Complex Securities∗

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Abstract

This paper proposes a dynamic equilibrium model to study the implications of the complexity of financial securities for investor behavior, asset prices, and welfare. The key assumption is that, unlike fund managers, non-professional investors cannot directly observe complex securities’ payoffs. In equilibrium, fund managers overinvest in complex securities as these potentially allow them to inflate investors’ expectations about their funds’ performance, thereby attracting more capital and thus more fees. This is so even though investors are not fooled in equilibrium. Overinvestment in complex securities drives up their price, leading to a “complexity premium” and a social welfare loss. The complexity premium creates incentives for “quants” to complexify simple securities. The supply of complex securities by quants is inverse-U shaped over time.

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1 Introduction

Complex financial securities, such as “structured” products (e.g., CDOs) or other complicated financial derivatives, are central to modern financial markets. Their importance was highlighted during the recent financial crisis, in which structured products seem to have played a key role. In this paper, I propose a dynamic equilibrium model to study the implications of the complexity of financial securities for investor behavior, asset prices, and welfare, as well as the supply of complex securities.

When analyzing complexity, it is important to consider financial intermediaries, such as professional fund managers. Indeed, with financial markets’ growing complexity and sophistication, their expertise has become essential for non-professional investors unable to handle complex investments on their own. With the need for delegation, however, complexity also creates problems of information asymmetry between experts and laymen. This paper emphasizes this effect, i.e., complexity can exacerbate agency problems between non-professional investors and the professionals who manage money on their behalf.

I consider a discrete-time infinite-horizon model with one risky and one risk-free security. The risky security’s per-period stochastic payoff is the sum of fundamental value and idiosyncratic noise, both unobservable. Thus, the agents update their estimates of the security’s fundamental value based on the payoff history. The model has two key ingredients.

The first ingredient is delegated portfolio management. There is a continuum of investment funds, each consisting of one risk-neutral fund manager and one risk-averse investor. The investor can invest directly in the risky-free security. To access the risky security, however, she must give capital to the manager, who forms a portfolio of the risky and risk-free securities and incurs a cost for holding the risky security. Importantly, the investor cannot directly observe the manager’s portfolio choice. The investor’s payment to the manager consists of a management fee (a fraction of asset under management) and a performance fee (a fraction of excess return on the fund’s portfolio).

Second, the risky security is “complex,” i.e., the investors do not “understand” its nature as well as the managers do. To capture this idea, I assume that, unlike fund managers, the investors cannot observe the risky security’s payoffs. This assumption can be interpreted as an extreme case of asymmetric financial expertise, i.e., the investors have a lower ability to process technical information. For instance, to fully understand the payoff values of a complex security (e.g., a CDO tranche), an investor may need to read and process the security’s prospectus and disclosure documents which are hundreds-of-pages long and filled with technical jargon. This vast volume of information and technical difficulty make the security’s payoffs effectively unobservable for the investor.1

To understand how complexity matters, note that the investor doesn’t observe two items the manager does: the fund portfolio and the complex security’s payoff. In such a situation, each investor needs to infer the security’s payoff based on the fund’s performance, the secu-

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1Brunnermeier and Oehmke (2009) refer to this type of problem as “information overload.”
rity’s price and her belief about the manager’s (unobserved) portfolio. Based on this inferred payoff, the investor forecasts the fund’s future performance, and decides how much capital to invest in the fund: the higher the inferred payoff, the more capital she invests in the fund, and hence the more fees the manager will collect in the future. This creates incentives for the manager to use complexity to “fool” the investor. That is, the manager may secretly overinvest in the complex security to boost the expected return on the portfolio (but inflicting excessive risk on the investor), in an effort to inflate the investor’s estimate of the security’s payoff, leading her to allocate more capital to the fund, which yields more fees. This leads to equilibrium outcomes which are different from the case of “simple” security, where the security’s payoffs are publicly observable.

The model predicts that a complex security’s price is higher (and thus the rate of return is lower) than that of a simple security with identical payoff distribution. The intuition is that fund managers are willing to pay a “complexity premium” because a complex security potentially allows them to influence investor beliefs and attract additional fees. Importantly, the investors are rational and are not fooled in equilibrium; however, managers pay the premium nonetheless. Indeed, as in Holmström (1999) or Stein (1989), given the investor’s belief that the manager will try to fool her (i.e., overinvest in the complex security), it is indeed optimal for the manager to do so for fear of being underestimated. Due to the security’s lower rate of return and the manager’s riskier portfolio choice, the investor’s capital investment in the fund is lower than in the simple security case. The analysis also has welfare implications. The allocation with the complex security is Pareto inferior to the one with the simple security: the investors achieve the same utility level, while the managers are worse off. Intuitively, this is because the managers end up attracting lower capital (and thus lower fees) as they cannot commit not to fool the investors using complexity.

To study the supply of complexity, I extend the model to allow simple and complex securities to coexist. I also introduce “quants,” i.e., agents who can transform the simple security into the complex one, and vice versa. The quants are effectively arbitrageurs exploiting the complexity premium. For instance, if the complexity premium is large enough, they buy the simple security, make it complex, and sell it at a profit. They incur a supply cost of complexity, which is proportional to the number of shares of complex security they supply. In equilibrium, the supply of complex security is such that the complexity premium equals the present value of future supply costs. I find that, due to learning, the supply of complex security first increases and then decreases over time. Early on, as the agents’ estimate of the security’s fundamental value is imprecise, it reacts much to new payoff realizations. Hence the managers have strong incentives to influence the investors’ estimate, leading to a high demand for the complex security. The resulting large complexity premium induces quants to supply a large amount of complex security. As learning goes on, the uncertainty about

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2This is consistent with the suggestions that CDO tranches were overpriced before 2007 (Coval, Jurek and Stafford 2009).

3This is consistent with the recent rise and fall of structured finance products (e.g., Coval, Jurek and Stafford 2008).
the security’s fundamental value decreases and the risk-averse investors increase investment in the funds, allowing the managers to earn more fees by influencing the investors’ estimate. This increases the complexity premium further, inducing the quants to increase the supply of the complex security. Over time, however, another effect working in the opposite direction becomes stronger. The investors’ estimate becomes more precise and reacts less to new payoff realizations, making complexity less attractive to the managers. This effect eventually dominates and the complexity premium decreases over time. The quants respond by reducing the supply of the complex security.

This paper’s contribution is also methodological as it proposes a tractable dynamic equilibrium asset pricing model with delegated portfolio management. In particular, the fund managers’ portfolios, asset prices and flows in and out of funds are endogenous and obtained in closed form. The model also allows us to discuss interactions between investors’ learning and the evolution of asset prices, both in steady state and for transitional dynamics. The latter analysis is important for understanding how financial markets react to financial innovation.

Standard asset-pricing models have often ignored the issue of complexity or delegation. Indeed, if all agents can process an unlimited amount of information instantly and at no cost, complexity is irrelevant. Also, standard models are typically based on neoclassical Arrow-Debreu models where agents invest directly in financial assets, and intermediaries play no role (Allen 2001).

Few theoretical papers study complexity in financial markets. Arora et al. (2009) show that, due to computational complexity, financial derivatives may amplify adverse selection between buyers and sellers. While using a different concept of complexity, this paper also shows that complexity exacerbates asymmetric information problems. In Carlin (2009), oligopolistic firms add complexity to the price structures of their products to increase their market power. In this paper too, complexity is a strategic tool for exploiting less-informed agents. In his model, the price and its complexity are directly chosen by the firms. In contrast, this paper’s main focus is on the equilibrium price formation in a competitive market given complexity of a financial security.

There is a growing theoretical literature discussing equilibrium implications of delegated portfolio management (Allen and Gorton 1993; Shleifer and Vishny 1997; Vayanos 2004; Cuoco and Kaniel 2007; He and Krishnamurthy 2009, 2010; Malliaris and Yan 2010; Vayanos and Woolley 2010). Kaniel and Kondor (2010), as in this paper, study equilibrium asset prices and trading strategies of fund managers with concerns for fund flows. While the flow-performance relationship is exogenous in their paper, in this paper it is endogenous and stems from learning. Berk and Green (2004) is also related in that the fund flows are endogenous and stem from learning. However, their model does not have asset prices and does not study explicitly the manager-investor relationship.

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4See also Caballero and Simsek (2010), in which complexity is modeled as banks’ limited knowledge of the interlinkages between the banks in a financial network.
This paper is also related to the literature on the effect of career concerns on asset prices (Dasgupta and Prat 2008; Dasgupta, Prat and Verardo 2010; Guerrieri and Kondor 2010). In these papers, fund managers try to influence the investors’ evaluation of their ability. This paper is methodologically similar in that the managers try to influence the investors’ estimates of the complex security’s value.

The paper proceeds as follows. Section 2 presents the model. I then characterize the equilibrium when the risky security is simple (Section 3), and when it is complex (Section 4). Section 5 studies the supply of simple and complex securities by quants. Section 6 concludes. The appendix contains the proofs.

2 Model

Time $t$ is discrete and goes from zero to infinity. There is a single risky security and a risk-free security. The risk-free security has an exogenous return $r > 0$, and can be thought of as a storage technology freely accessible to all agents. There are two classes of infinitely-lived competitive agents: risk-neutral fund managers and risk-averse investors. The investor can access the risky security only through the manager, who collects fees from the investor. The manager’s portfolio choice is unobservable to the investor. The manager’s portfolio choice, the investor’s investment and the risky security’s price are determined in equilibrium.

2.1 Risky security

Payoffs. At $t = 1, 2, ..., \ldots$, the risky security yields a stochastic payoff per share, $\delta_t = \bar{\delta}_t + u_t$, where $\bar{\delta}_t$ is the fundamental value, and $u_t$ the noise, i.i.d. across time, normally distributed with mean 0 and variance $1/\eta_u$. The fundamental value evolves according to $\bar{\delta}_t = \bar{\delta}_{t-1} + v_t$, where the noise $v_t$ is i.i.d. across time, normally distributed with mean 0 and variance $1/\eta_v$. The initial fundamental value, $\bar{\delta}_0$, is drawn by nature from a normal distribution with mean $\hat{\delta}_0$ and variance $1/\eta_0$. Nobody in the economy can observe $\bar{\delta}_t$, $u_t$ and $v_t$. I denote by $\mathcal{H}_t \equiv (\delta_1, \ldots, \delta_t)$ the payoff history up to time $t$.

Excess return. The security is traded in the market at a publicly observable market-clearing price, $P_t$. Its supply is normalized to 1. I denote by $R_{t+1} \equiv (\delta_{t+1} + P_{t+1}) - (1 + r)P_t$ the excess return on the risky security per share (i.e., the dollar-value excess return multiplied by $P_t$). The expected excess return conditional on $\mathcal{H}_t$ and $P_t$ is $\hat{R}_{t+1} \equiv E(R_{t+1}|\mathcal{H}_t, P_t)$, while the unconditional expectation is $\bar{R}_{t+1} \equiv E(R_{t+1})$.

Complexity. I compare the case of a “simple” security, where the investors can observe $\delta_t$, and that of a “complex” security, where they cannot. The managers can directly observe $\delta_t$ in both cases. This interpretation of complexity is motivated by the concept of “information overload” (Brunnermeier and Oehmke 2009). To fully understand the payoff values of a complex security (e.g., a CDO tranche), an investor may need to read the security’s prospectus or

5The difference in their risk preferences is only for tractability.
disclosure documents, hundreds of pages with technical jargon, and process this information. This makes the security’s payoff effectively unobservable for a non-professional investor.

Noisy demand. There are (unmodeled) noise traders who collectively buy $\epsilon_t$ shares of the risky security, where $\epsilon_t$ is unobservable and i.i.d. across time, normally distributed with mean 0 and variance $\sigma^2_\epsilon > 0$.6

2.2 Portfolio management

There is a measure-1 continuum of investment funds, each indexed by $j \in [0, 1]$. Fund $j$ consists of a single manager (manager $j$) and a single investor (investor $j$).7 Investor $j$ can invest capital in the fund, in which manager $j$ allocates the capital between the risky and the risk-free securities. I assume that each fund is “captive” in the following sense. First, investor $j$ cannot observe any activities in the other funds. Second, investor $j$ cannot invest in the other funds. These assumptions simplify the investors’ inference problem when the risky security is complex, while retaining the managers’ competitive behavior in the security market.

Portfolio. At time $t$, investor $j$ chooses her capital investment $X_{j,t}$ in fund $j$ (in dollar unit). Manager $j$ uses this capital to buy $\theta_{j,t}X_{j,t}$ shares of risky security and invests $(X_{j,t} - P_t \theta_{j,t}X_{j,t})$ in the risk-free security.8 Hence, the manager’s portfolio choice is summarized by $\theta_{j,t} \in (-\infty, \infty)$, which I call his risky portfolio. Throughout, I assume that the investor cannot observe $\theta_{j,t}$, while she can observe the total proceeds from the fund’s portfolio at time $t + 1$, $R_{t+1}\theta_{j,t}X_{j,t} + (1 + r)X_{j,t}$ (“fund performance”).

Fees. The investor pays the manager a management fee equal to a constant (exogenous) fraction of capital under management, $f_m X_{j,t}$, with $f_m \in (0, 1)$. She also pays him a performance fee equal to a constant (exogenous) fraction of the excess return on the portfolio, $f_p R_{t+1}\theta_{j,t}X_{j,t}$, with $f_p \in (0, 1)$. This specification is motivated by the so-called “2/20 rule,” the standard fee structure for hedge funds.9,10

Holding cost. I assume that the manager incurs a cost proportional to the number of shares of risky security in the portfolio: $\kappa \theta_{j,t}X_{j,t}$, where $\kappa > 0$. This cost can be interpreted

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6The variance $\sigma^2_\epsilon$ must be strictly positive, but all the results go through with $\sigma^2_\epsilon \to 0$.
7It is not important that there is only one investor in each fund. The results are similar even if there are multiple competitive investors in each fund.
8If $1 > P_t \theta_{j,t}$ the manager buys the risk-free security, while if $1 < P_t \theta_{j,t}$ he sells the security (i.e., borrow capital) to finance the purchase of the risky security.
9Dividing the fee into two parts is not only empirically plausible but also technically important in modeling, as is discussed in later sections. The presence of the management fee fixes the indeterminacy problem of $\theta_{j,t}$ and $X_{j,t}$: i.e., if $f_m = 0$, only the product $\theta_{j,t}X_{j,t}$ is determined in equilibrium. Moreover, the management fee effectively serves as a fixed cost of risky investment for the investor, and creates a link between the manager’s portfolio and the expected excess return on the risky security. The performance fee incentivizes the manager to make risky investments. Without it, in the simple security’s case, the manager would not hold the risky security due to a moral hazard problem caused by the unobservability of portfolio and the presence of holding cost (see the next paragraph).
10Ou-Yang (2003) shows that a similar fee arrangement is an optimal contract in a continuous-time delegated portfolio management problem.
as the manager’s personal cost of managing risky investments (e.g., disutility of labor). It prevents managers from choosing infinitely risky portfolio.

**Timing.** Each time $t$ is divided into two stages. In stage 1, the risky security’s payoff $\delta_t$ is realized, and the manager chooses (and fixes) portfolio $\theta_{j,t}$. In stage 2, the remaining events in time $t$ occur simultaneously. This makes the manager’s portfolio deterministic in equilibrium and simplifies the analysis.

### 2.3 Maximization problems

To focus on the problem of portfolio delegation, I assume that the managers cannot invest their own wealth in the securities, i.e., they consume their wealth within each period. Manager $j$’s utility at time $t$ is the difference between the fees revenue and the holding cost. Hence, his problem, $P_{m,j,t}$, at time $t$ is to choose $\theta_{j,t}$ to maximize

$$E \left[ \sum_{\tau=0}^{\infty} \beta^\tau \left( f_p R_{t+\tau+1} \theta_{j,t+\tau} X_{j,t+\tau} + f_m X_{j,t+\tau} - \kappa \theta_{j,t+\tau} X_{j,t+\tau} \right) \bigg| F_{j,t}^m \right],$$

(2.1)

where $\beta \in (0,1)$ is a discount factor, and $F_{j,t}^m$ is manager $j$’s information set at time $t$.

The investors face consumption-investment decision problems. At time $t$, investor $j$ allocates her wealth $W_{j,t}$ between consumption, $C_{j,t}$, investment in the fund, $X_{j,t}$, and investment in the risk-free security. Her problem, $P_{i,j,t}$, at time $t$ is to choose $C_{j,t}$ and $X_{j,t}$ to maximize

$$E \left[ -\sum_{\tau=0}^{\infty} \beta^\tau \exp(-\gamma C_{j,t+\tau}) \bigg| F_{j,t}^i \right],$$

(2.2)

where $\gamma > 0$ is a coefficient of absolute risk aversion, and $F_{j,t}^i$ is investor $j$’s information set at time $t$. Her dynamic budget constraint is

$$W_{j,t+1} = (1 - f_p) R_{t+1} X_{j,t} + f_m X_{j,t} + (1 + r)(W_{j,t} - C_{j,t}).$$

(2.3)

I.e., her wealth next period is the sum of the return from the fund (after fees) and the proceeds from her own investment in the risk-free security.

### 2.4 Learning

Since the security’s fundamental value $\bar{\delta}_t$ is unobservable, agents try to estimate it based on the payoff history $\mathcal{H}_t$. I denote the time-$t$ estimate of $\bar{\delta}_t$ as

$$\hat{\delta}_t \equiv E(\bar{\delta}_t | \mathcal{H}_t).$$

(2.4)
By standard Kalman filtering (see Appendix A), given a new observation of \( \delta_t \), \( \hat{\delta}_t \) is updated by the following rule.

\[
\hat{\delta}_t = \lambda_t \hat{\delta}_{t-1} + (1 - \lambda_t) \delta_t, \quad \text{where} \quad \lambda_t \equiv \frac{\text{Var}(\delta_t|\mathcal{H}_t)}{\text{Var}(\delta_t|\mathcal{H}_{t-1})}.
\] (2.5)

Importantly, \( \hat{\delta}_t \) is not only the estimate of \( \bar{\delta}_t \) but also that of future fundamental values, \( \bar{\delta}_{t+\tau}, \tau = 1, 2, ..., \) future payoffs, \( \delta_{t+\tau}, \tau = 1, 2, ..., \) and future values of \( \hat{\delta} \) itself (i.e., \( \hat{\delta}_t \) is a martingale). That is, \( E(\delta_{t+\tau}|\mathcal{H}_t) = E(\delta_{t+\tau+\tau}|\mathcal{H}_{t+\tau}) = E(\delta_{t+\tau+\tau}|\mathcal{H}_{t+\tau}) = \hat{\delta}_t \) holds for \( \tau = 1, 2, ...; \)

The updating factor \( \lambda_t \) evolves deterministically as

\[
\lambda_{t+1} = \frac{1}{2 + \rho - \lambda_t},
\] (2.6)

where \( \rho \equiv \eta_u/\eta_v \) is the ratio of the noise precisions (see Appendix A). It is clear from (2.6) that \( \lambda_t \) converges monotonically to

\[
\lambda^* \equiv 1 + \frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} + \rho}.
\] (2.7)

Note that \( 0 < \lambda^* < 1 \). The fact that \( \lambda^* < 1 \), together with (2.5), implies that the agents do not learn the true value of \( \bar{\delta}_t \) even in the long run. This is because the fundamental value \( \bar{\delta}_t \) itself is stochastic (i.e., \( \eta_v < \infty \)). The following assumption ensures that \( \lambda_t \) increases, and hence the precision of the agents’ information about \( \bar{\delta}_t \) also increases, over time.

**Assumption 1.** \( \lambda_1 < \lambda^* \).

This holds if, for instance, \( \eta_0 \) is low enough, i.e., if the agents are highly uncertain about \( \bar{\delta}_0 \) at \( t = 0 \).

### 2.5 Definition of Equilibrium

The equilibrium consists of price function \( P_t(\hat{\delta}_t, \epsilon_t) \), investor \( j \)'s investment \( X_{j,t} \) for \( j \in [0, 1] \) and manager \( j \)'s risky portfolio \( \theta_{j,t} \) for \( j \in [0, 1] \), such that

1. Given \( P_t(\hat{\delta}_t, \epsilon_t) \) and the others’ actions, investor \( j \) solves \( P^i_{j,t} \),
2. Given \( P_t(\hat{\delta}_t, \epsilon_t) \) and the others’ actions, manager \( j \) solves \( P^m_{j,t} \),
3. For all \( t \), the risky security’s market clears:

\[
\int_{0}^{1} \theta_{j,t} X_{j,t} dj + \epsilon_t = 1,
\] (2.8)

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11 These are checked as follows:
\[
E(\delta_{t+\tau}|\mathcal{H}_t) = E(\delta_{t+\tau-1} + \epsilon_{t+\tau}|\mathcal{H}_t) = E(\delta_{t+\tau-1}|\mathcal{H}_t) = \cdots = E(\delta_{t}|\mathcal{H}_t) = \hat{\delta}_t,
\]
\[
E(\delta_{t+\tau}|\mathcal{H}_{t+\tau}) = E(\delta_{t+\tau} + \epsilon_{t+\tau}|\mathcal{H}_{t+\tau}) = E(\delta_{t+\tau}|\mathcal{H}_{t+\tau}) = \hat{\delta}_{t+\tau},
\]
\[
E(\delta_{t+\tau}|\mathcal{H}_{t+\tau}) = E[E(\delta_{t+\tau}|\mathcal{H}_{t+\tau+\tau})|\mathcal{H}_t] = E(\delta_{t+\tau}|\mathcal{H}_t) = \hat{\delta}_t.
\]
4. Every agent has correct beliefs about the other agents’ actions,
5. Each agent updates the estimate of $\bar{\delta}_t$ by Kalman filtering.

3 Simple security

This section characterizes the equilibrium for the case of simple security, i.e., the investors can directly observe $\delta_t$. I conjecture (and later verify) the following.

1. There exists $a_t > 0$ and $b_t > 0$, which are non-stochastic and publicly known, such that the price function is
   \[ P_t(\hat{\delta}_t, \epsilon_t) = \frac{\hat{\delta}_t}{r} + \left( \frac{a_t}{1 + r} \right) \epsilon_t - b_t. \]  
   (3.1)

2. There exists $\theta_t > 0$, which is non-stochastic and publicly known, such that for all $j \in [0, 1]$, manager $j$ plays
   \[ \theta_{j,t} = \theta_t. \]  
   (3.2)

The intuition for conjecture (3.1) is simple. The first term of the RHS is the present value of expected future payoffs discounted at the risk-free rate. The second term reflects that the noise traders’ demand will increase the market-clearing price. The third term is a discount from the present value of expected future payoffs. As shown later, $b_t$ is the sum of risk premium and “fees premium,” the additional return compensating investors for the fees they pay. Conjecture (3.2) states that the manager will buy a deterministic (and time-varying) number of shares of risky security per dollar under management.

3.1 Investor’s optimization

To solve investor $j$’s problem, $P^j_{j,t}$, I conjecture that her value function is of the form
\[ V_t(W_{j,t}, \epsilon_t) = -\exp \left[ - A W_{j,t} - \frac{1}{2} B_t (1 - \epsilon_t)^2 - Z_t \right], \]  
(3.3)

where $A > 0$ is a constant, and $B_t$ and $Z_t$ are non-stochastic and time-varying.\(^\text{12}\) The Bellman equation is given by
\[ V_t(W_{j,t}, \epsilon_t) = \max_{C_{j,t}, X_{j,t}} \left[ -\exp(-\gamma C_{j,t}) + \beta E \left[ V_{t+1}(W_{j,t+1}, \epsilon_{t+1}) | \mathcal{F}^{j}_{t+1} \right] \right]. \]  
(3.4)

\(^\text{12}\)This conjecture implies that the investor achieves higher utility if her wealth $W_{j,t}$ is larger, and/or the residual supply of the risky security, $(1 - \epsilon_t)$, is larger. The term of the residual supply is quadratic. Intuitively, this is because the investor’s consumption level is increasing in the excess return on the fund’s portfolio, $R_{t+1} \theta_{j,t} X_{j,t}$, which involves a product of $R_{t+1}$ and $X_{j,t}$. Since, in equilibrium, $R_{t+1}$ and $X_{j,t}$ are both increasing and linear in the residual supply $(1 - \epsilon_t)$, the investor’s maximum utility will depend on $(1 - \epsilon_t)^2$. The terms $B_t$ and $Z_t$ are time-varying due to the investor’s learning. With learning, the investor’s estimate of $\hat{\delta}_t$ gets more precise over time, which changes her investment behavior and the utility she achieves.
The following Lemma verifies that the value function (3.3) indeed satisfies the Bellman equation (3.4) for some $A$, $B_t$ and $Z_t$.

