Consumption-Based Asset Pricing with Loss Aversion

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Abstract

In this paper, I incorporate loss aversion features in a consumption-based asset pricing model. I define new preferences with loss aversion that allow me to solve the asset pricing model with recursive utility in closed-form. I find that even small parameters of loss aversion increase risk prices substantially relative to the standard recursive utility model (level effect). This feature of my model improves on the calibration of the standard consumption-based asset pricing model with recursive utility. I also find that in the model with loss aversion, contrarily to the standard recursive utility model, risk prices vary with risk exposure (cross-sectional effect). This differentiating feature of my model is supported by the data in that it correctly predicts both a negative premium for skewness and a security market line, the excess returns as a function of the exposure to market risk, flatter than the CAPM.

Introduction

I incorporate loss aversion features in a consumption-based asset pricing model. Loss averse agents value consumption outcomes relative to a reference point and losses relative to the reference create more disutility than comparable gains. I suppose that agents are subject to loss aversion with regards to the value of the future consumption stream, in a recursive model of preferences. I obtain a tractable consumption-based asset pricing model with loss aversion that generates risk prices (the incremental excess returns that compensate for additional risk taking) that are starkly different from those of the standard recursive utility model. First, and most striking, risk prices vary with risk exposure in the model with loss aversion, contrarily to the standard recursive utility model (cross-sectional effect). Second, I find that even small parameters of loss aversion

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increase risk prices substantially (level effect). The standard recursive utility model, which allows to disentangle the risk aversion and the intertemporal elasticity of substitution, is central to the consumption-based asset pricing literature, notably the long-run risk models (Bansal and Yaron (2004), Hansen, Heaton and Li (2008), Bansal, Kiku and Yaron (2007, 2009) to name a few). This model has been successfully calibrated using moments on asset returns. The level effect that my model with loss aversion generates allows to match or improve on such calibration exercise. More interestingly, my model generates novel predictions for the cross section, and I find strong empirical support for these predictions.

Consider first the cross-sectional effect. In my model with loss aversion, the incremental excess returns are lower when additional risk is added to a risky asset than when it is added to a relatively safe asset: risk prices vary with the amount of risk exposure. In contrast, the standard recursive utility model generates risk prices that are constant across risk exposures. This cross-sectional effect truly differentiates my model from the standard recursive utility model. Two well known results in empirical finance offer support for my model. First, Black, Jensen, and Scholes (1972) and more extensively Frazzini and Pedersen (2010) show that the asset returns line (the excess returns as a function of beta, the exposure to market risk) is flatter than the CAPM, persistently over time, and for a wide class of assets (U.S. equities, 20 global equity markets, Treasury bonds, corporate bonds, and futures). The standard recursive utility model fails at predicting such results. My model with loss aversion on the other hand matches the results of Frazzini and Pedersen (2010), both qualitatively and quantitatively. Unlike previous models which address this central result in financial economics, my model does not require borrowing constraints or agent heterogeneity. Second, Harvey and Siddique (2000) show that assets with the same volatility but different skewness in their returns distribution yield different expected returns: they find a negative premium for skewness. My model with loss aversion predicts such a negative premium for skewness and can match the quantitative results of Harvey and Siddique (2010). In contrast, the standard recursive utility model generates returns that depend exclusively on the volatility, and therefore do not vary with the skewness of the returns distribution.

Let’s turn to the level effect. The excess returns are higher and the risk free rate is lower in my model with loss aversion than in the standard recursive utility model. Because the loss aversion specification increases the perceived risk aversion on part of the distribution of states, this level effect is to be expected. However, even if the risk aversion coefficient was set at the highest level ever perceived by the loss averse representative agent, the standard recursive utility model
would fail to match the excess returns and the risk free rate of my model with loss aversion. This feature of my model improves on the calibration of the standard consumption-based asset pricing model with recursive utility. Using the covariation between the market portfolio and stockholders’ consumption, I can match the equity premium, the value premium and the risk free rate for a risk aversion coefficient of $\gamma \leq 10$.

In Kahneman and Tversky (1979), the authors define and find empirical support for a model of preferences with loss aversion: agents value outcomes relative to a reference point and losses relative to the reference create more disutility than comparable gains. I incorporate such loss aversion features to a preference model with recursive utility. As in the model of Epstein and Zin (1989), the present value of the consumption stream depends on current consumption and next period’s value for future consumption. I suppose that agents are loss averse and thus suffer additional disutility if the realization of next period’s value disappoints (ie falls below their expectation). My model of loss aversion allows to find tractable solutions to the consumption-based asset pricing model with homogeneous agents. In this model, the agents appear to be more risk averse for disappointing outcomes, and they are thus expected to demand higher returns when taking risk, and to value risk-free assets more than in the traditional model without loss aversion. Accordingly, I find that my model generates a level effect as discussed above. Further, the discontinuity in risk aversion results in agents that are particularly averse to taking small risks around the reference point. This explains the cross-sectional results for asset prices that I derive.

Previous papers analyze the impact on asset prices of preferences with loss aversion (see Benartzi and Thaler (1995), Barberis et al. (2001), Barberis and Huang (2009) among others). I add to this literature by defining a new model of preferences with loss aversion that allows me to solve the asset pricing model with recursive utility in a tractable way. The advantage of using recursive preferences in consumption-based asset pricing models is well established, in particular for the long-run risk models. Combining behavioral models and recursive utility gives rise to interesting results. Some authors have adopted this approach before. Routledge and Zin (2010) present a model of generalized disappointment aversion, an extension to the disappointment aversion of Gul (1991). They analyze the asset pricing implications of Epstein-Zin preferences with generalized disappointment aversion and obtain closed form solutions and interesting results in a 2-state Markov economy. Bonomo et al (2011) extend the analysis to a 4-state Markov adapted from Bansal, Kiku and Yaron (2007). They match first and second moments on the market returns.
and risk free rate, predictability patterns and autocorrelations for realistic parameters. My model is close in spirit to the disappointment aversion model. However, my model is both more flexible, and more tractable, which allows me to find closed form solutions for more complex economies. Barberis and Huang (2009) use a recursive utility model with loss aversion narrow framed on the stock market returns and find closed form solutions for both partial and general equilibria. However, my modeling choices for loss aversion differ considerably from their model.

Beyond the contribution of developing a fully tractable consumption-based asset pricing model with loss aversion, my analysis of the cross-sectional risk prices is novel to the behavioral finance and the asset pricing literature.

The rest of the paper is organized as follows: In section 1, I model loss aversion in a recursive utility model of preferences. In section 2, I analyze the consumption-based asset pricing model and obtain closed-form solutions for my model of preferences with loss aversion. I then analyze the asset pricing implications of the model. The predictions of the model are brought to the data in section 3.

1 Preferences with Loss Aversion

I incorporate loss aversion features in a consumption-based asset pricing model. I define a new model of preferences with loss aversion that allows me to solve the asset pricing model in a tractable way. In this section, I define my model for loss aversion.

