Skin in the Game and Equilibrium Asset Prices *

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Abstract

I study the asset pricing implications of the presence of active capital. I define active capital as a form of ownership where some owners put skin in the game which makes firms more productive. In general equilibrium, active capital distorts risk sharing: aggregate risk is borne by levered active investors and regular ones are relieved from it. This effect tends to lower the risk price of all shocks. Dynamically, changes in the level of active capital, through deleveraging effects creates additional fluctuations in asset prices. I show that risk and risk aversion are negatively related to the supply of active capital. Fluctuations in active capital are therefore amplified by fluctuations in risk premium. This causes risk premium shocks to command a higher risk price. These two effects help explain the failure of cash-flow risk to explain asset returns and the importance of changes in risk premium. Additionally, it justifies the negative correlation of the quantity of active capital with the risk premium and its role for explaining the cross-section of returns.

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1 Introduction

A number of economic activities can be run more efficiently if some agents invest a significant fraction of their wealth in the enterprise. They can exert effort directly inside the firm (entrepreneurs, executives with variable compensation packages) or by exerting control or monitoring from outside the firm (private equity funds, venture capitalists, investment banks, hedge funds). The disproportionate risk these active providers of capital take face subjects them to large losses of wealth in adverse economic times. Additionally, new active capital does not always enter the market to replace such losses. The last financial crisis is an example of such period: bonuses dropped, hedge funds and investment banks lost an important fraction of their capital. Even though some new hedge funds were created and investment banks raised equity, the size of their industry relative to the aggregate economy remained smaller than its pre-crisis level\(^1\). Private equity activity was also largely impaired for an extended period of time\(^2\). Because of its important sensitivity to economic conditions, active capital has been pointed out as a potential source of amplification of economic fluctuations. This paper investigates how the aggregate quantity of active capital is determined jointly with asset prices. In particular, I study how various sources of fundamental fluctuations are transmitted to asset prices in presence of such a form of capital.

I present a dynamic general equilibrium model with a role for active capital. Agents are allowed to pick active capital as an alternative form of asset ownership. Active investors constrain themselves to a concentrated risky position in a firm, which makes it more productive. I represent this

\(^1\)Between december 2007 and march 2009, the hedge fund industry equity went from $1975bn to $973bn according to the Barclay Hedge database. For broker-dealers, He, Khang and Krishnamurthy (2010) estimate a change of trading assets from $2601bn to $1810bn using balance sheets of three pure broker-dealers.

activity by a *skin in the game* constraint on the portfolio shares of active agents. This constraint reproduces the high portfolio leverage typical of these investors and has been derived in various setups\(^3\). This framework allows me to study the joint determination of the quantity of active capital and asset prices in a variety of stochastic structures. I show that the effects of active capital on asset prices and the real economy crucially depend on the nature of fundamental risk in the economy.

Active capital affects asset prices through two channels: *distorted risk-sharing* and *deleveraging risk*. The static effect of active capital is a distortion of the risk-sharing arrangement in the economy. Active agents hold a disproportionate fraction of the risky assets. Therefore, the passive agents bear less risk in equilibrium. Because of this, they require a lower risk premium for the asset. This channel tells us that risks that can be borne by active agents will tend to have a lower price than those for which there is no active ownership. Diverging from the standard perfect risk sharing is optimal in this framework as it improves the productivity of firms. I show, however, that a competitive market yields an excess amount of active capital. Taxing firms that use this source of capital raises the welfare of all agents by improving risk sharing.

The second channel, *deleveraging risk*, is driven by the dependence of the quantity of active capital on economic fundamentals. For instance, if fundamental risk increases, the quantity of active capital decreases. We observe a deleveraging episode: some active investors switch back to passive investments. This requires selling assets to reduce their excess risky portfolio holdings. These assets arriving on the market have to be absorbed by the existing passive agents. This tends to lower prices further than the direct impact of the increase in risk. In this sense, active capital amplifies the fundamental volatility risk. Ex ante, this will tend to increase the price of this risk. The general insight is that shocks which affect the supply or demand of active capital are amplified and become more costly.

\(^3\)Holmstrom and Tirole (1997) derive such contracts inside and outside the firm as solution of a two-layered moral hazard problem with entrepreneurs and their monitors.
The determination of the quantity of active capital is key to understand characteristics of these two effects. Equilibrium in the active capital market equates the quantity of active capital demanded by firms with the number of investors willing to accept this particular portfolio. Firms demand active owners because it increases cash-flow. They trade off these productivity gains with the extra cost of active capital. I assume that the gains per fraction of active capital are not state dependent. Therefore the demand curve for active capital is constant over time. On the other hand, the supply of active capital is endogenously determined. Because all agents are ex ante identical, the extra returns paid to active capital must exactly compensate active agents for the extra risk they bear. The required compensation (cost of active capital) depends positively of risk aversion, the risk of the asset and how large is the deviation from the optimal portfolio. This points at two shocks that shift the amount of active capital: volatility and risk aversion shocks.

In general equilibrium, asset prices change with different levels of active capital. Market clearing implies that with more active agents, passive agents hold a smaller quantity of risky assets. For this to be consistent with optimization by passive agents, the asset must be more expensive. Therefore, it makes it more costly for active agents to buy their portfolio and they ask for more of a compensation. This feedback of activity on risk sharing makes the supply of active capital increasing in its price. Because deleveraging risk plays a role through variations in the quantity and not the price of active capital, the effects are more dramatic when the demand and supply are more elastic and when supply is more responsive to economic conditions.

My analysis provides a framework for understanding a number of asset pricing facts. I show that risk premia and the quantity of active capital are negatively related. This corresponds to the findings of Adrian, Moench and Shin (2010): they show that the aggregate risk premium covaries negatively with the balance sheet of financial intermediaries. Similarly, Haddad et al. (2011) find that fluctuations in buyout activity are strongly negatively correlated with a dynamic measure of the equity risk premium. In my model,
fluctuations in risk premium are a priced risk. Therefore, covariance with shocks to the quantity of active capital, as a measure of exposure to this risk, should help rationalize the cross-section of expected return. Adrian, Etula and Muir (2011) confirms this result: loadings on shocks to the leverage of broker-dealers explain the cross-section of equity expected returns. The model also provides insights regarding the sources of variation in prices. Since the "excess volatility puzzle" of Campbell and Shiller (1988), understanding the link between fundamental fluctuations and price fluctuations has been problematic. I show that changes in the quantity of active capital amplify the impact of some shocks (i.e. volatility) on prices. For the price of risks, the role of active capital can go in two directions: the prices of shocks that do not affect its quantity are lower relative to the standard endowment economy, whereas those that affect it can be larger. As cash-flow shocks fall in the first category, the model has the potential to explain the relative lack of success of approaches using measures of cash-flow risk to determine expected returns. Conversely, mild shocks to volatility can have a large impact on prices and command a high risk price, as they generate variation in the supply of active capital.

I discuss related work in section 2. Section 3 presents a static model showing how equilibrium in the active capital market is determined. I extend the model to a continuous time, infinite horizon framework in section 4. I derive the equilibrium conditions for the general case. In particular, I show the important role for disintermediation risk. Finally, section 5 discusses specializations of the model focusing on the difference between volatility shocks and cash-flow shocks.
2 Related Literature

3 Static model

In this section, I present a one-period model that illustrates the relation between ownership structure and asset prices. I first focus on the case of active capital localized in a small part of the economy. This partial equilibrium approach shows how the properties of returns affect the decision to provide active capital. I then move on to the possibility of active capital over all the economy. In this latter environment, the aggregate structure of capital ownership feeds back into asset prices.

3.1 Setup

The environment is a one period risky economy. Agents and firms take their investment and production decisions. Then uncertainty is realized and agents consume.

3.1.1 The ownership structure of firms

There is a continuum of firms indexed by $j \in [0, 1]$. Firms are of two types: a subset $\mathcal{A}$ of them has access to active capital whereas the remainder $\mathcal{P}$ does not.

Firms in $\mathcal{A}$ choose their ownership structure. They pick owners between two categories: active and passive. Active owners increase the output of the firm whereas passive ones leave it unaffected. Passive owners correspond to standard shareholders who do not participate in the decision-making of firms and just give a mandate to management to maximize the value of their shares. On the other hand, active owners represent investors that partake in the decision making of the firm so as to make it more profitable. These owners can be directly employed by the firm: entrepreneurs, managers, or employees with compensation indexed on the valuation of the firms. They can also
be outside investors who affect the firm, for instance, activist hedge funds, venture capitalists, private equity funds or investment banks for instance.

Formally, these firms pick which fraction \( m_j \) of their capital comes from active owners. I write \( M = \int_{j \in A} m_j dj / \int_{j \in A} dj \) the aggregate fraction of active capital for active firms. The firm’s output is then \((1 + \lambda(M)m_j)X\) where \( X \) is a positive random variable\(^4\). The function \( \lambda(.) \) represents the marginal return to active capital. I assume that it is decreasing and satisfies \( \lim_{M \to 0} \lambda(M) = +\infty \) and \( \lambda(1) = 0 \). Therefore, at the level of individual firms, the cash-flow is linear in the fraction of active capital. In aggregate, there are decreasing returns to scale for active capital. This represents the fact that for a given firm, as it is infinitesimal, the quality of active capital it receives does not depend of the quantity it chooses, but is decreasing in the total amount employed in the economy. I also study the case of constant returns, corresponding to a constant function \( \lambda(.) \).

Naturally, active capital is more costly to raise. For each share of the firm they purchase, active investors receive a fee \( fX \), senior to other payments. The residual cash-flow \( Y_j = (1 + \lambda(M)m_j - fm_j)X \) is then split between all investors in proportion to their contributions. The active fee, \( f \) is taken as given by the firm. It represents the equilibrium incremental cost of active relative to passive capital. In this static model it is a risky payoff received once uncertainty is resolved. In the dynamic case, as the payment is a flow, it is riskless. Hence it does not matter whether it is paid before or after the resolution of uncertainty. The objective of each firm is to maximize the amount, \( p_j \) of capital it raises. They trade off the additional cash-flow produced if they receive funding by active investors with the excess cost of this funding. On the other hand, firms in \( P \) do not take any decisions, they just all produce an identical risky amount \( X \).

\(^4\)The draw of \( X \) is the same for all firms.
3.1.2 The occupation decision of agents

There is a continuum of ex-ante identical agents indexed by $i \in [0, 1]$. They value risky consumption plans using a Von Neuman-Morgenstern utility function $u(.)$. They are all endowed with an equal fraction of all firms. Agents can choose one of two activities: passive or active investor. They choose their occupation taking as given asset prices and the active fee.

Passive investors are standard neoclassical agents. They have unrestricted access to the asset market and are free to buy and sell any claims. I note $\theta_i$ the number of shares of firm $j$ bought by agent $i$. Let us note this (endogenous) subset of investors as $P^*$.  

Active investors focus on a single firm and help increase its output. In order to commit to exert efforts to improve output, they have to put a large fraction of their wealth at stake in the project. This *skin in the game* alleviates moral hazard problems, as in Holmstrom (1979). Holmstrom and Tirole (1997) show that moral hazard problems can be present not only for the management of firms but also for outside investors who can affect output through their monitoring activities. I do not explicitly model the incentive problem, but rather focus on an exogenously specified contract. Active investors must take a disproportionately risky position in the firm: a fraction $\tilde{\theta}_A > 1$ of their wealth must be invested in shares of a given active firm. They finance this position through risk-free borrowing. I adopt a constraint proportional to wealth to reflect the idea that wealth dissolves incentives. An agent with ten times more wealth than another needs to invest more in a given project to be concerned about its outcome. To ensure that some investors choose to partake in this contract, the firm has to provide them an additional compensation. They receive the active fee for each unit of wealth they invested in the firm. I note $A^*$ this subset of investors.

