \xi - \gamma]} \{ F(\lambda) + \lambda W_0 \} \), which leads to the first order condition \( F'(\lambda^*) = -W_0 \). Using (101) leads to
\[
W_0 = E_0 \left( \int_0^\infty e^{-qs} H_s (\lambda^* e^{qs} X_s)^{-\frac{1}{\gamma}} ds \right) = E_0 \left( \int_0^\infty e^{-qs} H_s c_s^* ds \right).
\]

This last equation implies that \( \lambda^*, X_s^* \) and the associated consumption process \( c_t^* = (\lambda^* e^{qt} H_t X_t^*)^{-\frac{1}{\gamma}} \) satisfy (23) and (25). To show that the choice \( (\lambda^*, X_t^*, c_t^*) \) constitutes a feasible triplet, it remains to show that it also satisfies (24). By construction of \( X_t^* \) this will be the case as long as \( \lambda^* < \xi - \gamma \). This will indeed be the case as long as \( W_0 \) satisfies (29). To see this, note that \( \xi - \gamma \) is the unique solution of \( F'(\lambda^*) = -W_0 \), when \( W_0 \) is given by \( W_0 = \frac{1}{\phi - 1} K \xi \). Moreover, equation (100) implies that:
\[
F''(\lambda) = -K (\xi - \gamma)^{1-\frac{1}{\gamma}} \left( \frac{1}{\xi - \gamma} \right)^{\phi - 1} \lambda^{\phi - 2} + \frac{1}{\gamma} K \lambda^{-\frac{1}{\gamma} - 1}
\]
\[
= \frac{1}{\gamma} K \lambda^{-\frac{1}{\gamma} - 1} \left[ 1 - \left( \frac{\lambda}{\xi - \gamma} \right)^{\phi - \frac{1}{\gamma} - 1} \right] > 0.
\]
(102)

The above equation shows that \( F'(\lambda) \) is an increasing function of \( \lambda \) for \( 0 < \lambda < \xi - \gamma \) and hence the solution \( \lambda^* \) of equation \( F'(\lambda^*) = -W_0 \) is a decreasing function of \( W_0 \). Hence, as long as \( W_0 \) satisfies (29), then \( \lambda^* < \xi - \gamma \). Since the interior solution \( \lambda^* \) is smaller than \( \xi - \gamma \), equation (87) follows.
Combining the above Lemma with (84) implies that
\[ J(W_0) \leq \min_{\lambda > 0} \left[ F(\lambda) + \lambda W_0 \right] = F(\lambda^*) + \lambda^* W_0 = \]
\[ = E \left( \int_0^\infty e^{-(\rho+q)s} \left( (\lambda^* e^{\rho s} H_s X_s^*)^{1-\gamma} \right)^{1-\gamma} ds \right) \]
\[ = E \left( \int_0^\infty e^{-(\rho+q)s} \left( \frac{c_s^{1-\gamma}}{1-\gamma} ds \right) \right) \leq J(W_0). \]

The last inequality follows because \( c_s^* = (\lambda^* e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} \) is a feasible consumption process for problem for problem 2 and \( J(W_0) \) is the value function of the problem. The above three lines imply that equation (84) holds with equality as long as one chooses the optimal solution in the statement of the proposition. This concludes the proof of Proposition 2.

**Proof of Proposition 3.** The proof of this Proposition is just a special case of Section 6 in He and Pages (1993) and hence I give only a sketch and refer the reader to He and Pages (1993) for details. To start, define
\[ \tilde{V}(\lambda) = \min_{X_s \in D} E \left[ \int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left( \frac{c_s^{1-\gamma}}{1-\gamma} - \lambda c_s e^{\rho s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-q s} H_s X_s y_0 ds \right]. \] (103)

By equation (10) and equation (19) of Proposition 1
\[ V(W_0) = \min_{\lambda > 0} \left[ \tilde{V}(\lambda) + \lambda \left( W_0 - \frac{y_0}{r + q} \right) \right], \] (104)
since \( y_0 E \int_0^\infty H_s ds = \frac{y_0}{r + q} \). Next, for an arbitrary decreasing process \( X_t \) let \( Z_t \) be defined as \( Z_t \equiv \lambda e^{\rho s} H_s X_s \), and note that \( Z_0 = \lambda \). Applying Ito’s Lemma to \( Z_t \) gives:
\[ \frac{dZ_t}{Z_t} = (\rho - r) dt - \kappa dB_t + \frac{dX_t}{X_t}. \] (105)