In Kahneman and Tversky (1979), the authors provide various examples of agents’ choices over lotteries that are not compatible with the von Neumann- Morgenstern utility model of preferences. They propose a new theory of preferences, prospect theory. In prospect theory, agents value outcomes relative to a reference point. Furthermore, losses relative to the reference create more disutility than comparable gains: agents display first order risk aversion around the reference point (a kink in the preferences). These two features combined represent loss aversion.

In Gul (1991), the disappointment aversion preferences over lotteries display features similar to the prospect theory of Kahneman and Tversky (1979): losses relative to a reference point receive more weight than the gains, thus displaying a kink at the reference point and first order risk aversion. The reference point is endogenously determined recursively as the certainty equivalent of the lottery. Gul’s disappointment aversion model obeys an axiomatic structure of preferences in which only the independence axiom is relaxed relative to von Neumann- Morgenstern. The
axiomatic structure and the endogenous reference point are attractive features of disappointment aversion. However, this model does not allow for any flexibility in the choice of the reference point. Besides, it is not always tractable and it has been used mainly in the context of discrete outcomes sets (Markov chain economies).

I propose a new model of preferences that display loss aversion. It departs from the disappointment aversion model mainly through the choice of the reference point. In my model, the reference point is endogenously specified as an expectation of the future utility of consumption. I focus on a log-linear specification that allows me to obtain closed-form solutions when adapted to the consumption-based asset pricing models with unit intertemporal elasticity of substitution.

In this section, I define how I model loss aversion and how I incorporate it to the recursive utility model of Epstein and Zin (1989). For illustrative purposes, I describe in section 1.1 how I model loss aversion in a two-period model. In section 1.2, I extend the loss aversion specification to the multi-period, recursive utility model, and fully describe my choice of preferences. I then discuss my model and relate it to the existing literature.

1.1 Two-Period Model

For illustrative purposes, let’s first consider a two-period model. At period $t = 1$, the agent receives consumption $C$, the level of which is uncertain at period $t = 0$.

The standard CRRA model for this two-period setting is:

$$U_0 = \mathbb{E} \left( \frac{C^{1-\gamma}}{1-\gamma} \mid \mathcal{I}_0 \right)$$

where $\mathcal{I}_0$ is the information set at time $t = 0$ and $\gamma > 1$ is the coefficient of risk aversion.

1.1.1 With Loss Aversion

I modify the standard model by adding loss aversion. Below a reference point $Ref$, the agent is disappointed and thus receives less utility than in the standard model. She will behave as though more risk averse, with risk aversion $\bar{\gamma} \geq \gamma$. Above the reference point, the agent’s utility is unchanged and she behaves as though risk averse with risk aversion $\gamma$.

The model becomes:

$$U_0 = \mathbb{E} \left( f(C, Ref) \mid \mathcal{I}_0 \right)$$

where
\[
\begin{cases}
  f(C, \text{Ref}) \propto \frac{C^{1-\gamma}}{1-\gamma} & \text{for } C \leq \text{Ref} \\
  f(C, \text{Ref}) \propto \frac{C^{1-\bar{\gamma}}}{1-\bar{\gamma}} & \text{for } C \geq \text{Ref}
\end{cases}
\]

\( f(C, \text{Ref}) \) is the realized utility at period \( t = 1 \). It is a continuous function for all \( C \) and thus for \( C = \text{Ref} \).

Therefore, if

\[
\begin{cases}
  f(C, \text{Ref}) = a\frac{C^{1-\gamma}}{1-\gamma} & \text{for } C \leq \text{Ref} \\
  f(C, \text{Ref}) = b\frac{C^{1-\bar{\gamma}}}{1-\bar{\gamma}} & \text{for } C \geq \text{Ref}
\end{cases}
\]

then the scaling coefficients must be such that:

\[
\frac{b}{a} = \frac{1 - \gamma}{1 - \bar{\gamma}} \text{Ref}^{\bar{\gamma} - \bar{\gamma}}
\]

without loss of generality, I can set \( b = 1 \) or \( a = 1 \). In my model, the reference point is the agent’s expectation concerning future consumption (the choice of the reference point is discussed below). Therefore, I model a realized utility at period \( t = 1 \) that is increasing in the reference point \( \text{Ref} \). This is satisfied when \( a = 1 \) and:

\[
f(C, \text{Ref}) = \frac{1}{1 - \gamma} \left\{ \begin{array}{ll}
  C^{1-\gamma} & \text{for } C \leq \text{Ref} \\
  C^{1-\bar{\gamma}} \times \text{scaling factor} \text{Ref}^{\bar{\gamma} - \gamma} & \text{for } C \geq \text{Ref}
\end{array} \right.
\]

In figure 1, I illustrate how my model incorporates loss aversion into the standard CRRA two-period model. Above the reference point, the utility from consumption has the same curvature as in the standard CRRA model. Risk aversion \( \gamma \) is unchanged. Below the reference point, loss aversion generates a decrease in utility relative to the standard model. The curvature is stronger and the resulting risk aversion \( \bar{\gamma} \) is higher, \( \bar{\gamma} \geq \gamma \). The utility from consumption displays a kink at the reference point. The ratio of the slopes above and below the reference point is given by:

\[
\frac{1 - \gamma}{1 - \bar{\gamma}} \leq 1
\]

How much this ratio differs from 1 determines the degree to which the agent is loss averse in the model.

In my model, loss aversion is represented by a unique parameter \( \alpha \) where:

\[
\frac{1 - \gamma}{1 - \bar{\gamma}} = 1 - \alpha
\]
Figure 1: Loss Aversion in the Two-period Model
The loss aversion parameter $\alpha$ is in $[0, 1)$. In the limit case $\alpha = 0$, the agent displays no loss aversion and the model reverts to the standard CRRA model. In Kahneman and Tversky (1979), the authors estimate the ratio of the slopes at $1/2.25$, which corresponds to $\alpha = 0.55$.

1.1.2 Reference Point

I model a reference point endogenously determined by the agent’s expectation of outcomes given $\mathcal{I}_0$, the information at time $t = 0$. Koszegi and Rabin (2006) give a detailed argument as to why the reference point should be determined by the agent’s expectation of outcomes. In their model, the reference point is a stochastic expectation. As a special case, when all uncertainty is resolved at period $t = 1$, as in my model, the stochastic reference point of Koszegi-Rabin model collapses to a fixed point. Empirical evidence for the reference point as an expectation and the model of Koszegi-Rabin has been found in Post et al. (2008), Eil and Lien (2010), Sprenger (2010) and Crowford, Meng (2011). In Appendix A, I discuss my modelling choices for loss aversion, and relate them to the existing literature, in particular to the model of Koszegi and Rabin (2006). Because my model and the one of Koszegi and Rabin (2006) have similar features, the empirical evidence also validates my choice of the reference point as an endogenous expectation of outcomes.

I choose a log-linear specification for the reference point: in my model, the agent is disappointed and registers disutility from loss aversion when $\log C \leq \mathbb{E} (\log C \mid \mathcal{I}_0)$. When $\log C \geq \mathbb{E} (\log C \mid \mathcal{I}_0)$, the agent’s utility from consumption is unchanged. The threshold for consumption $C$ below which the agent registers disutility from loss aversion is $Ref = \exp (\mathbb{E} (\log C \mid \mathcal{I}_0))$.