The standard literature on contracting with moral hazard has emphasized

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5I focus on an equality constraint. However it is straightforward to check, both in this case and the dynamic one, that if agents have to hold at least $\tilde{\theta}_A$, the constraint will always bind.
that only idiosyncratic risk should be contracted upon. Here I assume that
the constraint is on the overall risk of the firm and that no hedging is possible.
In practice this assumption seems justified for a number of reasons. Aggre-
gate fluctuations are difficult to measure, and the exposure of individual firms
to it even more. Jointly with limitations in the completeness of contracting,
it appears complex to design purely idiosyncratic contracts in practice. Em-
pirically, hedging clauses or benchmarking is relatively rare in the contracts
of agents described above. One can argue that they could hedge on their own,
but it does not appear very common. Additionally, active agents appear to
receive disproportionately large losses in poor aggregate conditions, which
seems to confirm that their positions come with large loadings on aggregate
risk. He and Krishnamurthy (2008) and Brunnermeier and Sannikov (2010)
solve a dynamic moral hazard problem and find an equity constraint similar
to the one I use here, as they assume an impossibility to contract separately
on aggregate and idiosyncratic risk. In their frameworks, active investors are
a fixed exogenous group. They therefore focus on how the constraint applies
to this group. I focus on the constraint at the individual level and its effect
on the decision to become an active investor.

3.1.3 Equilibrium

The economy is a pure exchange economy with the addition of a market
for active capital. I extend the Walrasian notion of equilibrium by adding
price-taking in this market. Agents take the monitoring fee, asset prices and
payoffs as given in their investment decisions. Firms take the monitoring
fee, the aggregate level of monitoring and pricing as given when considering
their capital structure decisions. All cash-flows in the economy are linear
combinations of \( X \) and risk free cash flow. Therefore, the state-price density
can be characterized by only the relative price \( p_X \) of a payoff \( X \) to a constant
payoff 1.

Definition 3.1. Given \( f(\cdot), \tilde{\theta}_A, \lambda(\cdot), \mathcal{P} \) and \( A \), an equilibrium is constituted
of \( \{ \theta_i \}_{i,j \in [0,1]}, \{ b_i \}_{i \in [0,1]}, \mathcal{P}^*, \mathcal{A}^*, \{ m^j \}_{j \in [0,1]}, M, p_X \) and \( \{ p_j \}_{i \in [0,1]} \) such that:
(i) The allocation of active investors satisfies the budget constraint:

\[ \forall i \in A^*, b_{jA} + \tilde{\theta}_A p_j = \int_0^1 p_j dj \]

(ii) The occupation choice of active investors is optimal:

\[
U_m = \mathbb{E} \left[ u \left( \tilde{\theta}_A (Y_j + fX) + b_{jA} \right) \right] \geq \max_{(\{\theta_j\}, b)} \mathbb{E} \left[ u \left( \int_0^1 \theta^i Y_j dj + b \right) \right]
\]

s.t. \[ \int_0^1 p_j dj = \int_0^1 \theta^i p_j dj + b \]

(iii) The portfolio choice of passive investors is optimal:

\[
\forall i \in P^*, \{\{\theta^i_j\}_{j \in [0,1]}, b_i\} \in \arg \max_{(\{\theta^i\}, b)} \mathbb{E} \left[ u \left( \int_0^1 \theta^i Y_j dj + b \right) \right]
\]

s.t. \[ \int_0^1 p_j dj = \int_0^1 \theta^i p_j dj + b \]

(iv) The occupation choice of passive investors is optimal:

\[
\forall i \in P^*, \forall j \in [0,1], \mathbb{E} \left[ u \left( \int_0^1 \theta^i j Y_j dj' + b_i \right) \right] \geq \mathbb{E} \left[ u \left( \tilde{\theta}_A (Y_j + fX) + b_{jA} \right) \right]
\]

(v) The capital structure choice of firms is optimal:

\[
\forall j \in [0,1], m_j \in \arg \max_{m \in [0,1]} \int p_j = (1 + \lambda(M)m - fm)p_X
\]

(vi) The occupation market clears:

\[
M = \int_{j \in A} m_j dj = \tilde{\theta}_A \int_{i \in A^*} di
\]

(vii) The market for bonds clears:

\[
\int_0^1 b_i di = 0
\]
(viii) The market for assets clears:

\[ \forall j \in [0,1], \int_0^1 \theta_i^j di = \int_{i \in P^*} \theta_i^j di + \int_{i \in A^*} \bar{\theta}_A di = 1 \]

Given the symmetry of the problem, we can make a number of remarks that simplify the resolution of the equilibrium.

First, let us focus on the firm problem, condition (v). We can see that, because of constant return to active capital at the individual level, this problem is linear in \( m \). As I focus on interior equilibria, it implies that all active firms are indifferent between any level of active capital \( m \). The aggregate level of active capital is pinned down by their indifference condition:

\[ \lambda(M) = f \]

It tells us that the aggregate demand for active capital from firms does not depend on any characteristic of the cash-flow, \( X \), or preferences of the agents. Another further implication of indifference is that the dividend of a firm is unchanged by its level of active capital: \( \forall j \in A, Y^j_j = X \). It does not mean that there is no cash-flow created by the presence of active capital, but rather that all the value created is captured by the active investors. All firms have the same market price, active or not, so we can drop the index \( j \) on their price.

Let us now examine the agents’ optimality conditions. Since they all share the same preferences and endowment, conditions (ii) and (iv) are the same with the direction of the inequality reversed. Hence, where the equilibrium includes both active and passive agents, all agents must be indifferent over the choice of occupation. Linearity of the production function and the single shock structure means we do not need to keep track of with which firm each active investor contracts. The equilibrium conditions therefore collapses to:

\[
\mathbb{E} \left[ u \left( \bar{\theta}_A(1 + f)X + b_A \right) \right] = \max_{\{\theta^j, b\}} \mathbb{E} \left[ u \left( X \int_0^1 \theta^j_dj + b \right) \right]
\]

s.t. \( p = \int_0^1 \theta^j pdj + b \)
From this indifference condition, we learn that given asset prices, agents will supply active capital in a perfectly elastic way. We also can see directly that the equilibrium fee will in general be strictly positive. To see this, first consider the case of \( f = 0 \). In this case, equation (3.1) equates the utility of a given portfolio with that of the optimal portfolio: it is impossible. The right-hand side is necessarily at least as large as the left-hand side. Picking a negative \( f \) suppresses consumption from the agent, so it makes the inequality worse. This shows us that \( f \) has to be positive. As long as the utility function is unbounded, we know there will exist a unique solution to this equation.

The only ingredient we are missing now is the determination of asset prices. In the following sections, I first treat the case where active firms are negligible in size compared to the rest of the economy and then the other extreme where all firms have equal access to active capital. In the first situation, active capital does not have any feedback effect on equilibrium prices, whereas in the second, the level of active capital influences the equilibrium risk sharing in the economy and therefore influences prices.

### 3.2 Partial equilibrium

In this section, I consider the case where the set of active firms \( A \) is of measure 0.

From the point of view of passive investors, all firms are identical (they all pay the same dividend \( X \)). As the active firms are of infinitesimal size, so is the set of active investors. Therefore the mass of passive investors is as large as all the economy. For asset pricing, we are in the case of a pure exchange economy. All passive agents consume the dividend \( X \) and prices are determined by their marginal utility of consumptions. Because I use the risk-free asset as numéraire, the price is therefore:

\[
p = \frac{\mathbb{E}[Xu'(X)]}{\mathbb{E}[u'(X)]}
\]

Now that we know the asset price, the price \( f \) at which active capital is
supplied is uniquely pinned down by the indifference condition:

\[ E \left[ u \left( \tilde{\theta}_A (1 + f) X + b_A \right) \right] = E \left[ u \left( X \right) \right] \]

\[ \text{s.t. } p = \tilde{\theta}_A p + b_A \]

To understand the determinants of this fee, let us focus on an example. I assume exponential utility \( u(c) = -\exp(-\alpha c) \) and normally distributed cash-flow \( X \sim \mathcal{N}(\mu, \sigma^2) \). All the calculations are in appendix A.1. In this simple case we can derive equilibrium fee and the following comparative static:

**Proposition 3.2.** The active fee is given by:

\[ f = \frac{1}{\tilde{\theta}_A (\mu - \frac{1}{2} \alpha \sigma^2)} \frac{1}{2} \alpha (\tilde{\theta}_A - 1)^2 \sigma^2 \]

It is:

(i) increasing in absolute risk aversion \( \alpha \)

(ii) increasing in volatility \( \sigma^2 \)

To understand the equation, first note that the denominator corresponds to the utility provided by a payment of \( \tilde{\theta}_A X \). The numerator corresponds to the cost of deviating from the optimal portfolio. The intuition is that the effective extra payment will be determined by how “non-optimal” is the portfolio of an active investor. The degree of deviation is measured by the square distance of the contract portfolio \( \tilde{\theta}_A \) to the optimal 1. When volatility or risk aversion are higher this deviation is more costly. These comparative statics convey a key intuition for the rest of the results in the paper: larger volatility of the asset payoff decreases the supply of active capital. Note that here it is purely a partial equilibrium effect: the more risky the asset payoff, the less willing are the agents to depart from their optimal portfolio. We can summarize these results on figure 1 representing the equilibrium on the market for active capital.

Haddad et al. (2011) emphasize similar results in their study of leveraged buyouts. They argue that the illiquidity taken by investors in buyouts is a
potential channel explaining the empirical link between buyout activity and the aggregate risk premium. They also confirm the prediction that more risky firms, both idiosyncratically and systematically are less likely to be bought out.

3.3 General equilibrium

I now turn to the case where all firms have access to active capital: \( \mathcal{A} = [0, 1] \).

In this situation, it is more complex to determine the supply of active capital as active investors are not infinitesimal anymore. We cannot consider independently the problem of active and passive investors. In other words, the market clearing condition will link the portfolios of active and passive investors. Note \( \theta^* \) the total amount invested in stocks by passive investors. Remember the market clearing condition for the assets:

\[
\theta^* \int_{i \in \mathcal{P}^*} di + \tilde{\theta}_A \int_{i \in \mathcal{A}^*} di = 1
\]

and the market clearing condition for active capital:

\[
M = \tilde{\theta}_A \int_{i \in \mathcal{A}^*} di = 1
\]

Combining these, we get that \( \int_{i \in \mathcal{P}^*} di = (1 - M)/\theta^* \). Because agents either have to be active or passive we can conclude that:

\[
\begin{align*}
1 &= \int_{i \in \mathcal{P}^*} di + \int_{i \in \mathcal{A}^*} di \\
&= \frac{1 - M}{\theta^*} + \frac{M}{\theta_A}
\end{align*}
\]

Identically, using market clearing for bonds, we derive:

\[
0 = b^* \frac{1 - M}{\theta^*} + b_A \frac{M}{\theta_A}
\]

Equation (3.3) shows that mechanically, in a world with more active capital, passive agents have to take less risky portfolios. Because the consumption of
passive investor determines asset prices, the latter will change. This affects the tradeoff determining the fee $f$. To summarize, the supply of active capital is determined by the market clearing conditions (3.3) and (3.4) jointly with the indifference condition$^6$:

$p = \frac{\mathbb{E}[Xu'(\theta^*X + b^*)]}{\mathbb{E}[u'(\theta^*X + b^*)]}
= \mathbb{E}\left[u\left(\tilde{\theta}_A(1 + f)X + b_A\right)\right] = \mathbb{E}[u(\theta^*X + b^*)]
\quad \text{s.t. } p = \tilde{\theta}_Ap + b_A

From this nonlinear system, we can derive the supply curve $f^{\text{sup}}(M)$. For instance, going back to the example of the previous section, we can characterize it for the case of exponential utility and normal cash-flows:

**Proposition 3.3.** The fee $f^{\text{sup}}(M)$ is the unique solution of the system:

$$1 = 1 - \frac{M}{\theta^*} + \frac{M}{\tilde{\theta}_A}
\quad f = \frac{1}{\tilde{\theta}_A(\mu - \frac{1}{2}\alpha\sigma^2)} \frac{1}{2} \alpha(\tilde{\theta}_A - \theta^*)^2 \sigma^2$$

It is strictly increasing in $M$ and correspond to the partial equilibrium fee when $M = 0$.