**Lemma 1.** The value function (3.3) satisfies the Bellman equation (3.4) if:

\[
A = \gamma \left( \frac{r}{1 + r} \right),
\]

\[
B_t = A(1 - f_p) \frac{a_t}{1 + r},
\]

\[
Z_t = \frac{1}{1 + r} \left[ Z_{t+1} + \frac{1}{2} \log (1 + B_t \sigma^2) + \frac{1}{2} \left( \frac{B_t}{1 + B_t \sigma^2} \right) \right],
\]

In Lemma 1, $a_t$ has not been solved yet. I defer the derivation of $a_t$ until the next subsection, where the market clearing is imposed and the equilibrium price is identified. The investor’s optimal policy for her capital investment $X_{j,t}$ is as follows.

**Lemma 2.** For all $j \in [0, 1]$, investor $j$’s optimal investment policy is

\[
X_{j,t} = X_t \equiv \frac{(1 - f_p)\theta_t \hat{R}_{t+1} - f_m}{A (1 - f_p)^2 \theta_t^2 (\Gamma_t + \Psi_t)} + \frac{\Psi_t}{(\Gamma_t + \Psi_t) \theta_t},
\]

where

\[
\Gamma_t \equiv \text{Var} \left( \frac{\hat{\delta}_{t+1} + \hat{\delta}_{t+1}}{r} \mid \mathcal{H}_t \right) = \frac{1}{\eta u \lambda_{t+1}} \left( \frac{1 - \lambda_{t+1}}{r} + 1 \right)^2,
\]

and

\[
\Psi_t \equiv \frac{(a_{t+1} + 1)^2 \sigma^2}{1 + A (1 - f_p) \frac{a_{t+1}}{1 + r} \sigma^2}. \tag{3.10}
\]

Note that investor $j$’s optimal policy depends on her belief, $\theta_t$, about the manager’s action, not on the action itself, $\theta_{j,t}$, which she cannot observe. Note also that the investor’s policy depends on the unbiased estimate (i.e., conditional on history $\mathcal{H}_t$) of excess return $\hat{R}_{t+1}$. This is because the investor observes $\mathcal{H}_t$ regardless of the manager’s action. (This will not be the case for a complex security). The first term of (3.8) corresponds to the mean-variance solution, as standard in the CARA-normal setup: the numerator is the fund’s expected excess return (after fees), and the denominator is the adjustment due to risk aversion. Given Assumption 1, the variance factor $\Gamma_t$ is deterministically decreasing over time, as it is decreasing in $\lambda_{t+1}$. Hence, the upward pressure of $\Gamma_t$ on $X_{j,t}$ increases over time. This is intuitive. Investors being risk averse, they invest more capital in the fund as the uncertainty about $\bar{\delta}_t$ decreases due to learning. The second term of (3.8) is a “hedging demand” due to the presence of noise trading.\footnote{This results from Jensen’s inequality. That is, since the value function involves a quadratic (i.e., convex) function of the residual supply, $(1 - \epsilon_t)$, the expectation of the value function yields an additional component which is increasing in the variance of the residual supply, $\sigma^2$.} This term vanishes as $\sigma^2 \rightarrow 0$. (This results from Jensen’s inequality. That is, since the value function involves a quadratic (i.e., convex) function of the residual supply, $(1 - \epsilon_t)$, the expectation of the value function yields an additional component which is increasing in the variance of the residual supply, $\sigma^2$.)
3.2 Manager’s optimization

Because the investor observes $\mathcal{H}_t$ irrespective of the manager’s actions, the manager cannot influence the investor’s expectations about future excess returns, and hence her future capital investments. Moreover, by assumption, the manager does not face a consumption-saving decision problem. Thus, the manager’s problem is simply to maximize his within-period utility. Note that the manager’s choice of $\theta_{j,t}$ will not affect even $X_{j,t}$, since $\theta_{j,t}$ is unobservable to the investor. In other words, investor $j$ will invest $X_{j,t} = X_t$ (given by (3.8)) based on her belief that manager $j$ chooses $\theta_t$, regardless of the $\theta_{j,t}$ actually chosen. Therefore, manager $j$’s objective at time $t$ is

$$E\left(f_p R_{t+1} \theta_{j,t} X_t + f_m X_t - \kappa \theta_{j,t} X_t \bigg| \mathcal{F}^m_{j,t}\right).$$  \hspace{1cm} (3.11)

This problem can be simplified further. First, in a symmetric equilibrium, the market clearing condition (2.8) becomes

$$X_t = \frac{1 - \epsilon_t}{\theta_t}. \hspace{1cm} (3.12)$$

Second, using conjecture (3.1), the conditional expected excess return is\(^{14}\)

$$\hat{R}_{t+1} = \left(\frac{1+r}{r}\right)\hat{\delta}_t - b_{t+1} - (1+r)P_t.$$  \hspace{1cm} (3.13)

This equation states that the expected excess return is high if price $P_t$ is low and/or the estimate $\hat{\delta}_t$ is high. Combining (3.1) and (3.13), it is readily seen that $\hat{R}_{t+1} = \hat{R}_{t+1} - a_t \epsilon_t$ holds identically. This, together with (3.12), implies that maximizing (3.11) amounts to maximizing\(^{15}\)

$$\theta_{j,t} \left[ f_p (\hat{R}_{t+1} + a_t \sigma_t^2) - \kappa \right]. \hspace{1cm} (3.14)$$

Since (3.14) is proportional to $\theta_{j,t}$, the manager’s optimal action depends on the factor of

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\(^{14}\)Using conjecture (3.1), the excess return is written as $R_{t+1} \equiv \delta_{t+1} + P_{t+1} - (1+r)P_t = \delta_{t+1} + \left[\frac{\delta_{t+1}}{r} + \left(\frac{a_{t+1}}{1+r}\right) \epsilon_{t+1} - b_{t+1}\right] - (1+r)P_t$. So, since $E(\delta_{t+1} | \mathcal{H}_t) = E(\hat{\delta}_{t+1} | \mathcal{H}_t) = \hat{\delta}_t$, it follows that $\hat{R}_{t+1} \equiv E(\hat{R}_{t+1} \bigg| \mathcal{H}_t, P_t) = \left(\frac{1+r}{r}\right)\hat{\delta}_t - b_{t+1} - (1+r)P_t$.

\(^{15}\)The management fee $f_m X_t$ does not depend on $\theta_{j,t}$, and can be omitted from the objective. So, maximizing (3.11) is equivalent to maximizing $E \left( f_p R_{t+1} \theta_{j,t} X_t - \kappa \theta_{j,t} X_t \bigg| \mathcal{F}^m_{j,t}, P_t \right)$. This is rearranged as:

$$E \left[ E \left( f_p R_{t+1} \theta_{j,t} X_t - \kappa \theta_{j,t} X_t \bigg| \mathcal{F}^m_{j,t}, P_t \right) \bigg| \mathcal{F}^m_{j,t} \right] = E \left( f_p \hat{R}_{t+1} \theta_{j,t} X_t - \kappa \theta_{j,t} X_t \bigg| \mathcal{F}^m_{j,t} \right) = E \left[ f_p (\hat{R}_{t+1} - a_t \epsilon_t) - \kappa \right] \theta_{j,t} \left( \frac{1 - \epsilon_t}{\theta_t} \right) \left| \mathcal{F}^m_{j,t} \right] = \frac{\theta_{j,t}}{\theta_t} \left[ f_p (\hat{R}_{t+1} + a_t \sigma_t^2) - \kappa \right].$$

Omitting a constant $1/\theta_t$ yields (3.14). (Note that the manager takes $\theta_t$ as fixed.)
proportionality as follows:

$$\text{If } f_p(\bar{R}_{t+1} + a_t \sigma_e^2) - \kappa \begin{cases} > 0, & \text{then } \theta_{j,t} \to \infty, \\ < 0, & \text{then } \theta_{j,t} \to -\infty, \\ = 0, & \text{then } \theta_{j,t} \in (-\infty, \infty). \end{cases}$$ \quad (3.15)$$

In the conjectured symmetric equilibrium, $\theta_{j,t} = \theta_t > 0$ must hold for all $j \in [0, 1]$. Thus, conjecture (3.2) is correct (and investor $j$’s beliefs are consistent) if manager $j$ chooses $\theta_{j,t} = \theta_t$ while condition (3.15) holds with equality, i.e.,

$$\bar{R}_{t+1} = \frac{\kappa}{f_p} - a_t \sigma_e^2.$$ \quad (3.16)

Note that, under (3.16), manager $j$ is indifferent between all $\theta_{j,t}$. Hence, $\theta_{j,t} = \theta_t$ is also optimal for him. Condition (3.16) states that, in equilibrium, the expected excess return should equal the manager’s “cost-benefit ratio,” $\kappa/f_p$, adjusted by noise trading volatility. The value of $\theta_t$ is, at this stage, implicit in (3.16) since, as shown in Section 3.3, $\bar{R}_{t+1}$ depends on $\theta_t$. Once the relationship between $\bar{R}_{t+1}$ and $\theta_t$ is identified, the value of $\theta_t$ is obtained from (3.16).

### 3.3 Equilibrium

Finally, using market clearing, I derive the risky security’s equilibrium price $P_t$, its expected excess return $\bar{R}_{t+1}$, and the manager’s portfolio choice $\theta_t$.

First, I derive $\bar{R}_{t+1}$ as a function of $\theta_t$. From the investor’s optimal policy (3.8) and the market clearing condition (3.12), the conditional expected excess return is

$$\hat{R}_{t+1} = A(1 - f_p)(\Gamma_t + \Psi_t)(1 - \epsilon_t) + \frac{f_m}{(1 - f_p)\theta_t} - A(1 - f_p)\Psi_t.$$ \quad (3.17)

Taking expectations, the unconditional expected excess return is

$$\bar{R}_{t+1} = A(1 - f_p)\Gamma_t + \frac{f_m}{(1 - f_p)\theta_t}.$$ \quad (3.18)

The first term of the RHS is the risk premium. It is standard, except that it is adjusted downward by the performance fee rate $f_p$, as a higher $f_p$ leads the investor to lower her exposure to the risky security. The second term, the “fees premium,” is the additional return the investors demand to compensate the fees they pay. As expected, it is increasing in both $f_m$ and $f_p$. Importantly, the fees premium decreases with the manager’s risky portfolio $\theta_t$. Intuitively, this is because the management fee is effectively a fixed cost for the investor. Hence, an increase in $\theta_t$ leads to a lower average cost per share, which translates into a lower fees premium. Note that an increase in $\theta_t$ also leads to an increase in the risk borne by the investor, all else equal. However, the investor responds by decreasing her investment, $X_{j,t}$,
so that the total risk she bears remains the same. This is why the risk premium does not depend on \( \theta_t \).

Now I can identify the value of \( \theta_t \). I assume that the manager’s holding cost \( \kappa \) is large enough so that he does not buy an infinite amount of shares of risky security. This ensures existence.

**Assumption 2.** \( \frac{\kappa}{f_p} A(1 - f_p) \Gamma_t - a_t \sigma_t^2 > 0 \) for all \( t \).

Equating (3.18) with (3.16) yields

\[
\theta_t = \frac{f_m}{1 - f_p} - \frac{\kappa}{f_p - A(1 - f_p) \Gamma_t - a_t \sigma_t^2}.
\]

(3.19)

This solution has intuitive properties. For instance, a larger \( f_m \) is associated with a larger \( \theta_t \), since the managers increase risky investment to exploit the expected excess return \( \bar{R}_{t+1} \), which would otherwise have been higher due to the higher fees premium. A similar logic applies to \( \gamma \) (and hence \( A \)) for the risk premium.

Finally, I derive the equilibrium price. Equating (3.17) with (3.13) yields

\[
P_t = \frac{\hat{\delta}_t}{r} + \left[ \frac{A(1 - f_p)(\Gamma_t + \Psi_t)}{1 + r} \right] \epsilon_t - \frac{1}{1 + r} (b_{t+1} + \bar{R}_{t+1}).
\]

(3.20)

This implies that conjecture (3.1) is correct if and only if \( a_t = A(1 - f_p)(\Gamma_t + \Psi_t) \) and \( b_t = (b_{t+1} + \bar{R}_{t+1})/(1 + r) \). Thus, \( a_t \) and \( b_t \) must satisfy

\[
a_t = A(1 - f_p) \left[ \Gamma_t + \frac{(a_{t+1})^2 \sigma_t^2}{1 + A(1 - f_p)} \right],
\]

(3.21)

\[
b_t = \sum_{\tau=1}^{\infty} \left( \frac{1}{1 + r} \right)^\tau \bar{R}_{t+\tau}.
\]

(3.22)

That is, the equilibrium price is equal to, on average, the present value of expected future payoffs \( \left( \frac{\hat{\delta}_t}{r} \right) \) minus the present value of future expected excess returns (i.e., the present value of future risk premia and fees premia).

I can now characterize the steady state equilibrium, in which all variables are constant.

---

16Note also that without a management fee (i.e., \( f_m = 0 \)) the fees premium would be zero. The key to understand this is that an increase in the performance fee rate \( f_p \) has two opposite effects for the investor: a higher \( f_p \) implies a lower expected return, but is also a lower risk borne by the investor. In the CARA-normal setting, with no management fee, the lower-risk effect always dominates the lower-return effect, and hence the investor demands no fees premium. To see this mathematically, observe in (3.8) that, if \( f_m = 0, (1 - f_p) \) in the first term’s numerator would be offset by \( (1 - f_p)^2 \) in the denominator.

17This assumption is compatible with Assumption 1 since \( \kappa \) can be chosen arbitrarily large, or \( \sigma_t^2 \) and \( \gamma \) (and hence \( A \)) can be chosen arbitrarily small.
except for $X_t$ (which depends on $\epsilon_t$) and $P_t$ (which depends on $\hat{\delta}_t$ and $\epsilon_t$). For clarity, I focus on the limit case $\sigma^2 \rightarrow 0$, the general case being similar (Appendix C).

**Proposition 1.** A symmetric steady state equilibrium exists. For $\sigma^2 \rightarrow 0$:  
1) The risky security’s price is  
\[ P^*(\hat{\delta}_t, \epsilon_t) = \frac{\hat{\delta}_t}{r} + \left( \frac{a^*}{1+r} \right) \epsilon_t - \frac{n}{rf_p}, \]  
(3.23)  
2) The risky security’s expected excess return is  
\[ \bar{R}^* = \frac{n}{f_p}. \]  
(3.24)  
3) Each investor invests $X^*(\epsilon_t)$ dollars of capital in the fund, with  
\[ X^*(\epsilon_t) = \frac{1 - f_p}{f_m} \left( \frac{n}{f_p} - a^* \right) (1 - \epsilon_t). \]  
(3.25)  
4) Each manager buys $\theta^*$ shares of risky security per investor’s capital, with  
\[ \theta^* = \frac{f_m}{1-f_p} \left( \frac{n}{f_p} - a^* \right)^{-1}. \]  
(3.26)  
5) Each agent updates his/her estimate of $\hat{\delta}_t$ by  
\[ \hat{\delta}_t = \lambda^* \hat{\delta}_{t-1} + (1 - \lambda^*) \delta_t, \]  
(3.27)  
with  
\[ \lambda^* = 1 + \frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} + \rho} \quad \text{with} \quad \rho \equiv \frac{\eta_u}{\eta_v}, \]  
(3.28)  
\[ a^* = \frac{\gamma r(1-f_p)}{(1+r)\eta_u \lambda^*} \left( \frac{1 - \lambda^*}{r} + 1 \right)^2. \]  
(3.29)  

A detailed analysis of this equilibrium and of short-run transitional dynamics is deferred to the next section where I compare them with the complex security case’s.

### 4 Complex security

I now assume the risky security to be complex in the sense that the investors cannot observe its payoff $\delta_t$. With this assumption, each investor must infer $\delta_t$ based on her beliefs about the manager’s unobservable actions and from the observed fund performance and security prices. Thus the investor’s learning process differs from the simple security’s case: while she does infer $\delta_t$ correctly in equilibrium, this no longer holds off the equilibrium path. Hence,
the manager has an incentive to manipulate the investor’s inference of the payoff by secretly overinvesting in the complex security in a bid to attract more funds and earn more fees in the future. While investors are not fooled, this agency problem affects equilibrium positions and prices.

Much of the analysis in this section focuses on the investor’s beliefs and actions on off-the-equilibrium paths. Thus, it is useful to define the variables related to the investor’s beliefs separately from those of the manager. I denote the payoff history investor $j$ believes by 

$$H_{j,t}^i \equiv ( \delta_{j,t,1}, ..., \delta_{j,t} ),$$

where $\delta_{j,t}$ is the value of $\delta_t$ investor $j$ believes. Investor $j$’s estimate of $\delta_t$ is $\tilde{\delta}_{j,t} \equiv E(\delta_t|H_{j,t}^i)$, and her conditional expectation of the excess return is $\tilde{R}_{j,t+1}^i \equiv E(R_{t+1}|H_{j,t}^i, P_t)$.

As in the simple security’s case, I conjecture (and later verify) (3.1) and (3.2).

### 4.1 Investor’s optimization

The following Lemma shows that the investor’s value function (in equilibrium) is the same as in the simple security’s case, i.e., the values of $(A, B_t, Z_t)$ are (3.5), (3.6) and (3.7).

**Lemma 3.** The investor’s value function is identical to (3.3).

This holds because, as shown later, the investors correctly infer $\epsilon_t$ from the price in equilibrium. As a corollary, $a_t$ is also the same as in the simple security’s case, (3.21). However, as shown later, the value of $b_t$ is different, as is the equilibrium price.

**Lemma 4.** Investor $j$’s optimal investment policy is

$$X_{j,t} = \left( 1 - f_p \right) \theta_t \tilde{R}_{j,t+1}^i - f_m + \frac{\Psi_t}{(\Gamma_t + \Psi_t) \theta_t},$$

where $\Gamma_t$ and $\Psi_t$ are the same as (3.9) and (3.10), respectively.

The only difference from Lemma 2 is that the investor’s optimal policy depends on $\tilde{R}_{j,t+1}^i$, which is not necessarily equal to $\tilde{R}_{t+1}$ on off-the-equilibrium paths. In equilibrium, as shown below, the investor knows $H_t$ correctly and thus $\tilde{R}_{j,t+1}^i = \tilde{R}_{t+1}$ holds for all $j \in [0, 1]$.

### 4.2 Manager’s optimization

To verify that it is optimal for the manager to play $\theta_t$ deterministically, I check manager $j$’s one-shot deviation at an arbitrary time $t$. If the manager cannot gain by deviating from $\theta_t$ at time $t$ (i.e., choosing $\theta_{j,t} \neq \theta_t$) and reverting to $\theta_{t+1}, \theta_{t+2}, ...$ thereafter, it is optimal for him to play $\theta_t, \theta_{t+1}, \theta_{t+2}, ...$.\footnote{Although the time horizon of this model is infinite, the one-shot deviation principle is still applicable since the manager’s overall payoffs are a discounted sum of per-period profits, and the per-period profits are bounded. See Fudenberg and Tirole (1991).} Note that such a deviation does not affect prices or returns because each manager has measure 0.
I first determine the effect of a deviation \( \theta_{j,t} \) on the payoff \( \delta_{j,t+1}^i \) as inferred by investor \( j \). The value of \( \delta_{j,t+1}^i \) solves

\[
[\delta_{j,t+1}^i + P_{t+1} - (1 + r)P_t] \theta_{t} X_{j,t} = [\delta_{t+1}^i + P_{t+1} - (1 + r)P_t] \theta_{j,t} X_{j,t}. \tag{4.2}
\]

The RHS is the excess return on the fund’s portfolio, \( R_{t+1} \theta_{j,t} X_{j,t} \), which depends on the manager’s actual choice, \( \theta_{j,t} \), and the true payoff, \( \delta_{t+1} \). The LHS is the decomposition of that excess return as (wrongly) inferred by investor \( j \). It depends on her incorrect belief about the manager’s action, \( \theta_{t} \), and implies an erroneous inferred payoff, \( \delta_{j,t+1}^i \). Clearly, the source of the investor’s mistake is her inability to observe two variables: \( \delta_{t+1} \) and \( \theta_{j,t} \). Solving (4.2) for \( \delta_{j,t+1}^i \),

\[
\delta_{j,t+1}^i = \delta_{t+1} + \left( \frac{\theta_{j,t} - \theta_{t}}{\theta_{t}} \right) R_{t+1}. \tag{4.3}
\]

Hence, if manager \( j \) plays \( \theta_{j,t} > \theta_{t} \) and if \( R_{t+1} > 0 \), investor \( j \) overestimates \( \delta_{t+1} \).

Investor \( j \)'s error about \( \delta_{t+1} \) also biases her estimates of the security’s future fundamental values, i.e., \( \hat{\delta}_{j,t+\tau} \neq \hat{\delta}_{t+\tau} \) for \( \tau = 1, 2, \ldots \), and thereby her capital investments, i.e., \( X_{j,t+\tau} \neq X_{t+\tau} \) for \( \tau = 1, 2, \ldots \).

**Lemma 5.** If manager \( j \) deviates only at time \( t \) and chooses \( \theta_{j,t} \neq \theta_{t} \), then investor \( j \)'s capital investment at time \( t + \tau \), \( \tau = 1, 2, \ldots \), is

\[
X_{j,t+\tau} = X_{t+\tau} + X_{t+\tau}^+(\theta_{j,t}), \tag{4.4}
\]

where

\[
X_{t+\tau}^+(\theta_{j,t}) = \frac{1}{\alpha_{t+\tau} \theta_{t+\tau}} \left( 1 + \frac{r}{\theta_{t+\tau}} \right) (1 - \lambda_{t+1}) \left( \prod_{\nu=2}^{\tau} \lambda_{t+\nu} \right) \left( \frac{\theta_{j,t} - \theta_{t}}{\theta_{t}} \right) R_{t+1}. \tag{4.5}
\]

That is, given that \( R_{t+1} > 0 \), the investor invests additional capital \( X_{t+\tau}^+(\theta_{j,t}) > 0 \) at time \( t + \tau \), following the manager’s deviation \( \theta_{j,t} > \theta_{t} \) at time \( t \). From Lemma 4, this additional investment is caused by her overestimation about future excess returns, i.e., \( \hat{R}_{j,t+\tau+1} > \hat{R}_{t+\tau+1} \). The key for this to occur is the noise trading \( \epsilon_{t+\tau} \). Because \( \epsilon_{t+\tau} \) is unobservable, the investor cannot infer \( \hat{\delta}_{t+\tau} \) from the price \( P_{t+\tau} \), and hence does not realize that her estimate \( \hat{\delta}_{j,t+\tau} \) and her expectation \( \hat{R}_{j,t+\tau+1} \) are too high.\(^{19}\)

Now, manager \( j \) chooses \( \theta_{j,t} \), taking into account its effect on \( X_{t+\tau} \) through \( X_{t+\tau}^+(\theta_{j,t}) \).