The log-linear specification for the reference point is a natural choice for the consumption-based asset pricing model with unit intertemporal elasticity of substitution. However, the model can be solved for other choices of the reference point as an expectation. In particular, the conclusions to my model are largely unchanged by the choice of $Ref = \mathbb{E} (C \mid \mathcal{I}_0)$.

My model for the reference point is similar in spirit to the disappointment aversion model (in which the reference point is the endogenous certainty equivalent). However, in my model, the reference point is explicitly defined as an expectation whereas in the disappointment aversion model, it is the solution to a recursive problem. My model allows for greater flexibility in the choice of the reference point and is more tractable.
1.2 Multi-Period Model, Recursive Utility

I now consider a multi-period model with consumption stream \( \{C_t\} \). My choice for the standard preference model is the recursive utility model of Epstein and Zin (1989). In the asset pricing literature, this model of preferences in which the risk aversion coefficient is disentangled from the intertemporal elasticity of substitution allows to obtain realistic moments in the distribution of prices. In particular, it generates low and stable risk free rates along with high and volatile stock returns. Further, in contrast to the expected utility model, this model of preferences allows for the risks to long-run consumption to impact current prices.

At each period \( t \), the agent’s valuation for the future consumption stream is given by \( V_t \), which is defined recursively as:

\[
V_t = \left( (1 - \beta) C_t^{1-\rho} + \beta (h(V_{t+1}))^{1-\rho} \right)^{\frac{1}{1-\rho}}
\]

with \( \rho > 0 \) the inverse of the IES (intertemporal elasticity of substitution) and \( 0 < \beta < 1 \) represents the rate of time discount (\( \beta \) close to one represents a very low discount rate).

In the standard model, \( h \) is given by:

\[
h(V_{t+1}) = R_t V_{t+1} = \left( E_t \left( V_{t+1}^{1-\gamma} \right) \right)^{\frac{1}{1-\gamma}}
\]

where \( \gamma > 1 \) is the coefficient of risk aversion.

1.2.1 With Loss Aversion

In the recursive utility model, the consumption \( C_{t+1} \) of the two-period model is replaced by \( V_{t+1} \), the value of all future consumption.

Similarly to the two-period model of section (1.1), I modify the standard recursive utility model by defining \( h \) as:

\[
h(V_{t+1}) = R_t (V_{t+1}, R e f_t) = (E_t (f(V_{t+1}, R e f_t)))^{\frac{1}{1-\gamma}}
\]

where

\[
f(V_{t+1}, R e f_t) = \begin{cases} 
V_{t+1}^{1-\gamma} & \text{for } V_{t+1} \leq R e f_t \\
V_{t+1}^{1-\gamma} \times \underbrace{R e f_t^{\gamma-\gamma}}_{\text{scaling factor}} & \text{for } V_{t+1} \geq R e f_t
\end{cases}
\]

(3)
Eq. (3) is the multi-period extension to the two-period model of Eq. (1).

As in the two-period model, loss aversion is represented by one coefficient, $\alpha \in [0,1)$ where $\bar{\gamma} = \gamma + \frac{\alpha}{1-\alpha} (1-\gamma) \geq \gamma > 1$. When $\alpha = 0$, the agent displays no loss aversion and my model reverts to the standard recursive utility model. For $\alpha > 0$, the agent is loss averse and expects at time $t$ to experience a disutility at time $t + 1$ if the value of the future consumption stream $V_{t+1}$ is disappointing i.e. falls below her time $t$ reference point $Ref_t$. The agent expects the utility at time $t + 1$ to display a kink around the reference point $Ref_t$ with a ratio in slopes given by $(1 - \gamma) / (1 - \bar{\gamma}) = 1 - \alpha$.

Note that I did not include loss aversion on the contemporaneous consumption $C_t$. While doing so would be feasible, it would complicate the solution to the asset pricing model substantially. Besides, the one period discount rate is sufficiently low that most of the value in $V_t$ comes from the second term in $V_{t+1}$ and not from the first term in $C_t$. Simplifying the model by restricting the loss aversion specification to the second term in $V_{t+1}$ is a valid choice.

1.2.2 Reference Point

As in the two-period model, the agent’s reference point is an expectation of future outcomes. As discussed in Koszegi and Rabin (2006), the agent updates her reference point as an expectation when new information about future outcomes becomes available. However, the frequency with which the agent updates the reference point is a modelling choice.

Suppose that the agent updates her reference point at each period, so that the reference point at time $t$ is an expectation of outcomes at time $t + 1$ given the information $I_t$. As I show in section 2.1, this model is tractable and yields interesting results for asset prices. Whether or not the agent is disappointed depends solely on the shocks to the consumption process between $t$ and $t + 1$. In that case, the distribution of disappointment (and thus the risk prices) is constant if the consumption process is homoskedastic and varying if the consumption process is heteroskedastic. Time variation in the risk prices arises only when the consumption process has stochastic volatility (as in the standard recursive utility model).

Suppose now that the agent’s reference point at time $t$ depends on past expectations of the period $t + 1$ outcomes. For example, the agent partly updates the reference point at each period but does not modify the reference point all the way to the new expectation. As an example, the model for the reference point could be $Ref_{t,t+\tau} = (Ref_{t-1,t+\tau})^\xi (\mathbb{E}(outcomes_{t+1} | I_t))^{1-\xi}$...
with \( \xi \in [0, 1) \), and \( \text{Ref}_{t,t+\tau} \) is the reference point at time \( t \) for the outcomes at time \( t + \tau \).

The case \( \xi = 0 \) is the one described above. Dillenberger and Rozen (2011) argue for a history-dependent risk attitude (past disappointments and elation have an impact on risk aversion) which would support such a model of “sticky” updating of the reference point, and \( \xi > 0 \). On the other hand, price-dividend ratios are not predicted in the data by past consumption growth (this is also a critique of all habit models), which tends to suggest the degree of “stickiness” \( \xi \) is close to zero. When \( \xi > 0 \), the agent slowly upgrades the reference point following positive shocks to the consumption process and thus the risk of disappointment diminishes. Conversely, the reference point is slowly downgraded in a recession and thus the risk of disappointment increases. The risk prices are countercyclical, low following an expansion and high following a recession, even when the consumption process has constant volatility.

The need for asset pricing models with counter-cyclical risk prices is well illustrated in Melino and Yang (2003). In this paper, the authors show that, in a two-state economy, the empirical pricing kernel that matches asset prices displays higher risk prices in the bad state. Campbell and Cochrane (1999) introduce a habit model, in which time varying risk aversion obtains from an exogenous habit level, specified independently from the consumption process. Similarly, Barberis, Huang and Santos (2001) introduce time varying risk prices through a loss aversion parameter that is directly specified as time varying and counter-cyclical. In both cases, time varying risk aversion is exogenously enforced. In contrast, countercyclical risk prices endogenously obtain in my model with “sticky” updating of the reference point.

In order to concentrate on the price effects of loss aversion, independently from the time variation induced by “sticky” updating, I set \( \xi = 0 \) for the rest of the paper.