The reason why more active capital makes the fee larger is that it affects the risk-sharing arrangement. When there is more active capital, because of the market clearing condition (3.3) for the asset, the portfolio share of risky asset from passive agents has to decrease. It means that they bare less of the aggregate risk of the economy. Consequently the risk premium decreases and the price of the asset increases. The active agents need to borrow more to finance their skin in the game and therefore are worse off if the fee does

$^6$Because of Walras’ law, the budget constraint of the passive agents is redundant.
not change. The passive agents can on the other side save more and they are better off. This implies that, in order to make them indifferent, active agents need to be compensated more. Figure 2 shows this variations for a numerical example. The second part of the property confirms that this general effect always increases the cost of monitoring relative to a partial equilibrium situation.

In the general case, the intuition that the price decreases with the portfolio share is not always true. When the relative price of stocks relative to bonds decrease, investors move away from stock, this is the substitution effect. However, an income effect is also at play: because the endowment of agents is in stocks, they become richer and want to buy more stocks. This effect can generate some non-monotony in the demand curve. In appendix A.2, I detail some conditions under which this might happen and detail the solution of the model in those cases. In the example of exponential utility, there is no income effect. In the dynamic model I consider in the following sections, as I consider diffusion processes, all risks are local. This also gets rid of the income effect as risks and expected returns are negligible relative to wealth.

Before moving on to a continuous-time version of the model to explore dynamic pricing implications, I discuss welfare in this static case. This clarifies the notion of distorted risk-sharing and its optimality. I show that there is room for improving welfare relative to market outcomes.

### 3.4 Welfare analysis

The financial constraint in the model generates market incompleteness: active agents cannot choose freely their portfolio. This implies that in general, not only the first welfare theorem does not apply, but there will even exist allocations respecting the constraint that increase welfare. This is due to the presence of pecuniary externalities: the quantity of active investors affect risk sharing in the economy, and none of the agents internalize this effect. In a complete market framework, the fact that agents’ portfolio choice affect
prices does not affect welfare. The intuition for the source of inefficiency is that, with incomplete markets, marginal rates of substitutions are not equalized across agents. Changing prices affects different agents differently and generates some redistribution. Gromb and Vayanos (2002) study such welfare implications in an exogenously segmented market framework.

In my model, the externality comes from the effect of active investors on risk sharing. When an investor chooses his occupation, he does not internalize the fact that, by increasing his leverage, he decreases that of all other passive agents. Therefore they bear less risk and are better off. However, the price of the asset increases. This causes the other active agents to have to borrow more to finance their position, and makes them worse off. Of course in equilibrium the fee adjusts to make all agents indifferent. Finally there is a last effect: cash-flows are changed depending on the fee and the productivity of active capital. To study which effect dominates let us consider an intervention affecting the quantity of active capital.

Before describing the policy considered, note that there is another source of externality in the model. Productivity is affected by the quantity of active capital but firms do not internalize it. To focus on the effect of the financial constraint, I assume constant marginal productivity $\lambda$ of active capital, i.e., a perfectly elastic demand curve. Because we have seen that the general equilibrium supply curve is increasing, the equilibrium is still non-degenerate.

I study situations in which the fee $f$ for active capital is set exogenously and firms accept all the supply available at this cost. A simple way to implement this policy is impose a subsidy per unit of active capital equal to $(f - \lambda)X$ and rebate the proceeds as a lump sum to the firms. Because the market for supplying capital is unchanged, agents of both types are still indifferent. Increasing $f$ either makes all agents worse off or better off. The main results are in the following proposition:

**Proposition 3.4.** (i) Whenever the productivity $\lambda$ strictly exceeds the zero active capital fee $f_0$, it is optimal to have a non-zero level of active capital.
(ii) The market equilibrium dominates the allocation without active capital. The comparison is strict if they differ.

(iii) Decreasing the fee relative to the market equilibrium increases welfare.

The proof is in appendix A.3. Part (i) tells us that, when the proceeds from active capital exceed the cost of compensating the first active agent, it is optimal to introduce some active capital. Indeed, distorting allocation from the no active capital position has a second effect order on the risk sharing, but a first order effect on cash flows. From part (ii), we know that the market equilibrium cannot be worse than the allocation without active capital. Agents can always choose to have the same consumption as in the world without active capital. To do so they just do not trade and consume the payments of their claim. They optimally choose not to so they are better off than with no capital. Because the portfolio problem is strictly convex, we obtain the strict dominance.

Finally, from part (iii), we see that there is always too much active capital in the competitive equilibrium. To understand this result first notice that, at the competitive equilibrium a change in the quantity of active capital has no first-order effect on cash-flows as the fee and the productivity perfectly offset. All the welfare effect comes from changed pricing of the same cash-flow stream. Because active agents are away from their optimum portfolio, they get hit harder by an increase in price than the passive agents gain. Even when adjusting to make both classes of agents indifferent, the negative effect dominates.

4 Dynamic model

In the previous section, I showed how the supply of active capital depends on risk and risk aversion. Then, in general equilibrium, we saw that the level of active capital feeds back into asset prices. I now turn to a dynamic model to understand the intertemporal link between the level of active capital and
asset prices and focus on richer fundamental dynamics.

I consider a dynamic economy in continuous time. The notations correspond to the static model when possible. The economy is controlled by an exogenous set of $S$ state variables that follow a Markov diffusion:

$$ds_t = \mu_s(s_t) + \sigma_s(s_t)dZ_t$$

where $\{Z_t\}$ is a vector of $K$ independent brownian motions.

### 4.1 Firms

Passive firms in $\mathcal{P}$ do not take any decisions and their cash-flow stream is determined by:

$$\frac{dD_t}{D_t} = \mu_D(s_t)dt + \sigma_D(s_t)dZ_t$$

Active firms in $\mathcal{A}$ can use active capital. They choose their capital structure $m^j_t$ dynamically (I will drop the index $j$ to simplify notation), taking the whole process for the aggregate level of active capital $\{M_t = \int_\mathcal{A} m^j_t D^j_t dj\}$ as well as the active fee $\{f_t\}$ as given. The payment of the fee is different from the static model: it is now a flow $f_t m_t P_t$ per unit of capital provided. If the fraction of active capital is $m_t$ and the price of the firm is $P_t$, this amount to a flow of payments: $f_t m_t P_t$. The rest of the cash-flow is shared proportionally to the amount of capital invested.

Even if it is more costly, firms use active capital as it enhances the growth of their cash-flow stream. The marginal growth rate increase in response to an increase in the fraction of active capital is $\lambda(M_t)$. The level of active capital at a given instant affects the growth rate only in the next infinitesimal time period, but the level of cash-flow is permanently affected.

As before, firms maximize the level of capital they raise. Noting $\{S_t\}$ the stochastic discount factor of the economy, their problem at time $t$ is therefore:
\[ P_t = \sup_{\{m_t, \tau\} \leq t < \infty} \mathbb{E}_t \left[ \int_0^\infty \frac{S_{t+\tau}}{S_t} (D_{t+\tau} - f_{t+\tau}m_{t+\tau}P_{t+\tau}) \, d\tau \right] \]

s.t. \[ \frac{dD_{t+\tau}}{D_{t+\tau}} = (\mu_D(s_{t+\tau}) + \lambda(M_{t+\tau})m_{t+\tau}) \, dt + \sigma_D(s_{t+\tau})dZ_{t+\tau}, \quad 0 \leq \tau < T \]

The linearity of both the objective function and the dynamics in the current level of cash-flow tells us that the value function is linear in the level of cash-flow and that the optimal policy does not depend on it. In other words, \( P_t/D_t \) and \( m^*_t \) are deterministic functions of \( s_t \). By abuse of notation, I will write them \( P/D(s_t) = V(s_t) \) and \( m(s_t) \) respectively in the remainder of the paper.

One can wonder whether this problem generates a dynamically consistent policy function. The problem is actually time consistent as it can be expressed recursively. We can rewrite, \( \forall t < T \):

\[ \frac{P_t}{D_t} = \sup_{\{m_t, \tau\} \leq t < T - t} \mathbb{E}_t \left[ \int_0^{T-t} \frac{S_{t+\tau}}{S_t} \left( \frac{D_{t+\tau}}{D_t} - f_{t+\tau}m_{t+\tau} \right) \, d\tau + \frac{S_T}{S_t} \frac{D_T}{D_t} \tilde{V}_T \right] \]

s.t. \[ \frac{dD_{t+\tau}}{D_{t+\tau}} = (\mu_D(s_{t+\tau}) + \lambda(M_{t+\tau})m_{t+\tau}) \, dt + \sigma_D(s_{t+\tau})dZ_{t+\tau}, \quad 0 \leq \tau \leq T - t \]

\[ \tilde{V}_T = \sup_{\{m_t, \tau\} \leq T < \infty} \mathbb{E}_T \left[ \int_0^\infty \frac{S_{T+\tau}}{S_T} \left( \frac{D_{T+\tau}}{D_T} - f_{T+\tau}m_{T+\tau} \right) \, d\tau \right] \]

s.t. \[ \frac{dD_{T+\tau}}{D_{T+\tau}} = (\mu_D(s_{T+\tau}) + \lambda(M_{T+\tau})m_{T+\tau}) \, dt + \sigma_D(s_{T+\tau})dZ_{T+\tau}, \quad 0 \leq \tau < \infty \]

where \( \tilde{V}_{T+\tau} \) for \( \tau > 0 \) is defined similarly to \( \tilde{V}_T \). Examining the problems for \( \{\tilde{V}_{T+\tau}\}_{0 \leq \tau < \infty} \) shows that it exactly corresponds to the problem for \( \{P_{T+\tau}/D_{T+\tau}\}_{0 \leq \tau < \infty} \) which confirms time consistency.

The recursive structure of the problem allows us to write it in the form of a Hamilton-Jacobi-Bellman equation, as can be seen in appendix B.1. As is standard for this type of investment model, the choice of optimal active capital turns out to be static. The first-order condition to have an interior
optimum is, as it was for the static case:

$$\lambda(M_t) = f_t$$

The interpretation of these terms is actually slightly different from the static model. The payment to the active investors is now $m_t f_t P_t dt$. This allows us to interpret $m_t f_t dt$ as a number of new shares issued by the firm to pay its active investors. The firm equates the marginal cost of diluting the shares of its investors to the increase in the scale of the firm due to the presence of active capital. The linearity of the problem yields that exactly all the increase in firm size due to active capital compared to only active is used to issue new shares to pay the active capital fee. The capital structure decision here is essentially static. It corresponds exactly to the standard q-theory of investment except that here the price of the investment good, active capital, is quoted as a multiple of the firm price.

This condition pins down the aggregate demand for active capital but not individual policies. The indeterminacy can generate ex-post heterogeneity across firms in their size. However, because the dynamics of cash-flow are linear in the current level, and the way $m^j$ aggregates to $M$, one can see that aggregate dynamics are invariant to the distribution of individual firms policies.

### 4.2 Asset markets

I assume that all risks are traded on markets. To simplify notations, I assume a set of $K$ assets. The first one is a share of any of the firms. As noted before they all have the same price and cash-flow evolution so their shares have the same returns. The other $K - 1$ asset returns are in zero net supply, have unit variance and complete the market. Their expected excess returns can directly be inferred from the stochastic discount factor $\{S_t\}$ as they correspond to the risk prices. I note $\mu_{R,t}$ the vector of expected returns and $\sigma_{R,t}$ the vector of volatility of these assets. There is also a risk-free asset in zero net supply with instantaneous return $r_{f,t}$ given by the opposite of the drift of the stochastic
discount factor. In equilibrium they all are deterministic functions of the state $s_t$ of the economy.

4.3 Agents

There is a continuum of agents indexed by $i \in [0, 1]$ with identical preferences. I write $W_i^t$ their wealth at hand at time $t$ and $W_t = \int_0^1 W_i^t di$ the aggregate wealth in the economy.