\(^{19}\)Observe \( P_{t+\tau} \), investor \( j \) wrongly infers the value of \( \epsilon_{t+\tau} \) as \( \epsilon_{j,t+\tau} \), which solves

\[
\frac{\hat{\delta}_{j,t+\tau}}{a_{t+\tau} + \frac{1}{1 + r}} \epsilon_{j,t+\tau} \theta_{t+\tau} - b_{t+\tau} = \hat{\delta}_{t+\tau} + \frac{a_{t+\tau}}{1 + r} \epsilon_{t+\tau} - b_{t+\tau}. 
\]

The RHS is the price \( P_{t+\tau} \), and the LHS is its decomposition as (wrongly) inferred by investor \( j \). Solving this for \( \epsilon_{j,t+\tau} \),

\[
\epsilon_{j,t+\tau} = \epsilon_{t+\tau} - \frac{1}{a_{t+\tau}} \left( 1 + \frac{r}{\theta_{t+\tau}} \right) (\hat{\delta}_{j,t+\tau} - \hat{\delta}_{t+\tau}). 
\]

That is, if investor \( j \)'s estimate is too high, i.e., \( \hat{\delta}_{j,t+\tau} > \hat{\delta}_{t+\tau} \), she erroneously infers \( \epsilon_{j,t+\tau} < \epsilon_{t+\tau} \) and does not realize that \( \hat{\delta}_{j,t+\tau} \) is biased. The intuition for the high \( \hat{R}_{j,t+\tau+1} \) is as follows (see Appendix D for details).
Lemma 6. Manager $j$ chooses $\theta_{j,t}$ to maximize

$$\theta_{j,t}\left[f_p(\bar{R}_{t+1} + a_t \sigma^2_\epsilon) - \kappa\right] + (\theta_{j,t} - \theta_t)(\bar{R}_{t+1} \Omega_t - \phi_t),$$

where

$$\Omega_t \equiv \sum_{\tau=1}^{\infty} \beta^\tau \left(\frac{1+r}{r}\right)(1-\lambda_{t+1}) \left(\prod_{\nu=2}^{\tau} \frac{\lambda_{t+\nu}}{a_{t+\nu}}\right) \left(\frac{1}{a_{t+\tau}}\left( f_p\bar{R}_{t+\tau+1} + \frac{f_m}{\theta_{t+\tau}} - \kappa \right)\right),$$

and

$$\phi_t \equiv \beta \left(\frac{1 - \lambda_{t+1}}{r}\right) f_p a_{t+1} \sigma^2_\epsilon.$$

Compared to (3.14), its counterpart in the simple security’s case, (4.6) has a new term, the second one. It corresponds to the manager’s gain from influencing the investor’s estimates. In equilibrium, this term is zero as $\theta_{j,t} = \theta_t$ holds. However, on off-the-equilibrium paths, it is nonzero if the manager plays $\theta_{j,t} \neq \theta_t$. Importantly, this potential off-equilibrium gain will affect the equilibrium action. The variable $\Omega_t$ measures the sensitivity of the manager’s additional gain to a deviation at time $t$. Note that $\Omega_t$ and $\phi_t$ are deterministic and publicly known.

The manager’s optimal choice of $\theta_{j,t}$ is as follows:

If $f_p(\bar{R}_{t+1} + a_t \sigma^2_\epsilon) - \kappa + (\bar{R}_{t+1} \Omega_t - \phi_t) \begin{cases} > 0, & \text{then } \theta_{j,t} \to \infty, \\ < 0, & \text{then } \theta_{j,t} \to -\infty, \\ = 0, & \text{then } \theta_{j,t} \in (-\infty, \infty). \end{cases}$

In the conjectured symmetric equilibrium, $\theta_{j,t} = \theta_t > 0$ must hold for all $j \in [0,1]$. Thus, conjecture (3.2) is correct (and investor $j$’s beliefs are consistent) if manager $j$ plays $\theta_{j,t} = \theta_t$ while condition (4.9) is met with equality, i.e.,

$$\bar{R}_{t+1} = \frac{\kappa - f_p a_t \sigma^2_\epsilon + \phi_t}{f_p + \Omega_t}.$$  (4.10)

As in the simple security case, this condition requires that the expected excess return equals the manager’s cost-benefit ratio. The benefit (the denominator) is adjusted by $\Omega_t$, reflecting the additional marginal gain from deviating. The value of $\theta_t$, implicit in (4.10), will be obtained explicitly once the relationship between $\bar{R}_{t+1}$ and $\theta_t$ is identified in Section 4.3.

4.3 Equilibrium

I now derive $\bar{R}_{t+1}$, $\theta_t$ and $P_t$, from market clearing. The analysis is very similar to that of Section 3.3 because, in equilibrium, the investors infer the payoffs correctly as if they could...
observe them directly.

As discussed in Section 4.1, in equilibrium, the investor’s investment policy is identical to that of the simple security case, (3.8). Hence, following the same steps as in Section 3.3, \( \bar{R}_{t+1} \) is obtained as (3.18). Now, \( \theta_t \) can be identified. To ensure existence, I assume the following.\(^\text{20} \)

**Assumption 3.** \[
\frac{\kappa}{f_p + \Omega_t} - A(1 - f_p)\Gamma_t - \frac{f_p a_t \sigma^2}{f_p + \Omega_t} - \phi_t > 0 \text{ for all } t.
\]

Equating (3.18) with (4.10) yields

\[
\theta_t = \frac{f_m}{1 - f_p}\frac{1}{\frac{\kappa}{f_p + \Omega_t} - A(1 - f_p)\Gamma_t - \frac{f_p a_t \sigma^2}{f_p + \Omega_t} - \phi_t}, \tag{4.11}
\]

Here, the key difference from the simple security case’s solution (3.19) is the presence of \( \Omega_t \). A larger \( \Omega_t \) is associated with a larger \( \theta_t \), since the higher marginal gain from investing in the complex security induces the manager to invest more in the security. The third term of the RHS’s denominator vanishes as \( \sigma^2 \to 0 \).

Finally, the equilibrium price is obtained as in Section 3.3. Thus, conjecture (3.1) is correct if and only if \( a_t \) and \( b_t \) satisfy (3.21) and (3.22), respectively (with different values of \( \bar{R}_{t+1} \) in (3.22)).

For clarity, as before, I characterize the steady state equilibrium in the limit case \( \sigma^2 \to 0 \), the general case being similar (see Appendix F).

**Proposition 2.** A symmetric steady state equilibrium exists. For \( \sigma^2 \to 0 \):

1) The risky security’s price is

\[
P^*(\hat{\delta}_t, \epsilon_t) = \frac{\hat{\delta}_t}{r} + \left( \frac{a^*}{1 + r} \right) \epsilon_t - \frac{\kappa}{r(f_p + \Omega^*)}, \tag{4.12}
\]

with

\[
\Omega^* = \left( \frac{1 + r}{r} \right) \left[ f_p + \left( \frac{f_m}{1 - f_p} \right) \frac{1}{a^* \theta^*} - \frac{\kappa}{a^*} \right] \left[ \frac{1}{\beta(1 - \lambda^*)} - \frac{1}{\rho + (1 - \lambda^*)} \right]^{-1}. \tag{4.13}
\]

2) The risky security’s expected excess return is

\[
\bar{R}^* = \frac{\kappa}{f_p + \Omega^*}. \tag{4.14}
\]

3) Each investor invests \( X^*(\epsilon_t) \) dollars of capital in the fund, with

\[
X^*(\epsilon_t) = \frac{1 - f_p}{f_m} \left( \frac{\kappa}{f_p + \Omega^*} - a^* \right)(1 - \epsilon_t). \tag{4.15}
\]

\(^{20}\)This can hold since \( \kappa \) can be chosen large, or \( \sigma^2 \) and \( \gamma \) (and hence \( A \)) can be chosen small.
4) Each manager buys $\theta^*$ shares of risky security per investor’s capital, with

$$
\theta^* = \frac{f_m}{1 - f_p} \left( \frac{\kappa}{f_p + \Omega^* - a^*} \right)^{-1},
$$

(4.16)

where $(\theta^*, \Omega^*)$ are the unique positive solutions to equations (4.13) and (4.16).

5) Each agent updates his/her estimate of $\delta_t$ by

$$
\hat{\delta}_t = \lambda^* \hat{\delta}_{t-1} + (1 - \lambda^*)\delta_t,
$$

(4.17)

with

$$
\lambda^* = 1 + \frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} + \rho} \quad \text{with} \quad \rho \equiv \frac{\eta_u}{\eta_v},
$$

(4.18)

$$
a^* = \frac{\gamma r (1 - f_p)}{(1 + r) \eta_u \lambda^*} \left( \frac{1 - \lambda^*}{r} + 1 \right)^2.
$$

(4.19)

Comparing the simple security case (Proposition 1), the only difference is the presence of $\Omega^*$, i.e, the simple security case corresponds to $\Omega^* = 0$. The following Corollary conducts this comparison for the same realizations of $\delta_t$ and $\epsilon_t$. It is also shown that the two equilibrium outcomes would coincide if the fundamental value of the security, $\bar{\delta}_t$, were constant over time.

**Corollary 1. In the complex security case, relative to the simple security case,**

1) *The price $P^*$ is higher,*

2) *The expected excess return $\bar{R}^*$ is lower,*

3) *The investor’s capital investment $X^*$ is lower,*

4) *The manager’s portfolio is riskier (i.e., $\theta^*$ is larger).*

*These differences vanish when $\eta_v \to \infty.*

The main cause of these differences is that the manager can get an additional component of marginal gain from investing in the complex security (represented by $\Omega^*$), since it potentially enables him to inflate the investor’s expectation about fund performance and hence collect additional fees in the future. This leads the manager to invest more in the complex security than in the simple one (i.e., larger $\theta^*$). The investor is rational and her expectation is not inflated in equilibrium; nevertheless, the manager invests more in the complex security since it is optimal for him given the investor’s belief that he overinvests in the security. The resulting higher aggregate demand for the complex security is settled by a higher market-clearing price. The associated expected excess return, $\bar{R}^*$, is lower for the complex security. This is because, on the one hand, a higher $\theta^*$ leads to a lower fees premium required by the investors (see (3.18)). Also, on the other hand, the manager’s total marginal gain from investing in the risky security, i.e., $f_m \bar{R}^*$ for the simple and $(f_m + \Omega^*)\bar{R}^*$ for the complex, must be equal to the common holding cost $\kappa$ in equilibrium (see (3.16) and (4.10)). The investors optimally decrease their capital investments in response to the lower expected excess return and the
managers’ riskier portfolio choice. In the steady state equilibria considered in Propositions 1 and 2, all these differences would vanish if \( \bar{\delta} \) were constant over time, i.e., \( \eta_v \to \infty \) leads to \( \lambda^* \to 1 \) and thus \( \Omega^* \to 0 \). This is because, in the long run, the investors would perfectly learn the true value of \( \bar{\delta} \) (almost surely) and thus the manager would not be able to manipulate the investor’s expectations using the complex security even on off-the-equilibrium paths.\(^{21}\)

An important issue is the implication of complexity for the agents’ welfare. The following Corollary states that complexity leads to a social welfare loss.

**Corollary 2.** The complex security case is Pareto inferior to the simple security case. The investor’s utility level is the same in these two cases; the manager’s utility level is lower in the complex security case.

The intuition is as follows. In the complex security case, the excess return \( R_{t+1} \) is lower than the simple security case. Therefore, the total excess return from the fund’s portfolio, \( R_{t+1} \theta_t X_t \), is also lower in the complex security case because, in equilibrium, \( \theta_t X_t \) is equal to the residual supply \( (1 - \epsilon_t) \) whether the security is simple or complex. But, since the investor invests lower \( X_t \) in the complex security case, she pays a lower management fee, \( f_m X_t \). The lower portfolio return and the lower management fee just offset, and the investors achieve the same utility in the two cases. For the manager, however, a larger \( \theta_t \) has only negative effects on his fees revenue. The complex security’s lower return leads to a lower performance fee, \( f_p R_{t+1} \theta_t X_t = f_p R_{t+1} (1 - \epsilon_t) \), while the management fee \( f_m X_t \) is lower too as \( X_t \) is lower. The holding cost is, in equilibrium, equal to \( \kappa (1 - \epsilon_t) \) in both cases and is irrelevant for the utility comparison.

**Remark.** (“Optimal hiding price”? ) It should be clear from the above argument that, for the investor to have a biased estimate on off-the-equilibrium paths, it is crucial that she observes the price \( P_{t+\tau} \). If she could not observe \( P_{t+\tau} \), she would make a decision based on the expectation of \( P_{t+\tau} \), so \( X_{j,t+\tau} \) would depend only on the unconditional \( \bar{R}_{t+\tau+1} \) and become deterministic. If that were the case, there would be no possibility that the manager misleads \( X_{j,t+\tau} \), even on off-the-equilibrium paths. Thus, given that the manager is worse off with the complex security, it would be optimal for the manager to “hide” the complex security’s price, i.e., prevent the price from being observed by the investor, since it would allow him to commit not to fool the investor. I assume that the managers cannot hide the price in this model. But, if they could, it would be a Pareto improvement and would replicate the simple security case’s allocation. This may be an interesting observation, as it might open a door for investigating the reason why complex financial products are often traded in opaque OTC markets.\(^{22}\)

\(^{21}\)Even if \( \eta_v \to \infty \), the equilibrium outcomes are different on the transitional path where the investors have not yet fully learnt the value of \( \bar{\delta} \). The transitional dynamics are discussed in Section 4.4.\(^{22}\)However, note that the reason why hiding price is welfare improving in this model is that, in equilibrium, the price does not convey information superior to each investor’s. That is, even if the investors cannot observe the price, they are not worse off. With more realistic setting with asymmetric information or differential information where the price conveys useful information, it would be questionable that hiding price is Pareto improving.
4.4 Transitional dynamics

So far, I have focused on the steady state equilibria where the agents have already learnt the information about $\delta_t$ as much as they can, and hence the precision of their information is constant over time (i.e., $\lambda_t = \lambda^*$). Now, I focus on the “short-run” transitional path of the economy, where the agents are still in early stages of learning about $\delta_t$. On this path, by Assumption 1, the precision of the agents’ information increases over time. This will generate dynamic properties of the equilibrium unique to the complex security case. This exercise is particularly important for understanding financial markets’ reaction to the introduction of a new, highly uncertain financial product.23

Figure 1 shows the transitional dynamics of the economy. Panel (a) is an example of the dynamics of $P_t$. The price is highly volatile in the beginning, because, in early stages of learning, the agents change their estimate $\hat{\delta}_t$ heavily depending on the stochastic realizations of $\delta_t$. This price volatility decreases over time, but never vanishes since the agents never learn the true value of stochastic $\delta_t$.

To illustrate the model’s core mechanism, panel (b) plots the dynamics of the “average price,” where the noise $u_t$ and $v_t$ as well as the noisy demand $\epsilon_t$ are artificially set to zero.24 Interestingly, the complex security’s average price first rises and then falls over time. Understanding this pattern amounts to understanding the dynamics of $\Omega_t$, which is plotted in panel (d), since $\Omega_t$ moves ( inversely) along $R_{t+1}$ by (4.10), which then moves along ( inversely) with $P_t$ through $b_t$ (see (3.22)). Intuitively, $\Omega_t$ is inverse-U shaped over time because the investors’ learning has two opposite effects on the manager’s deviation payoff.

1. As the uncertainty about $\delta_t$ decreases (i.e., $\lambda_t$ increases), the risk-averse investor invests more capital $X_t$, as shown in panel (e). This allows the manager to earn more fees for a given level of deviation. This effect has an upward pressure on $\Omega_t$.

2. As the uncertainty about $\delta_t$ decreases (i.e., $\lambda_t$ increases), the investor’s estimate $\hat{\delta}_t^i$ does not react much to new payoff realizations. Thus, the manager’s deviation has smaller influence on $\hat{\delta}_t^i$, and hence on $X_t$. This effect has a downward pressure on $\Omega_t$.

For the parameter values used in Figure 1, the first effect dominates the second in early periods but they are reversed later, leading to the inverse-U shape. The inverse-U pattern is not a general result. For instance, with a smaller $\gamma$ (say, $\gamma = 0.02$) the second effect dominates the first even in early periods, hence $\Omega_t$ and the complex security’s average price simply decrease over time.25

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23The idea of this analysis is to simulate the dynamical systems obtained in Sections 3 and 4 numerically. Since the time horizon of this model is infinite, I conduct the simulation as an approximation. I assume that the simulated economy “reaches” the steady state equilibrium (specified in Propositions 1 and 2) at period $T$, where $T$ is very large. More precisely, I set $\Omega_{T+1} = \Omega^*$ and then solve $(\Omega_0, ..., \Omega_T)$ by backward induction using the law of motion (F.3) in Appendix F (along with the other variables). In the simulations, I set $T = 100,000$ and show the first 200 periods on the graphs.

24Note that I do not assume $\eta_u \to \infty$ or $\eta_v \to \infty$. I present a special case where the realizations of stochastic noise are all zero, with the same set of parameter values as in panel (a).

25The simulation result is not shown here, but is available upon request.
Figure 1: Transitional dynamics. The parameter values are: $r = 0.3, f_m = 0.02, f_p = 0.2, \kappa = 2, \gamma = 0.25, \beta = 1/(1 + r), \eta_a = 0.01, \eta_b = 10^5, \bar{\eta}_0 = 0.1, \sigma^2 = 10^{-4}, \bar{\delta}_0 = 15, \hat{\delta}_0 = 15.$
In panel (e), the investors invest smaller capital in earlier periods due to higher uncertainty about the security’s payoffs. The level of investment is lower for the complex security case due to the lower expected return on the fund’s portfolio. In panel (f), the manager’s portfolio is riskier in earlier periods because the managers invest more in the risky security to exploit the expected excess return that otherwise would have been higher due to a high risk premium. The portfolio is riskier in the complex security case as $\Omega_t$ induces the managers to invest more in the complex security.

5 The Supply of Complexity

In the previous sections, I focused on the effects of complexity, taking the presence of complexity as given. However, it is also important to understand the supply of complexity. In this section, I consider a more realistic setting where simple and complex securities coexist, and the securities are endogenously supplied by a new set of agents called “quants.” The quants not only trade the securities but also transform a simple security into a complex security, as well as the complex into the simple, with an associated supply cost of complexity. The quants effectively act as arbitrageurs: if the complex security’s price is sufficiently higher than the simple’s, they try to make profits by buying the simple, transforming it into the complex and selling it. The prices of simple and complex securities, the investor’s capital investment, the manager’s portfolio choice (between simple, complex and risk-free securities) and the quants’ supply of securities are determined in equilibrium.

5.1 Setup

The model setup is very similar to the one presented in Section 2. Unless otherwise noted, the assumptions regarding the model setup are maintained.

Risky securities. The simple security and the complex security are indexed by $k = s, c$, respectively. The total number of shares of the simple and complex securities is fixed over time, and is normalized to 1. The complex security’s supply at time $t$ is denoted by $\xi_t \in [0, 1]$, equivalently, the simple security’s supply is $(1 - \xi_t)$, where $\xi_t$ is endogenously determined by the quants’ optimization. I assume that the simple and complex securities have identical payoff distributions (while the payoff realizations may be different). Formally, security $k$’s payoff at time $t$ is $\delta_t^k = \bar{\delta}_t + u_t^k$, where the noise $u_t^k$ is i.i.d. across time and between $k = s, c$, normally distributed with mean 0 and variance $1/\eta_u$. The fundamental value $\bar{\delta}_t$ is common for $k = s, c$, and its properties are the same as in Section 2. The precision $\eta_u$ is also common for $k = s, c$. I denote by $\mathcal{H}_t \equiv ( (\delta_t^s, \delta_t^c), ..., (\delta_t^s, \delta_t^c) )$ the payoff history up to time $t$, and by $\mathcal{H}_{j,t}^i \equiv ( (\delta_t^i, \delta_{t,j}^i), ..., (\delta_t^j, \delta_{t,j}^i) )$ the history that investor $j$ believes. Note that the simple security’s payoffs in $\mathcal{H}_{j,t}^i$ are the true values since they are publicly observable. The time-$t$ estimate of $\bar{\delta}_t$ is denoted by $\hat{\delta}_t \equiv E(\bar{\delta}_t|\mathcal{H}_t)$, and its counterpart for investor $j$ is $\hat{\delta}_{j,t}^i \equiv E(\bar{\delta}_t|\mathcal{H}_{j,t}^i)$. Security $k$ is traded in the market at a market-clearing price $P^k_t$, and
its excess return is denoted by \( R^k_{t+1} = (\delta^k_{t+1} + P^k_{t+1}) - (1 + r)P^k_t \). The conditional expected excess return on security \( k \) is \( \tilde{R}^k_{t+1} = E(\tilde{R}^k_{t+1} | H_t, P^s_t, P^c_t) \), while its counterpart for investor \( j \) is \( \tilde{R}^j_{j,t+1} = E(R^j_{j,t+1} | H_{j,t}, P^s_t, P^c_t) \).

Learning about \( \delta_t \). Noting that the payoffs \( \delta^s_t \) and \( \delta^c_t \) are independent realizations from identical payoff distributions, \( \hat{\delta}_t \) is updated by the following Kalman filtering.

\[
\hat{\delta}_t = \lambda_t \hat{\delta}_{t-1} + (1 - \lambda_t) \left( \frac{\delta^s_t + \delta^c_t}{2} \right),\quad \text{where} \quad \lambda_t \equiv \frac{\text{Var}(\hat{\delta}_t | H_t)}{\text{Var}(\hat{\delta}_t | H_{t-1})}.
\] (5.1)

The updating factor \( \lambda_t \) evolves deterministically over time according to (2.6) and converges to (2.7), with different notation of \( \rho \): in this section, \( \rho \) is defined as \( \rho \equiv 2 \eta_u / \eta_v \). This difference in \( \rho \) comes from the fact that the speed of learning about \( \hat{\delta}_t \) is twice faster than in the previous sections since the agents update their estimates based on two independent payoff realizations each period. As before, I assume \( \lambda_t < \lambda^* \) to ensure that \( \lambda_t \) increases over time.

Quants. There is a measure-1 continuum of quants, each indexed by \( \ell \in [0, 1] \). They are risk neutral, competitive and infinitely lived. They can trade both simple and complex securities in the markets. In addition, they can transform the simple security into the complex security, as well as the complex into the simple. I assume for simplicity that the quants cannot hold these securities and receive the payoffs: they can only buy, transform and sell (i.e., supply) these securities within each period. Let \( Q_{\ell,t} \) be the number of shares of complex security that quant \( \ell \) supplies to the market at time \( t \). The consistency of the aggregate supply requires \( \int_0^1 Q_{\ell,t} d\ell = \xi_t \). Each quant has measure 0, and takes \( \xi_t \) as given. Transformation of the securities is costless. However, quant \( \ell \) needs to incur a supply cost of complex security at time \( t \), \( \frac{\mu}{1 + r} Q_{\ell,t} \), where \( \mu > 0 \).\(^{26}\) Quant \( \ell \)'s per-period profit is\(^{27}\)

\[
\pi_{\ell,t} = (Q_{\ell,t} - Q_{\ell,t-1})(P^c_t - P^s_t) - \frac{\mu}{1 + r} Q_{\ell,t},
\] (5.2)

where \( Q_{\ell,-1} = 0 \) for all \( \ell \) (i.e., there is no complex security before the initial date). Each quant maximizes the present value of his future profits. I.e., quant \( \ell \)'s problem, \( \mathcal{P}_{\ell,t} \), at time \( t \) is to choose \( Q_{\ell,t} \) to maximize

\[
\Pi_{\ell,t} \equiv \sum_{\tau=0}^{\infty} \left( \frac{1}{1 + r} \right)^\tau \pi_{\ell,t+\tau}.
\] (5.3)

Portfolio management. At time \( t \), investor \( j \) invests \( X_{j,t} \) dollars of capital in fund \( j \). Manager \( j \) chooses \( \theta^k_{j,t} \in (-\infty, \infty), k = s, c \), to buy \( \theta^k_{j,t} X_{j,t} \) shares of security \( k \) at time \( t \).

---

\(^{26}\)The division by \((1 + r)\) is a normalization. It is equivalent to assume that he pays \( \mu \) at time \( t + 1 \).