As in the two-period model, I choose a log-linear specification for the reference point: at time \( t \), the agent expects to be disappointed and to register disutility from loss aversion at period \( t + 1 \) if:

\[
\log V_{t+1} \leq \mathbb{E} (\log V_{t+1} | I_t)
\]

In this set-up, the threshold for the value function \( V_{t+1} \) to be disappointing is:

\[
\text{Ref}_t = \exp \left[ \mathbb{E} (\log V_{t+1} | I_t) \right]
\]

\(^{1}\)Another example would be \( \text{Ref}_{t,t+\tau} = \left( \prod_{i=0}^{T} \mathbb{E} (\text{outcomes}_{t+i+\tau} | I_{t-\tau}) \right)^{\xi^T} \sum_{i=0}^{T} \xi^i \) with \( \xi \in [0, 1) \), \( T \geq 0 \). In this case, the expectations of only the past \( T \) periods impact the reference point when \( \xi > 0 \).
1.2.3 Characteristics of the Model

The modified recursive utility model with loss aversion can be rewritten as:

\[ V_t = \left( (1 - \beta) C_t^{1 - \rho} + \beta (h(V_{t+1}))^{1 - \rho} \right)^{\frac{1}{1 - \rho}} \tag{4} \]

\[ h(V_{t+1}) = \mathcal{R}_t \left( \frac{V_{t+1}}{V_t} \right) = \mathbb{E}_t \left( \frac{V_{t+1}^{1 - \gamma}}{V_t^{1 - \gamma}} \right)^{\frac{1}{1 - \gamma}} \]

\[ \log V_{t+1} = \log V_{t+1} - \alpha \max(0, \log V_{t+1} - \mathbb{E}_t(\log V_{t+1})) \]

\[ \rho > 0, \beta \in [0, 1), \alpha \in [0, 1), \bar{\gamma} = \gamma + \frac{\alpha}{\alpha - 1} (\gamma - 1) \geq \gamma > 1 \]

where \( \alpha \) is the coefficient of loss aversion. When \( \alpha = 0 \), my model reverts to the standard recursive utility model.

My choice function \( h \) has the following properties: 1) if the outcome \( V_{t+1} \) is certain, then \( h(V_{t+1}) = V_{t+1} \); 2) it is increasing (first order stochastic dominance); 3) it is concave (second order stochastic dominance); and 4) it is homogeneous of degree one (and therefore \( V_t \) is homogeneous of degree one in \((C_t, V_{t+1})\)).

These characteristics of my model allow me to use most of the results from Epstein and Zin (1989), notably the unicity of the solution to the optimization problem. Because of the concavity in the preferences, the use of first-order conditions at the optimum is justified.

Notice that because at time \( t \), \( V_t \) is increasing in \( V_{t+1} \) (first order stochastic dominance), my model of preferences is time consistent.

This is the model of preferences I use in the consumption-based asset pricing model analyzed in the rest of the paper.

Notice that in my model, the agent is loss averse over the outcomes on the whole of the future consumption stream value. There is no narrow framing in my model. Rabin (2000) points out that in lab experiments, agents tend to reject small favourable gambles. For expected utility agents to reject small favourable gambles, as is observed in the data, the risk aversion coefficient has to be so high that extremely favourable large gambles are also rejected. Barberis, Huang and Thaler (2006) analyze which preferences allow for agents to reject a small gamble \( G_S = [(550, \frac{1}{2}), (-500, \frac{1}{2})] \) while also accepting a large gamble \( G_L = [(20,000,000, \frac{1}{2}), (-10,000, \frac{1}{2})] \). They show that first order risk aversion (a kink in the preferences) can justify such behaviour. However, first order risk aversion alone is not sufficient. Indeed if all risks are evaluated together, they argue that

\[ \text{Proof of these properties is provided in Appendix B.} \]
the diversification effect of gambles with independent risk dominates and the agent behaves as if second order risk averse. They propose to combine first order risk aversion with narrow framing, a process in which new gambles are evaluated independently from all other risky revenue sources. Barberis, Huang and Santos (2001) and Barberis and Huang (2009) apply this model to asset pricing. Because stock market returns have low correlation with consumption growth, the impact of the loss aversion specification is much stronger if the agent narrow frames on the stock market returns, and even a small coefficient of loss aversion can increase stock returns substantially. I make the more conservative choice of a consumption-based model without narrow framing, with the view that the results I obtain could only be enhanced if I introduced some narrow framing.

Koszegi and Rabin (2009) propose a model without narrow framing in which agents remain first order risk averse for choices over small and large gambles. In their model, agents are loss averse in the news about future outcomes. This type of modelization may be adapted to my model and as an extension to this paper, it would be interesting to add loss aversion over the updating of the reference point.

The advantage of using recursive preferences in consumption-based asset pricing models is well established, in particular for the long-run risk models. Combining behavioral models and recursive utility gives rise to interesting results. Some authors have adopted this approach before.

Routledge and Zin (2010) present a model of generalized disappointment aversion, an extension to the disappointment aversion of Gul (1991). They analyze the asset pricing implications of Epstein-Zin preferences with generalized disappointment aversion and obtain closed form solutions and interesting results in a 2-state Markov economy. Bonomo et al (2011) extend the analysis to a 4-state Markov adapted from Bansal, Kiku and Yaron (2007). They match first and second moments on the market returns and risk free rate, predictability patterns and autocorrelations for realistic parameters. As I have discussed before, my model is close in spirit to the disappointment aversion model. However, my model is both more flexible, which allows me to analyze such modelling choices as the “sticky” updating of the reference point, and more tractable, which allows me to find tractable solutions for more complex economies.

Barberis and Huang (2009) use a recursive utility model with loss aversion narrow framed on the stock market returns and find closed form solutions for both partial and general equilibria. My model differs from theirs in two crucial ways. First, I do not opt for narrow framing on financial risks. This is a more conservative choice which makes the results I obtain all the more
robust. Second, Barberis and Huang (2009) choose the constant risk free rate as the reference point for market returns. As discussed in section (1.1), there is substantial empirical evidence for the reference as an expectation. My model, in which the reference point is endogenously determined as an expectation, reflects this evidence.

2 Consumption-Based Asset Pricing Model

I assume that all agents have identical preferences with loss aversion, given by Eq. (4), and differ only in their wealth. Because preferences are homothetic, the representative agent assumption is justified.

I suppose that the optimal consumption follows a log-normal process with time varying drift and volatility:

$$\log C_{t+1} - \log C_t = \mu + GX_t + \sigma_t HW_{t+1}$$

$$X_{t+1} = AX_t + \sigma_t BW_{t+1}$$

$$\sigma_{t+1} = (1 - a) + a\sigma_t + B_\sigma W_{t+1}$$

The three-dimension vector of shocks \( \{W_t\} \) is iid \( \mathcal{N}(0, I) \). The first shock is the immediate consumption shock. The second shock does not impact consumption at first but its effect builds up over time and impacts consumption in the long-run. The third shock impacts the volatility of the consumption growth process. \( A \) and \( a \) are contracting (all eigen values have module strictly less than one). \( \{\sigma_t\} \) is scalar with mean value one and stationary distribution \( \mathcal{N}(1, B_\sigma B'_\sigma) \). The log consumption growth has a slowly moving drift with mean value \( \mu \) and time varying component \( GX_t \), which follows an AR(1) with high persistence \( GA \). The log consumption growth process has a slowly moving volatility determined by \( \sigma_t H \) with mean \( H \) and \( \sigma_t B \) with mean \( B \). To simplify the model, volatility shocks are supposed to be independent from expected consumption shocks: \( B_\sigma B' = B_\sigma H' = 0 \). For illustrative purposes, some solutions are presented in the particular case \( B_\sigma = 0 \) (consumption process with constant volatility).