With this definition of \( Z_t \) one can solve the maximization problem inside (103) and rewrite \( \tilde{V}(\lambda) \) as
\[ \tilde{V}(Z_0) = \min_{X_s \in D} E \left[ \int_0^\infty e^{-(\rho+q)s} \left( \frac{\gamma}{1-\gamma} Z_s^{1-\gamma} + y_0 Z_s \right) ds \right]. \] (106)

From this point on, one can use similar arguments to He and Pages (1993), and treat (106) as a singular stochastic control problem over the set of decreasing processes \( X_t \). As He and Pages (1993) show, the optimal solution is to always decrease \( X_t \) appropriately, so as to keep \( Z_t \) in the interval \((0, Z]\). \( Z \) is a free boundary that is determined next.

Using this conjecture for the optimal policy one can now proceed as He and Pages (1993) to
establish that \( \tilde{V}(Z) \) satisfies the ordinary differential equation:

\[
\frac{\kappa^2}{2} \tilde{V}_{ZZ}Z^2 + (\rho - r) \tilde{V}_Z Z - (\rho + q) \tilde{V} + \frac{\gamma}{1 - \gamma} Z^{1-\frac{1}{\gamma}} + y_0 Z = 0 \text{ for all } Z \in (0, \bar{Z}).
\]

The general solution to this equation is

\[
\tilde{V}(Z) = C_1 Z^\phi + C_2 Z^{\phi^-} + K \frac{\gamma}{1 - \gamma} Z^{1-\frac{1}{\gamma}} + \frac{y_0}{r + q} Z, \tag{107}
\]

where \( K \) is given in (28), \( \phi \) in (27) and \( \phi^- \) in (93) and \( C_1, C_2 \) are arbitrary constants. By arguments similar to He and Pages (1993), one can set \( C_2 = 0 \) (since \( \phi^- < 0 \)). Hence it remains to determine \( C_1 \) and the free boundary \( \bar{Z} \). As most singular stochastic control problems, one can employ a “smooth pasting” and “high contact” principle, namely by determining \( C_1 \) and \( \bar{Z} \) so that \( \tilde{V}_Z(\bar{Z}) = 0, \tilde{V}_{ZZ}(\bar{Z}) = 0 \). Using the “smooth pasting” and “high contact” conditions, along with the general solution in (107) and \( C_2 = 0 \), one can solve for \( C_1 \) and \( \bar{Z} \) to obtain

\[
\bar{Z}^{\frac{1}{\gamma}} = \frac{1}{K} \frac{y_0}{r + q} \left( \frac{\phi - 1}{\frac{1}{\gamma} + \phi - 1} \right), \tag{108}
\]

\[
C_1 = -\frac{\frac{1}{r + q}}{\phi \bar{Z}^{\phi^-} \left[ \frac{1}{\gamma} + \phi - 1 \right]}, \tag{109}
\]

The next steps to verify that the conjectured policy is indeed optimal are identical to He and Pages (1993) and are left out.

To conclude the proof, note that so far the calculations were true for an arbitrary \( y_0 \). To determine the \( y_0 \) that will safeguard that \( c_t \geq \xi \) observe that \( c_t = Z^{-\frac{1}{\gamma}} \) by equation (18). Since the optimal policy is to control \( X_t \) so as to “keep” \( Z_t \) in the interval \( (0, \bar{Z}) \) it follows that the minimum level of consumption is given by \( \bar{Z}^{-\frac{1}{\gamma}} \). Hence, in order to guarantee condition \( c_t \geq \xi \) it suffices to determine \( y_0 \) so that

\[
\xi = \bar{Z}^{-\frac{1}{\gamma}} = \frac{1}{K} \frac{y_0}{r + q} \left( \frac{\phi - 1}{\frac{1}{\gamma} + \phi - 1} \right).
\]