\(^{27}\)This is checked as follows. At time \( t - 1 \), quant \( \ell \) has supplied \( Q_{\ell,t-1} \) shares of complex security to the market. Suppose that he supplies \( z \) more shares of complex security to the market at time \( t \), where \( z \) can be positive or negative. This amounts to buying \( z \) shares of simple security at price \( P^s_t \), transforming it into complex security and selling at price \( P^c_t \). So, his profit at time \( t \) is \( \pi_{\ell,t} = -P^s_tz + P^c_tz - \frac{\mu}{1 + r} (Q_{\ell,t-1} + z) \). Noting that \( Q_{\ell,t} = Q_{\ell,t-1} + z \) by the definition of \( z \), it follows that \( \pi_{\ell,t} = (P^c_t - P^s_t)(Q_{\ell,t} - Q_{\ell,t-1}) - \frac{\mu}{1 + r} Q_{\ell,t} \).
The remaining capital, \((X_{j,t} - P_t^s\theta_{j,t}^s, X_{j,t} - P_t^c\theta_{j,t}^c, X_{j,t})\), is invested in the risk-free security. The investor cannot observe \(\theta_{j,t}^s\), but she can observe, at time \(t+1\), the excess return on security \(k\), \(R_{t+1,j,t}^kX_{j,t}\), separately for each security.\(^{28}\) The manager incurs a holding cost \(\kappa > 0\) per share of risky security in his portfolio, which is common for \(k = s, c\). The fee consists of a management fee, \(f_mX_{j,t}\), and a performance fee, \(f_p(R_{t+1,j,t}^s\theta_{j,t}^s + R_{t+1,j,t}^c\theta_{j,t}^c)X_{j,t}\).

Differently from the previous sections, there is only one stage in each period, i.e., all the events in time \(t\) occur simultaneously. Also, there is no noise trading in this section. These settings make the model very easy to solve. However, without noise trading, the equilibrium I characterize in this section is not a Nash equilibrium but a self-confirming equilibrium (Fudenberg and Levine 1993). A self-confirming equilibrium requires weaker conditions than Nash equilibrium. It allows each player to have incorrect beliefs about how his opponents would play in contingencies that do not arise when the players follow the equilibrium strategies. On the equilibrium path, however, every agent is required to have correct beliefs about the opponents’ actions. I discuss this issue later when I specify the equilibrium I look for in this section.

Maximization problems of investors and managers. Investor \(j\) allocates her wealth between consumption and investments in fund \(j\) and the risk-free security. Her problem, \(P_{j,t}^I\), at time \(t\) is to choose \(C_{j,t}\) and \(X_{j,t}\) to maximize (2.2), subject to her dynamic budget constraint:

\[
W_{j,t+1} = (1 - f_p)(R_{t+1,j,t}^s\theta_{j,t}^s + R_{t+1,j,t}^c\theta_{j,t}^c)X_{j,t} - f_mX_{j,t} + (1 + r)(W_{j,t} - C_{j,t}). \tag{5.4}
\]

Manager \(j\)'s utility at time \(t\) is the difference between his fees revenue and the holding costs. Hence, his problem, \(P_{j,t}^M\), at time \(t\) is to choose \(\theta_{j,t}, k = s, c\), to maximize

\[
E \left[ \sum_{\tau=0}^{\infty} \beta^\tau \left[ f_p(R_{t+\tau+1,j,t+\tau}^s\theta_{j,t+\tau}^s + R_{t+\tau+1,j,t+\tau}^c\theta_{j,t+\tau}^c)X_{j,t+\tau} + f_mX_{j,t+\tau} + \kappa(\theta_{j,t+\tau}^s + \theta_{j,t+\tau}^c)X_{j,t+\tau} \right] \right] F_{j,t}^m \tag{5.5}
\]

Definition of equilibrium. The equilibrium consists of price function \(P(\hat{\delta}_t)\), aggregate supply of complex security \(\xi_t\), investor \(j\)’s capital investment \(X_{j,t}\) for \(j \in [0, 1]\), manager \(j\)’s risky portfolio \(\theta_{j,t}^k\) for \(j \in [0, 1]\), \(k = s, c\), and quant \(\ell\)'s supply of complex security \(Q_{\ell,t}\) such that

1. Given \(P(\hat{\delta}_t)\), \(\xi_t\) and the other agents’ actions, investor \(j\) solves \(P_{j,t}^I\).
2. Given \(P(\hat{\delta}_t)\), \(\xi_t\) and the other agents’ actions, manager \(j\) solves \(P_{j,t}^M\).
3. Given \(P(\hat{\delta}_t)\), \(\xi_t\) and the other agents’ actions, quant \(\ell\) solves \(P_{\ell,t}^Q\).
4. For all \(t\), each quant’s supply of complex security sums up to aggregate supply:

\[
\int_0^1 Q_{\ell,t}d\ell = \xi_t. \tag{5.6}
\]

\(^{28}\)Due to the assumption that the investor observes the security’s excess return separately, the manager’s choice of \(\theta_{j,t}^k\) cannot influence the investor’s estimation of \(\hat{\delta}_t\), while \(\theta_{j,t}^c\) can.
5. For all $t$, the markets for the simple and complex securities clear:

\[
\int_0^1 \theta_{j,t}^s X_{j,t} dj = 1 - \xi_t, \tag{5.7}
\]

\[
\int_0^1 \theta_{j,t}^c X_{j,t} dj = \xi_t, \tag{5.8}
\]

6. Every agent has correct beliefs about the other agents’ actions on the equilibrium path,

7. Each agent updates the estimate of $\delta_t$ by Kalman filtering.

**Conjectures.** I conjecture (and later verify) the following.

1. There exists $b_t^s > 0$ and $b_t^c > 0$, which are non-stochastic and publicly known, such that for $k = s, c$, the price function is

\[
P_k(t)(\hat{\delta}_t) = \frac{\hat{\delta}_t}{r} - b_t^k. \tag{5.9}
\]

2. The aggregate supply $\xi_t$ is non-stochastic, publicly known, and satisfies $0 < \xi_t < 1$.

3. There exists $\theta_t^s > 0$ and $\theta_t^c > 0$, which are non-stochastic, publicly known and satisfy

\[
\frac{\theta_t^s}{\theta_t^c} = \frac{1 - \xi_t}{\xi_t}, \tag{5.10}
\]

such that for all $j \in [0, 1]$ and $k = s, c$, manager $j$ plays

\[
\theta_{j,t}^k = \theta_t^k. \tag{5.11}
\]

Conjecture (5.9) is the same as in the previous sections except that here there is no term with noise trading. The conjecture that $\xi_t$ is non-stochastic is a corollary of conjecture (5.9): given (5.9), $P(t)$ becomes a non-stochastic problem with a non-stochastic solution $Q(t)$, leading to non-stochastic $\xi_t$.\(^{29}\) The relationship (5.10) is a consequence of the symmetric-equilibrium versions of the market clearing conditions (5.7) and (5.8).

**Investor’s off-equilibrium beliefs.** In order to focus on the equilibrium in which the manager can potentially fool the investor (without introducing noise trading), I specify the investor’s off-equilibrium beliefs as follows.

1. If investor $j$’s estimate of $\hat{\delta}_t$ disagrees with the one implied by the price functions (5.9), i.e., if $\hat{\delta}_{j,t} \neq r(P_t^s + b_t^s) = r(P_t^c + b_t^c)$, then she believes that some investors in other funds have the estimates different from hers because these funds’ managers have deviated from

\(^{29}\)The requirement of interior solution, $0 < \xi_t < 1$, is for simplicity of the analysis. Allowing the corner solutions (i.e., $\xi_t = 0$ or $\xi_t = 1$ for some $t$) is an obvious generalization and can be solved, but little additional economic insight is obtained.
the equilibrium strategy at time \( t \). Also, she believes that the prices reflect the average of all the investors’ estimates, i.e.,
\[
\int_0^1 \hat{\delta}_{i,t}^v dv = r(P_t^s + b_t^s) = r(P_t^c + b_t^c),
\]
and that all the other agents will play the equilibrium strategy from time \( t + 1 \) on.

2. Investor \( j \) believes that manager \( j \) will play the equilibrium strategy (5.11) even if the manager’s estimate of \( \bar{\delta}_t \) disagrees with the one implied by the price functions (5.9).

The first point ensures that investor \( j \) will not attribute the discrepancy between her estimate and the one implied by the prices to manager \( j \)’s deviation.\(^{30,31}\) The second point implies that the investor has incorrect beliefs about the manager’s actions when her estimate disagrees with the prices. That is, the equilibrium I look for in this section is a self-confirming equilibrium (Fudenberg and Levine 1993), not a Nash equilibrium.\(^{32}\) Of course, this self-confirming equilibrium satisfies the definition of equilibrium. On the equilibrium path, the investor does not observe anything that contradicts her beliefs. Her action is best response to her beliefs and the others’ actions, which she correctly believes on the equilibrium path.

### 5.2 Optimizations

**Investor’s optimization.** I conjecture that investor \( j \)’s value function is of the form
\[
V_t(W_{j,t}) = -\exp(-AW_{j,t} - Z_t),
\]
where \( A > 0 \) is a constant and \( Z_t \) is non-stochastic. The Bellman equation is given by
\[
V_t(W_{j,t}) = \max_{C_{j,t},X_{j,t}} \left[ -\exp(-\gamma C_{j,t}) + \beta E \left[ V_{t+1}(W_{j,t+1}) | \mathcal{F}_{j,t}^t \right] \right] .
\]

\(^{30}\)There is also an equilibrium in which the investor always revises her estimate in favor of the prices when her estimate disagrees with them. In this case, the investor’s estimates will not be influenced by the manager’s deviation. In Section 4, this type of equilibrium did not exist due to the presence of noise trading.

\(^{31}\)Note that, on off-the-equilibrium paths, the investor has no particular reason to revise her estimate \( \hat{\delta}_{j,t}^i \) to \( \hat{\delta}_t \) implied by the prices because the prices do not convey information superior to the investor’s. Of course, if they instead revealed superior information then the investor would definitely revise her estimate in favor of the prices. This is the case in asymmetric information models à la Grossman and Stiglitz (1980), where uninformed investors try to infer information reflected in the price. Similarly, in differential information models à la Grossman (1976), the price aggregates the investors’ private signals that are collectively more useful than each investor’s private signal, hence the investors should act in favor of the price. However, in this model, no investor has information superior to the other groups’ investors, and they do not have private information collectively useful. In the conjectured equilibrium, the price simply aggregates the estimates of all the investors, each of whom learns about the same \( \delta_k^t \) from observing fund performance. Also note that investor \( j \) has no reason to believe that manager \( j \) has deviated, as it might be other groups’ managers or investors that have deviated.

\(^{32}\)If it were a Nash equilibrium, investor \( j \) would have to believe that manager \( j \) would no longer play the equilibrium strategy if the manager’s estimate — which investor \( j \) believes is the same as her estimate given that manager \( j \) and investor \( j \) observe the same fund performance — disagrees with the one implied by the prices, since it will be no longer optimal for the manager to follow the equilibrium strategy. The self-confirming equilibrium I look for in this section can be thought of as the counterpart of Nash equilibrium in Section 4 in the sense that, on off-the-equilibrium path where manager \( j \) one-shot deviates at time \( t \), investor \( j \) correctly knows the manager’s actions from time \( t + 1 \) onwards.
Lemma 7. The value function (5.12) satisfies the Bellman equation (5.13) if

\[
A = \gamma \left( \frac{r}{1 + r} \right), \tag{5.14}
\]

\[
Z_t = \frac{1}{1 + r} \left[ Z_{t+1} + \frac{1}{2} A^2 (1 - f_p)^2 \Gamma_t - \log \beta + r \log r - (1 + r) \log(1 + r) \right], \tag{5.15}
\]

where

\[
\Gamma_t = \frac{\mathrm{Var}(R_{t+1}^s \theta_{j,t}^s + R_{t+1}^c \theta_{j,t}^c | \mathcal{F}_t)}{(\theta_t^s + \theta_t^c)^2} = \frac{1}{2 \eta u} \left[ \frac{1}{\lambda_{t+1}} \left( \frac{1 - \lambda_{t+1}}{r} + 1 \right)^2 + (1 - 2 \xi_c)^2 \right]. \tag{5.16}
\]

Lemma 8. For all \( j \in [0,1] \), investor \( j \)'s optimal investment policy is

\[
X_{j,t} = \frac{(1 - f_p)(\hat{R}_{j,t+1}^s \theta_t^s + \hat{R}_{j,t+1}^c \theta_t^c) - f_m}{A(1 - f_p)^2(\theta_t^s + \theta_t^c)^2 \Gamma_t}. \tag{5.17}
\]

That is, the investor’s optimal policy is given by a simple mean-variance solution: the numerator is the expected excess return from the fund (after fees), and the denominator represents the adjustment due to risk aversion.

Manager’s optimization. As in Section 4, the optimality of the manager’s equilibrium strategy (5.11) is checked by considering manager \( j \)'s one-shot deviation: at an arbitrary time \( t \), manager \( j \) chooses \((\theta_{j,t}^s, \theta_{j,t}^c) \neq (\theta_t^s, \theta_t^c)\), and reverts thereafter to \((\theta_{t+1}^s, \theta_{t+1}^c), (\theta_{t+2}^s, \theta_{t+2}^c), \ldots\). The underlying idea is very similar to that of Section 4. That is, potentially (off-equilibrium), the manager’s one-shot overinvestment in the complex security inflates the investor’s perception about \( \delta_{t+1}^c \). This biases the investor’s estimate of \( \delta_{t+\tau} \), and hence her investment, \( X_{j,t+\tau} \), for \( \tau = 1, 2, \ldots \).

Lemma 9. If manager \( j \) deviates only at time \( t \) and chooses \((\theta_{j,t}^s, \theta_{j,t}^c) \neq (\theta_t^s, \theta_t^c)\), then investor \( j \)'s capital investment at time \( t + \tau \), \( \tau = 1, 2, \ldots \), is

\[
X_{j,t+\tau} = X_{t+\tau} + X_{t+\tau}^+(\theta_{j,t}^c), \tag{5.18}
\]

where

\[
X_{t+\tau}^+(\theta_{j,t}^c) \equiv \frac{1}{a_{t+\tau} (\theta_{t+\tau}^s + \theta_{t+\tau}^c)} \left( \frac{1 - \lambda_{t+\tau+1}}{r} + 1 \right) \left( \frac{1 - \lambda_{t+1}}{2} \right) \left( \prod_{\nu=2}^{\tau} \lambda_{t+\nu} \right) \left( \frac{\theta_{j,t}^c - \theta_t^c}{\theta_t^c} \right) R_{t+1}^c, \tag{5.19}
\]

\[
a_{t} \equiv A(1 - f_p) \Gamma_t. \tag{5.20}
\]

I.e., given that \( R_{t+1}^c > 0 \), the investor will invest an additional capital in the fund following the manager’s deviation in the complex security, \( \theta_{j,t}^c > \theta_t^c \). The investor invests additional capital because her expectations about the excess return on the risky securities is too high. In Section 4, a similar result is obtained due to the presence of noise trading. In this section, this result follows from the specification of the investor’s off-equilibrium beliefs.
Now, manager $j$’s problem is to choose $\theta_{j,t}^k$, $k = s, c$, to maximize his lifetime utility (5.5), taking into account the effect of $\theta_{j,t}^k$ on $X_{t+\tau}$ through $X_{t+\tau}^\pm(\theta_{j,t}^k)$. The following Lemma provides an equivalent problem for the manager.

**Lemma 10.** Manager $j$ chooses $\theta_{j,t}^s$ and $\theta_{j,t}^c$ to maximize

$$
f_p(\hat{R}_{t+1}^s \theta_{j,t}^s + \hat{R}_{t+1}^c \theta_{j,t}^c) \xi_t - \kappa(\theta_{j,t}^s + \theta_{j,t}^c) \xi_t + (\theta_{j,t}^c - \theta_{j,t}^s) R_{t+1}^c \Omega_t,
$$

where

$$
\Omega_t \equiv \left(1 - \frac{\lambda_{t+1}}{2}\right) \sum_{r=1}^\infty \beta^r \left(1 - \frac{\lambda_{t+\tau+1}}{r} + 1\right) \left(\prod_{\nu=2}^r \lambda_{t+\nu}\right) \left[f_p + \frac{f_m}{(1 - f_p) a_{t+\tau} (\theta_{t+\tau}^s + \theta_{t+\tau}^c)} - \frac{1}{a_{t+\tau}}\right].
$$

(5.22)

The first two terms of (5.21) correspond to the manager’s utility at time $t$. The third term corresponds to the additional (off-equilibrium) expected gain from time $t + 1$ onwards that results from his time-$t$ deviation. The deterministic variable $\Omega_t$ measures the sensitivity of the manager’s additional gain to his time-$t$ deviation.

The manager’s optimal choice of $\theta_{j,t}^k$, $k = s, c$, is as follows.

If $f_p \hat{R}_{t+1}^s \xi_t - \kappa \xi_t$

- $> 0$, then $\theta_{j,t}^s \rightarrow \infty$,
- $< 0$, then $\theta_{j,t}^s \rightarrow -\infty$,
- $= 0$, then $\theta_{j,t}^s \in (-\infty, \infty)$.

If $f_p \hat{R}_{t+1}^c \xi_t - \kappa \xi_t + \hat{R}_{t+1}^c \Omega_t$

- $> 0$, then $\theta_{j,t}^c \rightarrow \infty$,
- $< 0$, then $\theta_{j,t}^c \rightarrow -\infty$,
- $= 0$, then $\theta_{j,t}^c \in (-\infty, \infty)$.

(5.23)

(5.24)

In the conjectured symmetric equilibrium, $\theta_{j,t}^k = \theta_t^k > 0$ must hold for all $j \in [0, 1]$ and $k = s, c$. Thus, conjecture (5.11) is correct (and investor $j$’s beliefs are consistent) if manager $j$ plays $\theta_{j,t}^k = \theta_t^k$ for $k = s, c$ while conditions (5.23) and (5.24) both hold with equality, i.e.,

$$
\begin{cases}
\hat{R}_{t+1}^s = \hat{R}^s \equiv \frac{\kappa}{f_p}, \\
\hat{R}_{t+1}^c = \frac{\kappa}{f_p} \left(\frac{f_p \xi_t}{f_p \xi_t + \Omega_t}\right).
\end{cases}
$$

(5.25)

As in the previous sections, this equilibrium condition requires that the expected excess return on each security is equalized to the manager’s cost-benefit ratio of the security. Note that the condition implies that, if $\Omega_t > 0$, $\hat{R}_{t+1}^c < \hat{R}_{t+1}^s$ holds in equilibrium. This is because, in equilibrium, the manager must be indifferent between investing in these two securities. The marginal cost of investment for the manager, $\kappa$, is common for both securities. However, the complex security delivers an additional marginal gain represented by $\Omega_t$. Hence, $\hat{R}_{t+1}^c < \hat{R}_{t+1}^s$ is required to make the manager indifferent. The values of $\theta_t^s$ and $\theta_t^c$ are implicit in (5.25).
at this stage, and will be obtained explicitly once the relationship between $\hat{R}_{k,t+1}^k$ and $\theta_t^k$ is identified in Section 5.3.

Quant’s optimization. Using conjecture (5.9), quant $\ell$’s per-period profit (5.2) is

$$
\pi_{\ell,t} = (Q_{\ell,t} - Q_{\ell,t-1})(b_{\ell}^* - b_{\ell}^t) - \frac{\mu}{1 + r} Q_{\ell,t}.
$$

(5.26)

From this, his objective (5.3) can be written as

$$
\Pi_{\ell,t} = -Q_{\ell,t-1}(b_{\ell}^* - b_{\ell}^t) + \frac{Q_{\ell,t}}{1 + r} (MR_t - \mu) + \frac{Q_{\ell,t+1}}{(1 + r)^2} (MR_{t+1} - \mu) + \cdots,
$$

(5.27)

where

$$
MR_t \equiv [(1 + r)b_{\ell}^* - b_{\ell}^{t+1}] - [(1 + r)b_{\ell}^t - b_{\ell}^{t+1}].
$$

(5.28)

As is shown in Section 5.3, the “marginal revenue,” $MR_t$, for the quant is a function of $\xi_t$.

Since I am looking for an equilibrium in which $0 < \xi_t < 1$ for all $t$, a necessary condition for this equilibrium is that every quant has no incentive to buy or sell another share of complex security for all $t$. From (5.27), this requires that, for all $t$,

$$
MR_t = \mu.
$$

(5.29)

Note that (5.27) and (5.29), together with $Q_{t,-1} \equiv 0$, imply that $\Pi_{t,0} = 0$ for all $t \in [0,1]$. That is, the quant’s discounted lifetime profits is zero at $t = 0$. At this stage, the equilibrium value of $\xi_t$ is implicit in (5.29). It will be explicitly obtained in Section 5.3 where the relationship between $MR_t$ and $\xi_t$ is identified.

5.3 Equilibrium

I consider the market clearing conditions and derive $\hat{R}_{t+1}^k$, $\theta_t^k$, $P_t^k$ and $\xi_t$.

Determination of $\theta_s^t$ and $\theta_c^t$. To ensure existence, I assume the following.\(^{33}\)

**Assumption 4.** \(\frac{\kappa}{f_p} \left[ \frac{f_p \xi_t + (1 - \xi_t) \Omega_t}{f_p \xi_t + \Omega_t} \right] - a_t > 0 \) for all $t$.

From the market clearing conditions (5.7) and (5.8), the relationship (5.10) and the investor’s optimal investment (5.17),

$$
(1 - \xi_t)\hat{R}_{t+1}^* + \xi_t\hat{R}_{t+1}^c = A(1 - f_p)\Gamma_t + \frac{f_m}{(1 - f_p)(\theta_t^s + \theta_t^c)}.
$$

(5.30)

This relationship is the counterpart of (3.18): the LHS is the “average” expected excess return on the risky securities, and the RHS is the sum of the risk premium and the fees premium. Meanwhile, from the manager’s optimality condition (5.25), the average expected

\(^{33}\)This can be satisfied since $\kappa$ can be chosen large, or $\gamma$ (and hence $a_t$) can be chosen small.
excess return is also written as
\[
(1 - \xi_t)\hat{R}^s_{t+1} + \xi_t\hat{R}^c_{t+1} = \frac{\kappa}{f_p} \left[ \frac{f_p\xi_t + (1 - \xi_t)\Omega_t}{f_p\xi_t + \Omega_t} \right].
\] (5.31)

Thus, taking \(\xi_t\) as given, \(\theta^s_t\) and \(\theta^c_t\) are uniquely pinned down by equating (5.30) with (5.31) and using the relationship (5.10).

\[
\begin{align*}
\left\{ \begin{array}{l}
\theta^s_t = \frac{(1 - \xi_t) \left( \frac{f_m}{1 - f_p} \right)}{\frac{f_p\xi_t + (1 - \xi_t)\Omega_t}{f_p\xi_t + \Omega_t}} - \alpha_t,
\theta^c_t = \frac{\xi_t \left( \frac{f_m}{1 - f_p} \right)}{\frac{f_p\xi_t + (1 - \xi_t)\Omega_t}{f_p\xi_t + \Omega_t}} - \alpha_t.
\end{array} \right.
\] (5.32)

**Determination of \(P^s_t\) and \(P^c_t\).** The equilibrium prices are obtained following the similar steps as in Section 3. Plugging conjecture (5.9) into the definition of \(R^k_{t+1}, k = s, c\), the counterpart of (3.20) is
\[
P^k_t = \frac{\hat{\delta}_t}{r} - \frac{1}{1 + r} (b^k_{t+1} + \hat{R}^k_{t+1}).
\] (5.33)

This implies that \(b^k_t\) in conjecture (5.9) is determined by the following difference equations.
\[
\begin{align*}
b^s_t &= \frac{1}{1 + r} (b^s_{t+1} + \hat{R}^s_{t+1}) = \frac{1}{1 + r} \left( b^s_{t+1} + \frac{\kappa}{f_p} \right),
\quad b^c_t = \frac{1}{1 + r} (b^c_{t+1} + \hat{R}^c_{t+1}) = \frac{1}{1 + r} \left[ b^c_{t+1} + \frac{\kappa}{f_p} \right].
\end{align*}
\] (5.34)

**Determination of \(\xi_t\).** From (5.28) and (5.34), MR\(_t\) is written as a function of \(\xi_t\):
\[
\text{MR}_t(\xi_t) = \frac{\kappa}{f_p} \left( \frac{\Omega_t}{f_p\xi_t + \Omega_t} \right).
\] (5.35)

Since \(\text{MR}'_t(\xi_t) < 0\), there is no multiplicity of equilibrium caused by strategic complementarity among the quants. Now, \(\xi_t\) is pinned down from (5.35) and (5.29).
\[
\xi_t = \frac{1}{f_p} \left( \frac{\kappa}{\mu f_p} - 1 \right) \Omega_t.
\] (5.36)

That is, the aggregate supply of complex security is proportional to \(\Omega_t\) with a constant factor of proportionality. I assume that the parameter values are such that \(0 < \xi_t < 1\) for all \(t\) (see Appendix L for details). It is intuitive that \(\xi_t\) is increasing in \(\Omega_t\). By the same logic as in Section 4, in the absence of quants, a large \(\Omega_t\) would lead to a large price gap between the simple and complex securities ("complexity premium"). This potential gap is exploited by the quants who supply (i.e., sell) a large amount of the complex security.
Proposition 3. A symmetric steady state equilibrium exists. In this equilibrium,

1) The prices of simple and complex securities are, respectively,

\[ \hat{P}_s^- (\hat{\delta}_t) = \frac{\delta_t}{r} - \frac{\kappa}{r f_p} \text{ and } \hat{P}_c^- (\hat{\delta}_t) = \frac{\delta_t}{r} - \frac{\kappa}{r f_p} + \frac{\mu}{r}. \]  \hspace{1cm} (5.37)

2) The expected excess return on simple and complex securities are, respectively,

\[ \hat{R}_s^- = \frac{\kappa}{f_p} \text{ and } \hat{R}_c^- = \frac{\kappa}{f_p} - \mu. \]  \hspace{1cm} (5.38)

3) The number of shares of complex security supplied by the quants is \( \xi^* \), and that of simple security is \( (1 - \xi^*) \).