Agents rank consumption streams according to the stochastic differential utility of Duffie and Epstein (1992). It is a continuous-time version of the recursive preferences of Epstein and Zin (1989). Write $J_t$ the utility of the agent at time $t$ and $f(C, J)$ the aggregator. The utility value $J_t$ is defined recursively by:

$$J_t = \mathbb{E}_t \left[ \int_t^\infty f(C_s, J_s) ds \right]$$

For the aggregator $f$, we use the standard function:

$$f(C, J) = \beta \frac{\gamma J}{\rho} \left[ \frac{C^{\rho}}{\gamma J^{\frac{\rho}{\gamma}}} - 1 \right]$$

$\beta$ is the rate of time preference. $\Gamma = 1 - \gamma$ is the relative risk aversion (RRA) of the agent. $\psi = \frac{1}{1-\rho}$ is the intertemporal elasticity of substitution (IES). When $\Gamma = \frac{1}{\psi}$, or equivalently $\gamma = \rho$, the utility function reduces to the standard power utility specification. An important remark is that these preferences are homogenous of degree $\gamma$.

Agents can choose whether they are active or passive investors. I write $\mathcal{A}_t^*$ and $\mathcal{P}_t^*$ these sets of investors at each time. Agents can move freely between occupations. They can choose their consumption $C_t$ without any constraint.

Passive investors can choose freely their portfolio $\theta_t^*$. Their wealth evolution is then:

$$dW_i^t = \left( W_i^t(\theta^*_t (\mu_{R,t} + r_{f,t}) - C_t) + W_i^t \theta^*_t \sigma_R dZ_t \right) dt + W_i^t \theta^*_t \sigma_R dZ_t$$
Active investors are constrained to choose a portfolio \( \tilde{\theta} = [\tilde{\theta}_A, 0, \ldots, 0] \) that only consists of a position in shares of the firm. I assume \( \tilde{\theta}_A > 1 \). This constrains them to hold more of the asset than anybody would hold in a world without active investors. However, as a compensation for taking this extra risk, they receive the fee \( fdt \) for each unit of the firms they own. The evolution of their wealth is driven by:

\[
(4.2) \quad dW^i_t = \left( W^i_t (\tilde{\theta}' (\mu_{R,t} - r_{f,t}) + r_{f,t} + \tilde{\theta}_A f_t) - C_t \right) dt + W^i_t \tilde{\theta}' \sigma_R dZ_t
\]

Because all these dynamics are linear in wealth and the homogeneity of utility functions, one can see that all agents will face the same tradeoff, irrespectively of their current wealth, when choosing their occupation. Additionally, agents in one occupation will all have the same consumption-wealth ratio that I write \( c^i_t = C^i_t / W^i_t \). In order to have an interior equilibrium for the level of active capital, agents, given their wealth, have to be indifferent between activities at each point in time. Let us note \( F_t / \gamma \) the utility of an agent with wealth one at date \( t \). The utility of an agent with wealth \( W^i_t \) is then:

\[
J_t = \frac{(W^i_t)^\gamma}{\gamma} F_t
\]

\( F_t \) is actually a deterministic function of the state variables that we note \( F(s_t) \). In particular \( F(.) \) has to be the value function of an agent being active or passive for some interval of time. The following proposition formalizes this idea.

**Proposition 4.1.** The fee \( f_t \) is such that the value function per unit of wealth \( F(s_t) \) solves the Hamilton-Jacobi-Bellman problems:

(i) **Passive investor:**

\[
0 = \max_{c \geq 0, \theta \in \mathbb{R}^K} f(\gamma^{1/\gamma} c, F) + \frac{\mathbb{E}[d(W^{\gamma} F)]}{W^{\gamma} dt}
\]

s.t. \( dW_t = (W_t (\theta' (\mu_{R,t} - r_{f,t}) + r_{f,t}) - C_t) dt + W_t \theta' \sigma_R dZ_t \)
(ii) **Active investor:**

\[
0 = \max_{c \geq 0} f(\gamma^{1/\gamma}c, F) + \frac{\mathbb{E}[d(W^\gamma F)]}{W^\gamma dt}
\]

s.t. \(dW_t = \left(W_t(\tilde{\theta}'(\mu_{R,t} - r_{f,t}) + r_{f,t} + \tilde{\theta}_A f_t) - C_t\right)dt + W_t \tilde{\theta}' \sigma_R dZ_t\)

I derive these problems in appendix B.2.1. This proposition helps us sidestep an issue with the continuous time model: a few of the agents switch very often. The aggregate level of active capital is a function of the state variables. As they follow a diffusion, the fraction of agents that are active will also follow a brownian motion. For instance, one can prove that it is impossible that all agents follow stopping time strategies to change their activities. This problem is not present in the discrete time version of the model. In appendix, I show how all aggregate quantities in a discretized version of the model converge to the one I derive here. In particular proposition 4.1 holds in the limit, as it characterizes agents indifference, not their actual occupation trajectory.

Examining the problems of proposition 4.1, we can simply derive the consumptions and portfolios of agents as well as the activity fee:

**Proposition 4.2.** At equilibrium:

(i) All agents (active and passive) have the same consumption wealth ratio, determined by:

\[
c = \beta_{(\rho-1)F}^{\rho/(\rho-1)}
\]

(ii) The portfolio \(\theta^*\) of passive agents is:

\[
\theta^* = \frac{1}{1 - \gamma} (\sigma_R \sigma'_R)^{-1}(\mu_R - r_f) + \frac{1}{1 - \gamma} (\sigma_R \sigma'_R)^{-1}\sigma_R \sigma'_s F_s
\]

(iii) The activity fee \(f\) is given by:

\[
\tilde{\theta}_A f = (1 - \gamma) \left(\tilde{\theta} - \theta^*\right)' \sigma_R \sigma'_R \left(\tilde{\theta} - \theta^*\right)
\]
The proof is in appendix B.2.2. Points (i) and (ii) are the standard portfolio results for a passive agent with recursive preferences. It is interesting to note that active agents do not choose a distinct consumption policy from passive ones. The reason for this result is that the only determinant of consumption is the marginal utility of wealth. It has to be the same, active or passive, as agents are indifferent between occupations at all levels of wealth.

Finally, one can see that the fee $f$ is the product of three terms: the relative risk aversion $(1 - \gamma)$, the return variance $\sigma_R \sigma_R'$ and the distance of the active portfolio to the unconstrained portfolio $(\bar{\theta} - \theta^*)$. This reflects the intuition from the static model: the more risk averse agents are, and the more risky is the asset, the less agents are willing to take the extra risk of the active position. One can note that it is purely a compensation for the extra risk taken by the agent, and that it does not depend of the covariance of returns with the state variables. It is because, in a diffusion framework, hedging demands are linear in the amount of risk, and therefore the already larger returns from the levered position exactly offset the additional loading on the state variables the agent has to take.

If the subset of active firms is infinitesimal, as in the static case, $\theta^* = [1, 0, \ldots, 0]$ and the fee does not depend of the amount of active capital supplied. In the case where a substantive fraction of firms can be active, the market clearing condition creates a link between $\theta^*$ and the fraction of active capital $M$. This correspond to the general equilibrium effect of the static model and I detail it in section 4.4.

In this dynamic framework, a second dependency appears. The price volatility $\sigma_R$ does not have to coincide with the cash-flow volatility $\sigma_D$. As seen before the price cash-flow ratio can change and in this sense we have "excess volatility". The price of the asset tomorrow depends on the exogenous changes in the evolution of cash-flow, but also in the endogenous level of active capital, through its effect on the stochastic discount factor. The fluctuations in active capital generate price volatility, which feeds into $\sigma_R$. I detail this intertemporal link in section 4.5.
4.4 The equilibrium link of active capital and risk sharing

I now focus only on the case $\mathcal{A} = [0, 1]$. As for the static case, because of the market clearing condition, the amount of active capital influences the holdings of passive agent, and therefore the stochastic discount factor. Now that there can be wealth heterogeneity, we need to weigh agents by their wealth. The market clearing condition for the assets becomes:

$$
\int_{i \in \mathcal{P}^*} w^i \theta^* di + \int_{i \in \mathcal{A}^*} w^i \tilde{\theta} di = [1, 0, \ldots, 0]
$$

All but the first asset are in zero net supply and not owned by the active investors. Therefore the passive investors have to endogenously choose a zero position in those assets too. By abuse of notation, I write $\theta^* = [\theta^*, 0, \ldots, 0]$. We can, use the fact that the total wealth is split between active and passive investors to obtain a condition linking $M$ and $\theta^*$:

$$
\frac{M}{\theta_A} + \frac{1 - M}{\theta^*} = 1
$$

It is exactly the same equation as the static model with a slightly different interpretation. Each term is the fraction of total wealth owned by each class of agents.

Looking jointly at this market clearing condition and the problem of passive investors, we have again the general equilibrium feedback of the amount of active capital into asset prices. When there is a large amount of active capital, more risk is in the hand of active investors. This lowers the risk borne by passive investors and therefore the risk premium as well. The asset is more costly, active investors are worse off. To compensate them the fee needs to increase.

4.5 Dynamic interactions in the active capital market

A second equilibrium feedback of the amount of active capital is the intertemporal effect. In a dynamic environment, the volatility of asset returns is not
caused only by the volatility of cash-flows, but also by changes in the pricing environment. A simple application of Ito’s lemma gives us:

\[
\frac{dR}{R} = \frac{D_t}{P_t} dt + \frac{dD_t}{D_t} + \frac{dV_t}{V_t} + <dD_t, dV_t> \frac{P_t}{P_t}
\]

The volatility comes from fluctuations in cash-flows \( dD_t \) as well as fluctuations in the price cash-flow ration \( dV_t \). Any changes in the properties of the stochastic discount factor affect this valuation ratio. These changes are amplified by fluctuation in the fraction of active investors.

To see this, consider some change in the volatility of the cash flow tomorrow. If we happen to be in a high volatility state, the price will be lower than in a high volatility state for three reasons. First, future cash-flow are more risky and as such are discounted more. Second, because the environment is more risky, the risk price for cash-flow shocks is larger. Finally, because of the more risky environment there will be less active capital, and as such the risk price will be even larger as passive investor bear more of the risk.

This third channel, present only with an active sector, creates additional price volatility. It makes it more costly to be an active investor today, and therefore it depresses the supply of active investors today.

It is important to notice that this amplification channel is present here only for shocks that affect the valuation ratio and not for cash-flow shocks. Indeed, because the active capital sector can adjust frictionlessly its size, there is no difference between being in a large or small economy.

### 4.6 Individual policies and the wealth distribution

Before studying the solution of the model for various dynamics, let us consider possible wealth distribution evolutions in the population. Let us first recapitulate properties of wealth trajectories and then examine a couple of possible implementations of the equilibrium.

At each point in time, active investors get on average a higher wealth increase. This comes through two channels: they take more leverage and therefore get extra asset returns, and additionally they receive the fee. They
are also more exposed to shocks than passive investors. Therefore, in case of
good returns they gain relatively more wealth and in case of bad returns lose
relatively more. These differences in wealth evolution will create dispersion in
wealth across agents. However, as agents can choose their occupation at each
point in time, and preferences are homogenous this wealth heterogeneity does
not create heterogeneity in behavior. The individual policies are not pinned
down by the equilibrium conditions and many occupation choice trajectories
are consistent with the evolution of aggregate quantities. To get a better
idea of how much switching is necessary at equilibrium, let us consider two
simple implementations.

Remember that we have a continuum of agents indexed by \(i\) on \([0, 1]\)
and note \(w^i_t = W^i_t/W_t\) their fraction of total wealth at time \(t\). A fraction
\(M(s_t)/\bar{\theta}_A\) has to be in the active sector. A first implementation is that the
agents with the lowest indices are active. To determine which agents are
active, define implicitly the threshold \(I_t\) by:

\[
\int_0^{I_t} w^i_t \, di = \frac{M(s_t)}{\bar{\theta}_A}
\]

There always exists a unique solution to this equation as all individual wealth
fractions \(w^i_t\) clearly stay strictly positive and they integrate to 1 which is
strictly larger than the right hand side. Also, because all individual wealths
\(w^i_t\) and \(M(s_t)\) follow a diffusion, so does \(I_t\). Using properties of diffusion
processes, we can derive local properties of career dynamics in this imple-
mentation:

**Proposition 4.3.** At each point in time \(t\):

(i) For almost every agent \(i\), there almost surely exists \(\varepsilon > 0\) such that \(i\)
does not change occupation on \([t, t + \varepsilon)\),

(ii) Agent \(I_t\), for any interval \([t, t + \varepsilon)\), almost surely changes occupation at
a set of times that is uncountable, without isolated points.\(^7\)

\(^7\)We also know that this set is of measure 0 and is fractal with Hausdorff dimension
1/2.
The almost every statement is in the sense of the canonical measure on \([0,1]\), or that implied by \(w_i\) as they are equivalent. Actually, it concerns the set \([0,1]\setminus I_t\). The almost surely statement is in the sense of the probability measure implied by the filtration \(\{\mathcal{F}_t\}\) generated by the brownian motions of the economy. This proposition is a direct consequence of the properties of the zeros of the Brownian motion that can be found for instance in Morters and Peres (2010).