In section 2.1, I analyze the consumption-based asset pricing model and obtain tractable solutions for the model of preferences of Eq. (4). In section 2.2, I analyze the asset pricing implications.

---

3Discussing the possible impact of heterogeneity in preferences is not in the scope of this paper, but would be worth exploring. I cannot use the equilibrium existence, representative agent and PDE solutions of Duffie and Lyons (1992) and Skiadas and Schroder (1999) since the preferences are not continuously differentiable in the interior domain.
of the model. I find that the loss aversion specification has a level effect: risk prices are higher than in the standard recursive utility model. This feature of my model is brought to the data in section 3.1. Further, the loss aversion specification has a cross-sectional effect: depending on the risk exposure of the asset, the impact of loss aversion is more or less intense. I test this feature, which allows to differentiate my model from the standard recursive utility model, in section 3.2 and section 3.3.

To illustrate the results, I use the consumption process parameters of Hansen, Lee, Polson and Yae (2011). In the specific case $B_\sigma = 0$, I use the consumption process parameters of Hansen, Heaton and Li (2008). Their empirical set up is discussed in section 3.

2.1 Solving the Consumption-Based Asset Pricing Model

Following the methodology of Hansen, Heaton, Lee and Roussanov (2007), the model is first solved in closed-form for a unit elasticity of intertemporal substitution (case $\rho = 1$). A first-order Taylor expansion around $\rho = 1$ allows to analyze the model for $\rho \neq 1$.

Let’s write $\log C = c$, $\log V = v$ , $\log \overline{V} = \overline{v}$ and $v^1 = v |_{\rho=1}$.

When $\rho = 1$, the model becomes:

$$v_t = (1 - \beta) c_t + \frac{\beta}{1 - \gamma} \log \mathbb{E}_t [\exp (1 - \gamma \overline{v}_{t+1})]$$

$$\overline{v}_{t+1} = v_{t+1} - \alpha \max (0, v_{t+1} - \mathbb{E}_t (v_{t+1}))$$

Because $\overline{v}$ is increasing in $v$, this recursive problem trivially follows Blackwell conditions, and thus admits a unique solution.

As a first-order Taylor expansion around $\rho = 1$, $v_t = v^1_t + (\rho - 1) Dv^1_t$ with $Dv^1_t \leq 0$ for all $t$.

With a higher elasticity of intertemporal substitution ($\rho$ less than one), the future consumption stream has more immediate value and thus the value function increases. The opposite occurs when the elasticity decreases.

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5The details of the calculation are given in Appendix D
Let’s assume that the optimal consumption follows the stochastic volatility process of Eq. (5). I approximate the unique solution for $v^1$ with a closed-form solution and find:

$$v_t^1 - c_t \approx p + Q X_t + q_1 \sigma_t + q_2 \sigma_t^2$$

(7)

The dependence on $\{X_t\}$ is exactly determined in the recursive problem of Eq. (6) and only the dependence on $\{\sigma_t\}$ is an approximation in Eq. (7). In Figure 2, I compare the approximate closed-form solution and the exact solution to the recursive problem. The distribution of $\sigma_t$ illustrates that, for realistic volatility levels, the closed-form solution of Eq. (7) is a good approximation of the exact solution. For the rest of this section, the results are derived using the approximate closed-form solution of Eq. (7).

The value function of Eq. (7) has the same functional form as in the standard recursive utility model: the functional dependence in the state variables $\{X_t, \sigma_t\}$ is unchanged. The dependence in the state variable $\{X_t\}$ is given by $Q X_t$ with $Q = \beta G (I - \beta A)^{-1}$ unchanged from the standard recursive utility model. The solution for $Q$ shows that the log value to consumption ratio is above average in good times ($G X_t > 0$) and below average in bad times ($G X_t < 0$). The higher the persistence of the consumption growth drift (the higher the module of the eigen values of $A$), the stronger the impact of the time varying $X_t$ on the value function. If the persistence is high, the value function varies more between good times and bad times. This directly translates into higher risk prices. Increasing $\beta$ and thus the importance of the risky future consumption stream relative to immediate consumption also increases the value of $|Q|$ and thus the risk prices.

As in the standard recursive utility model, I find that $|q_1|$ and $|q_2|$, which determine the value function dependence in the volatility $\sigma_t$, are increasing in the rate of time discount $\beta$ (a lower rate of time discount increases the relative importance of the future consumption stream and thus the impact of its level of volatility), in the persistence of the volatility process $\alpha$ (the higher the persistence, the more relevant the current value of $\sigma_t$ and thus the higher its impact on the value function), in the risk aversion coefficient $\gamma$ (a higher risk aversion increases the relative importance of the volatility) and in the volatility of the consumption process given by $|H|$, $|B|$ and $|B_{\sigma}|$ (these are multipliers for the state variable $\{\sigma_t\}$). I find that $p$ is increasing in $\mu$ (a higher mean consumption growth translates into higher value of the consumption process) and $\beta$.

---

6 The details of the calculation are in Appendix F.
7 Therefore, the solution is exact in the constant volatility case.
Figure 2: Approximation of the Value Function
On the left axis, the exact (dotted line) and closed-form approximate (bold line) solutions for $v_t^1 - c_t$ are displayed for $X_t = 0$, at its mean value. On the right axis, the distribution of $\sigma_t$ illustrates how close the approximate solution is to the exact solution for realistic values of the volatility.
I use the parameters of Hansen, Lee, Polson and Yae (2011) for the consumption process and $\beta = 0.99$, $\gamma = 10$, $\alpha = 0.55$. 
(a higher $\beta$, ie a lower time discount rate, increases the value) and decreasing in the persistence of the volatility process $\alpha$, in the risk aversion coefficient $\gamma$ (a higher risk aversion lowers the value of the risky consumption process) and in the volatility of the consumption process given by $|H|$, $|B|$ and $|B_\sigma|$ (higher risk levels lower the value of the consumption process).\textsuperscript{8}

The mean value-to-consumption ratio and the dependence on the volatility of the consumption process are impacted by the loss aversion specification. In Figure 3, I display the value of the consumption stream (relative to consumption) as it varies with volatility both in the model with loss aversion and in the standard recursive utility model. I find that the mean value function is lower than in the standard recursive utility model, either with risk aversion $\gamma$ or risk aversion $\bar{\gamma}$. Even though the agent behaves as though risk averse with risk aversion $\gamma \leq \bar{\gamma}$ on the non-disappointing outcomes, the model with loss aversion reduces the mean value of the consumption stream to a level that could not be achieved by the standard recursive utility model unless for a level of risk aversion much higher than $\bar{\gamma}$. I also find that the value function’s dependence in the volatility of the consumption process is greater for the loss averse agent. The pro-cyclical variations in the value function are thus amplified in the model with loss aversion relative to the standard recursive utility model.