4) Each investor invests \( X^* \) dollars of capital in the fund, with

\[ X^* = \frac{1 - f_p}{f_m} \left[ \frac{\kappa}{f_p} \left[ \frac{f_p\xi^* + (1 - \xi^*)\Omega^*}{f_p\xi^* + \Omega^*} \right] - a^* \right]. \]  \hspace{1cm} (5.39)

5) Each manager buys \( \theta^* = \left[ \frac{(1 - \xi^*)}{\xi^*} \right] \theta^* \) shares of simple security and \( \theta_c^* \) shares of complex security per investor’s capital.

6) Each agent updates his/her estimate of \( \tilde{\delta}_t \) by

\[ \tilde{\delta}_t = \lambda^* \tilde{\delta}_{t-1} + (1 - \lambda^*) \left( \frac{\tilde{\delta}_t + \delta_t}{2} \right), \]  \hspace{1cm} where \( \lambda^* = 1 + \frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} + \rho} \) with \( \rho \equiv \frac{2\eta_u}{\eta_v}. \)  \hspace{1cm} (5.40)

The values of \((\xi^*, \Omega^*, \theta^*, a^*)\) are the solutions to the following system of equations:

\[
\begin{align*}
\xi^* &= \frac{1}{f_p} \left( \frac{\kappa}{f_p} - 1 \right) \Omega^*, \\
\Omega^* &= \frac{1}{2} \left[ \frac{1}{r} \left( \frac{1 + \rho - \lambda^*}{2 + \rho - \lambda^*} \right) + 1 \right] f_p + \frac{f_m \xi^*}{(1 - f_p) a^* \theta^*} - \frac{\kappa}{a^*} \left[ \frac{1}{\beta (1 - \lambda^*)} - \frac{1}{\rho + (1 - \lambda^*)} \right]^{-1}, \\
\theta_c^* &= \frac{\kappa}{f_p} \left[ \frac{f_p \xi^* + (1 - \xi^*) \Omega^*}{f_p \xi^* + \Omega^*} \right] - a^*, \\
a^* &= \frac{\gamma r (1 - f_p)}{2 (1 + r) \eta_u} \left[ \frac{1}{\lambda^*} \left( \frac{1 - \lambda^*}{r} + 1 \right)^2 + (1 - 2\xi^*) \right].
\end{align*}
\]  \hspace{1cm} (5.41)

The quants effectively act as arbitrageurs and competitively exploit the price gap between the simple and complex securities. As a result, the price gap is equalized to the present value of the quant’s future supply costs of complexity, i.e., \( P_{cs} - P_{ss} = \mu/r. \)
5.4 Transitional dynamics

As in Section 4.4, I simulate the transitional path of the economy where the precision of the agents’ information increases over time. Figure 2 shows the transitional dynamics of the economy.

Figure 2: Transitional dynamics. The parameter values are: \( r = 0.2, f_m = 0.02, f_p = 0.2, \kappa = 0.05, \mu = 0.1, \gamma = 0.05, \beta = 1/(1 + r), \eta_u = 0.05, \eta_e = 70, \eta_0 = 1, \delta_0 = 10, \widehat{\delta}_0 = 10. \)

Panel (a) is the aggregate supply of complex security \( \xi_t. \) For this set of parameter values, \( \xi_t \) is inverse-U shaped over time. As is clear from (5.36), this is because \( \xi_t \) is proportional to \( \Omega_t, \) which is inverse-U shaped over time. The intuition for the inverse-U shape of \( \Omega_t \) is the same as in Section 4.4. Interestingly, this pattern is consistent with the recent rise and fall

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of issuance of structured financial products (see Coval, Jurek and Stafford 2008).35

Panel (b) shows that the average prices are constant over time, as opposed to Section 4.4. Also, the price gap between the securities is equal to $\mu/r$, the present value of the quant’s future supply costs, since in every period the quants competitively exploit the price gap.

In panel (c), the investors invest smaller capital in earlier periods due to higher uncertainty about the security’s payoffs. In panel (d), the manager’s portfolio is riskier in earlier periods because the managers invest more in the risky securities to exploit the (average) excess return that would otherwise have been higher due to a higher risk premium. In this example $\theta_c^t > \theta_s^t$ holds, but this is not a general result. The relative size of $\theta_c^t$ and $\theta_s^t$ is determined by $\xi_t$ through the relationship (5.10). Hence, if $\xi_t < 0.5$ (which occurs if, for instance, $\mu$ is large) then $\theta_c^t < \theta_s^t$ holds.

6 Conclusion

This paper proposes a dynamic equilibrium model to study the implications of financial securities’ complexity for investor behavior, asset prices, and welfare. Complexity creates agency problems in delegated portfolio management, leading to excessively risky portfolio, lower capital investments in the fund, and welfare loss. Complex security’s price is higher than the simple one’s since the managers pay a premium for complexity (“complexity premium”) as it allows them to manipulate the investors’ beliefs. This is so even though investors are not fooled in equilibrium. Also, to study the supply of complexity, I present a model with “quants,” who can trade and transform simple and complex securities to exploit the complexity premium. The dynamic pattern of complex security’s supply replicates the recent rise and fall of structured finance products.

For future work, it would be interesting to extend this model by allowing the investors to invest in other funds as well, so that their capital flows in and out across funds. This setup is more realistic, and may yield some more insights. For instance, with some additional structure (e.g., heterogeneous abilities of managers), it would create reputational concerns for the managers, which might lead to interesting signaling equilibria (e.g., only a part of managers engage in complex investments).

In this paper, complexity arises for complexity’s sake, i.e., it is created by quants purely to obfuscate the security’s payoffs. This extreme way of modeling is useful for understanding one aspect of complexity. However, of course, it is also important to consider more traditional, “positive” views on complex securities. For instance, complexity may be a minor side-effect associated with sophisticated financial technique achieving a better risk sharing. Exploring this potential trade-off would be important and interesting, and is left for future research.

35If the investors are less risk-averse (say, $\gamma = 0.02$), then $\xi_t$ simply decreases over time. The simulation result is available upon request.
Appendix

A Derivation: Dynamics of $\lambda_t$ (Eq. (2.6))

Let $\eta_t^- \equiv 1/\text{Var}(\delta_t | \mathcal{H}_{t-1})$ be the precision of the agents’ estimate of $\delta_t$ before observing $\delta_t$, and $\eta_t^+ \equiv 1/\text{Var}(\delta_t | \mathcal{H}_t)$ be the one after observing $\delta_t$. By standard Kalman filtering, a new observation of $\delta_t$ will update the estimate of $\delta_t$ and its precision as follows:

$$
\begin{align*}
\hat{\delta}_t &= \lambda_t \delta_{t-1} + (1 - \lambda_t) \delta_t, \quad \text{where} \quad \lambda_t \equiv \frac{\eta_t^-}{\eta_t^+}, \quad \text{(A.1)} \\
\eta_t^+ &= \eta_t^- + \eta_u. \quad \text{(A.2)}
\end{align*}
$$

Take $\text{Var}(\cdot | \mathcal{H}_t)$ to the payoff at time $t+1$, $\delta_{t+1} = \delta_t + \epsilon_{t+1}$. By normality and independence,

$$
\frac{1}{\eta_{t+1}} = \frac{1}{\eta_t^-} + \frac{1}{\eta_v} \iff \eta_{t+1} = \frac{\eta_t^- \eta_v}{\eta_t^- + \eta_v} \iff \eta_{t+1} = \frac{(\eta_t^- + \eta_u) \eta_v}{\eta_t^- + \eta_u + \eta_v}. \quad \text{(A.3)}
$$

Meanwhile, from the definition of $\lambda_t$,

$$
\lambda_t = \frac{\eta_t^-}{\eta_t^- + \eta_u} \iff \eta_t^- = \eta_u \frac{\lambda_t}{1 - \lambda_t}. \quad \text{(A.4)}
$$

Plugging (A.4) into (A.3) and rearranging,

$$
\eta_u \frac{\lambda_{t+1}}{1 - \lambda_{t+1}} = \left(\frac{\eta_u}{1 - \lambda_t} + \eta_u\right) \eta_v \iff \lambda_{t+1} = \frac{1}{2 + \rho - \lambda_t}. \quad \text{(A.5)}
$$

B Proofs of Lemmas 1, 2, 3 and 4

In order to obtain the FOCs for $X_{j,t}$ and $C_{j,t}$, first I need to write $E[V_{t+1}(W_{j,t+1}, \epsilon_{t+1}) | \mathcal{F}_{j,t}^t]$ explicitly as a function of $X_{j,t}$ and $C_{j,t}$. Using conjecture (3.1), $V_{t+1}(W_{j,t+1}, \epsilon_{t+1})$ can be written as

$$
\begin{align*}
V_{t+1}(W_{j,t+1}, \epsilon_{t+1}) &= - \exp \left[ - AW_{j,t+1} - \frac{1}{2} B_{t+1}(1 - \epsilon_{t+1})^2 - Z_{t+1} \right] \\
&= - \exp \left[ - A(1 - f_p) \theta_{j,t} X_{j,t} \left( \delta_{t+1} + \frac{\hat{\delta}_{t+1}}{r} - b_{t+1} - (1 + r) P_t \right) \right] \quad \text{(i)} \\
&\times \exp \left[ A(1 - f_p) \theta_{j,t} X_{j,t} \frac{a_{t+1}}{1 + r} (1 - \epsilon_{t+1}) - \frac{1}{2} B_{t+1}(1 - \epsilon_{t+1})^2 \right] \quad \text{(ii)} \\
&\times \exp \left[ A f_m X_{j,t} - A(1 + r)(W_{j,t} - C_{j,t}) - Z_{t+1} - A(1 - f_p) \theta_{j,t} X_{j,t} \frac{a_{t+1}}{1 + r} \right]. \quad \text{(iii)}
\end{align*}
$$

(B.1)

Since (i), (ii) and (iii) in (B.1) are independent with each other, the expectation of $V_{t+1}(W_{j,t+1}, \epsilon_{t+1})$ is simply a product: $E[V_{t+1}(W_{j,t+1}, \epsilon_{t+1}) | \mathcal{F}_{j,t}^t] = -E[(i) | \mathcal{F}_{j,t}^t] E[(ii) | \mathcal{F}_{j,t}^t] E[(iii) | \mathcal{F}_{j,t}^t]$. In the following, I calculate $E[(i) | \mathcal{F}_{j,t}^t]$, $E[(ii) | \mathcal{F}_{j,t}^t]$ and $E[(iii) | \mathcal{F}_{j,t}^t]$. 

35
First, I calculate $E[(i)|F_{j,t}^i]$. The expectation and the variance of the terms inside the brackets of (i) are

$$E\left[ -A(1-f_p)\theta_{j,t}X_{j,t}\left( \delta_{t+1} + \frac{\hat{\theta}_{t+1}}{r} - b_{t+1} - (1+r)P_t \right) \right] = -A(1-f_p)\theta_{j,t}\bar{R}_{j,t+1}^i, \quad (B.2)$$

$$\text{Var}\left[ -A(1-f_p)\theta_{j,t}X_{j,t}\left( \delta_{t+1} + \frac{\hat{\theta}_{t+1}}{r} - b_{t+1} - (1+r)P_t \right) \right] = A^2(1-f_p)^2\theta_t^2X^2_{j,t}\Gamma_t, \quad (B.3)$$

where

$$\Gamma_t \equiv \text{Var}\left( \delta_{t+1} + \frac{\hat{\theta}_{t+1}}{r} | \mathcal{H}_t \right) = \frac{1}{\eta_u\lambda_{t+1}} \left( \frac{1-\lambda_{t+1}}{r} + 1 \right)^2. \quad (B.4)$$

Note that $\theta_{j,t}$ becomes $\theta_t$ since the investor believes that $\theta_{j,t} = \theta_t$. Therefore, $E[(i)|F_{j,t}^i]$ is

$$E[(i)|F_{j,t}^i] = \exp \left[ E\left[ -A(1-f_p)\theta_{j,t}X_{j,t}\left( \delta_{t+1} + \frac{\hat{\theta}_{t+1}}{r} - b_{t+1} - (1+r)P_t \right) \right] \right. + \frac{1}{2} \text{Var}\left[ -A(1-f_p)\theta_{j,t}X_{j,t}\left( \delta_{t+1} + \frac{\hat{\theta}_{t+1}}{r} - b_{t+1} - (1+r)P_t \right) \right] \bigg]$$

$$= \exp \left[ -A(1-f_p)\theta_{t}X_{j,t}\bar{R}_{j,t+1}^i + \frac{1}{2} A^2(1-f_p)^2\theta_t^2X^2_{j,t}\Gamma_t \right]. \quad (B.5)$$

Second, I calculate $E[(ii)|F_{j,t}^i]$. This can be computed by directly applying the following fact of normal calculus. If $Y \sim N(\mu, \sigma^2)$ then

$$E\left[ \exp \left( -mY - \frac{n}{2}Y^2 \right) \right] = \exp \left[ -\frac{1}{2} \log(1+n\sigma^2) - \frac{1}{1+n\sigma^2} \left( m\mu - \frac{n}{2} \sigma^2 + \frac{n}{2} \mu^2 \right) \right].$$

Setting $Y = (1-\epsilon_t)$, $m = -A(1-f_p)\theta_{j,t}X_{j,t}\frac{\alpha_{t+1}}{1+r}$, $n = B_{t+1}$, $\mu = 1$ and $\sigma^2 = \sigma^2_t$, it follows that

$$E[(ii)|F_{j,t}^i] = \exp \left[ -\frac{1}{2} \log(1+B_{t+1}\sigma^2_t) - \frac{1}{1+B_{t+1}\sigma^2_t} \left[ -A(1-f_p)\theta_{j,t}X_{j,t}\frac{\alpha_{t+1}}{1+r} \right] \right. \left. - \frac{1}{2} A^2(1-f_p)^2\theta^2_tX^2_{j,t}\left( \frac{\alpha_{t+1}}{1+r} \right)^2 \sigma^2_t + \frac{1}{2} B_{t+1} \right]. \quad (B.6)$$

---

**36The derivation of $\Gamma_t$ is as follows. Since**

$$\text{Var}(\delta_{t+1}|\mathcal{H}_t) = \text{Var}(\delta_{t+1} + u_{t+1}|\mathcal{H}_t) = \frac{1}{\eta_u} + \frac{1}{\eta_u} = \frac{1}{\eta_u} \left( 1 - \lambda_{t+1} \right) + \frac{1}{\eta_u} = \frac{1}{\eta_u} \lambda_{t+1},$$

$$\text{Var}(\hat{\delta}_{t+1}|\mathcal{H}_t) = \text{Var}[\lambda_{t+1}\hat{\theta}_t + (1 - \lambda_{t+1})\delta_{t+1}|\mathcal{H}_t] = (1 - \lambda_{t+1})^2 \text{Var}(\delta_{t+1}|\mathcal{H}_t) = \frac{1}{\eta_u} \left( 1 - \lambda_{t+1} \right)^2,$$

$$\text{Cov}(\delta_{t+1}, \hat{\delta}_{t+1}|\mathcal{H}_t) = \text{Cov}[\delta_{t+1}, \lambda_{t+1}\hat{\theta}_t + (1 - \lambda_{t+1})\delta_{t+1}|\mathcal{H}_t] = (1 - \lambda_{t+1}) \text{Var}(\delta_{t+1}|\mathcal{H}_t) = \frac{1}{\eta_u} \left( 1 - \lambda_{t+1} \right),$$

it follows that

$$\Gamma_t = \text{Var}(\delta_{t+1}|\mathcal{H}_t) + \left( \frac{1}{r} \right)^2 \text{Var}(\hat{\delta}_{t+1}|\mathcal{H}_t) + \frac{2}{r} \text{Cov}(\delta_{t+1}, \hat{\delta}_{t+1}|\mathcal{H}_t)$$

$$= \frac{1}{\eta_u} \left( \frac{1}{\lambda_{t+1}} \right) \left[ 1 + \left( \frac{1}{r} \right)^2 (1 - \lambda_{t+1})^2 + \frac{2}{r} (1 - \lambda_{t+1}) \right] = \frac{1}{\eta_u} \lambda_{t+1} \left( \frac{1 - \lambda_{t+1}}{r} + 1 \right)^2.$$
Finally, I calculate $E[(iii)|F^t_{j,t}]$. This is done by just setting $\theta_{j,t} = \theta_t$. I.e.,

$$E[(iii)|F^t_{j,t}] = \exp \left[ A f_m X_{j,t} - A(1 + r)(W_{j,t} - C_{j,t}) - Z_{t+1} - A(1 - f_p)\theta_t X_{j,t} \frac{a_{t+1}}{1 + r} \right]. \quad (B.7)$$

So, multiplying (B.5), (B.6) and (B.7), $E[V_{t+1}(W_{j,t+1}, \epsilon_{t+1})|F^t_{j,t}]$ can be written as

$$E[V_{t+1}(W_{j,t+1}, \epsilon_{t+1})|F^t_{j,t}] = - \exp \left[ -A(1 - f_p)\theta_t X_{j,t} \tilde{R}_{j,t+1} + A f_m X_{j,t} - A^2(1 - f_p)^2\theta_t X_{j,t} \Psi_t \left[ \frac{(1 + r)B_{t+1}}{a_{t+1} A(1 - f_p)} \right] + \frac{1}{2} A^2(1 - f_p)^2 \theta_t^2 X_{j,t}^2 (\Gamma_t + \Psi_t) - \frac{1}{2} \log(1 + B_{t+1} \sigma^2_t) - \frac{1}{2} \left( \frac{B_{t+1}}{1 + B_{t+1} \sigma^2_t} \right) - A(1 + r)(W_{j,t} - C_{j,t}) - Z_{t+1} \right]. \quad (B.8)$$

where

$$\Psi_t \equiv \left( \frac{a_{t+1}}{1 + r} \right)^2 \sigma^2_t \frac{1}{1 + B_{t+1} \sigma^2_t}. \quad (B.9)$$

From (B.8), the FOC for $X_{j,t}$ is

$$(1 - f_p)\theta_t \tilde{R}_{j,t+1} - f_m + A(1 - f_p)^2 \theta_t \Psi_t \left[ \frac{(1 + r)B_{t+1}}{a_{t+1} A(1 - f_p)} \right] - A(1 - f_p)^2 \theta_t^2 X_{j,t} (\Gamma_t + \Psi_t) = 0. \quad (B.10)$$

This implies that the investor’s optimal policy for investment is

$$X_{j,t} = \frac{(1 - f_p)\theta_t \tilde{R}_{j,t+1} - f_m}{A(1 - f_p)^2 \theta_t^2 (\Gamma_t + \Psi_t)} + \frac{\Psi_t}{(\Gamma_t + \Psi_t)} \theta_t \left[ \frac{(1 + r)B_{t+1}}{a_{t+1} A(1 - f_p)} \right]. \quad (B.11)$$

Plugging back the optimal $X_{j,t}$ (B.11) into (B.8),

$$E[V_{t+1}(W_{j,t+1}, \epsilon_{t+1})|F^t_{j,t}] = - \exp \left[ -\frac{1}{2} A^2(1 - f_p)^2 \theta_t^2 X_{j,t}^2 (\Gamma_t + \Psi_t) - \frac{1}{2} \log(1 + B_{t+1} \sigma^2_t) - \frac{1}{2} \left( \frac{B_{t+1}}{1 + B_{t+1} \sigma^2_t} \right) - A(1 + r)(W_{j,t} - C_{j,t}) - Z_{t+1} \right].$$

In the symmetric equilibrium, the market clearing implies $X_{j,t} = X_t = (1 - \epsilon_t)/\theta_t$. Also, I conjecture that $a_t = A(1 - f_p)(\Gamma_t + \Psi_t)$. I verify this later when obtaining the equilibrium price. (See (3.21) in the text.) Using these, the above expression becomes

$$E[V_{t+1}(W_{j,t+1}, \epsilon_{t+1})|F^t_{j,t}] = - \exp \left[ -\frac{1}{2} A(1 - f_p)a_t (1 - \epsilon_t)^2 - \frac{1}{2} \log(1 + B_{t+1} \sigma^2_t) - \frac{1}{2} \left( \frac{B_{t+1}}{1 + B_{t+1} \sigma^2_t} \right) - A(1 + r)(W_{j,t} - C_{j,t}) - Z_{t+1} \right]. \quad (B.12)$$

To simplify notation, define

$$\zeta_t \equiv - \log \beta + \frac{1}{2} A(1 - f_p)a_t (1 - \epsilon_t)^2 + \frac{1}{2} \log(1 + B_{t+1} \sigma^2_t) + \frac{1}{2} \left( \frac{B_{t+1}}{1 + B_{t+1} \sigma^2_t} \right) + Z_{t+1}. \quad (B.13)$$

Then, since $X_{j,t}$ is already chosen optimally, the Bellman equation is now

$$V_t(W_{j,t}, \epsilon_t) = \max_{Z_{j,t}} \left[ - \exp(-\gamma C_{j,t}) - \exp \left[ -\zeta_t - A(1 + r)(W_{j,t} - C_{j,t}) \right] \right]. \quad (B.14)$$

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The FOC for \( C_{j,t} \) gives the investor’s optimal consumption policy:

\[
\gamma \exp(-\gamma C_{j,t}) - \exp(-\zeta_t)A(1 + r) \exp\left[-A(1 + r)(W_{j,t} - C_{j,t})\right] = 0
\]

\[\iff \quad C_{j,t} = \frac{A(1 + r)}{\gamma + A(1 + r)} W_{j,t} + \frac{1}{\gamma + A(1 + r)} \left[\zeta_t + \log \frac{\gamma}{A(1 + r)}\right].\]  
(B.15)

Plugging this optimal \( C_{j,t} \), the Bellman equation can be written as

\[
-ZW_{j,t} - \frac{1}{2} B_t(1 - \epsilon_t)^2 - Z_t = -\frac{\gamma A(1 + r)}{\gamma + A(1 + r)} W_{j,t} - \frac{\gamma}{\gamma + A(1 + r)} \zeta_t
\]

\[\times \left[\exp\left[-\frac{\gamma}{\gamma + A(1 + r)} \log \frac{\gamma}{A(1 + r)}\right] + \exp\left[\frac{A(1 + r)}{\gamma + A(1 + r)} \log \frac{\gamma}{A(1 + r)}\right]\right].\]  
(B.16)