In other words, the proposition tells us that most agents do not change job on any finite interval. Only the agents at the border will go back and forth an infinite amount of time. As this agent only represents an infinitesimal fraction of aggregate wealth, his back and forth do not affect aggregate dynamics. This implementation of the equilibrium has the undesirable effect of not admitting a stationary distribution of wealth. To see this consider the wealth of agent 0 relative to any other agents. As he is always active, his wealth has a larger drift than any other agent and therefore their ratio of their wealth tends to increase, and in the limit diverges to infinity.

An alternative implementation that insures that the wealth distribution do not diverge is to make the group of active agents change over time. For instance, given \(\zeta > 0\), we can choose that the subset \([\zeta t, \zeta t + I'_t]\) mod 1 where \(I'_t\) is now defined by:

\[
\int_{\zeta t}^{\zeta t + I'_t} \frac{w_i}{I_t} (i \mod 1) \, di = \frac{M(s_t)}{\bar{\theta}_A}
\]

As the group of active agent cycles through the population, no individual wealth can drift permanently over the other ones. Relative to the proposition, we now have two agents switching occupation at each point in time. Agent \(\zeta t \mod 1\) becomes passive, agent \((\zeta t + I'_t) \mod 1\) switches an infinite number of times.

Finally, note that in both implementations considered here, neither the wealth distributions nor the threshold \(I_t\) are deterministic functions of the state \(s_t\). Indeed, past shocks affect the relative wealth evolutions of the two groups and modify the fraction of agents to include in order to obtain the
equilibrium fraction of active capital.

5 Examples

I now solve completely the model for two cases. I first focus on an i.i.d. economy where the size of the active sector stays constant. There are no dynamic effects in this situation, and we can solve the model in closed form. This example shows the role of the active sector in the presence of cash-flow shocks only.

The second example focuses on the more realistic situation of shocks to the discount factor. They are generated by variation in the riskiness of the economy. I show how these fluctuations are amplified by generating fluctuations in active capital. As a result, most of the asset expected returns come as a compensation for discount rate shocks and not for cash flow shocks.

Before addressing these two cases, I first detail how to solve the model.

5.1 Solving the model

Here I summarize the set of equations characterizing an equilibrium. I focus on Markovian equilibria characterized entirely by the exogenous variables. One can show that it actually pins down the unique equilibrium of the economy. All equilibrium objects will therefore be functions of the $S$ state variables.

I start with a guess for the risk-free rate $r_f(s)$ and the risk prices $\{rp_k(s)\}_{k=1}^K$. The valuation ratio of firms $V(s)$ is pinned down by the HJB. Using Ito’s lemma, we can infer the return dynamics for the firm share. The return of the other assets are derived straight from the stochastic discount factor. Now that we have $\mu_R(s)$ and $\sigma_R(s)$, we solve the portfolio problem of a passive agent and obtain $\theta^*(s)$ and its consumption-wealth ratio $c(s)$. Using the market clearing condition, we obtain $M(s)$ and the first order condition of the firm’s problem gives $f(s)$.
We need exactly $K + 1$ conditions to check in order to make sure that we have an equilibrium. The first $K − 1$ are the fact that passive investors do not have any demand for the $K − 1$ assets in zero net supply. The indifference condition across position gives one more condition. Finally checking that all dividends are consumed provides one last condition. This last condition corresponds to checking that $c(s) = 1/V(s)$. The last equilibrium condition is market clearing in the risk-free asset, but because we checked market clearing in both the other assets and the consumption good, this last condition is redundant.

5.2 An i.i.d. economy

I now solve the model in an i.i.d. framework. There are no state variables, and therefore all the equilibrium quantities are just scalars. The cash-flow evolution is simply:

$$\frac{dD}{D} = (\mu_D + \lambda(M)m)dt + \sigma_D dZ$$

Applying the results from the previous sections, I derive the equilibrium:

**Proposition 5.1.** There is a unique equilibrium where:

- **The portfolio share** $\theta^*$ **of the passive agents** is the unique solution of the equation:
  $$\lambda \left( \frac{1 - \theta^*}{1 - \frac{\theta^*}{\theta_A}} \right) = \frac{(1 - \gamma)(\bar{\theta}_A - \theta^*)^2}{\theta_A}$$

- **The equilibrium fraction of monitoring capital** $M$ **is**:
  $$M = \frac{1 - \theta^*}{1 - \frac{\theta^*}{\theta_A}}$$

- **The risk price** $rp$ **is**:
  $$rp = (1 - \gamma)\sigma_D \theta^*$$
• The stock price cash-flow ratio is:

\[ V = \left( r_f + \sigma_D r_p - \mu_D \right)^{-1} \]

The proof is in appendix C.1. We can see that almost all equilibrium quantities are tied to the portfolio share \( \theta^* \). As described earlier, \( \theta^* \) represent how much of the risk of the economy is taken by passive agents. A low \( \theta^* \) corresponds to a low level of risk taken up by passive agents, and therefore they ask for a lower risk price and risk premium for the asset. The market clearing condition implies naturally that there should be more active capital to take on the risk and therefore it corresponds to larger \( M \).

With this simple characterization of the equilibrium we can derive comparative statics with respect to the parameters of the model. In particular we can see the role of risk and risk aversion for the equilibrium quantity of active capital:

**Proposition 5.2.** The portfolio share \( \theta^* \), the risk price \( r_p \), the expected excess return \( \mu_R - r_f \), the fee \( f \) and the fraction of passive capital \( 1 - M \) are:

1. **increasing in cash-flow volatility \( \sigma_D \)**
2. **increasing in relative risk aversion \( 1 - \gamma \)**

In an economy with a lot of risk or with very risk averse agents, active capital is a very costly form of ownership and therefore is less used. One can notice that the results for the risk price and the expected excess return do not appear to rely on the existence of active capital. Active capital actually affects very differently the level of the risk premium and the sensitivity of the risk premium to the volatility level and the risk aversion. Focusing for instance on the risk price, one can note that in an economy without active capital, it would be equal to \( r_p^0 = (1 - \gamma) \sigma_D \). Because \( \theta^* < 1 \), the risk price is always lower in the economy with active capital. However the elasticity of the risk price with respect to the coefficient of relative risk aversion or
the volatility are always larger in the economy with active capital. These elasticities are exactly 1 in the case of no active capital and strictly larger than 1 with active capital. For the case of cash-flow volatility we get for instance:

$$\frac{\partial \log(rp)}{\partial \log(\sigma_D)} = 1 + \frac{\partial \log(\theta^*)}{\partial \log(\sigma_D)} > 1$$

The extra term comes from the loss of active active capital in the high volatility case.

These comparative statics hint at the idea that even though cash flow fluctuations are transmitted as such to prices, the impact of fluctuations in risk and risk aversion is amplified. To show this feature in a dynamic framework instead of comparative statics, I now turn to a model with time-varying risk.

### 5.3 An economy with time-varying risk

I now introduce one state variable in the economy: the volatility of cash-flow. We have $s_t = \sigma_t^2$ and the dynamics are:

$$\frac{dD_t}{D_t} = (\mu_D + \lambda(M_t)m_t)dt + \sigma_t dZ^D_t$$

$$d\sigma_t^2 = -\kappa(\sigma_t^2 - \sigma_0^2)dt + \nu \sigma_t dZ^\sigma_t$$

where $Z^D$ and $Z^\sigma$ are independent. In order for $\sigma_t^2$ to stay positive, I impose the parameter restriction $2\kappa \sigma_0^2 > \nu^2$. The fundamental square volatility $\sigma_t^2$ follows a Feller square root process. It is a stationary mean reverting process with half life $\log(2)/\kappa$. In appendix C.3, I specialize the general results of the previous section to characterize the equilibrium.

**Proposition 5.3.** An equilibrium verifies:

- The portfolio share $\theta^*$ of the passive agents is the unique solution of the equation:

$$\lambda \left( \frac{1 - \theta^*}{1 - \frac{\sigma_0^2}{\theta_A}} \right) = \frac{(1 - \gamma)(\theta_A - \theta^*)^2 \sigma_t^2}{\theta_A} \left[ 1 + \nu^2 \left( \frac{V'}{V} \right)^2 \right]$$
• The equilibrium fraction of monitoring capital $M$ is:

$$M = \frac{1 - \theta^*}{1 - \frac{\theta^*}{\theta_A}}$$

• The risk price for cash flow shocks is $r_{p_D}$ is:

$$r_{p_D} = (1 - \gamma)\sigma_t \theta^*$$

• The risk price for volatility shocks $r_{p_\sigma}$ is:

$$r_{p_\sigma} = \nu \sigma_t \frac{V'}{V} \left[ (1 - \gamma)\theta^* + \gamma \frac{\rho - 1}{\rho} \right]$$

• The stock price cash-flow ratio is:

$$V = \left( r_f + \sigma_t r_{p_D} + r_{p_\sigma} \nu \sigma_t \frac{V'}{V} - \mu_D + \kappa \frac{V''}{V} (\sigma_t^2 - \sigma_0^2) - \frac{1}{2} \frac{V'''}{V} \nu^2 \sigma_t^2 \right)^{-1}$$

5.3.1 The active capital market

The active capital market equilibrium is similar to that in the economy without time-varying volatility. The direct effect of an increase in volatility is to increase $\theta^*$ or equivalently decrease the level $M$ of active capital. Compared to the economy with constant volatility, there is now a multiplier on fundamental volatility. Now, cash-flow shocks are not the only source of return volatility. When the $\sigma_t$ fluctuates, the price cash-flow ratio changes, which is reflected by some additional return volatility.

Two reasons cause the price cash-flow ratio to fluctuate with volatility. The first one is the standard one: when there is more volatility, the aggregate endowment is more risky so cash-flow are discounted more. But fluctuations in active capital amplify this effect. Now when volatility increase, there are less active investors and passive investors have to bear a larger fraction of aggregate risk. Therefore they ask for an even larger compensation. In this sense, future fluctuations in active capital lower the supply of active capital today.
5.3.2 Risk prices and expected returns

As for the determination of its level, active capital affects risk prices in two ways: through its current level and through future fluctuations.

The current level of active capital is characterized by the portfolio position of passive agents $\theta^*$. As active investors take on a disproportionately large fraction of aggregate risk, passive investors bear less risk. This is reflected in lower risk prices. For the price of cash-flow risk $rp_D$, this is the only impact of active capital: the price is the product of risk aversion, fundamental volatility and the portfolio share $\theta^*$. The price of volatility risk $rp_\sigma$ is not directly proportional to $\theta^*$. The compensation comes from the exposure of wealth to volatility shocks, which is lowered because of $\theta^* < 1$ and also for the exposure of the continuation utility $F$ to changes in volatility which is unaffected by the contemporaneous portfolio position. This second component explains why the price of cash-flow shock is lowered relatively more than that of volatility shocks by active capital.

A second force tends to increase the risk price of volatility: active capital increases the elasticity $V'/V$ of the asset price with respect to volatility. As explained before, this is because high volatility is twice bad news for a passive agent: the environment is more risky and he has to bear more risk. This latter effect is largest when changes in volatility have the largest effect on the amount of active capital. This corresponds to very elastic supply and demand for active capital. An elastic demand corresponds to constant return to scale for active capital. The supply is endogenous and is more elastic for large values of $\tilde{\theta}_A$ and small values of $\theta^*$.