Solving for \( y_0 \) gives

\[
y_0 = \xi (r + q) K \frac{1}{\phi - 1} \frac{1 + \phi - 1}{\phi - 1}.
\]

One can now substitute that level of \( y_0 \) into (109), (108) and use the resulting expressions to obtain from (107) the following expression for \( \tilde{V}(Z) \):

\[
\tilde{V}(Z) = -\frac{K \xi^{1-\gamma}}{\gamma \phi (\phi - 1)} \left( \frac{Z}{\xi^{\gamma}} \right) + K \frac{\gamma}{1 - \gamma} Z^{1-\frac{1}{\gamma}} + \frac{y_0}{r + q} Z.
\]
Evaluating this expression at $Z_0 = \lambda$ and using equation (104) gives equation (32), which shows that the “constant income” policy of the current proposition attains the upper bound of Proposition 2.

**Proof of Lemma 2.** First note that $\lim_{\gamma \to \infty} \left( \frac{\gamma}{\xi} \right) = 1$. To show the result, it suffices to show that $\frac{d\left( \frac{\gamma}{\xi} \right)}{d\gamma} < 0$. Differentiating $\frac{\gamma}{\xi}$ with respect to $\gamma$ gives

$$
\frac{d\left( \frac{\gamma}{\xi} \right)}{d\gamma} = \frac{(r + q)}{\left( \phi - 1 \right) \left( \rho + q + (\phi - 1) \frac{\kappa^2}{2} \right)} \frac{B}{\left( \gamma - 1 \right) \frac{\kappa^2}{2} + \gamma (r + q) + \rho - r}^2.
$$

where

$$
B \equiv (\phi - 1) \left( \rho - r \right) - (r + q) + (\phi - 1) \frac{\kappa^2}{2} - (\phi - 1) \frac{1}{\gamma} \frac{\kappa^2}{2} - \left[ \gamma (\phi - 1) + 1 \right] \frac{1}{\gamma} \frac{\kappa^2}{2}.
$$

Since $\phi > 1$ and $r + q > 0$, it follows that $\frac{d\left( \frac{\gamma}{\xi} \right)}{d\gamma} < 0$, as long as $(\phi - 1) (\rho - r) - (r + q) + (\phi - 1) \frac{\kappa^2}{2} < 0$. Since $\phi$ solves the quadratic equation $\frac{\kappa^2}{2} \phi^2 + \left( \rho - r - \frac{\kappa^2}{2} \right) \phi - (\rho + q) = 0$, it follows that $(\phi - 1) (\rho - r) - (r + q) + (\phi - 1) \frac{\kappa^2}{2} = - (\phi - 1) \frac{\kappa^2}{2} < 0$. ■

**Proof of Proposition 4.** The proof of this proposition proceeds in steps. The first two Lemmas establish that the proposed transfer policy will make it possible for an agent who follows the optimal consumption process of proposition 4 to satisfy the intertemporal budget constraint. The proof then continues to show that the wealth process associated with the optimal consumption process of proposition 4, along with the portfolio process (36), will lead to non-negative levels of wealth at all times. Finally, it is shown that the consumption policy of proposition 4, along with the portfolio choice (36), are optimal for an agent who is faced with transfers given by (34) and attain the upper bound of proposition 2.

**Lemma 10** Let $K$ and $\phi$ be given by (28) and (27) and for any $0 < \lambda < \xi^{-\gamma}$ let $Z_t = \lambda e^{\alpha_s} H_s X_s^\ast$. Then

$$
\int_0^\infty E_t \left( \int_t^\infty e^{-q(s-t)} H_s X_s^\ast dG_s - \int_t^\infty e^{-q(s-t)} H_s X_s^\ast Z_s^{-\frac{1}{2}} ds \right) dX_t^\ast = 0. 
$$

(110)

**Proof of Lemma 10.** It will simplify notation to let

$$
\eta \equiv -K\xi \left( \phi - 1 + \frac{1}{\gamma} \right).
$$

The first step is to compute

$$
\frac{E_t \int_t^\infty e^{-q s} H_s X_s^\ast dG_s}{e^{-q t} H_t X_t^\ast} = \eta \frac{E_t \int_t^\infty e^{-q s} H_s dX_s^\ast}{e^{-q t} H_t X_t^\ast}.
$$