Taking log,

\[
-ZW_{j,t} - \frac{1}{2} B_t(1 - \epsilon_t)^2 - Z_t = -\frac{\gamma A(1 + r)}{\gamma + A(1 + r)} W_{j,t} - \frac{\gamma}{\gamma + A(1 + r)} \zeta_t
\]

\[+ \log \left[\exp\left[-\frac{\gamma}{\gamma + A(1 + r)} \log \frac{\gamma}{A(1 + r)}\right] + \exp\left[\frac{A(1 + r)}{\gamma + A(1 + r)} \log \frac{\gamma}{A(1 + r)}\right]\right].\]  
(B.17)

Comparing the coefficients of \( W_{j,t} \), the value of \( A \) is pinned down:

\[\gamma A(1 + r) = \frac{(1 + r)}{1 + r}. \quad \iff \quad A = \gamma \left(\frac{r}{1 + r}\right).\]  
(B.18)

Plugging (B.18) into (B.17) and rearranging,

\[\zeta_t - \frac{1}{2}(1 + r)B_t(1 - \epsilon_t)^2 - (1 + r)Z_t + r \log r - (1 + r) \log(1 + r) = 0.\]  
(B.19)

Plugging the definition of \( \zeta_t \) (B.13) back into this,

\[-\log \beta + \frac{1}{2} A(1 - f_p)a_t(1 - \epsilon_t)^2 + \frac{1}{2} \log(1 + B_{t+1}\sigma_t^2) + \frac{1}{2} \left(\frac{B_{t+1}}{1 + B_{t+1}\sigma_t^2}\right) + Z_{t+1}
\]

\[-\frac{1}{2}(1 + r)B_t(1 - \epsilon_t)^2 - (1 + r)Z_t + r \log r - (1 + r) \log(1 + r) = 0.\]  
(B.20)

Comparing the coefficients of \((1 - \epsilon_t)^2\) in (B.20),

\[\frac{1}{2} A(1 - f_p)a_t = \frac{1}{2} (1 + r) B_t \quad \iff \quad B_t = A(1 - f_p) \frac{a_t}{1 + r}.\]  
(B.21)

The remaining deterministic terms in (B.20) imply

\[Z_t = \frac{1}{1 + r} \left[Z_{t+1} + \frac{1}{2} \log(1 + B_{t+1}\sigma_t^2) + \frac{1}{2} \left(\frac{B_{t+1}}{1 + B_{t+1}\sigma_t^2}\right) - \log \beta + r \log r - (1 + r) \log(1 + r)\right].\]  
(B.22)

Using (B.21), the investor’s optimal investment (B.11) becomes

\[X_{j,t} = \frac{(1 - f_p)\theta_t \hat{R}_{t+1} - f_m}{A(1 - f_p)^2 \theta_t^2 (\Gamma_t + \Psi_t)} + \frac{\Psi_t}{(\Gamma_t + \Psi_t) \theta_t}.\]  
(B.23)
and (B.9) becomes
\[
\Psi_t = \frac{\left(\frac{a_{t+1}}{1 + \tau}\right)^2 \sigma^2_c}{1 + A(1 - f_p)\left(\frac{a_{t+1}}{1 + \tau}\right)\sigma^2_c}.
\]  
(B.24)

\[\square\]

C Proof of Proposition 1

First I show that there exists a unique steady state value of \(a_t\), denoted as \(a^*\), for a general value of \(\sigma^2_c\). From (3.21), this amounts to show that there exists a unique \(a^* > 0\) such that

\[a^* = \Phi(a^*),\]  
(C.1)

where

\[\Phi(a) \equiv A(1 - f_p) \left[ \Gamma^* + \frac{\left(\frac{a}{1 + \tau}\right)^2 \sigma^2_c}{1 + A(1 - f_p)\left(\frac{a}{1 + \tau}\right)\sigma^2_c} \right] \quad \text{with} \quad \Gamma^* = \frac{1}{\eta_a \lambda^*} \left(\frac{1 - \lambda^*}{r} + 1\right)^2,\]  
(C.2)

where \(\Gamma^*\) is the unique long-run steady state value of \(\Gamma_t\). It is easy to check that \(\Phi'(. > 0), \Phi''(.) > 0, \Phi(0) = A(1 - f_p)\Gamma^* > 0\) and \(\lim_{a \to \infty} \Phi(a) = \frac{1}{1 + \tau} < 1\). Therefore, there exists a unique \(a^* > 0\) such that \(a^* = \Phi(a^*)\). The solution is, by solving (C.1),

\[a^* = \frac{(1 + r)^2}{2r \sigma^2_c} \left[-h + \sqrt{h^2 + \frac{4r \Gamma^* \sigma^2_c}{(1 + r)^2}}\right] \quad \text{with} \quad h \equiv \frac{1}{A(1 - f_p)} - A(1 - f_p)\Gamma^* \sigma^2_c.\]  
(C.3)

Then, from (3.22) and (3.16), there exists a unique steady state value of \(b_t\):

\[b^* = \frac{\tilde{R}^*}{r} = \frac{1}{r} \left(\frac{\kappa}{f_p} - a^* \sigma^2_c\right),\]  
(C.4)

which is positive by Assumption 2. Then, \(\theta^*\) and \(X^*\) are readily obtained from (3.19) and (3.12), respectively.

The limit equilibrium is obtained by taking \(\sigma^2_c \to 0\). It is straightforward to check that \(\Psi_t \to 0 \forall t\), \(a^* \to A(1 - f_p)\Gamma^*\) (by l’Hôpital’s rule) and \(b^* \to \frac{\kappa}{r f_p}\) when \(\sigma^2_c \to 0\).  
\[\square\]

D Proof of Lemma 5

To calculate the impact of the manager’s deviation on his lifetime utility, I need to study how \(X_{j,t+\tau}\), \(\tau = 1, 2, \ldots\), will react to this deviation, since the manager’s utility at time \(t + \tau\) depends on \(X_{j,t+\tau}\). This amounts to study, as is clear from Lemma 4, how \(\tilde{R}_{j,t+\tau+1}\) will react to this deviation. This, in turn, amounts to identify how \(\mathcal{H}_{j,t+\tau}\) will differ from \(\mathcal{H}_{t+\tau}\) after the manager’s deviation at time \(t\).

The investor’s inferred history \(\mathcal{H}_{j,t+\tau}\) is identified as follows. As discussed in the text, \(\delta^j_{t+1}\) is obtained as (4.3). Then, what happens to her inferred payoffs afterwards? From time \(t + 1\) on, manager \(j\) reverts to the equilibrium strategy, \(\theta_{t+1}, \theta_{t+2}, \ldots\). This means that investor \(j\) will correctly know the manager’s risky security holding from time \(t + 1\) on, \(\theta_{t+1}X_{j,t+1}, \theta_{t+1}X_{j,t+2}, \ldots\). This, in turn, implies that the investor will correctly infer the payoff values from time \(t + 2\) onwards, \(\delta_{t+2}, \delta_{t+3}, \ldots\), from observing the prices and fund performances. Also, the investor correctly infers the payoff values up to time \(t\) since the manager plays the equilibrium strategy up to time \(t - 1\). In sum, following manager \(j\)’s deviation at time \(t\), investor \(j\) misperceives
the payoff value once and only at time $t + 1$. I.e., for $\tau = 1, 2, \ldots$, $\mathcal{H}^{\ell}_{j,t+\tau} = (\delta_1, \ldots, \delta_{t}, \delta_{t+1}, \delta_{t+2}, \ldots, \delta_{t+\tau})$.

Now I derive $\delta^j_{t,t+\tau}$. Since investor $j$ knows the true history up to time $t$ (i.e., $\mathcal{H}^{j}_{j,t} = \mathcal{H}_{t}$), $\delta^j_{t,t} = \delta_{t}$ still holds at time $t$. At time $t + 1$, however, she misperceives the value of $\delta_{t+1}$ as $\delta^j_{t,t+1}$. So, from (2.5),

$$
\begin{align*}
\hat{\delta}_{j,t+1} &= \lambda_{t+1}\hat{\delta}_{j,t} + (1 - \lambda_{t+1})\delta^j_{t,t+1} = \lambda_{t+1}(\hat{\delta}_{j,t} + (1 - \lambda_{t+1})\delta^j_{t,t+1}) + (1 - \lambda_{t+1})((\delta^j_{t,t+1} - \delta_{t+1}) \\
&= \hat{\delta}_{t+1} + (1 - \lambda_{t+1})(\delta^j_{t,t+1} - \delta_{t+1}).
\end{align*}
$$

Because she correctly knows the payoff values from time $t + 2$ on,

$$
\begin{align*}
\hat{\delta}_{j,t+2} &= \lambda_{t+2}\hat{\delta}_{j,t+1} + (1 - \lambda_{t+2})\delta_{t+2} = \lambda_{t+2}\hat{\delta}_{t+1} + (1 - \lambda_{t+2})\delta_{t+2} + \lambda_{t+2}(1 - \lambda_{t+1})(\delta^j_{t,t+1} - \delta_{t+1}) \\
&= \hat{\delta}_{t+2} + \lambda_{t+2}(1 - \lambda_{t+1})(\delta^j_{t,t+1} - \delta_{t+1}).
\end{align*}
$$

Repeating this, at time $t + \tau$, $\tau = 1, 2, \ldots$,

$$
\begin{align*}
\hat{\delta}_{j,t+\tau} &= \hat{\delta}_{t+\tau} + \lambda_{t+\tau}\lambda_{t+\tau - 1} \cdots \lambda_{t+2}(1 - \lambda_{t+1})(\delta^j_{t,t+1} - \delta_{t+1}) \\
&= \hat{\delta}_{t+\tau} + (1 - \lambda_{t+1})\left(\prod_{\nu=2}^{\tau} \lambda_{t+\nu}\right)(\delta^j_{t,t+1} - \delta_{t+1}).
\end{align*}
$$

(D.1)

Here, I set $\prod_{\nu=2}^{t+1} \lambda_{t+\nu} \equiv 1$ by abuse of notation.

Equation (D.1) shows that, at time $t + \tau$, investor $j$ has a biased estimate $\hat{\delta}_{j,t+\tau} \neq \delta_{t+\tau}$. She observes the price $P_{t+\tau}$, but, since she cannot observe $\epsilon_{t+\tau}$, she will wrongly infer the composition of the price function. I.e, she infers $\epsilon_{t+\tau}$ as $\epsilon^j_{t,t+\tau}$, whose value is defined in the following identity.

$$
P_{t+\tau} = \frac{\delta_{t+\tau}}{r} + \left(\frac{a_{t+\tau}}{1 + r}\right)\epsilon_{t+\tau} - b_{t+\tau} \equiv \frac{\hat{\delta}_{t+\tau}}{r} + \left(\frac{a_{t+\tau}}{1 + r}\right)\epsilon^j_{t,t+\tau} - b_{t+\tau} \tag{D.2}
$$

Decomposition of LHS inferred by investor $j$

$$
\iff \epsilon^j_{t,t+\tau} = \epsilon_{t+\tau} - \frac{1}{a_{t+\tau}}\left(\frac{1 + r}{r}\right)(\hat{\delta}_{j,t+\tau} - \hat{\delta}_{t+\tau}). \tag{D.3}
$$

Clearly, $\epsilon^j_{t,t+\tau} < \epsilon_{t+\tau}$ if $\hat{\delta}_{j,t+\tau} > \hat{\delta}_{t+\tau}$. Due to the assumption that there is no informational externalities between funds, there is no way the investor learns that her inference is incorrect. This undershoot of the inference on $\epsilon_{t+\tau}$ will lead to an overshoot of the expected excess return. This is checked by noting that $\hat{R}_{t+\tau+1} = \hat{R}_{t+\tau+1} - a_{t+\tau}\epsilon_{t+\tau}$ holds identically: it follows that $\hat{R}^j_{t,t+\tau+1} = \hat{R}_{t+\tau+1} - a_{t+\tau}(\epsilon_{t+\tau} - \epsilon^j_{t,t+\tau})$. So,

$$
\begin{align*}
\hat{R}^j_{t,t+\tau+1} &= \hat{R}_{t+\tau+1} + a_{t+\tau}(\epsilon_{t+\tau} - \epsilon^j_{t,t+\tau}) \\
&= \hat{R}_{t+\tau+1} + \left(\frac{1 + r}{r}\right)(1 - \lambda_{t+1})\left(\prod_{\nu=2}^{\tau} \lambda_{t+\nu}\right)\left(\frac{\theta_{j,t} - \theta_t}{\theta_t}\right)R_{t+1}. \tag{D.4}
\end{align*}
$$

Thus, following the manager’s deviation, $\theta_{j,t} > \theta_t$, investor $j$ will become over-optimistic about the excess return at time $t + \tau$ if $R_{t+1} > 0$, while she becomes over-pessimistic if $R_{t+1} < 0$. 

40
Now, from (D.4) and (3.8), the investor’s capital investment at \( t + \tau \) is calculated as:

\[
X_{j,t+\tau} = \frac{1}{a_{t+\tau} \theta_{t+\tau}} \left[ \hat{R}_{j,t+\tau+1}^i - \frac{f_m}{(1 - f_p) \theta_{t+\tau}} \right] + \frac{\Psi_{t+\tau}}{(1 - f_p) \theta_{t+\tau} + \Psi_{t+\tau}} \\
= X_{t+\tau} + \frac{1}{a_{t+\tau} \theta_{t+\tau}} (\hat{R}_{j,t+\tau+1}^i - \hat{R}_{j,t+\tau+1}) \\
= X_{t+\tau} + \frac{1}{a_{t+\tau} \theta_{t+\tau}} \left( \frac{1 + \tau}{\tau} \right) (1 - \lambda_{t+1}) \left( \prod_{j=2}^{\tau} \lambda_{t+\nu} \right) \frac{\theta_{j,t}}{\theta_t} R_{\tau+1} \\
= X_{t+\tau} + X_{t+\tau}^+ (\theta_{j,t}). \quad \text{(D.5)}
\]

\[\square\]

\section*{E Proof of Lemma 6}

There are several points to note for setting up manager \( j \)'s problem.

1. Since the manager chooses \( \theta_{j,t} \) in stage 1 of time \( t \), \( X_{j,t} \) should be in his expectation operator.

2. Since the manager has played the equilibrium strategy up to time \( t - 1 \), investor \( j \) correctly knows \( H_t \), hence \( \hat{R}_{j,t+1}^i = \hat{R}_{t+1}^i \). Moreover, since the investor cannot observe \( \theta_{j,t} \), she invests capital based on her (wrong) belief that \( \theta_{j,t} = \theta_t \). Thus, \( X_{j,t} \) can be replaced by \( X_t \), the equilibrium value. Of course, \( X_{j,t+\tau} \) will depend on \( \theta_{j,t} \), so should not be replaced by \( X_{t+\tau} \).

3. The management fee at time \( t \), \( f_m X_t \), can be omitted from the problem. Of course, the management fee at time \( t + \tau \) should not be omitted.

4. The terms involving \( X_{t+\tau} \) (not \( X_{j,t+\tau} \)) can be omitted from the problem. This is innocuous since these terms are additively separated from \( X_{t+\tau}^+ (\theta_{j,t}) \), which is relevant for the problem at time \( t \).

Taking these into account, an equivalent problem for manager \( j \) is to choose \( \theta_{j,t} \in (-\infty, \infty) \) to maximize

\[
E \left( \sum_{\tau=1}^{\infty} \beta^\tau \left[ f_p R_{t+\tau+1} \theta_{t+\tau} X_{t+\tau}^+ (\theta_{j,t}) + f_m X_{t+\tau}^+ (\theta_{j,t}) - \kappa \theta_{t+\tau} X_{t+\tau}^+ (\theta_{j,t}) \right] F_{j,t}^m \right). \quad \text{(E.1)}
\]

In the following, I compute (i), (ii) and (iii) explicitly.

Calculation of (i):

Conditional on \( F_{j,t}^m \) (i.e., in stage 1 of time \( t \)), the manager does not know \( \epsilon_t \), \( P_t \), \( X_t \) and \( P_{t+1} \). Conditional on \( F_{j,t+1}^m \), he knows \( \epsilon_t \), \( P_t \), \( X_t \), but not \( P_{t+1} \). Thus,

\[
E \left( f_p R_{t+1} \theta_{j,t} X_t \mid F_{j,t}^m \right) = E \left[ E \left( f_p R_{t+1} \theta_{j,t} X_t \mid F_{j,t+1}^m \right) \mid F_{j,t}^m \right] = E \left( f_p \hat{R}_{t+1} \theta_{j,t} X_t \mid F_{j,t}^m \right) \\
= E \left( f_p \hat{R}_{t+1} \theta_{j,t} \frac{1 - \epsilon_t}{\theta_t} \mid F_{j,t}^m \right) = f_p \theta_{j,t} E \left[ (\hat{R}_{t+1} - a_t \epsilon_t) (1 - \epsilon_t) \mid F_{j,t}^m \right] \\
= f_p \theta_{j,t} (\hat{R}_{t+1} + a_t \sigma_t^2). \quad \text{(E.2)}
\]
Calculation of (ii):

\[
E(X_t | \mathcal{F}^m_{j,t}) = E \left( \frac{1 - \epsilon_t}{\theta_t} \Big| \mathcal{F}^m_{j,t} \right) = \frac{1}{\theta_t}. \tag{E.3}
\]

Calculation of (iii):

First, focus on the expectation of the terms inside the small square brackets:

\[
E \left( f_p R_{t+\tau+1} \theta_{t+\tau} X^+_{t+\tau} (\theta_{j,t}) + f_m X^+_{t+\tau} (\theta_{j,t}) - \kappa \theta_{t+\tau} X^+_{t+\tau} (\theta_{j,t}) \Big| \mathcal{F}^m_{j,t} \right)
\]

\[
= E \left[ (f_p \tilde{R}_{t+\tau+1} \theta_{t+\tau} + f_m - \kappa \theta_{t+\tau}) R_{t+\tau+1} \frac{1}{\alpha_{t+\tau} \theta_{t+\tau}} \left(1 + \frac{r}{\tau} \right) (1 - \lambda_{t+\tau}) \left( \prod_{\nu=2}^{\tau} \lambda_{t+\nu} \right) \left( \frac{\theta_{j,t} - \theta_t}{\theta_t} \right) \bigg| \mathcal{F}^m_{j,t} \right]
\]

\[
= \left( \frac{\theta_{j,t} - \theta_t}{\theta_t} \right) \left(1 + \frac{r}{\tau} \right) (1 - \lambda_{t+1}) \left( \prod_{\nu=2}^{\tau} \lambda_{t+\nu} \right) \frac{1}{\alpha_{t+\tau} \theta_{t+\tau}} E \left[ (f_p \tilde{R}_{t+\tau+1} \theta_{t+\tau} + f_m - \kappa \theta_{t+\tau}) R_{t+\tau+1} \bigg| \mathcal{F}^m_{j,t} \right]. \tag{iv}
\]

Calculation of (iv):

\[
E \left[ (f_p \tilde{R}_{t+\tau+1} \theta_{t+\tau} + f_m - \kappa \theta_{t+\tau}) R_{t+\tau+1} \bigg| \mathcal{F}^m_{j,t} \right]
\]

\[
= E \left[ (f_p (\tilde{R}_{t+\tau+1} - \alpha_{t+\tau} \epsilon_{t+\tau}) \theta_{t+\tau} + f_m - \kappa \theta_{t+\tau}) R_{t+\tau+1} \bigg| \mathcal{F}^m_{j,t} \right]
\]

\[
= (f_p \tilde{R}_{t+\tau+1} \theta_{t+\tau} + f_m - \kappa \theta_{t+\tau}) \tilde{R}_{t+\tau+1} - f_p \alpha_{t+\tau} \epsilon_{t+\tau} E \left[ (\epsilon_{t+\tau} \tilde{R}_{t+\tau+1} \bigg| \mathcal{F}^m_{j,t} \right]. \tag{v}
\]

Calculation of (v):

\[
E(\epsilon_{t+\tau} \tilde{R}_{t+\tau+1} \bigg| \mathcal{F}^m_{j,t} \right) = \text{Cov}(\epsilon_{t+\tau}, R_{t+\tau+1} | \mathcal{F}^m_{j,t})
\]

\[
= \text{Cov} \left[ \epsilon_{t+\tau}, \delta_{t+1} + \left( \frac{\alpha_{t+1}}{1 + r} \right) \epsilon_{t+1} - b_{t+1} \right] - (1 + r) \left[ \frac{\delta_t}{r} + \left( \frac{a_t}{1 + r} \right) \epsilon_t - b_t \right] \bigg| \mathcal{F}^m_{j,t} \right]
\]

\[
= \left\{ \begin{array}{ll}
\left( \frac{a_{t+1}}{1 + r} \right) \sigma^2_{\epsilon_{t+1}}, & \text{if } \tau = 1, \\
0, & \text{if } \tau = 2, 3, \ldots
\end{array} \right. \tag{v}
\]

So,

\[
\text{(iv)} = \left\{ \begin{array}{ll}
(f_p \tilde{R}_{t+\tau+1} \theta_{t+\tau} + f_m - \kappa \theta_{t+\tau}) \tilde{R}_{t+\tau+1} - f_p \alpha_{t+\tau} \epsilon_{t+\tau} \left( \frac{a_{t+1}}{1 + r} \right) \sigma^2_{\epsilon_{t+1}}, & \text{if } \tau = 1, \\
(f_p \tilde{R}_{t+\tau+1} \theta_{t+\tau} + f_m - \kappa \theta_{t+\tau}) \tilde{R}_{t+\tau+1}, & \text{if } \tau = 2, 3, \ldots
\end{array} \right. \tag{E.7}
\]

Plugging (E.7) into (E.4), multiplying \(\beta^\tau\) and rearranging,

\[
\beta^\tau E \left( f_p R_{t+\tau+1} \theta_{t+\tau} X^+_{t+\tau} (\theta_{j,t}) + f_m X^+_{t+\tau} (\theta_{j,t}) - \kappa \theta_{t+\tau} X^+_{t+\tau} (\theta_{j,t}) \bigg| \mathcal{F}^m_{j,t} \right)
\]

\[
= \beta^\tau \left( \frac{\theta_{j,t} - \theta_t}{\theta_t} \right) \frac{1 + r}{r} (1 - \lambda_{t+1}) \left( \prod_{\nu=2}^{\tau} \lambda_{t+\nu} \right) \frac{1}{\alpha_{t+\tau} \theta_{t+\tau}} (f_p \tilde{R}_{t+\tau+1} \theta_{t+\tau} + f_m - \kappa \theta_{t+\tau}) \tilde{R}_{t+\tau+1}
\]

\[
- \beta \left( \frac{\theta_{j,t} - \theta_t}{\theta_t} \right) \frac{1 - \lambda_{t+1}}{r} f_p a_{t+1} \sigma^2_{\epsilon_{t+1}}. \tag{E.8}
\]
Summing this up over $\tau$, (iii) is
\[
(iii) = \sum_{\tau=1}^{\infty} \beta^\tau E\left(f_p R_{t+\tau} \theta_{t+\tau} X_{t+\tau+1}^+ (\theta_{t+\tau}) + f_m X_{t+\tau+1}^+ (\theta_{t+\tau}) - \kappa \theta_{t+\tau} X_{t+\tau+1}^+ (\theta_{t+\tau}) | X_{j,t}^m\right)
\]
\[
= \left(\frac{\theta_{j,t} - \theta_t}{\theta_t}\right) R_{t+1} \left[ \sum_{\tau=1}^{\infty} \beta^\tau \left(1 + \frac{1}{\tau}\right) (1 - \lambda_{t+\tau}) \left(\prod_{\nu=2}^{\tau} \lambda_{t+\nu}\right) \frac{1}{a_{t+\tau}} \left(f_p R_{t+\tau+1} + \frac{f_m}{\theta_{t+\tau}} - \kappa\right) \right]
\]
\[
- \beta \left(\frac{\theta_{j,t} - \theta_t}{\theta_t}\right) \left(\frac{1 - \lambda_{t+1}}{\theta_t}\right) f_p a_{t+1} \sigma_e^2
\]
\[
= \left(\frac{\theta_{j,t} - \theta_t}{\theta_t}\right) (R_{t+1} \Omega_t - \phi_t),
\]
where
\[
\Omega_t \equiv \sum_{\tau=1}^{\infty} \beta^\tau \left(1 + \frac{1}{\tau}\right) (1 - \lambda_{t+\tau}) \left(\prod_{\nu=2}^{\tau} \lambda_{t+\nu}\right) \frac{1}{a_{t+\tau}} \left(f_p R_{t+\tau+1} + \frac{f_m}{\theta_{t+\tau}} - \kappa\right)
\]
and
\[
\phi_t \equiv \beta \left(1 - \lambda_{t+1}\right) f_p a_{t+1} \sigma_e^2.
\]