One can see these effects on figure 3. I compare the result of the model to a model without active capital. In this case, we can see that the dominant effect for the volatility risk price and risk premium is an the increase in price volatility.
5.3.3 Asset prices through a financial crisis

I describe here a succession of events in the model that is reminiscent of a financial crisis. In normal times, active capital absorbs some of the risk in the economy. The compensation for cash-flow risk of assets is not very large and an important component of expected returns is compensation for future variation in expected returns (due to changes in volatility). When there is a large positive shock to volatility, active capital owners (the financial sector) lose a lot of their wealth, but the overall supply of capital dropped because this activity is risky. Not many agents are willing to become active and pick up their assets. The level of active capital drops in the economy and aggregate growth slows down. Furthermore, passive agents have to bear more of the economic fluctuations and therefore the level of all risk premia is even higher (and prices are further depressed). Once volatility is at its highest, there is no active capital and so no active capital fluctuations as well. Now most of expected returns are compensation for cash-flow risk. Over time, mean-reversion naturally brings back volatility to its pre-crisis level and we get back to the world of large fluctuations in risk premia and an important premium for volatility shocks. These different regimes can be followed on a numerical example on figure 4. Figure 5 shows the fraction of expected returns due to compensation for volatility shocks. We see the importance of volatility risk premium shoot up in times of low volatility.

It is important to note that these fluctuation in risk prices, in particular the larger risk price for volatility shocks do not reflect a lower utility for agents. Actually it is easy to see that agents in the economy with active capital are always better off: they can always choose to stay in autarky and have the same consumption path as in the model without active capital. The reason for these amplification is not that high volatility times are worse than without active capital, but rather than low volatility times are much better than without active capital. Therefore the fall is harder for agents when volatility increases.
6 Conclusions

In this paper, I showed that asset pricing and the provision of active capital are strongly tied. Variations in risk premia affect the supply of active capital. This generates the documented negative relation between expected returns and the amount of active capital. The quantity of active capital feeds back into risk sharing in the economy. As active investors deleverage to get out of their positions in times of high volatility, passive investors have to bear the risk exactly when they do not want to. This feedback is a source of amplification for shocks to risk premium. On the other hand, cash flow shocks, as they are borne by active agents, are less costly for passive investors and command a lower risk premium.

These results call for further empirical investigation of the macroeconomic role of active capital. Because active capital takes many forms, inside and outside the firm, it is an important challenge to build aggregate measures in order to study its dynamics. For this a better understanding of the heterogeneity in sources of capital would be necessary. Another important question to study is the development of the active technology at a lower frequency. The last few decades have seen an important expansion of the financial industry. Understanding the interaction of investment in physical and financial technology could be crucial for this question.

For asset pricing, active capital can help understand why cash-flow shocks are not very important for expected returns. On the other hand fluctuations in risk premium command a high risk price. This is due to their amplification as they cause variations in the quantity of active capital. The positive side is that there might not be a need for large fluctuations in volatility or risk aversion to explain the level and fluctuations of the risk premium. However, it also shows that the fundamental source of these fluctuations might be hard to detect in the data.
References


A Proofs for the static model

A.1 Case of exponential utility

The utility of a passive agent with portfolio $\theta$ is:

$$
U(\theta) = \mathbb{E}[-\exp(-\alpha(1-\theta)p - \alpha \theta X)] = -\exp\left(-\alpha(1-\theta)p - \alpha \theta \mu + \frac{1}{2} \alpha^2 \theta^2 \sigma^2\right) \\
= -\exp\left(\kappa + \frac{1}{2} \alpha^2 \sigma^2 (\theta - \theta^*)^2\right)
$$

where the last equality is obtained by noticing that the expression in the exponential is a second degree polynomial in $\theta$ whose optimum $\theta^*$ is given by the standard formula:

$$
\theta^* = \frac{\mu - p}{\alpha \sigma^2}
$$

An constrained agent has portfolio $\tilde{\theta}_A$ and receives an excess $\tilde{\theta}_AfX$. His utility is therefore:

$$
U_A = -\exp\left(\kappa + \frac{1}{2} \alpha^2 \sigma^2 (\tilde{\theta}_A - \theta^*)^2 - \alpha \tilde{\theta}_Af(\mu - \frac{1}{2} \alpha \sigma^2)\right)
$$

In order to have $U(\theta^*) = U_A$, the fee has to be:

$$
f = \frac{1}{\tilde{\theta}_A(\mu - \frac{1}{2} \alpha \sigma^2)} \frac{1}{2} \alpha (\tilde{\theta}_A - \theta^*)^2 \sigma^2
$$

A.1.1 Partial equilibrium

If active capital is possible only for an infinitesimal number of firms, market clearing imposes $\theta^* = 1$ for the portfolio of passive agents. The price is then equal to $p = \mu - \alpha \sigma^2$. It is direct to see that the equilibrium fee $f$ is increasing both in volatility $\sigma$ and risk aversion $\alpha$.

A.1.2 General equilibrium

The supply curve is the possible couples of values $(f, M)$. Market clearing imposes a negative relationship between $M$ and $\theta^*$, where the only parameter
involved is $\tilde{\theta}_A$. We see that $f$ is decreasing in $\theta^*$ so the supply curve is increasing. Additionally, we see that increases in $\alpha$ or $\sigma$ increase the fee given a level $M$ of active capital. This tells us that the supply is decreasing in volatility and risk aversion.

A.2 Non-increasing supply curve

A.3 Welfare analysis

I solve for the equilibrium given the fee $f$ imposed to firms and then compute the derivative with respect to $f$.

The payoff of the residual claim is $(1 + (\lambda - f)M)X$ and its price is $p$. Note $p_X$ the price of a claim to $X$. We have $p = (1 + (\lambda - f))p_X$.

- **Passive agent:**

  $$U = \max_{\theta, b} \mathbb{E}[u(\theta(1 + (\lambda - f)M)X + b)]$$

  s.t. $p = \theta p + b$ \ [\lambda]

  F.O.C.:

  \begin{align*}
  (b) & \quad \lambda = \mathbb{E}[u'(\theta(1 + (\lambda - f)M)X + b)] \\
  (\theta) & \quad \lambda p = \mathbb{E}[(1 + (\lambda - f)M)Xu'(\theta(1 + (\lambda - f)M)X + b)]
  \end{align*}

  Note $C$ his consumption.

- **Active agent:**

  $$U_A = \max_{\theta, b_A} \mathbb{E}[u(\theta(1 + (\lambda - f)M + f)X + \theta_AfX + b_A)]$$

  s.t. $p = \theta p + b_A$ \ [\lambda_A]

  $\theta = \theta_A$ \ [\mu_A]

  F.O.C. (Both constraints are binding and multipliers are positive):

  \begin{align*}
  (b) & \quad \lambda_A = \mathbb{E}[u'(\theta_A(1 + (\lambda - f)M + f)X + b_A)] \\
  (\theta) & \quad \lambda_A p - \mu_A = \mathbb{E}[(1 + (\lambda - f)M)Xu'(\theta_A(1 + (\lambda - f)M + f)X + b_A)]
  \end{align*}
Note $C_A$ his consumption.

- **Partial derivatives:**
  
  $$\frac{\partial U}{\partial p} = (1 - \theta)E[u'(C)] \geq 0$$
  
  $$\frac{\partial U_A}{\partial p} = (1 - \theta_A)E[u'(C_A)] \leq 0$$
  
  $$\frac{\partial U}{\partial f} = -\theta M E[Xu'(C)] \leq 0$$
  
  $$\frac{\partial U_A}{\partial f} = (1 - M)\theta_A E[Xu'(C_A)] \geq 0$$
  
  $$\frac{\partial U}{\partial M} = (\lambda - f)\theta E[Xu'(C)] \text{ (sign of } \lambda - f)$$
  
  $$\frac{\partial U_A}{\partial M} = (\lambda - f)\theta_A E[Xu'(C_A)] \text{ (sign of } \lambda - f)$$

- **Indifference condition**
  
  $$\frac{dU}{df} = \frac{dU_A}{df}$$

  Expanding:
  
  $$\frac{\partial U}{\partial f} + \frac{dp}{df} \left[ \frac{\partial U}{\partial p} + \frac{dM}{dp} \frac{\partial U}{\partial M} \right] = \frac{\partial U_A}{\partial f} + \frac{dp}{df} \left[ \frac{\partial U_A}{\partial p} + \frac{dM}{dp} \frac{\partial U_A}{\partial M} \right]$$

  Or:
  
  $$\frac{dp}{df} = \frac{\frac{\partial U}{\partial f} - \frac{\partial U_A}{\partial f}}{\frac{\partial U}{\partial p} + \frac{dM}{dp} \frac{\partial U}{\partial M} - \frac{\partial U_A}{\partial p} - \frac{dM}{dp} \frac{\partial U_A}{\partial M}}$$

- **Case $f = \lambda$:**
  
  $$\frac{dp}{df} = \frac{\frac{-\partial U}{\partial f} + \frac{\partial U_A}{\partial f}}{\frac{\partial U}{\partial p} + \frac{dM}{dp} \frac{\partial U}{\partial M}} > 0$$

  An increase in $f$ causes an increase in the price $p$. We want to know if $dU/df \geq 0$. This is equivalent to:
  
  $$\frac{dp}{df} \geq \frac{\frac{\partial U}{\partial f}}{\frac{\partial U}{\partial p}}$$
The right-hand side is:

\[-\frac{\partial U}{\partial f} = \frac{\theta M}{1 - \theta} \mathbb{E}[Xu'(C)] = \frac{\theta M}{1 - \theta} \mathbb{P}_X\]

The left-hand side is between \(-\frac{\partial U}{\partial f}\) and \(-\frac{\partial U}{\partial A}\)\(^8\). The first term is exactly the right-hand side and the other one is:

\[-\frac{\partial U}{\partial A} = \frac{\theta_A(1 - M)}{\theta_A - 1} \mathbb{E}[Xu'(C_A)] < \frac{(1 - M)\theta_A}{\theta_A - 1} \mathbb{P}_X = \frac{\theta M}{1 - \theta} \mathbb{P}_X\]

where the last equality is obtained by noticing that:

\[\frac{1 - \theta}{\theta M} + \frac{1 - \theta_A}{(1 - M)\theta_A} = (M(1 - M))^{-1} \left[ (1 - \theta) (1 - \frac{M}{\theta}) + (1 - \theta_A) \frac{M}{\theta_A} \right] = (M(1 - M))^{-1} \left[ \frac{1 - M}{\theta} + \frac{M}{\theta_A} - (1 - M) - M \right] = 0\]

We can then conclude that \(\frac{dp}{df} < -\frac{\partial U}{\partial f}\)\(^9\). Equivalently we have showed that \(\frac{dU}{df} < 0\). An increase in the fee decreases welfare. There is always too much active capital in the competitive equilibrium.

**B Proofs for the dynamic model**

**B.1 Firm problem**

We can write the HJB for the firm value \(V_t\). After dividing by \(S_tD_tV_t\), I get:

\[0 = \max_m \frac{1}{V} - fm + \mathbb{E}[d(SDV)]/SDVdt\]

\(^8\)This is just saying that if \(a, b, c, d > 0\) verify \(a/b < c/d\), then \(a/b < (a+c)/(b+d) < c/d\).