(112)
Applying integration by parts and using the definition of $Z_t$ gives

$$E_t \left( \int_t^\infty e^{-qs} H_s dX_s \right) = \frac{1}{\lambda} \left[ -e^{-(\rho+q)t} Z_t + E_t \left( \int_t^\infty (r + q) e^{-(\rho+q)s} Z_s ds \right) \right].$$

(113)

Using (113) in equation (112) gives

$$E_t \left( \int_t^\infty e^{-qs} H_s X_s dG_s \right) = \eta \left[ (r + q) \frac{E_t \left( \int_t^\infty e^{-(\rho+q)(s-t)} Z_s ds \right)}{Z_t} - 1 \right].$$

(114)

By using a logic similar to equations (92)-(94),

$$E_t \left( \int_t^\infty e^{-(\rho+q)(s-t)} Z_s ds \right) = -\frac{1}{\phi} \frac{\xi^{-\gamma}}{r + q \left( \frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1}} + \frac{1}{r + q} Z_t,$$

(115)

where $\phi$ is defined in equation (27). Plugging back (115) into (114) gives

$$E_t \left( \int_t^\infty e^{-qs} H_s X_s dG_s \right) = -\eta \left( \frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1}.$$

(116)

To conclude the proof, note that equations (90) and (95) imply that

$$\frac{E_t \left( \int_t^\infty e^{-qs} H_s X_s Z_s^{-\frac{1}{\gamma}} ds \right)}{e^{-qt} H_t X_t^*} = \frac{E_t \left( \int_t^\infty e^{-(\rho+q)(s-t)} Z_s^{1-\frac{1}{\gamma}} ds \right)}{Z_t} = \frac{1}{\phi} \frac{1 - \gamma}{\gamma} K \xi^{1-\gamma} \left( \frac{Z_t}{\xi} \right)^{\phi} + K Z_t^{1-\frac{1}{\gamma}}.$$

(117)

Combining (117) with (116) gives:

$$E_t \left( \int_t^\infty e^{-qs} H_s X_s dG_s - \int_t^\infty e^{-qs} H_s X_s Z_s^{-\frac{1}{\gamma}} ds \right) = -\eta \left( \frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} - \frac{1}{\phi} \frac{1 - \gamma}{\gamma} K \xi^{1-\gamma} \left( \frac{Z_t}{\xi} \right)^{\phi} + K Z_t^{1-\frac{1}{\gamma}}.$$

Since $dX_t^* \neq 0$ when and only when $Z_t = \xi^{-\gamma}$, equation (110) amounts to checking that:

$$-\frac{\eta}{\phi} - \left( \frac{1}{\phi} \frac{1 - \gamma}{\gamma} + 1 \right) K \xi = 0$$

which follows easily from the definition of $\eta$. 

**Lemma 11** Let $Z_s$ be as in the statement of the proposition 4 and let $G_t$ be as in (34). Then the consumption policy:

$$c_s^* = (Z_s)^{-\frac{1}{\gamma}}$$

(118)

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satisfies:

\[ E \int_0^\infty e^{-qs} H_s X_s^* c_s^* ds = W_0 + \int_0^\infty e^{-qs} H_s (X_s^* - 1) dG_s \]  

(119)

**Proof of Lemma 11.** Taking any \( \lambda \in (0, \xi^{-\gamma}] \), using the definition of \( X_t^* \), and equation (110), the same reasoning behind (65) leads to

\[
E \left( \int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left( \frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s^* c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s (X_s^* - 1) dG_s \right) + \lambda W_0 = (120)
\]

\[ = E \left[ \int_0^\infty e^{-(\rho+q)s} \frac{\gamma}{1-\gamma} \left( e^{\rho s} \lambda H_s X_s^* \right)^{\frac{\gamma-1}{\gamma}} ds + \int_0^\infty e^{-(\rho+q)s} \left( e^{\rho s} \lambda H_s X_s^* \right)^{1-\frac{1}{\gamma}} \left( 1 - \frac{1}{X_s^*} \right) ds \right] + \lambda W_0 \]  

(121)

Hence the \( \lambda^* \) that minimizes (32) (and hence minimizes [121]) also minimizes (120). But since \( \lambda \) minimizes (120), the same argument as in He and Pages (1993) (Proof of Theorem 1) leads to (119).