Using the above expressions for (i), (ii) and (iii), and canceling out $1/\theta_t$, problem (E.1) reduces to
\[
\max_{\theta_{j,t} \in (-\infty, \infty)} f_p \theta_{j,t} \left(R_{t+1} + a_t \sigma_e^2\right) - \kappa \theta_{j,t} + (\theta_{j,t} - \theta_t) (R_{t+1} \Omega_t - \phi_t).
\]

**F Proof of Proposition 2**

The existence of $a^*$ is proved as in the proof of Proposition 1. Now, in order to obtain $\Omega^*$, I derive the law of motion for $\Omega_t$. First, rearrange $\Omega_t$ as follows.
\[
\Omega_t \equiv \sum_{\tau=1}^{\infty} \beta^\tau \left(1 + \frac{1}{\tau}\right) (1 - \lambda_{t+\tau}) \left(\prod_{\nu=2}^{\tau} \lambda_{t+\nu}\right) \frac{1}{a_{t+\tau}} \left(f_p R_{t+\tau+1} + \frac{f_m}{\theta_{t+\tau}} - \kappa\right)
\]
\[
= (1 - \lambda_{t+1}) \sum_{\tau=1}^{\infty} \beta^\tau \left(\prod_{\nu=2}^{\tau} \lambda_{t+\nu}\right) M_{t+\tau},
\]
where
\[
M_{t+\tau} \equiv \left(1 + \frac{1}{\tau}\right) \left[f_p \left(\frac{\Gamma_{t+\tau}}{\Gamma_{t+\tau+\theta_{t+\tau}} + \psi_{t+\tau}}\right) + \left(1 - f_p\right) \frac{1}{a_{t+\tau} \theta_{t+\tau}} - \kappa \frac{1}{a_{t+\tau}}\right].
\]

Then, noting that $\prod_{\nu=2}^{\infty} \lambda_{t+\nu} \equiv 1$,
\[
\frac{\Omega_t}{1 - \lambda_{t+1}} = \beta M_{t+1} + \beta^2 \lambda_{t+2} M_{t+2} + \beta^3 \lambda_{t+3} M_{t+3} + \beta^4 \lambda_{t+4} M_{t+4} + \cdots,
\]
\[
\frac{\Omega_{t+1}}{1 - \lambda_{t+2}} = \beta M_{t+2} + \beta^2 \lambda_{t+3} M_{t+3} + \beta^3 \lambda_{t+4} M_{t+4} + \cdots.
\]

From these two equations,
\[
\frac{\Omega_t}{1 - \lambda_{t+1}} = \beta M_{t+1} + \beta \lambda_{t+2} \frac{\Omega_{t+1}}{1 - \lambda_{t+2}} = \beta \left[M_{t+1} + \frac{\Omega_{t+1}}{\rho + (1 - \lambda_{t+1})}\right].
\]
The second equality follows from $\lambda_{t+2}/(1-\lambda_{t+2}) = 1/|\rho + (1-\lambda_{t+1})|$. Plugging back the definition of $M_{t+1}$, the law of motion for $\Omega_t$ is

$$
\Omega_t = \beta(1-\lambda_{t+1}) \left[ \left( \frac{1+r}{r} \right) f_p \left( \frac{\Gamma_{t+1}}{\Gamma_{t+1} + \Psi_{t+1}} \right) + \left( \frac{f_m}{1-f_p} \right) \frac{1}{a_{t+1} \theta_{t+1}} - \frac{\kappa}{a_{t+1}} + \frac{\Omega_{t+1}}{\rho + (1-\lambda_{t+1})} \right].
$$

(F.3)

Using this, I show that there exist unique steady state values $\Omega^* > 0$ and $\theta^* > 0$. From (F.3),

$$
\Omega^* = \beta(1-\lambda^*) \left[ \left( \frac{1+r}{r} \right) f_p \left( \frac{\Gamma^*}{\Gamma^* + \Psi^*} \right) + \left( \frac{f_m}{1-f_p} \right) \frac{1}{a^* \theta^*} - \frac{\kappa}{a^*} + \frac{\Omega^*}{\rho + (1-\lambda^*)} \right].
$$

(F.4)

Solving this for $\Omega^*$,

$$
\Omega^* = \left( \frac{1+r}{r} \right) f_p \left( \frac{\Gamma^*}{\Gamma^* + \Psi^*} \right) + \left( \frac{f_m}{1-f_p} \right) \frac{1}{a^* \theta^*} - \frac{\kappa}{a^*} + \frac{\Omega^*}{\rho + (1-\lambda^*)} \right]^{-1} \equiv J(\theta^*).
$$

(F.5)

It is readily checked that $J'(\cdot) < 0$ and $\lim_{\theta \to 0} J(\theta) = \infty$. On the other hand, rearranging (4.11),

$$
\Omega^* = \frac{\kappa - f_p a^* \sigma^2 + \phi^*}{A(1-f_p)\Gamma^* + \frac{f_m}{(1-f_p)\theta^*}} - f_p \equiv K(\theta^*).
$$

(F.6)

Note that $K'(\cdot) > 0$ and $K(0) = -f_p < 0$. For the existence of solution, I assume $\lim_{\theta \to \infty} J(\theta) < \lim_{\theta \to \infty} K(\theta)$, i.e.,

$$
\left( \frac{1+r}{r} \right) f_p \left( \frac{\Gamma^*}{\Gamma^* + \Psi^*} \right) - \frac{\kappa}{a^*} \right] \left[ \frac{1}{\beta(1-\lambda^*)} - \frac{1}{\rho + (1-\lambda^*)} \right]^{-1} < \frac{\kappa - f_p a^* \sigma^2 + \phi^*}{A(1-f_p)\Gamma^*} - f_p.
$$

(F.7)

This assumption is easily satisfied. To check this, consider the case $\sigma^2 \to 0$. In this case, this assumption becomes

$$
\left( \frac{1+r}{r} \right) \left( f_p - \frac{\kappa}{a^*} \right) \left[ \frac{1}{\beta(1-\lambda^*)} - \frac{1}{\rho + (1-\lambda^*)} \right]^{-1} < \frac{\kappa}{a^*} - f_p.
$$

(F.8)

So, if $\kappa > f_p a^*$, this assumption is satisfied; but $\kappa > f_p a^*$ is already assumed by Assumption 3. Then, $\theta^*$ is uniquely pinned down by $J(\theta^*) \equiv K(\theta^*)$. The rest of the Proposition is proved as in Proposition 1. \qed

G Proof of Corollary 2

The investor’s value function is the same for the simple- and the complex security cases. So, the investor’s utility is the same if her wealth $W_t$ is the same for both cases. In the symmetric equilibrium (hence subscript $j$ is omitted), the wealth is determined by the dynamic budget constraint:

$$
W_{t+1} = (1-f_p)R_{t+1}\theta_t X_t - f_m X_t + (1+r)(W_t - C_t).
$$

(G.1)

(i)

(ii)

In the following, I compute the equilibrium values of (i) and (ii) of (G.1) and show that $W_t$ is the same in the two cases.
(i): In equilibrium, the market clearing implies \( \theta_t X_t = 1 - \epsilon_t \). Using this, (i) is

\[
(1 - f_p) R_{t+1} \theta_t X_t - f_m X_t = (1 - f_p) R_{t+1} (1 - \epsilon_t) - \frac{f_m}{\theta_t} (1 - \epsilon_t)
\]

\[
= (1 - f_p) \left[ \delta_{t+1} + \frac{\theta_{t+1}^e}{r} + \frac{\alpha t + \epsilon_{t+1} - (1 + \frac{r}{r}) \hat{\theta}_t - a_t \epsilon_t}{1 + \frac{r}{r}} (1 - \epsilon_t) \right] + A (1 - f_p)^2 \Gamma_t (1 - \epsilon_t) + \left( \frac{f_m}{\theta_t} - \frac{f_m}{\theta_t} \right) (1 - \epsilon_t).
\]

(G.2)

The terms depending on \( \theta_t \) cancel out. Thus (i) does not depend on \( \theta_t \), and is common for the two cases.

(ii): From the optimal policy for consumption,

\[
(1 + r)(W_t - C_t) = (1 + r) \gamma W_t - \frac{1 + r}{\gamma + A (1 + r)} \left[ \zeta_t + \log \frac{\gamma}{A (1 + r)} \right] = W_t - \frac{\zeta_t - \log r}{\gamma},
\]

(G.3)

where the second equality follows from \( A = \gamma [r / (1 + r)] \quad \rightarrow A (1 + r) = \gamma r \). Since \( \zeta_t \) does not depend on \( \theta_t \), this term is also common for simple and complex cases.

Therefore, both (i) and (ii) are common for the two cases, meaning that the investor’s wealth \( W_t \) (and the utility) is the same in the two cases. Hence the investors attain the same utility in these cases.

On the other hand, in equilibrium, the manager’s per-period utility is

\[
f_p R_{t+1} \theta_t X_t + f_m X_t - \kappa \theta_t X_t = f_p R_{t+1} (1 - \epsilon_t) + \frac{f_m}{\theta_t} (1 - \epsilon_t) - \kappa (1 - \epsilon_t)
\]

\[
= f_p \left[ \delta_{t+1} + \frac{\theta_{t+1}^e}{r} + \frac{\alpha t + \epsilon_{t+1} - (1 + \frac{r}{r}) \hat{\theta}_t - a_t \epsilon_t}{1 + \frac{r}{r}} (1 - \epsilon_t) \right] + A f_p (1 - f_p)^2 \Gamma_t (1 - \epsilon_t) + \left( \frac{f_m}{1 - f_p} \right) (1 - \epsilon_t) - \kappa (1 - \epsilon_t).
\]

-Decreasing in \( \theta_t \)

(G.4)

I.e., the manager’s consumption depends on a term decreasing in \( \theta_t \). Since \( \theta_t \) is larger for the complex security case for all \( t \), the manager is worse off in the complex security case.

\[\Box\]

H Proofs of Lemma 7 and Lemma 8

Using the conjectured value function (5.12),

\[
E \left[ V_{t+1} (W_{j,t+1}) \left| F^*_{j,t} \right. \right] = E \left[ - \exp(-A W_{j,t+1} - Z_{t+1}) \left| F^*_{j,t} \right. \right]
\]

\[
= - \exp \left[ - A E (W_{j,t+1} \left| F^*_{j,t} \right. ) + \frac{1}{2} A^2 \text{Var}(W_{j,t+1} \left| F^*_{j,t} \right. ) - Z_{t+1} \right]
\]

\[
= - \exp \left[ - A (1 - f_p)(\hat{R}^{j^i}_{t+1} \theta_{t}^e + \hat{R}^{i^e}_{j,t+1} \theta_{t}^i) X_{j,t} + Af_m X_{j,t} - A(1 + r)(W_{j,t} - C_{j,t}) + \frac{1}{2} A^2 (1 - f_p)^2 (\theta_{t}^e + \theta_{t}^i)^2 \Gamma_t X_{j,t}^2 - Z_{t+1} \right].
\]

(H.1)

where

\[
\Gamma_t = \frac{\text{Var}(\hat{R}^{j^i}_{t+1} \theta_{t}^e + \hat{R}^{i^e}_{j,t+1} \theta_{t}^i) \left| F^*_{j,t} \right. )}{(\theta_{t}^e + \theta_{t}^i)^2} = \frac{1}{2 \eta_u} \left[ \frac{1}{1 - \lambda} \frac{1 - \lambda}{r} + 1 \right]^2 + (1 - 2 \xi t)^2.
\]

(H.2)

The derivation of \( \Gamma_t \) is in Appendix I. From (H.1), the FOC for \( X_{j,t} \) is

\[
A (1 - f_p)(\hat{R}^{j^i}_{t+1} \theta_{t}^e + \hat{R}^{i^e}_{j,t+1} \theta_{t}^i) - Af_m - A^2 (1 - f_p)^2 (\theta_{t}^e + \theta_{t}^i)^2 \Gamma_t X_{j,t} = 0.
\]

(H.3)
This yields the investor’s optimal policy for investment:

\[ X_{j,t} = \frac{(1 - f_p)\left(\overline{R}^s_{j,t+1}\theta_s^i + \overline{R}^c_{j,t+1}\theta_c^i\right) - f_m}{A(1 - f_p)^2(\theta_s^i + \theta_c^i)^2} \] (H.4)

In the symmetric equilibrium \(X_{j,t} = X_t \forall j\), the market clearings imply \(\theta_s^i X_t = 1 - \zeta_t\) and \(\theta_c^i X_t = \zeta_t\), hence \(X_t = 1/(\theta_s^i + \theta_c^i)\). Plugging this and (H.4) into (H.1),

\[ E\left[ V_{t+1}(W_{j,t+1})I^{f_{j,t}} \right] = -\exp\left[-\left(1+r\right)\left(W_{j,t} - C_{j,t}\right) - Z_{t+1} - \frac{1}{2}A^2(1 - f_p)^2\Gamma_t \right]. \]

To simplify notation, define

\[ \zeta_t \equiv -\log \beta + Z_{t+1} + \frac{1}{2}A^2(1 - f_p)^2\Gamma_t. \] (H.5)

Then, the Bellman equation is now

\[ V_t(W_{j,t}) = \max_{C_{j,t}} \left[ -\exp(-\gamma C_{j,t}) - \exp\left[-\zeta_t - A(1+r)(W_{j,t} - C_{j,t})\right] \right]. \] (H.6)

The FOC for \(C_{j,t}\) gives the investor’s optimal consumption policy:

\[ \gamma \exp(-\gamma C_{j,t}) - \exp(-\zeta_t) A(1+r) \exp\left[-\left(1+r\right)(W_{j,t} - C_{j,t})\right] = 0 \]

\[ \iff C_{j,t} = \frac{A(1+r)}{\gamma + A(1+r)} W_{j,t} + \frac{1}{\gamma + A(1+r)} \zeta_t + \gamma \log \frac{\gamma}{A(1+r)}. \] (H.7)

Plugging the optimal \(C_{j,t}\) into the Bellman equation,

\[ -\exp(-AW_{j,t} - Z_t) = -\exp\left[-\frac{\gamma A(1+r)}{\gamma + A(1+r)} W_{j,t} - \frac{\gamma}{\gamma + A(1+r)} \zeta_t + \gamma \log \frac{\gamma}{A(1+r)} \right] \]

\[ \times \left[ \exp\left[-\frac{\gamma}{\gamma + A(1+r)} \log \frac{\gamma}{A(1+r)} \right] + \exp\left[\frac{A(1+r)}{\gamma + A(1+r)} \log \frac{\gamma}{A(1+r)} \right] \right]. \] (H.8)

Taking log,

\[ -AW_{j,t} - Z_t = -\frac{\gamma A(1+r)}{\gamma + A(1+r)} W_{j,t} - \frac{\gamma}{\gamma + A(1+r)} \zeta_t + \log \left[ \exp\left[-\frac{\gamma}{\gamma + A(1+r)} \log \frac{\gamma}{A(1+r)} \right] + \exp\left[\frac{A(1+r)}{\gamma + A(1+r)} \log \frac{\gamma}{A(1+r)} \right] \right]. \] (H.9)

Comparing the coefficients of \(W_{j,t}\), the value of \(A\) is pinned down:

\[ A = \frac{\gamma A(1+r)}{\gamma + A(1+r)} \iff A = \gamma \left(\frac{r}{1+r}\right). \] (H.10)

Plugging (H.10) into (H.9) and rearranging,

\[ \zeta_t - (1+r)Z_t + r \log r - (1+r)\log(1+r) = 0. \] (H.11)

Plugging the definition of \(\zeta_t\) (H.5) back into this,

\[ -\log \beta + Z_{t+1} + \frac{1}{2}A^2(1 - f_p)^2\Gamma_t - (1+r)Z_t + r \log r - (1+r)\log(1+r) = 0. \] (H.12)
Rearranging this,

\[ Z_t = \frac{1}{1+r} \left[ Z_{t+1} + \frac{1}{2} A^2 (1 - f_p)^2 T_t - \log \beta + r \log r - (1 + r) \log(1 + r) \right]. \] (H.13)

\[ \square \]

I Derivation: \( \Gamma_t \) (Eq. (5.16))

\[
\text{Var}(R_{t+1}^s \theta_{s,t} + R_{t+1}^c \theta_{s,t} | F_{j,t}^i) = \theta_t^2 \text{Var}(R_{t+1}^s | F_{j,t}^i) + \theta_t^2 \text{Var}(R_{t+1}^c | F_{j,t}^i) + 2 \theta_t^2 \text{Cov}(R_{t+1}^s, R_{t+1}^c | F_{j,t}^i). \] (I.1)

(i):

\[
\text{Var}(R_{t+1}^s | F_{j,t}^i) = \text{Var}(\delta_{t+1}^s + F_{t+1}^s - (1 + r) F_{j,t}^i) = \text{Var}(\delta_{t+1}^s + \left( \frac{\delta_{t+1}^s}{r} - b_{t+1}^s \right) - (1 + r) F_{j,t}^i). \] (I.2)

(iv):

\[
\text{Var}(\delta_{t+1}^s | F_{j,t}^i) = \text{Var}(\delta_{t+1} + u_{t+1} | F_{j,t}^i) = \text{Var}(\delta_t | F_{j,t}^i) + \frac{1}{\eta_u} = \frac{1}{\eta_{t+1}} + \frac{1}{\eta_u} = \frac{1}{2\eta_u} \left( \frac{1}{\lambda_{t+1}} + 1 \right). \] (I.3)

The last equality follows from \( \eta_u = 2\eta_u [\lambda_t / (1 - \lambda_t)] \). By symmetry,

\[
\text{Var}(\delta_{t+1}^s | F_{j,t}^i) = \frac{1}{2\eta_u} \left( \frac{1}{\lambda_{t+1}} + 1 \right). \] (I.4)

(v):

\[
\text{Var}(\delta_{t+1} + \hat{\delta}_{t+1} | F_{j,t}^i) = \text{Var}\left( \lambda_{t+1} \hat{\delta}_t + (1 - \lambda_{t+1}) \frac{\delta_{t+1}^s + \delta_{t+1}^c}{2} | F_{j,t}^i \right) = \left( \frac{1 - \lambda_{t+1}}{2} \right)^2 \left[ \text{Var}(\delta_{t+1}^s | F_{j,t}^i) + \text{Var}(\delta_{t+1}^c | F_{j,t}^i) + 2 \text{Cov}(\delta_{t+1}^s, \delta_{t+1}^c | F_{j,t}^i) \right] = \text{Var}(\delta_{t+1} | F_{j,t}^i) = 1/\eta_{t+1}. \] (I.5)

(vi):

\[
\text{Cov}(\delta_{t+1}^s, \hat{\delta}_{t+1} | F_{j,t}^i) = \text{Cov}\left( \delta_{t+1}^s, \lambda_{t+1} \hat{\delta}_t + (1 - \lambda_{t+1}) \frac{\delta_{t+1}^s + \delta_{t+1}^c}{2} | F_{j,t}^i \right) = \left( \frac{1 - \lambda_{t+1}}{2} \right) \text{Var}(\delta_{t+1}^s | F_{j,t}^i) + \left( \frac{1 - \lambda_{t+1}}{2} \right) \text{Cov}(\delta_{t+1}^s, \delta_{t+1}^c | F_{j,t}^i) = \frac{1}{2\eta_u} \frac{1 - \lambda_{t+1}}{\lambda_{t+1}}. \] (I.6)

By symmetry,

\[
\text{Cov}(\delta_{t+1}^s, \hat{\delta}_{t+1} | F_{j,t}^i) = \frac{1}{2\eta_u} \frac{1 - \lambda_{t+1}}{\lambda_{t+1}}. \] (I.7)
So, (i) is:

\[
\text{Var}(R_{t+1}^s | F_{j,t}^i) = \frac{1}{2\eta_u} \left( \frac{1}{\lambda_{t+1}} + 1 \right) + \frac{1}{r} \left( \frac{1}{2\eta_u} \left( 1 - \lambda_{t+1} \right)^2 + \left( \frac{2}{r} \right) \frac{1}{2\eta_u} \left( 1 - \lambda_{t+1} \right) \right)
\]

\[
= \frac{1}{2\eta_u} \left( \frac{1}{\lambda_{t+1}} \left( \frac{1 - \lambda_{t+1}}{r} + 1 \right)^2 + 1 \right).
\] (I.8)

By symmetry, (ii) is the same as (i):

\[
\text{Var}(R_{t+1}^c | F_{j,t}^i) = \frac{1}{2\eta_u} \left[ \frac{1}{\lambda_{t+1}} \left( \frac{1 - \lambda_{t+1}}{r} + 1 \right)^2 + 1 \right].
\] (I.9)

(iii):

\[
\text{Cov}(R_{t+1}^s, R_{t+1}^c | F_{j,t}^i) = \text{Cov} \left( \delta_{t+1}^s + \frac{\delta_{t+1}}{r}, \delta_{t+1}^c + \frac{\delta_{t+1}}{r} | F_{j,t}^i \right)
\]

\[
= \frac{1}{r} \text{Cov} \left( \delta_{t+1}^s, \delta_{t+1}^c | F_{j,t}^i \right) + \frac{1}{r} \text{Cov} \left( \delta_{t+1}^c, \frac{\delta_{t+1}}{r} | F_{j,t}^i \right) + \frac{1}{r} \text{Cov} \left( \frac{\delta_{t+1}}{r}, \frac{\delta_{t+1}}{r} | F_{j,t}^i \right)
\]

\[
= \frac{1}{2\eta_u} \left( \frac{1 - \lambda_{t+1}}{\lambda_{t+1}} \right) + \left( \frac{2}{r} \right) \frac{1}{2\eta_u} \left( \frac{1 - \lambda_{t+1}}{r} \right) + \left( \frac{1}{r} \right) \frac{1}{2\eta_u} \left( 1 - \lambda_{t+1} \right)^2 = \frac{1}{2\eta_u} \left[ \frac{1}{\lambda_{t+1}} \left( \frac{1 - \lambda_{t+1}}{r} + 1 \right)^2 - 1 \right].
\] (I.10)

Therefore, finally,

\[
\text{Var}(R_{t+1}^s \theta_{t+1}^s + R_{t+1}^c \theta_{t+1}^c | F_{j,t}^i)
\]

\[
= \theta_{t+1}^s \left( \frac{1}{2\eta_u} \left( \frac{1 - \lambda_{t+1}}{r} + 1 \right)^2 + 1 \right) + \theta_{t+1}^c \left( \frac{1}{2\eta_u} \left( \frac{1 - \lambda_{t+1}}{r} + 1 \right)^2 + 1 \right)
\]

\[
+ 2\theta_{t+1}^s \theta_{t+1}^c \left( \frac{1}{2\eta_u} \left( \frac{1 - \lambda_{t+1}}{r} + 1 \right)^2 - 1 \right)
\]

\[
= (\theta_{t+1}^s + \theta_{t+1}^c)^2 \Gamma_1,
\] (I.11)

where

\[
\Gamma_1 = \frac{1}{2\eta_u} \left[ \frac{1}{\lambda_{t+1}} \left( \frac{1 - \lambda_{t+1}}{r} + 1 \right)^2 + (1 - 2\xi_t)^2 \right].
\] (I.12)