\(^9\)The strict inequality is clear as soon as \(M \neq 0\).
Because of the homogeneity of the problem, the price-cash flow ratio $V_t$ is just a function of the state variables $s_t$. I note it $V_t = V(s_t)$. Plugging into the HJB equation we obtain:

$$0 = \max_m \frac{1}{V} - fm + \frac{\mathbb{E}[dS]}{Sdt} + \frac{\mathbb{E}[dD]}{Ddt} + \frac{\mathbb{E}[dV]}{Vdt} + \frac{\mathbb{E}[dD]}{Sdt} + \frac{\mathbb{E}[dV]}{Sdt} + \frac{\mathbb{E}[dD]}{Sdt}$$

The activity level only appears in the flow term $-fm$ and the drift of the size of the firm $\mathbb{E}[dD]/D = \mu_D(s) + \lambda(M)(m)$. The first-order condition is

(B.1) $$f(s) = \lambda(M(s))$$

**B.2 Agent problem**

**B.2.1 Obtaining the HJB**

$$0 = \max_{C \geq 0, \theta \in \Theta} f(C, J) + \mathbb{E}(dJ)/dt$$

$$0 = \max_{c \geq 0, \theta \in \mathbb{R}} f(\gamma^{1/\gamma}c, F) \frac{W^\gamma}{\gamma} + \mathbb{E} \left[ d \left( \frac{W^\gamma}{\gamma} F \right) \right]/dt$$

**B.2.2 Solving for consumption and the fee**

First compute the various derivatives of the value function $J$: 44
\[ J = \frac{W^\gamma F}{\gamma} \]
\[ J_W = \gamma \frac{W^{\gamma-1} F}{\gamma} \]
\[ J_{WW} = \gamma(\gamma - 1) \frac{W^{\gamma-2} F}{\gamma} \]
\[ J_{Ws} = \gamma \frac{W^{\gamma-1} F_s}{\gamma} \]

Let us focus first on the HJB of the passive agent:

\[ 0 = \max_{C, \theta} f(C, J) + \frac{\mathbb{E}[dJ]}{dt} \]

Expanding gives:

\[ 0 = \max_{C, \theta} f(C, J) \]

\[ + J_W [W(\theta'(\mu_R - r_f) + r_f) - C] \]

\[ + J'_s \mu_s \]

\[ + \frac{1}{2} J_{WW} W^2 \theta' \sigma_R \sigma'_R \theta \]

\[ + W \theta' \sigma_R \sigma'_s J_{Ws} \]

\[ + \frac{1}{2} (\sigma'_s \sigma_s) * J_{ss} \]

where * is the elementwise multiplication. The first-order condition for consumption is:

\[ f_C(C, J) = J_W \]

The first-order condition for the portfolio is:

\[ \theta^* = -\frac{J_W W}{J_{WW} W^2} (\sigma_R \sigma'_R)^{-1} (\mu_R - r_f) - (\sigma_R \sigma'_R)^{-1} \sigma_R \sigma_s \frac{J_{Ws} W}{J_{WW} W^2} \]

Plugging in using the formulas for the derivatives, we get:

\[ c = C/W = \beta^\frac{1}{(\nu - 1)} F^\frac{1}{(\nu - 1)\gamma} \]

\[ \theta^* = \frac{1}{1 - \gamma} (\sigma_R \sigma'_R)^{-1} (\mu_R - r_f) + \frac{1}{1 - \gamma} (\sigma_R \sigma'_R)^{-1} \sigma_R \sigma'_s F_s \]
Now we can turn to the problem of the passive agent. It is:

\[
0 = \max_{C, \theta} f(C, J) \\
+ J_W \left[ W(\tilde{\theta}'(\mu_R - r_f) + r_f + \tilde{\theta}_A f) - C \right] \\
+ J_s' \mu_s \\
+ \frac{1}{2} J_{WW} W^2 \tilde{\theta}' \sigma_R \sigma'_R \tilde{\theta} \\
+ W \tilde{\theta}' \sigma_R \sigma'_s J_{Ws} \\
+ \frac{1}{2} (\sigma'_s \sigma_s) * J_{ss}
\]

The first order condition for consumption is clearly the same as for the active agent which proves that they pick the same consumption wealth ratio. To solve for the fee, just substract this HJB from that of the passive investor. It gives:

\[
J_W \tilde{\theta}_A f = \left[ J_W W \theta^{**'} (\mu_R - r_f) + \frac{1}{2} J_{WW} W^2 \theta^{**'} \sigma_R \sigma'_R \theta^* + W \theta^{**'} \sigma_R \sigma'_s J_{Ws} \right] \\
- \left[ J_W W \tilde{\theta}' (\mu_R - r_f) + \frac{1}{2} J_{WW} W^2 \tilde{\theta}' \sigma_R \sigma'_R \tilde{\theta} + W \tilde{\theta}' \sigma_R \sigma'_s J_{Ws} \right]
\]

The right-hand side is the difference of two values of a affine-quadratic form where one of the evaluation point is the optimum. It is therefore exactly quadratic, and equal to, after dividing by \( J_W \):

\[
\tilde{\theta}_A f = (1 - \gamma)(\theta^* - \tilde{\theta})' \sigma_R \sigma'_R (\theta^* - \tilde{\theta})
\]

which concludes the proof.
C Solutions of the examples

C.1 Solution of the i.i.d. model

The stochastic discount factor follows:
\[
\frac{dS}{S} = -r_f dt - rp dZ
\]

- **Price of the firm:** The HJB is:
\[
\frac{1}{V} = -\frac{\mathbb{E}[d(SD)]}{SD} = -\frac{\mathbb{E}[d(D)]}{D} - \frac{\mathbb{E}[d(S)]}{S} - \frac{<dS, dD>}{SD} = -\mu_D + r_f + \sigma_D rp
\]
Or \( V = 1/(r_f + \sigma_D rp - \mu_D) \) which is the standard Gordon growth formula.

- **Stock return:** Because the volatility of stock return is \( \sigma_D \) we get immediatly the dynamics of returns:
\[
\sigma_R = \sigma_D \ 
\mu_R - r_f = \sigma_D rp
\]

- **Value function of the passive investor:**
\[
0 = \max_{c, \theta} f(C, J) + W J W [\theta(\mu_R - r_f) + r_f - C/W] + \frac{1}{2} W^2 J W W \theta^2 \sigma^2_R
\]
Using the homogeneity: \( J = W^{\gamma} F/\gamma \), we get:
\[
0 = \max_{c, \theta} f(\gamma^{1/\gamma} c, F) + \gamma F [\theta(\mu_R - r_f) + r_f - c] + \frac{1}{2} F \gamma (\gamma - 1) \theta^2 \sigma^2_R
\]
The optimal portfolio is given by the standard Merton formula:
\[
\theta^* = \frac{\mu_R - r}{(1 - \gamma) \sigma^2_R} = \frac{rp}{(1 - \gamma) \sigma_D}
\]
And the consumption wealth ratio is:
\[
c = \beta^{\frac{1}{\gamma - 1}} F \gamma^{\frac{\mu}{\gamma - 1}}
\]
• **Amount of active capital and activity fee:** We need to have:

\[
\frac{M}{\theta_A} + \frac{1 - M}{\theta^*} = 1
\]

Which gives:

\[
M = \frac{1 - \theta^*}{1 - \frac{\theta^*}{\theta_A}}
\]

For the fee, we have:

\[
f = \lambda(M) = \lambda \left( \frac{1 - \theta^*}{1 - \frac{\theta^*}{\theta_A}} \right)
\]

The indifference condition between occupations is:

\[
\tilde{\theta}_A f = (1 - \gamma)(\tilde{\theta}_A - \theta^*)^2 \sigma_D^2
\]

The first equation for \( f \) is increasing in \( \theta^* \) whereas the second one is decreasing. The boundary conditions on the function \( \lambda \) insure the existence of a solution.

### C.2 Solution of the model with jumping volatility

#### C.2.1 Stochastic discount factor

The stochastic discount factor evolution has a drift, a diffusion and a jump component:

\[
\frac{dS_t}{S_t} = -r_c(s_t) - rp(s_t)dZ_t + \sum_{j \neq s_t} (S_{stj} - 1) dN^j_t
\]

where \( dN^j_t \) is a counter of jumps to state \( j \). The risk-free rate in state \( i \) given by:

\[
r_f(i) = -\mathbb{E} \left[ \frac{dS_t}{S_t dt} | s_t = i \right]
\]

\[
= r_c(i) - \sum_{j \neq i} \Lambda_{ij} (S_{ij} - 1)
\]
C.2.2 Firm’s HJB

The firm HJB is:

\[ 0 = \max_m \frac{1}{V} - \frac{f}{m} + \frac{\mathbb{E}[d(SDV)]}{SDV dt} \]

It is clear that the first order condition is still \( \lambda(M) = f \) and that the linearity allows us to solve as if \( m = 0 \). Expanding, we get:

\[ 0 = \frac{1}{V_i} + \mu_D - r_{c,i} - rp_i \sigma_i + \sum_{j \neq i} \Lambda_{ij} \left(S_{ij} \frac{V_j}{V_i} - 1\right) \]

Rearranging and plugging in for the risk-free rate, we obtain the Gordon growth formula:

\[ \frac{1}{V_i} = -\mu_D + r_{f,i} + rp_i \sigma_i + \sum_{j \neq i} \Lambda_{ij} S_{ij} \left(1 - \frac{V_j}{V_i}\right) \]

C.2.3 Stock return dynamics

The return evolution for a firm share is given by:

\[
\frac{dR_t}{R_t} = \frac{D_t}{P_t} dt + \frac{dP_t}{P_t} + \frac{dV_t}{V_t} = \frac{1}{V_t} dt + \frac{dD_t}{D_t} + \frac{dV_t}{V_t}
\]

If the current state is \( i \), we obtain:

\[
\frac{dR}{\tilde{R}} = \left(1 + \frac{\mu_D}{V_i}\right) dt + \sigma_i dZ_t + \sum_{j \neq i} \left(\frac{V_j}{V_i} - 1\right) dN^j
\]

Expected returns in state \( i \) are given by:

\[
\mathbb{E} \left[ \frac{dR}{R dt} \right] = \frac{1}{V_i} + \mu_D + \sum_{j \neq i} \Lambda_{ij} \left(\frac{V_j}{V_i} - 1\right)
\]

\[
= r_{f,i} + rp_i \sigma_i + \sum_{j \neq i} \Lambda_{ij} \left(\frac{V_j}{V_i} - 1\right) (1 - S_{ij})
\]

where the second line is obtained by plugging in the valuation equation for the firm.
C.2.4 Return dynamics for jump insurance claims

In state $i$ there are claims to insure against jump to each other state $j \neq i$. The claims are free and they pay 1 if there is a switch to state $j$ and a flow $\tilde{\mu}_j = -\Lambda_{ij}S_{ij}dt$.

We need to check that the price of this claim is indeed 0:

$$E_t \left( \frac{S_{t+dt}}{S_t} X_{t+dt} \right) = E_t \left( X_{t+dt} \right) + E_t \left( \frac{dS_t}{S_t} X_{t+dt} \right)$$

$$= -\Lambda_{ij}S_{ij}dt + \Lambda_{ij}dt + \Lambda_{ij}(S_{ij} - 1)dt$$

$$= 0$$

C.2.5 Passive agent portfolio problem

The passive agents can invest fractions of his wealth $\theta_R$ in the stock and $\{\theta_j\}_{j \neq i}$ in the insurance claims. At equilibrium he will choose to invest $\theta^*$ in the stock and 0 in all the insurance claims.

The law of motion for his wealth if the current state is $i$ is given by:

$$\frac{dW_t}{W_t} = (\theta_R(\tilde{\mu}_{R,i} - r_{f,i}) + \sum_{j \neq i} \theta_j \tilde{\mu}_j + r_{f,i} - c)dt$$

$$+ \theta_R \sigma_i dZ_t$$

$$+ \sum_{j \neq i} \left( \theta_R \left( \frac{V_j}{V_i} - 1 \right) + \theta_j \right) dN^j$$

The HJB equation for his consumption-portfolio problem is:

$$0 = \max_{c,\theta_R,\{\theta_j\}_{j \neq i}} f(cW, J) + \frac{E[dJ]}{dt}$$

where $J$ is a function of current wealth $W_t$ and the state $s_t$. 