Lemma 11 has asserted that the consumption policy (118) satisfies the intertemporal budget constraint (119). It remains to show that this consumption policy along with the portfolio policy (36) will lead to a process for financial wealth that satisfies \( W_t \geq 0 \). To that end let \( \eta \) be given as in (111) and define:

\[
W^* (Z_t) = -K \left( \xi^{-\gamma} \right)^{-\frac{1}{\gamma}} \left( \frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} + KZ_t^{-\frac{1}{\gamma}} \]  

(122)

It is straightforward to verify the following facts about \( W^* (Z_t) \):

\[
\frac{\kappa^2}{2} Z^2 W_{ZZ}^2 + (\rho - r + \kappa^2) Z W_Z^2 - (r + q) W + (Z)^{-\frac{1}{\gamma}} = 0
\]  

(123)

\[
W^* (\xi^{-\gamma}) = 0, W^* (Z) \geq 0 \text{ for all } Z \in (0, \xi^{-\gamma}]
\]  

(124)

\[
W^*_{Z} (\xi^{-\gamma}) = -K \xi \left( \phi - 1 + \frac{1}{\gamma} \right) (\xi^{-\gamma})^{-1} = \frac{\eta}{\xi^{-\gamma}}
\]  

(125)

The next step is to verify that \( W^* (Z_t) \) is the stochastic process for the financial wealth of the agent. To see this, use the definition of \( c_s^* \) (equation [118]) along with the definitions of \( dG_t, W_t^* \) (equations [34] and [122] respectively) and apply Ito’s Lemma to obtain:

\[
d \left( \int_0^t c_s^* ds - \int_0^t dG_s + W_t^* \right) =
\]
the agent equal to \( Q \) lower bound to the value function \( V \) the consumption policy (118) is also feasible, the payoff associated with that policy also provides a way to show that \( V \) faced with the transfer process (34). Accordingly, the policies given by (118) and (36) are feasible for an agent who is that is associated with that policy pair. Moreover, by equation (124) the financial wealth process given by (118) and a portfolio policy given by (36). Accordingly, it is the financial wealth process Hence the process \( W \) Integrating gives

\[
\int_0^t c_s^* ds + W_t^* = W_0 - D_0 + \int_0^t dG_s + \int_0^t qW_s^* dt + \int_0^t r (W_s^* - \pi_s^*) dt + \int_0^t \pi_s^* \frac{dP_s}{P_t}.
\]

Hence the process \( W_t^* \) satisfies the equation (12) for an agent who chooses a consumption policy given by (118) and a portfolio policy given by (36). Accordingly, it is the financial wealth process that is associated with that policy pair. Moreover, by equation (124) the financial wealth process is non-negative. Accordingly, the policies given by (118) and (36) are feasible for an agent who is faced with the transfer process (34).

Verifying the optimality of the stated policy pair is simple. According to proposition 1

\[
V(W_0) = \min_{\lambda > 0, \ X_s \in D} \left[ E \left( \int_0^\infty e^{-q(s)} \max c_s \left( \frac{1}{\xi - \gamma} - \lambda e^{q_s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-q_s} H_s X_s dG_s \right) \right] 
\]

where

\[
Q(W_0) \equiv \min_{\lambda > 0} \left[ E \left( \int_0^\infty e^{-q(s)} \max c_s \left( \frac{1}{\xi - \gamma} - \lambda e^{q_s} H_s X_s^* c_s \right) ds + \lambda \int_0^\infty e^{-q_s} H_s X_s^* dG_s \right) \right].
\]

One can use now Lemma 11 to illustrate that the consumption policy (118) leads to a payoff for the agent equal to \( Q(W_0) \) which is an upper bound to the value function of the agent \( V(W_0) \). Since the consumption policy (118) is also feasible, the payoff associated with that policy also provides a lower bound to the value function \( V(W_0) \). Hence this policy must be optimal. Finally, the easiest way to show that

\[
D_0 = K \xi^{\frac{1}{\gamma}} + \phi - 1 \left( \frac{\lambda^*}{\xi - \gamma} \right)^{\phi-1},
\]