#### J Proof of Lemma 9

Manager \( j \) deviates at time \( t \) and chooses \( \theta_{j,t}^k \neq \theta_{t}^k, k = s, c \). At time \( t + 1 \), investor \( j \) believes that manager \( j \) purchased \( \theta_{t}^s X_{j,t} \) shares of simple security and \( \theta_{t}^c X_{j,t} \) shares of complex security at time \( t \). The investor directly observes \( \delta_{t+1}^s \), as well as the prices. So, observing the excess return on the simple security, he correctly infers the manager’s actual choice of \( \theta_{t}^s \), even if \( \theta_{j,t}^s \neq \theta_{t}^s \). However, the situation is different for the complex security, since the investor cannot directly observe \( \delta_{t+1}^c \). From the observed values of \( P_{t+1}^c \), \( P_{t+1}^c \) and the excess return \( R_{t+1}^c \theta_{t+1}^c X_{j,t} \), investor \( j \) wrongly infers that the realized value of \( \delta_{t+1}^c \) is \( \delta_{j,t+1}^c \), whose value solves

\[
\frac{\delta_{j,t+1}^c + P_{t+1}^c \theta_{t}^c X_{j,t}}{R_{t+1}^c \theta_{t+1}^c X_{j,t}} = \frac{\delta_{t+1}^c + P_{t+1}^c (1 + r) \theta_{t}^c X_{j,t}}{R_{t+1}^c \theta_{t+1}^c X_{j,t}}.
\] (J.1)
\[\delta_{j,t+1}^c = \delta_{t+1}^c + \left( \frac{\theta_{t+1}^c - \theta_{t}^c}{\theta_{t}^c} \right) R_{t+1}^c. \]  

From time \(t+1\) on, manager \(j\) reverts to the equilibrium strategy. This implies that the investor will correctly infer the payoff values from time \(t+2\) onwards, \(\delta_{t+2}^c, \delta_{t+3}^c, \ldots\), from observed values of the prices and the excess returns. That is, investor \(j\) misperceives complex security’s payoff value once and only at time \(t+1\): i.e., for \(\tau = 1, 2, \ldots\), \(H_j^t \cap T + \tau = \{ (\delta_1^j, \delta_2^j, \ldots, \delta_t^j, \delta_{t+1}^j, \delta_{t+2}^j, \delta_{t+3}^j, \ldots, \delta_{t+\tau}^j, \delta_{t+\tau+1}^j) \}.\)

The investor’s one-off misperception about \(\delta_{t+1}^c\) will have a permanent effect on her estimates in the future. By the same argument in the proof of Lemma 5, for \(\tau = 1, 2, \ldots\), the investor’s estimate becomes

\[\hat{\delta}_{j,t+\tau}^c = \hat{\delta}_{t+\tau}^c + \left( \frac{1 - \lambda_{t+1}}{2} \right) \left( \prod_{\nu=2}^{\tau} \lambda_{t+\nu} \right) \left( \frac{\theta_{t+1}^c - \theta_{t}^c}{\theta_{t}^c} \right) R_{t+1}^c. \]  

At time \(t+\tau\), investor \(j\) observes the prices, \(P_{t+\tau}^s\) and \(P_{t+\tau}^c\), and realizes that her estimate, \(\hat{\delta}_{j,t+\tau}^c\), disagrees with the other investors’ average, \(\hat{\delta}_{t+\tau}^c\). From investor \(j\)’s point of view, the excess return on the simple security in the next period will be

\[R_{t+\tau+1}^s = \delta_{t+\tau+1}^s + P_{t+\tau+1}^s - (1 + r)P_{t+\tau}^s = \delta_{t+\tau+1}^s + \frac{\hat{\delta}_{t+\tau+1}^s}{r} - b_{t+\tau+1}^s - (1 + r)P_{t+\tau}^s. \]  

Similarly for the complex security:

\[R_{t+\tau+1}^c = \delta_{t+\tau+1}^c + \frac{\hat{\delta}_{t+\tau+1}^c}{r} - b_{t+\tau+1}^c - (1 + r)P_{t+\tau}^c. \]  

Here, note that investor \(j\) believes that \(P_{t+\tau+1}^c\) will depend on the market’s aggregate estimate, \(\hat{\delta}_{t+\tau+1}^c\), not her own estimate, \(\hat{\delta}_{j,t+\tau+1}^c\), as she has measure 0 and thus has no impact on the prices. At time \(t+\tau\), investor \(j\) believes that \(\hat{\delta}_{t+\tau+1}^c\) will be

\[\hat{\delta}_{t+\tau+1}^c = \lambda_{t+\tau+1}^c \hat{\delta}_{t+\tau}^c + (1 - \lambda_{t+\tau+1}^c) \frac{\delta_{t+\tau+1}^s + \delta_{t+\tau+1}^c}{2}. \]  

So, her expectation about the excess return is

\[\hat{R}_{j,t+\tau+1}^c = \hat{\delta}_{j,t+\tau}^c + \frac{1}{r} \left[ \lambda_{t+\tau+1}^c \hat{\delta}_{t+\tau}^c + (1 - \lambda_{t+\tau+1}^c) \delta_{t+\tau}^1 \right] - \frac{b_{t+\tau+1}^c}{r} - (1 + r)P_{t+\tau}^c \]

\[= \hat{R}_{t+\tau+1}^c + \left( \frac{1 - \lambda_{t+\tau+1}^c}{r} + 1 \right) \left( \delta_{j,t+\tau}^1 - \hat{\delta}_{t+\tau}^1 \right). \]  

Note that this holds both for \(k = s, c\). This is due to the assumption that \(\hat{\delta}_t\) is common for both securities.

Finally, investor \(j\)’s capital investment at time \(t+\tau\) is calculated as follows. First, note that

\[[(1 - \xi_1)\hat{R}_{j,t+\tau+1}^s + \xi_1 \hat{R}_{j,t+\tau+1}^c] - [(1 - \xi_1)\hat{R}_{t+\tau+1}^s + \xi_1 \hat{R}_{t+\tau+1}^c] = \left( \frac{1 - \lambda_{t+\tau+1}^c}{r} + 1 \right) \left( \delta_{j,t+\tau}^1 - \hat{\delta}_{t+\tau}^1 \right). \]
Using this, the investor’s capital investment at time $t + \tau$ is

\[
X_{j,t+\tau} = X_{t+\tau} + \left[ (1 - \xi_t) \hat{R}_{t+\tau+1}^s \xi_t \hat{R}_{t+\tau+1}^c - (1 - \xi_t) \hat{R}_{t+\tau+1}^s \xi_t \hat{R}_{t+\tau+1}^c \right] a_{t+\tau}(\theta_{t+\tau}^s + \theta_{t+\tau}^c) \\
= X_{t+\tau} + \frac{1}{a_{t+\tau}(\theta_{t+\tau}^s + \theta_{t+\tau}^c)} \left( \frac{1 - \lambda_{t+\tau+1}}{r} \right) \left( \frac{1 - \lambda_{t+\tau+1}}{1} \right) \left( \delta_{t+\tau} \right) \\
= X_{t+\tau} + \frac{1}{a_{t+\tau}(\theta_{t+\tau}^s + \theta_{t+\tau}^c)} \left( \frac{1 - \lambda_{t+\tau+1}}{r} \right) \left( \frac{1 - \lambda_{t+\tau+1}}{2} \right) \left( \prod_{\nu=2}^{\tau} \lambda_{t+\nu} \right) \left( \frac{\theta_{t+\tau}^c - \theta_{t+\tau}^c}{\theta_{t+\tau}^c} \right) R_{t+\tau+1}^c \\
\equiv X_{t+\tau} + X_{t+\tau}^c(\theta_{t+\tau}^c). \tag{J,8}
\]

## K Proof of Lemma 10

There are several points to note for setting up manager $j$’s problem.

1. Differently from the single-security models, manager $j$ chooses $\theta_{j,t}^k$, observing the prices and $X_{j,t}$. So $X_{j,t}$ can be out of his expectation operator.

2. Since the manager has played the equilibrium strategy up to time $t - 1$, investor $j$ knows $\mathcal{H}_t$, hence $\hat{R}_{j,t+1}^k = \hat{R}_{t+1}^k$ for $k = s, c$. Moreover, since the investor cannot observe $\theta_{j,t}^k$, she invests capital based on her (wrong) belief that $\theta_{j,t}^k = \theta_t^k$ for $k = s, c$. Thus, $X_{j,t}$ can be replaced by $X_t$. Of course, $X_{j,t+\tau}$ should not be replaced by $X_{t+\tau}$ as it depends on $\theta_{t+\tau}^c$.

3. The management fee at time $t$, $f_m X_t$, can be omitted from the problem as it does not depend on $\theta_{j,t}^k$.

4. The terms involving $X_{t+\tau}$ can be omitted from the problem. This is innocuous since these terms are additively separated from $X_{t+\tau}^c(\theta_{j,t}^c)$, which is relevant for the manager’s problem at time $t$.

Taking these points into account, an equivalent maximization problem is

\[
\max_{\theta_{j,t}^s, \theta_{j,t}^c \in (-\infty, \infty), k = s, c} f_p(\hat{R}_{t+\tau+1}^s \theta_{t+\tau+1} + \hat{R}_{t+\tau+1}^c \theta_{t+\tau+1}) X_t - \kappa(\theta_{j,t}^s + \theta_{j,t}^c) X_t \\
+ E \left[ \sum_{\tau=1}^{\infty} \beta^\tau \left[ f_p(\hat{R}_{t+\tau+1}^s \theta_{t+\tau+1} + \hat{R}_{t+\tau+1}^c \theta_{t+\tau+1}) + f_m - \kappa(\theta_{t+\tau}^s + \theta_{t+\tau}^c) \right] X_{t+\tau}^c(\theta_{j,t}^c) \right] F_{j,t}^m \tag{K.1}
\]

This problem can be simplified as follows. First, note that

\[
E \left[ f_p(\hat{R}_{t+\tau+1}^s \theta_{t+\tau+1} + \hat{R}_{t+\tau+1}^c \theta_{t+\tau+1}) X_{t+\tau}^c(\theta_{j,t}^c) \right] = f_p \left[ a_{t+\tau}(\theta_{t+\tau}^s + \theta_{t+\tau}^c) + \frac{f_m}{1 - f_p} \right] E \left[ X_{t+\tau}^c(\theta_{j,t}^c) \right] F_{j,t}^m.
\]
Thus, (i) in (K.1) is

\[ \begin{align*}
(i) &= \sum_{\tau=1}^{\infty} \beta^\tau \left[ f_p a_{t+\tau}(\theta_{t+\tau}^c + \theta_{t+\tau}^e) + f_p \frac{f_m}{1-f_p} + f_m - \kappa(\theta_{t+\tau}^c + \theta_{t+\tau}^e) \right] E \left[ X_{t+\tau}^+(\theta_{t+\tau}^c) X_{t+\tau}^m \right] \\
&= \left( \frac{\theta_{t+\tau}^c - \theta_{t}^c}{\theta_{t}^c} \right) \tilde{R}_{t+1}^c \sum_{\tau=1}^{\infty} \beta^\tau \left( 1 - \frac{1}{2} \right) \left( 1 - \frac{\lambda_{t+\tau+1}}{r} \right) + 1 \left( \frac{\tau}{\nu} \lambda_{t+\tau} \right) \\
&\quad \times \left[ f_p + \frac{f_m}{1 - f_p} a_{t+\tau}(\theta_{t+\tau}^c + \theta_{t+\tau}^e) - \kappa a_{t+\tau} \right]. \quad \text{(K.2)}
\end{align*} \]

So, a problem equivalent to (K.1) is

\[ \max_{\theta_{j,t}^e, j=\infty, k=s,c} f_p (\tilde{R}_{t+1}^c + \tilde{R}_{t+1}^e) X_t - \kappa(\theta_{t+\tau}^c + \theta_{t+\tau}^e) X_t + \left( \frac{\theta_{t+\tau}^c - \theta_{t}^c}{\theta_{t}^c} \right) \tilde{R}_{t+1}^c \Omega_t, \quad \text{(K.3)} \]

where

\[ \begin{align*}
\Omega_t &= \left( \frac{1 - \lambda_{t+1}}{2} \right) \sum_{\tau=1}^{\infty} \beta^\tau \left( 1 - \frac{1}{2} \right) \left( 1 - \frac{\lambda_{t+\tau+1}}{r} \right) + 1 \left( \frac{\tau}{\nu} \lambda_{t+\tau} \right) \left[ f_p + \frac{f_m}{1 - f_p} a_{t+\tau}(\theta_{t+\tau}^c + \theta_{t+\tau}^e) - \kappa a_{t+\tau} \right]. \quad \text{(K.4)}
\end{align*} \]

Plugging the market clearing condition \( X_t = 1/(\theta_{t}^c + \theta_{t}^e) = \xi_t/\theta_{t}^e \) into (K.3) and canceling out \( 1/\theta_{t}^c \), the manager’s problem reduces to

\[ \max_{\theta_{j,t}^e, j=\infty, k=s,c} f_p (\tilde{R}_{t+1}^c + \tilde{R}_{t+1}^e) \xi_t - \kappa(\theta_{t+\tau}^c + \theta_{t+\tau}^e) \xi_t + \left( \frac{\theta_{t+\tau}^c - \theta_{t}^c}{\theta_{t}^c} \right) \tilde{R}_{t+1}^c \Omega_t. \quad \text{(K.5)} \]

\[ \square \]

L Proof of Proposition 3

I show that there exists a long-run steady state value of \( \Omega_t, \Omega^* > 0 \). First, note that the steady-state risk premium \( a^* \) is a quadratic function of \( \Omega^* \):

\[ a^* = A(1 - f_p) \Omega^* = \frac{A(1 - f_p)}{2\eta_u} \left[ \frac{1}{\lambda^*} \left( 1 - \frac{\lambda^*}{r} \right) + 1 \right] + (1 - 2g\Omega^*)^2 \equiv a^*(\Omega^*). \quad \text{(L.1)} \]

For the second equality, I used \( \xi^* = g\Omega^* \), where \( g \equiv (1/f_p)[(\kappa/mf_p) - 1] > 0 \). Define

\[ \bar{a} \equiv \lim_{\Omega \to 0} a^*(\Omega) = \frac{A(1 - f_p)}{2\eta_u} \left[ \frac{1}{\lambda^*} \left( 1 - \frac{\lambda^*}{r} \right) + 1 \right] > 0. \quad \text{(L.2)} \]

Then, \( a^*(\Omega^*) \) can be written as

\[ a^*(\Omega^*) = n_1 \Omega^2 - n_2 \Omega^* + \bar{a}, \quad \text{(L.3)} \]

where \( n_1 \) and \( n_2 \) are positive constants. From (5.32), in the long run, \( \theta_{t}^e \) satisfies

\[ \theta_{t}^e = \frac{\xi^* \left( \frac{f_m}{1 - f_p} \right) \Omega^*}{\frac{\kappa}{f_p} \left[ f_p \xi^* + (1 - \xi^*)\Omega^* \right] - a^*(\Omega^*)} = \frac{g\Omega^* \left( \frac{f_m}{1 - f_p} \right) \Omega^*}{\frac{\kappa}{f_p} \left[ f_p g\Omega^* + (1 - g\Omega^*)\Omega^* \right] - a^*(\Omega^*)} \]

\[ = \frac{g \left( \frac{f_m}{1 - f_p} \right) J(\Omega^*)^{-1}}, \quad \text{(L.4)} \]
where
\[
J(\Omega^*) \equiv \frac{\kappa}{\bar{f}_p} \left[ f_p g + (1 - g\Omega^*) \right] \frac{1}{\Omega^*} - a^*(\Omega^*) = \frac{\kappa}{\bar{f}_p} \left( 1 - g\Omega^* \right) \frac{1}{\Omega^*} - \frac{n_1 \Omega^* - n_2 \Omega^* + \bar{a}}{\Omega^*} \tag{L.5}
\]

Assuming \( \frac{\kappa}{\bar{f}_p} > \bar{a} \), it is readily checked that \( J'(\Omega) < 0 \), \( \lim_{\Omega \to 0} J(\Omega) = \infty \) and \( \lim_{\Omega \to \infty} J(\Omega) = -\infty \).

Meanwhile, following similar steps as the proof of Proposition 2, it can be shown that
\[
\Omega_t = \beta(1 - \lambda_{t+1}) \left[ \frac{1}{2} \left( \frac{1 + \rho - \lambda_t}{\rho} \right) + 1 \right] + \left[ f_p + \frac{f_m \xi t+1}{(1 - f_p) a t+1 \theta t+1} - \frac{\kappa}{a t+1} \right] + \frac{\Omega_{t+1}}{\rho + (1 - \lambda_{t+1})} \tag{L.6}
\]

Hence, the long-run value of \( \Omega_t \) satisfies
\[
\Omega^* = \beta(1 - \lambda^*) \left[ \frac{1}{2} \left( \frac{1 + \rho - \lambda^*}{\rho} \right) + 1 \right] + f_p + \frac{f_m \xi^*}{(1 - f_p) a^* \theta^*} = \frac{\kappa}{a^*} + \frac{\Omega^*}{\rho + (1 - \lambda^*)} \tag{L.7}
\]

where
\[
\psi = \frac{1}{\beta(1 - \lambda^*)} - \frac{1}{\rho + (1 - \lambda^*)} \left[ \frac{1}{\frac{1}{2} \left( \frac{1 + \rho - \lambda^*}{\rho} \right) + 1} \right] > 0 \tag{L.8}
\]

is a constant. Using \( \xi^* = g\Omega^* \) and solving (L.7) for \( \theta^* \),
\[
\theta^* = \frac{g \left( \frac{f_m}{1 - f_p} \right)}{\psi a^*(\Omega^*) + \frac{\kappa - f_p a^*(\Omega^*)}{\Omega^*}} = g \left( \frac{f_m}{1 - f_p} \right) K(\Omega^*)^{-1}, \tag{L.9}
\]

where
\[
K(\Omega^*) = \psi a^*(\Omega^*) + \frac{\kappa - f_p a^*(\Omega^*)}{\Omega^*} = \psi a^*(\Omega^*) + \frac{\kappa - f_p a^*(\Omega^*)}{\Omega^*} + \psi a^*(\Omega^*)
= \psi n_1 \Omega^* - (f_p n_1 + \psi n_2) \Omega^* + f_p \left( \frac{\kappa}{\bar{f}_p} - \bar{a} \right) \frac{1}{\Omega^*} + \left( f_p n_2 + \psi \bar{a} \right) \tag{L.10}
\]

Under the assumption \( \frac{\kappa}{\bar{f}_p} > \bar{a} \), it is easy to see that \( \lim_{\Omega \to 0} K(\Omega) = \infty \) and \( \lim_{\Omega \to \infty} K(\Omega) = \infty \). (Note that \( K() \) is a U-shape function.) It is already shown that \( \lim_{\Omega \to \infty} J(\Omega) = -\infty \). So, if \( \lim_{\Omega \to 0} J(\Omega) > \lim_{\Omega \to 0} K(\Omega) \) holds, then there exists \( \Omega^* > 0 \) such that \( J(\Omega^*) = K(\Omega^*) \). This is shown as follows:
\[
\lim_{\Omega \to 0} J(\Omega) \rightarrow \lim_{\Omega \to 0} K(\Omega) \Rightarrow \lim_{\Omega \to 0} \left[ \frac{1 \kappa}{f_p} a^*(\Omega) - \frac{\kappa g}{f_p(f_p g + 1)} \right] = \lim_{\Omega \to 0} \left[ \frac{\kappa}{f_p} - a^*(\Omega) - \frac{\kappa g}{f_p(f_p g + 1)} \Omega^* \right]
= \frac{\kappa}{f_p} - \bar{a} \frac{1}{f_p} > 1 \tag{L.11}
\]

Now, given \( \Omega^* \), it is straightforward to obtain \( \xi^* \), \( \Gamma^* \), \( a^* \) and \( \theta^* \). The value of \( \xi^* \) can be less than 1 by choosing parameter values appropriately. For instance, by choosing \( \mu \) large to make \( g \) close to 0, \( \xi^* = g\Omega^* \) can be made arbitrarily close to 0.

\[\square\]
M Notation

\[ \delta_t \] Risky security’s payoff. \( \delta_t = \tilde{\delta}_t + u_t, \; u_t \sim N(0, \eta_u^{-1}) \).

\[ \tilde{\delta}_t \] Fundamental value of the payoff. \( \tilde{\delta}_t = \tilde{\delta}_{t-1} + v_t, \; v_t \sim N(0, \eta_v^{-1}) \).

\[ \delta_{i,t} \] Value of \( \delta_t \) that investor \( j \) believes.

\[ \mathcal{H}_t \] Payoff history. \( \mathcal{H}_t \equiv (\delta_1, \ldots, \delta_t) \).

\[ \mathcal{H}_{i,t} \] Payoff history that investor \( j \) believes. \( \mathcal{H}_{i,t} \equiv (\delta_{i,1,t}, \ldots, \delta_{i,t}) \).

\[ \hat{\delta}_t \] Unbiased estimate of \( \tilde{\delta}_t \). \( \hat{\delta}_t \equiv E(\tilde{\delta}_t | \mathcal{H}_t) \).

\[ \hat{\delta}_{j,t} \] Investor \( j \)’s estimate of \( \tilde{\delta}_t \). \( \hat{\delta}_{j,t} \equiv E(\tilde{\delta}_t | \mathcal{H}_{j,t}) \).

\[ \lambda_t \] Kalman filtering: \( \hat{\delta}_t = \lambda_t \hat{\delta}_{t-1} + (1 - \lambda_t) \delta_t \) and \( \hat{\delta}_{j,t} = \lambda_t \hat{\delta}_{j,t-1} + (1 - \lambda_t) \delta_{j,t} \).

\[ R_{t+1} \] Risky security’s excess return (per share). \( R_{t+1} \equiv (\delta_{t+1} + P_{t+1}) - (1 + r)P_t \).

\[ \hat{R}_{t+1} \] Unconditional expected excess return. \( \hat{R}_{t+1} \equiv E(R_{t+1}) \).

\[ \hat{R}_{t+1}^j \] Conditional expected excess return. \( \hat{R}_{t+1}^j \equiv E(R_{t+1} | \mathcal{H}_t, P_t) \).

\[ \epsilon_t \] Noise trading. \( \epsilon_t \sim N(0, \sigma^2) \).

\[ \sigma^2 \] Variance of noise trading.

\[ F_{i,t} \] Investor \( j \)’s information set.

\[ F_{m,t} \] Manager \( j \)’s information set.

\[ X_{j,t} \] Investor \( j \)’s capital investment in the fund. (In equilibrium, \( X_{j,t} = X_t \; \forall \; j \).)

\[ \theta_{j,t} \] Manager \( j \)’s risky portfolio. (In equilibrium, \( \theta_{j,t} = \theta_t \; \forall \; j \).)

\[ f_m \] Management fee rate.

\[ f_p \] Performance fee rate.

\[ \kappa \] Manager’s holding cost of risky security.

\[ \gamma \] Coefficient of absolute risk aversion.

\[ A, B_t, Z_t \] Undetermined coefficients of the investor’s value function.

\[ a_t, b_t \] Undetermined coefficients of the price function.

\[ \Gamma_t \] Conditional variance of \( R_{t+1} \) (ignoring \( \epsilon_{t+1} \)). \( \Gamma_t \equiv \text{Var}(\delta_{t+1} + \frac{\delta_{t+1}}{\tau} | \mathcal{H}_t) \).

\[ \Psi_t \] Deterministic term resulting from Jensen’s inequality (\( \Psi_t \to 0 \) as \( \sigma^2 \to 0 \)).

\[ \Omega_t \] Sensitivity of the manager’s additional gain from the future to his time-

\[ Q_{\ell,t} \] Quant \( \ell \)’s supply of complex security.

\[ \xi_t \] Aggregate supply of complex security. \( \xi_t \equiv \int_0^1 Q_{\ell,t} d\ell \).

\[ \mu \] Quant’s supply cost of complex security.
References