Let’s now turn to the first order approximation around $\rho = 1$. In the particular case $B_\sigma = 0$ (constant volatility case), I find:\textsuperscript{9}

$$Dv_1^1 = q + RX_t + X_t' SX_t$$

The solution for $Dv^1$ has the same functional form as in the standard recursive utility model: the functional dependence in the state variable $\{X_t\}$ is unchanged. I find that the solution for $S$ is unchanged from the standard recursive utility model, but $R$ and $|q|$ are lower than in the standard recursive utility model. Therefore the impact of a change in the elasticity of intertemporal substitution is reduced by the loss aversion specification. In Figure 4, I display the mean log value-to-consumption ratio as it varies with $\rho$. Observe that in the model with loss aversion, the dependence on the elasticity of intertemporal substitution is limited. For the results that follow, I conduct the analysis in the case $\rho = 1$.

\textsuperscript{8}The solutions for $p$, $q_1$ and $q_2$ are in Appendix F.

\textsuperscript{9}The details of the calculation are in Appendix E.
$v_1^l - c_t$ in the model with loss aversion and in the standard recursive utility model with risk aversion $\gamma$ and $\hat{\gamma}$ are plotted on the left axis. Because the dependence on the state variable $\{X_t\}$ is the same with and without loss aversion, I plot the value functions for $X_t = 0$, the mean value. The bold line is the value function with loss aversion and the dotted lines are the ones for the standard recursive utility model with risk aversion $\gamma$ (higher plot) and $\hat{\gamma}$ (lower dotted line). On the right axis, I plot the distribution of $\sigma_t$.

I use the parameters from Hansen, Lee, Polson and Yae (2011) for the consumption process and $\beta = 0.99$, $\gamma = 10$, $\rho = 1$, $\alpha = 0.55$, $\xi = 0$. 

Figure 3: Value Function
Figure 4: Value Function dependence on $\rho$

Average $v_t - c_t$ in the model with loss aversion and in the standard recursive utility model. The bold line is the value function with loss aversion $\alpha = 0.55$ and the dotted line the value function in the standard recursive utility model with risk aversion $\gamma$. The intermediary plots are for the model with loss aversion $\alpha = 0.10$ and $\alpha = 0.25$.

I use the parameters from Hansen, Heaton and Li (2008) for the consumption process and $\beta = 0.99$, $\gamma = 10$. 
Let’s now turn to the asset pricing implications of the model. Because of the concavity of the preferences, using the first order conditions at the optimum is justified.

At time $t$, all uncertain returns $R_{t+1}$ must satisfy the Euler Equation:\footnote{The details of the calculation are in Appendix C.}

$$
\mathbb{E}_t \left[ f_t \left( R_{t+1}, \frac{C_{t+1}}{C_t}, V_{t+1} \right) \right] = 1
$$

(8)

where $f_t$ satisfies:

for $\log V_{t+1} \leq \mathbb{E}_t (\log V_{t+1})$:

$$
f_t \left( R_{t+1}, \frac{C_{t+1}}{C_t}, V_{t+1} \right) = \beta R_{t+1} \left( \frac{V_{t+1}}{R_{t+1}} \left( \frac{C_{t+1}}{C_t} \right)^{\rho - \bar{\gamma}} \right)^{-\rho}
$$

(9)

This is the functional form of the Euler Equation in the standard recursive utility model with risk aversion $\bar{\gamma}$. This directly follows from the specification of loss aversion in the preferences: below the reference point, the preferences are standard with risk aversion $\bar{\gamma} = \gamma$.

for $\log V_{t+1} \geq \mathbb{E}_t (\log V_{t+1})$:

$$
f_t \left( R_{t+1}, \frac{C_{t+1}}{C_t}, V_{t+1} \right) = \beta \left( \frac{\exp \mathbb{E}_t (v_{t+1})}{R_{t+1} (V_{t+1})} \right)^{\gamma - \bar{\gamma}}
$$

(10)

$$
\times \left( (1 - \alpha) R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{\rho - \gamma} \left( \frac{V_{t+1}}{R_{t+1}} \left( \frac{C_{t+1}}{C_t} \right)^{\rho - \gamma} \right)^{\rho - \gamma}
\right)

\times \left( + \alpha \mathbb{E}_t \left( R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{\rho - \gamma} \left( \frac{V_{t+1}}{R_{t+1}} \left( \frac{C_{t+1}}{C_t} \right)^{\rho - \gamma} \right)^{\rho - 1} \right) \left( \frac{V_{t+1}}{R_{t+1} (V_{t+1})} \right)^{1 - \gamma} \right)
$$

At time $t$, the prospect of receiving a return $R_{t+1}$ impacts both $V_{t+1}$ and $\mathbb{E}_t (\log V_{t+1})$. For this reason, above the reference point, the functional form of the Euler Equation, $f_t$, derives from both the change in the value of the future consumption stream due to the change in $V_{t+1}$ and from the change in the value of the future consumption stream due to the change in the reference point. The impact of the change in the reference point is to lower the effective risk aversion of the model for $\log V_{t+1} \geq \mathbb{E}_t (\log V_{t+1})$ and thus to reduce the risk prices. The scaling factor $\left( \frac{\exp \mathbb{E}_t (v_{t+1})}{R_{t+1} (V_{t+1})} \right)^{\gamma - \bar{\gamma}}$ guarantees the continuity of $f_t$. Because $\left( \exp \mathbb{E}_t (v_{t+1}) / R_{t} (V_{t+1}) \right)^{\gamma - \bar{\gamma}} \leq 1$, the scaling term increases the risk prices.

Notice that if $\alpha = 0$, the Euler Equation of Eq. (8), (9) and (10) reverts to the standard model with risk aversion $\gamma$. The starkly different pricing effects that I obtain for the model
with loss aversion in section 2.2 derive from the modification of the Euler Equation. Similarly to
the standard recursive utility model, my consumption-based asset pricing model is a two-factor
model: the covariations of cash-flows with the consumption growth and with the shocks to the
value function determine prices. Introducing loss aversion in the model does not generate a new
price factor. However, the pricing effects of these two factors are greatly impacted by the changes
in the functional form of the Euler-Equation.

In the following sections, I explicitly derive and obtain closed-form solutions for risk prices
generated by the Euler Equation of Eq. (8), (9) and (10) in the case with unit intertemporal
elasticity of substitution.

2.2 Risk Prices

In this section, I derive the expected returns and risk prices for assets with cash-flows that are
correlated with the consumption process. I present three main results. First, and most striking,
I find that the loss aversion specification does not impact the risk prices identically across risk
exposure: for small exposure to risk, the risk prices are higher than for large exposures to risk
(cross-sectional effect). Second, the risk prices are considerably increased by the loss aversion
specification relative to the standard recursive utility model (level effect) for assets with small
exposure to the consumption shocks (such as the market portfolio). Third, the loss aversion
specification reduces the risk free rate.