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is to observe that the intertemporal budget constraint implies that

\[
E_{t_0} \left( \int_{t_0}^{\infty} e^{-q(s-t_0)} \frac{H_s}{H_{t_0}} c_s^* ds \right) = E_{t_0} \left( \int_{t_0}^{\infty} e^{-q(s-t_0)} \frac{H_s}{H_{t_0}} dG_s \right),
\]

where \( t_0 \) is the first time that \( X_{t_0} \geq 1 \) (or equivalently the first time that \( W_{t_0} = 0 \) and \( \lambda^* e^{\rho t_0} H_{t_0} = \xi^{-\gamma} \)). A few manipulations can be used to show that

\[
E_{t_0} \left( \int_{t_0}^{\infty} e^{-q(s-t_0)} \frac{H_s}{H_{t_0}} c_s^* ds \right) = \frac{N(\xi^{-\gamma}, 1)}{\xi^{-\gamma}} = K \xi \frac{1 + \phi - 1}{\phi - 1}
\]

where \( N \) is defined and computed in (97) and (96). Finally, since there are no transfers between 0 and \( t_0 \):

\[
D_0 = E \left( e^{-\gamma t_0} H_{t_0} \right) K \xi \frac{1 + \phi - 1}{\phi - 1} = \frac{1}{\lambda^*} E \left( e^{-q+q} t_0 \lambda^* e^{\rho t_0} H_{t_0} \right) K \xi \frac{1 + \phi - 1}{\phi - 1} =
\]

\[
= \frac{\xi^{-\gamma}}{\lambda^*} E \left( e^{-q+q} t_0 \right) K \xi \frac{1 + \phi - 1}{\phi - 1} = \left( \frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi - 1} K \xi \frac{1 + \phi - 1}{\phi - 1}
\]

where the proof of \( E \left( e^{-q+q} t_0 \right) = \left( \frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi} \) is identical to the one given in Oksendal (1998), Chapter 10.

**Proof of Proposition 5.** Take any transfer process \( G_t \) such that the resulting consumption process of the agent satisfies \( c_t \geq \xi \). Proposition 1 implies then that there exists a cumulative multiplier process \( X_t^G \) and a constant \( \lambda^G \) such that \( c_t = (\lambda^G e^{\rho t} H_t X_t^G)^{-\frac{1}{\gamma}} \geq \xi \). Letting \( X_t^* \equiv \min \left[ 1, \frac{\xi^{-\gamma}/\lambda^G}{\max_{0 \leq s \leq t} \left( e^{\rho s} H_s \right)} \right] \), and \( P = E \left( \int_0^{\infty} e^{-qs} H_s c_s ds \right) \) gives

\[
P = E \left( \int_0^{\infty} e^{-qs} H_s \left( \lambda^G e^{\rho s} H_s X_s^G \right)^{-\frac{1}{\gamma}} ds \right) \geq E \left( \int_0^{\infty} e^{-qs} H_s \left( \lambda^G e^{\rho s} H_s X_s^* \right)^{-\frac{1}{\gamma}} ds \right)
\]

since\(^{32}\) \( X_s^* (\lambda^G) \geq X_s^G \). Equation (99) implies that

\[
E \left( \int_0^{\infty} e^{-qs} H_s \left( \lambda^G e^{\rho s} H_s X_s^* \right)^{-\frac{1}{\gamma}} ds \right) = K \xi \frac{1 - \gamma}{(\phi - 1)} \left( \frac{\lambda^G}{\xi^{-\gamma}} \right)^{\phi - 1} + K \left( \lambda^G \right)^{\frac{1}{\gamma}}.
\]