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Expanding, we get:

\[ 0 = \max_{c, \theta_R, \{\theta_j\}_{j \neq i}} f(cW, J) \]

\[ + W J_W (\theta_R (\bar{\mu}_{R,i} - r_{f,i}) + \sum_{j \neq i} \theta_j \bar{\mu}_j + r_{f,i} - c) \]

\[ + \frac{1}{2} W^2 J_{WW} \theta_R^2 \sigma_i^2 \]

\[ + \sum_{j \neq i} \Lambda_{ij} \left[ J \left( W \left[ 1 + \theta_R \left( \frac{V_j}{V_i} - 1 \right) + \theta_j \right], j \right) - J(W, i) \right] \]

Using the homogeneity of preferences and the linearity of dynamics we can rewrite \( J(W, i) = F_i W^{\gamma}/\gamma \). The derivatives are \( J_W = \gamma J/W \) and \( J_{WW} = \gamma(\gamma - 1) J/W^2 \). The aggregator also reduces: \( f(cW, J) = f(\gamma^{1/c} c, F) W^{\gamma}/\gamma \).

Plugging in those results and dividing by \( J \), we get:

\[ 0 = \max_{c, \theta_R, \{\theta_j\}_{j \neq i}} \beta \gamma \left( \frac{c^\rho}{F_i^{\rho/\gamma}} - 1 \right) \]

\[ + \gamma (\theta_R (\bar{\mu}_{R,i} - r_{f,i}) + \sum_{j \neq i} \theta_j \bar{\mu}_j + r_{f,i} - c) \]

\[ + \frac{1}{2} \gamma(\gamma - 1) \theta_R^2 \sigma_R^2 \]

\[ + \sum_{j \neq i} \Lambda_{ij} \left[ \left( 1 + \theta_R \left( \frac{V_j}{V_i} - 1 \right) + \theta_j \right)^{\gamma} F_j^\rho F_i - 1 \right] \]

The first-order conditions are:

**Consumption** \( c \): \( \beta \gamma \frac{c^{\rho-1}}{F_i^{\rho/\gamma}} = \gamma \) or equivalently:

\[ c = \beta^{1/\rho} F_i^{\rho/\gamma (\rho-1)} \]

**Insurance claims** \( \theta_j(= 0) \):

\[ -\bar{\mu}_j = \Lambda_{ij} \left[ 1 + \theta^* \left( \frac{V_j}{V_i} - 1 \right) \right]^{\gamma-1} F_j^\rho F_i \]

\[ S_{ij} = \left[ 1 + \theta^* \left( \frac{V_j}{V_i} - 1 \right) \right]^{\gamma-1} F_j^\rho F_i \]
Stock position \( \theta_R = \theta^* \)

\[
\theta^* = \frac{1}{(1-\gamma)\sigma^2_i} \left[ \frac{1}{V_i} + \mu_D + -r_{f,i} \sum_{j\neq i} \Lambda_{ij} S_{ij} \left( \frac{V_j}{V_i} - 1 \right) \right]
\]

\[
= \frac{1}{(1-\gamma)\sigma^2_i} r p_i \sigma_i
\]

As in the i.i.d. model we can express the risk price as a simple function of the portfolio share:

\[ r p_i = (1-\gamma)\sigma_i \theta^* \]

### C.2.6 Active agent indifference condition

The law of motion for the portfolio of the active agent is:

\[
\frac{dW_t}{W_t} = (\tilde{\theta}_A(\tilde{\mu}_{R,i} - r_{f,i} + f) + r_{f,i} - c) dt
\]

\[
+ \tilde{\theta}_A \sigma_i dZ_t
\]

\[
+ \sum_{j\neq i} \left( \tilde{\theta}_A \left( \frac{V_j}{V_i} - 1 \right) \right) dN_j
\]

Similarly to the passive agent’s problem, the HJB reduces to:

\[
0 = \max_c \beta \frac{\gamma}{\rho} \left[ \frac{c}{F^\rho} - 1 \right]
\]

\[
+ \gamma(\tilde{\theta}_A(\tilde{\mu}_{R,i} - r_{f,i} + f) + r_{f,i} - c)
\]

\[
+ \frac{1}{2} \gamma (\gamma - 1) \tilde{\theta}_A^2 \sigma^2_R
\]

\[
+ \sum_{j\neq i} \Lambda_{ij} \left[ \left( 1 + \tilde{\theta}_A \left( \frac{V_j}{V_i} - 1 \right) \right)^\gamma \frac{F_j}{F_i} - 1 \right]
\]

The first-order condition for consumption is the same as before. Substracting the two HJB, we obtain the equation determining the fee:

\[
\tilde{\theta}_A f = (\theta^* - \tilde{\theta}_A) + \frac{1}{2} (\theta^* - \tilde{\theta}_A^2) (\gamma - 1) \sigma^2_i
\]

\[
+ \frac{1}{\gamma} \sum_{j\neq i} \Lambda_{ij} \left[ \left( 1 + \theta^* \left( \frac{V_j}{V_i} - 1 \right) \right)^\gamma - \left( 1 + \tilde{\theta}_A \left( \frac{V_j}{V_i} - 1 \right) \right)^\gamma \right] \frac{F_j}{F_i}
\]
C.2.7 Market clearing conditions

In addition to these conditions, all the cash-flow have to be consumed. Therefore \( c = 1/V \). To obtain the risk-free rate, just plug everything into the HJB of the passive agent.

We can simplify the formula for the price of the firm:

\[
\frac{1}{V_i} = -\mu_D + r_f + (1-\gamma)\sigma_i\theta^*_i + \sum_{j \neq i} \Lambda_{ij} \left[ 1 + \theta^*_i \left( \frac{V_j}{V_i} - 1 \right) \right] \gamma^{-1} \left( \frac{V_j}{V_i} \right)^{\gamma(1-\rho)/\rho} \left( 1 - \frac{V_j}{V_i} \right)
\]

C.3 Solution of the model with time-varying volatility

C.3.1 Stochastic discount factor

The stochastic discount factor is:

\[
\frac{dS}{S} = -r_f(\sigma_i^2) - r_D(\sigma_i^2)dZ^p - r_D(\sigma_i^2)dZ^\sigma
\]

C.3.2 Firm problem

The HJB is:

\[
0 = \frac{1}{V} + \frac{\mathbb{E}[d(SDV)]}{SDVdt}
\]

We are solving for the function \( V(\sigma_i^2) \). We obtain an ODE:

\[
-\frac{1}{V} = \frac{\mathbb{E}[dS]}{Sdt} + \frac{\mathbb{E}[dD]}{Ddt} + \frac{\mathbb{E}[dV]}{Vdt} + \frac{\langle dS, dD \rangle}{SDdt} + \frac{\langle dS, dV \rangle}{SVdt}
\]

\[
= -rf + \mu_D - \kappa(\sigma_i^2 - \sigma_0^2)\frac{V'}{V} + \frac{1}{2} \nu^2 \sigma_i^2 \frac{V''}{V}
\]

\[
- r_D\sigma_i - r_D\nu\sigma_i \frac{V'}{V}
\]

Or:

\[
\frac{1}{V} = -\mu_D + r_f + r_D\nu\sigma_i \frac{V'}{V} + r_D\sigma_i + \kappa \frac{V'}{V} (\sigma_i^2 - \sigma_0^2) - \frac{1}{2} \frac{V''}{V} \nu^2 \sigma_i^2
\]
C.3.3 Stock return

The stock return for the firm is given by:

\[
\frac{dR}{R} = \left( \frac{1}{V} + \mu_D - \kappa \frac{V'}{V} (\sigma_t^2 - \sigma_0^2) + \frac{1}{2} \frac{V''}{V} \sigma_t^2 \right) dt + \sigma_t \left( dZ^D + \nu \frac{V'}{V} dZ^\sigma \right)
\]

The expected return is:

\[
\mathbb{E}_t \left[ \frac{dR}{R dt} \right] = \frac{1}{V} + \mu_D - \kappa \frac{V'}{V} (\sigma_t^2 - \sigma_0^2) + \frac{1}{2} \frac{V''}{V} \nu \sigma_t^2 = r_f + r p D \sigma_t + r p \nu \sigma_t \frac{V'}{V}
\]

C.3.4 Return dynamics for volatility insurance

Zero-cost asset that pays off \( r p \sigma dt + dZ^\sigma \). We need to check that its price is indeed zero:

\[
\mathbb{E} \left[ \frac{S_t+dt}{S_t} X_{t+dt} \right] = \mathbb{E} [X_{t+dt}] + \mathbb{E} \left[ \frac{dS_t}{S_t} X_{t+dt} \right] = r p \sigma dt - r p \sigma dt = 0
\]

C.3.5 Portfolio problem of a passive agent

The passive agent invests a fraction of his wealth \( \theta_R \) in the stock and \( \theta_\sigma \) in the volatility asset. The law of motion for his wealth is then:

\[
dW = W(\theta_R(\mu_R - r_f) + \theta_\sigma r p_\sigma + r_f - c) dt + W \theta_R \sigma_t dZ^D + W(\theta_\nu \frac{V'}{V} \sigma_t + \theta_\sigma) dZ^\sigma
\]

The HJB of his problem is:

\[
0 = \max_{c, \theta_R, \theta_\sigma} f(cW, J) + \mathbb{E} \left[ \frac{dJ}{J} \right]
\]

As usual we expand and divide by \( J \), using the fact that the solution can be written \( F(\sigma_t^2) W^\gamma / \gamma \):
We can now take first-order conditions and plug in equilibrium values: $\theta_R = \theta^*$ and $\theta_\sigma = 0$.

Consumption $c$: It is still the same formula:

$$c = \beta^{-1} F^{\frac{\theta - 1}{\gamma}}$$

Volatility insurance $\theta_\sigma = 0$:

$$0 = \gamma rp_\sigma + \gamma(\gamma - 1)\theta^* \nu \frac{V'}{V} \sigma_t + \gamma F' \nu \sigma_t$$

$$rp_\sigma = \left[(1 - \gamma)\theta^* \frac{V'}{V} - \frac{F'}{F}\right] \nu \sigma_t$$

Stock position $\theta_R = \theta^*$

$$0 = (\mu_R - r_f) + (\gamma - 1) \left[\theta^* \sigma_t^2 + \theta^* \nu^2 \left(\frac{V'}{V}\right)^2 \sigma_t^2\right] + \frac{F'}{F} \nu^2 \frac{V'}{V} \sigma_t^2$$

Plugging in for $\mu_R - r_f$ as a function of risk premia and arranging terms, we get:

$$(1 - \gamma)\theta^* \sigma_t^2 - rp_D \sigma_t = \nu \sigma_t \frac{V'}{V} \left[rp_\sigma + (\gamma - 1)\theta^* \nu \frac{V'}{V} \sigma_t + \frac{F'}{F} \nu \sigma_t\right]$$

$$= 0$$
as we recognize the equation for the volatility risk price on the right-hand side. This gives us:

\[ r p_D = (1 - \gamma) \theta^* \sigma_t \]

### C.3.6 Active agent indifference condition

The law of motion of wealth for the active agent is:

\[
dW = W(\tilde{\theta}_A(\mu - r_f + f) + r_f - c)dt + W\tilde{\theta}_A \sigma_t \left( dZ^D + \nu \frac{V'}{V} dZ^\sigma \right)
\]

We obtain a similar HJB, except for the fee. As the dependency in \( \theta_R \) is a polynomial of degree 2 and \( \theta^* \) is the optimum, we obtain immediately:

\[
\tilde{\theta}_A f = \frac{1}{2}(1 - \gamma)(\tilde{\theta}_A - \theta^*)^2 \sigma_t^2 \left[ 1 + \nu^2 \left( \frac{V'}{V} \right)^2 \right]
\]

### C.3.7 Market clearing

As before we have, from the firms’ FOC:

\[
f = \lambda(M) = \lambda \left( \frac{1 - \theta^*}{1 - \frac{\theta^*}{\tilde{\theta}_A}} \right)
\]

which with the indifference condition of agents clears the market for active capital.

Consumption has to equal cash-flow so \( c = 1/V \). Finally, the risk-free rate \( r_f \) is pinned down by plugging in all other values in the HJB of any agent.
Figure 1: **Equilibrium of the active capital market**, case of infinitesimal $\Delta$.
Figure 2: **Supply of active capital**, case of $A = [0, 1]$
Figure 3: Solution to the model with active capital (red) and without (blue)
Figure 4: Solution to the model with active capital (red) and without (blue)
Figure 5: Fraction of expected returns due to volatility risk premium
(red: active capital; blue: without active capital)