Let’s consider an asset with time $t + 1$ return $R_{t+1}$, which is uncertain at time $t$ and follows
the log-normal process:

$$\log R_{t+1} = \left( \bar{r}_t - \frac{1}{2} |\Delta|^2 - \frac{1}{2} |\bar{\Delta}|^2 \right) + \Delta W_{t+1} + \bar{\Delta} W_{t+1}$$

(11)

where $\{W_{t+1}\}$ are the shocks to the consumption process and $\{\bar{W}_{t+1}\}$ are independent shocks.
$\bar{r}_t$ is the log expected return of the asset. The covariances between the returns and the con-
sumption shocks and between the returns and other idiosyncratic shocks are determined by the
exposure $(\Delta, \bar{\Delta})$. I analyze in this section how the expected returns vary with the risk exposure.

Applying the Euler Equation of Eq. (8), (9) and (10) to this return yields $\bar{r}_t$, the log expected
return of the asset. Notice that the exposure to shocks that are independent from the shocks to
the consumption process is not priced. The log expected returns is a function of the exposure to
the priced shocks $\{W_{t+1}\}$ only:
\[ \tilde{r}_t (\Delta) = -\log \mathbb{E}_t \left[ f_t \left( \exp\left( \Delta W_{t+1} - \frac{1}{2} |\Delta|^2 \right), \frac{C_{t+1}}{C_t}, V_{t+1} \right) \right] \]  

(12)

Increasing the risk exposure \( \Delta \) has a price, which is reflected in a change in \( \tilde{r}_t \). The cost of an additional increment of risk is defined as the risk price. For a given exposure to risk \( \Delta \), I define the risk price \( RP_t (\Delta) \) as:

\[ RP_t (\Delta) = \frac{\partial \tilde{r}_t (\Delta)}{\partial \Delta} \]  

(13)

In the rest of this section, I derive the closed-form solutions for the risk-free rate, the excess returns and the risk prices for the model of preferences with unit elasticity of intertemporal substitution as in Eq. (6), and the consumption process of Eq. (5).

**Risk Free Rate**

Let’s first consider the risk free asset, \( \tilde{r}_t (0) = r f_t \).\(^{11}\)

A second order approximation around \( QB = 0 \), and \( H = 0 \) in the constant volatility case simplifies the closed-form solution of the risk free rate to:

\[ r f_t^1 \approx -\log \beta + \mu + G X_t - \frac{1}{2} |H|^2 + (1 - \gamma) (H + QB) H' \]

\[ + \alpha \left\{ -\frac{1}{2\sqrt{2\pi}} |H + QB|^{-\frac{1}{2}} \left( 1 - \frac{1}{2\sqrt{2\pi}} \right) \right\} \]

\[ + \frac{1}{2} \alpha \left( 1 - \frac{1}{2} \right) (1 - \tilde{\gamma}) (H + QB) H' \]

loss aversion terms

(14)

The usual results for the risk free rate obtain: it is procyclical, it is increasing in the mean consumption growth \( \mu \) (when the expected consumption growth is high, agents are less inclined to save), decreasing in \( \beta \) (with a lower rate of time discount the agents are more willing to substitute between immediate and future consumption and thus to save), decreasing in the risk aversion \( \gamma \) and in the amount of risk determined by \( |H| \) and \( |QB| \) (the precautionary savings effect dominates over the substitution effect).

The extra terms due to the loss aversion specification are both negative: the effect of loss aversion is to decrease the risk free rate. In this model, as in the reference model, the precautionary savings effect dominates over the substitution effect and the risk free rate is lowered by the loss aversion specification. Further, the negative impact on the risk free rate of an increase in \( \gamma \) or

\(^{11}\)The closed-form solution for the risk free rate is in Appendix E for the constant volatility case and in Appendix F for the general case.
in the amount of risk is amplified by the loss aversion specification. This result extends to the general case $B_\sigma \neq 0$ with stochastic volatility. I plot in Figure 5 the annual risk free rate as a function of the volatility $\sigma_t$.

Observe that the risk free rate in the model with loss aversion is lower than in the standard recursive utility model with risk aversion $\bar{\gamma}$, even though the agent behaves as though risk averse with risk aversion $\gamma \leq \bar{\gamma}$ on the non-disappointing outcomes. The standard recursive utility model tends to overvalue the risk free rate. Therefore, the model with loss aversion improves on the calibration of the risk free rate, even when compared to the standard recursive utility model with risk aversion $\gamma$.

**Risk Prices**  I now analyze the expected excess returns $\bar{r}_t(\Delta) - r_f t$ and risk prices $RP_t(\Delta)$ for $\Delta \neq 0$.

In Figure 6, Figure 7 and Figure 8, I display $\{\bar{r}_t(\Delta) - r_f t\}$ and $\{RP_t(\Delta)\}$, as functions of the exposure $\Delta$ and of the volatility of the underlying consumption process $\{\sigma_t\}$, for the model with loss aversion and for the standard recursive utility model with risk aversion $\gamma$ and $\bar{\gamma}$. These graphs illustrate the fundamental difference in risk prices between the model with loss aversion and the standard recursive utility model. While in the standard model the excess returns yield a risk price that is constant across risk exposures, in my model, the risk prices vary with the risk exposure $\Delta$. The risk prices display an asymmetrical bell shape.

First, the risk prices are higher for negative exposure (hedges) than for positive ones. Hedges generate positive returns when the shocks are negative and the agent is disappointed, and are thus mostly priced in a model with high risk aversion $\bar{\gamma}$. In contrast, assets with positive risk exposure generate positive returns when the agent is not disappointed, and are thus mostly priced in a model with risk aversion $\gamma$.

Second, the risk prices are highest for small risk exposure: the agent’s effective risk aversion is higher for small amounts of risk and lower for large amounts of risk. In section 3.2 and section 3.3, I show that this differentiating feature of my model is supported by the empirical evidence on asset prices.

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12The closed-form solutions for the returns and risk prices are in Appendix E for the constant volatility case and in Appendix F for the general case.

13Only the dependence on the volatility $\{\sigma_t\}$ differs between the two models. The dependene on the time varying drift $\{X_t\}$ is the same. I analyze the excess returns and risk prices for $X_t = 0$ at its mean value.

14This result of my model is also supported by the empirical evidence on small versus large gambles (see section 1.2).
The annual risk free rate with loss aversion and without loss aversion (standard recursive utility model for risk aversion $\gamma$ and $\bar{\gamma}$) are plotted on the left axis. Because the dependence on the state variable $\{X_t\}$ is the same with and without loss aversion, I plot the risk free rates for $X_t = 0$, the mean value. On the right axis, I plot the distribution of $\sigma_t$. I use the parameters from Hansen, Lee, Polson and Yae (2011) for the consumption process and $\beta = 0.999$, $\gamma = 10$, $\alpha = 0.55$. 

Figure 5: Risk Free Rate
To better understand how these effects arise, let’s consider the excess returns in the constant volatility case.