Combining (101) and (102) implies that the right hand side of the above equation is decreasing in \( \lambda^G \) whenever \( \lambda^G \leq \xi^{-\gamma} \). Since \( c_0 = (\lambda^G)^{-\frac{1}{\gamma}} \geq \xi \) this implies furthermore

\[
E \left( \int_0^{\infty} e^{-qs} H_s \left( \lambda^G e^{\rho s} H_s X_s^* \right)^{-\frac{1}{\gamma}} ds \right) \geq K \xi \frac{1 - \gamma}{(\phi - 1)} \left( \frac{\lambda^G}{\xi^{-\gamma}} \right)^{\phi - 1} + K \xi = K \xi \left( 1 + \frac{1}{\phi - 1} \right)
\]

\[
= K \xi \left( \frac{\phi - 1}{\phi - 1} \right).
\]

\(^{32}\)This is an implication of the Skorohod equation. See Karatzas and Shreve (1991).
Combining (126) and (127) concludes the proof. ■

**Proof of Lemma 3.** Take any feasible choice of $S_t, L_0$ that satisfies (43), (44), and (46) and fix the associated processes for $c_t, W_0$. Then that combination of $c_t, W_0$ is a feasible choice for the consumer who solves problem 4.

To see this, note first that $W_0 \geq W^{\min}$ by (43).

Furthermore, the dynamic completeness of markets implies that any combination of $c_t, W_0$ is feasible for problem 4 as long as it satisfies the requirements $^{33}$

$$E_t \int_t^b e^{-q(t-t^b)} \left( \frac{H_t}{H_{t^b}} \right) c_t dt + E_t e^{-q \tau} \left( \frac{H_0}{H_{t^b}} \right) W_0 \leq E_t \int_t^b e^{-q(t-t^b)} \left( \frac{H_t}{H_{t^b}} \right) Y dt, \quad (128)$$

and

$$W_t = E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) (c_u - Y) du + E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) W_0 \geq 0 \text{ for all } t \in [t^b, 0]. \quad (129)$$

To show that the combination of $c_t, W_0$ that is associated with problem 3 satisfies (128), use (51) and Ito’s Lemma to compute $d \left( e^{-q t} H_t W_t \right)$, integrate and use the fact that $W_{t^b} = 0, W_t \geq 0$ to obtain

$$E_t \int_t^b e^{-q(t-t^b)} \left( \frac{H_t}{H_{t^b}} \right) c_t dt + E_t e^{-q \tau} \left( \frac{H_0}{H_{t^b}} \right) W_0 = \quad (130)$$

$$E_t \int_t^0 e^{-q(t-t^b)} \left( \frac{H_t}{H_{t^b}} \right) Y dt - E_t \int_t^0 e^{-q(t-t^b)} \left( \frac{H_t}{H_{t^b}} \right) dS_t.$$

Using $W_0 = W_0^- + L_0$ and (44) inside (130) implies (128).

Finally, it remains to show (129). Since the processes $c_t, W_t$ associated with $S_t, L_0$ satisfy $W_t \geq 0$, it follows that

$$E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) (c_u - Y) du + E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) dS_u + E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) W_0^- \geq 0. \quad (131)$$

Adding $E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) L_0$ to both sides of the inequality (131) and re-arranging gives

$$E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) (c_u - Y) du + E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) W_0$$

$$\geq E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) L_0 - E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) dS_u \geq 0,$$

where the last inequality follows from (46). Since any attainable combination of $c_t, W_0$ for problem 3 is feasible for 4, this implies that the value function in problem 4 must be at least as high as the respective value function of problem 3. ■

**Proof of Lemma 4.** If $c_t^*, W_t^*$ are optimal consumption and wealth processes that solve

\[^{33}\text{For a proof, see e.g. Karatzas and Shreve (1998), Chapter 3 or He and Pages (1993).}\]
Adding and subtracting \( e^{-(r+q)(0-t)} W_{\text{min}} \) on the right-hand side of (132) implies that for any \( t > \chi \)

\[
W_t^* = E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) (c_u^* - Y) \, du + E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) W_0^* 
+ e^{-(r+q)(0-t)} W_{\text{min}} - \frac{1 - e^{-(r+q)(0-t)}}{r + q} Y \tag{134}
\]

The first two terms on the right-hand side of equation (134) are non-negative (since \( W_0^* - W_{\text{min}} \geq 0 \)), while the sum of the last two terms on the right-hand side of equation (134) is positive\(^{34} \).

**Proof of Proposition 6.** The fact that the proposed policy satisfies (43) follows from \( W_0^* \geq 0 \). The requirement (44) follows by the construction of \( \chi \), while the requirement (46) follows from the fact that \( dS_t \) is deterministic (see remark in the text).

To conclude the proof, it suffices to show that \( V_{t^b} \geq J_{t^b} \). To that end, let \( c_t^*, W_0^* \) denote the optimal consumption and wealth processes that solve problem 4. Consider an agent faced with the policy pair \(<dS_t, L_0>\) in the statement of the proposition. To show \( V_{t^b} \geq J_{t^b} \), it suffices to show that \( c_t^*, W_0^* \) remain feasible choices for this agent.

To that end, note that the wealth process \( W_t \) for an agent who chooses \( c_t = c_t^* \) and \( W_0 = W_0^* - L_0 \) in the presence of the policy pair \(<dS_t, L_0>\) given in the proposition, is given by

\[
W_t = E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) (c_u^* - Y \times 1_{\{t < \chi\}}) \, du + E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) W_0^*,
\]

where \( 1_{\{t < \chi\}} \) is an indicator function that takes the value 1 if \( t < \chi \) and zero otherwise. Clearly,

\(^{34}\)Note that

\[
e^{-(r+q)(0-t)} W_{\text{min}} - \frac{1 - e^{-(r+q)(0-t)}}{r + q} Y = \]

\[
e^{(r+q)(t-\chi)} \left( e^{-(r+q)(0-\chi)} W_{\text{min}} - \frac{e^{(r+q)(\chi-t)} - e^{-(r+q)(0-\chi)}}{r + q} Y \right) > 0,
\]

where the last inequality follows by equation (133) and \( t > \chi \).
$W_0 \geq 0$ for all $t \geq \chi$, since $c^*_t \geq 0$ and $W_{0-} \geq 0$. For $t < \chi$, observe that

$$W_t = E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) c^*_u du - E_t \int_t^\chi e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) Y du + E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) W_{0-}$$

$$= E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) (c^*_u - Y) du + E_t \int_\chi^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) Y du + E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) W_{0-}$$

$$= E_t \int_t^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) (c^*_u - Y) du + E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) (W_{0-} + W^{\min}) \geq 0,$$

where the last line in (135) follows from $E_t \int_\chi^0 e^{-q(u-t)} \left( \frac{H_u}{H_t} \right) Y du = E_t e^{-q(0-t)} \left( \frac{H_0}{H_t} \right) W^{\min}$, the definition $W_{0-} + W^{\min} = W_0$, and the fact that the wealth process in problem 4 is non-negative. Clearly, $W_t = 0$ at $t = t^b$, so that the pair $c_t = c^*_t$ and $W_{0-} = W^*_0 - L_0$ also satisfies the intertemporal budget constraint at $t = t^b$. This verifies that the pair $c_t = c^*_t$ and $W_{0-} = W^*_0 - L_0$ is a feasible pair for the agent solving the problem (50) - (52), which implies $V_t \geq J_t$. Combining this with Lemma 3 concludes the proof. 

**Proof of Lemma 5.** Writing out the Bellman equation for an agent and proceeding as in Kobila (1993) leads to the optimality condition \( \int_0^\infty (V_N I W - \xi - \gamma) dN_I = 0 \). Combining this optimality condition with the first order condition for consumption \( V_N I W = (c^{\mathcal{N}} - \gamma) \) yields the result.

**Proof of Proposition 7.** Since an agent who has experienced an idiosyncratic shock can always set $N_{0+} - N_0 = \frac{\sum}{r+q}$, the value function of an agent who has experienced an idiosyncratic shock is bounded. Therefore, as $\theta \to 0$, the objective in equation (56) converges to the objective of problem 1 (taking into account equation [55]). Also, in light of Lemma 5, any consumption plan that is feasible (for a consumer that has not experienced an idiosyncratic shock) under problem 5 is feasible under problem 1 and vice versa. By the theorem of the maximum the value function of problem 5 converges to the value function of problem 1 as $\theta \to 0$. 

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References