If $|\Delta| \gg |H|$, and $|\Delta| \gg |QB|$, I obtain the approximation:

$$r_t^1(\Delta) - rf_t^1 \approx |\Delta| \gg 0 (H + (\gamma - 1) (QB + H)) \Delta'$$

(15)

$$- \log \left\{ \begin{aligned}
\Phi \left( -\frac{\Delta(QB + H)}{|QB + H|} \right) & \exp (\alpha (1 - \tilde{\gamma}) (QB + H) \Delta') \\
+ (1 - \alpha) \Phi \left( \frac{\Delta(QB + H)}{|QB + H|} \right) & + \alpha \Phi ((1 - \gamma) |QB + H|) \exp (- (1 - \gamma) (QB + H) \Delta')
\end{aligned} \right\}$$

loss aversion terms

The first term in Eq. (15) is equal to the excess returns in the standard recursive utility model. In the standard model, risk prices are constant and equal to $H + (\gamma - 1) (QB + H)$. They are increasing in the coefficient of risk aversion $\gamma$, in the level of risk (given by $|H|$ and $|B|$), in the persistence of the consumption process and in $\beta$.

The extra term due to loss aversion modifies the excess returns and introduces non-linearity in the risk exposure.

When $\Delta (QB + H)' \to +\infty$, I find that $r_t^1(\Delta) - rf_t^1 \sim H \Delta'$ and thus the risk prices tend toward a lower value than in the standard recursive utility model with risk aversion $\gamma$. When $\Delta (QB + H)' \to +\infty$, the relevant Euler Equation for pricing the cash-flows is given by Eq. (10). The second term in Eq. (10) dominates when the risk exposure is large and yields a constant risk price of $H$ when $\rho = 1$. The direct contribution to the value function of the reference point (see Appendix A) pushes the effective risk aversion down when the agent is far from the disappointment threshold. For this reason, when the risk exposure is large and positive, the risk prices are lower in my model than in the standard recursive utility model.

When $\Delta (QB + H)' \to -\infty$, I find that $r_t^1(\Delta) - rf_t^1 \sim ((H + (\tilde{\gamma} - 1) (QB + H)) \Delta')$. The risk prices tend from below toward a value of $H + (\tilde{\gamma} - 1) (QB + H)$, which is the same as in the standard recursive utility model with risk aversion $\tilde{\gamma}$. When $\Delta (QB + H)' \to -\infty$, the relevant Euler Equation for pricing the cash-flows is given by Eq. (9) and the agent behaves as in the standard recursive utility model with risk aversion $\tilde{\gamma}$. However, the fact that the agent behaves with risk aversion $\gamma < \tilde{\gamma}$ when she is not disappointed pushes the effective risk aversion slightly below $\tilde{\gamma}$ when the exposure is large and negative.

If the risk exposure $|\Delta|$ is small, a second order approximation around $QB = 0$, $H = 0$ and $\Delta = 0$ simplifies the solution for the excess returns to:
\[ r_t^1 (\Delta) - r_f^1 \approx |\Delta| \leq 1 (H + (\gamma - 1)(QB + H)) \Delta' \]
\[ + \alpha \left\{ \frac{1}{\sqrt{2\pi}} \frac{\Delta(QB + H)'(1 - \frac{\alpha}{\sqrt{2\pi}} \frac{H(QB + H)'}{|QB + H|})}{|QB + H|} \right\} \]
\[ + \frac{1}{2} \alpha (\tilde{\gamma} - 1)(1 - \frac{\alpha}{\pi}) \Delta (QB + H)' + \frac{1}{2} \alpha \left( \frac{\Delta(QB + H)'}{|QB + H|} \right)^2 \]

The new terms due to loss aversion have positive loadings on \( \Delta \) and thus push the risk prices higher. The dominant term due to loss aversion is \( \frac{\alpha}{\sqrt{2\pi}} \frac{\Delta(QB + H)'}{|QB + H|} \). It is the only first order term for \(|H|, |QB| \) and \(|\Delta| \) close to zero. Therefore, even for small values of loss aversion \( \alpha \), the contribution of the loss aversion specification to the excess returns is significant. Further, the contribution of the loss aversion model dominates the one from the standard recursive utility model which explains why the risk prices in the loss aversion model are higher than the ones in the standard model with risk aversion \( \tilde{\gamma} \).

The volatility level \( \sigma_t \) also impacts risk prices, as evidenced in Figure 6, Figure 7 and Figure 8.

For the first two shocks, the shocks to immediate and long-term consumption, I find that, as in the standard recursive utility model, risk prices increase with the amount of risk in the model (as determined by \( \sigma_t \)). I also find a stronger bell shape in the risk prices when the volatility of the process is high. When the volatility is so low that the consumption process is essentially always at the kink, only assets with extreme risk exposure (not displayed in the graph) would start to display a decrease in risk prices. When the volatility is low, the model with loss aversion yields risk prices that are very low (the consumption process is close to deterministic) but above risk prices in the standard recursive utility model with both risk aversion \( \gamma \) and \( \tilde{\gamma} \). In contrast, when the consumption process is volatile, assets with large exposure are priced away from the kink in the preferences and the bell shape is more pronounced. Notice also that, for small risk exposure (\( \Delta \) close to zero), the risk prices increase with the volatility of the consumption process \( \sigma_t \) at a faster rate in the model with loss aversion than in the standard recursive utility model. Risk prices for the consumption shocks are therefore more strongly counter-cyclical in my model with loss aversion than in the standard recursive utility model.

Positive volatility shocks increase the risk of the consumption process and thus decrease the present value of the consumption stream. Positive exposure to such shocks serves as a hedge which is reflected in the negative risk prices. Notice that the loss aversion specification makes the
agent particularly sensitive to the volatility risk and generates risk prices that are above those of the recursive utility model with risk aversion $\gamma$ for all levels of risk exposure. In contrast to the risk prices for the shocks to consumption, the bell shape in the volatility-risk prices is more pronounced when the volatility is low. When the volatility is very low, assets with small volatility risk exposure are entirely priced around the kink in the preferences and thus yield high risk prices. In contrast, assets with large volatility risk exposure are priced far from the kink and thus yield lower risk prices. When the volatility is high, on the other hand, the risk prices for both small and large exposure are determined in a large range around the kink. Depending on the risk aversion coefficient, this contrasting effect is strong enough to generate risk prices for small volatility-risk exposure that are decreasing in the volatility of the consumption process. For $\gamma = 10$, the volatility risk prices are pro-cyclical in the model with loss aversion for assets with small exposure to risk, contrarily to the standard recursive utility model, and counter-cyclical for assets with large exposure to risk. For higher coefficients of risk aversion ($\gamma \geq 20$), this effect becomes smaller and the volatility risk prices become counter-cyclical for most risk exposures. However, compared to the standard recursive utility model, they vary less with the underlying volatility. Therefore, my model predicts volatility-risk prices that either vary very little with the business cycle or are actually counter-cyclical. This is an interesting feature of my model and it would be worth exploring its empirical application in an extension to this paper.

In this section, I exposed three main results. First, and most striking, the loss aversion model generates a cross-sectional effect on risk prices: risk prices are no longer constant across risk exposure. This prediction allows to test my model against the standard recursive utility model, which I do in section 3.2 and section 3.3. Second, the loss aversion generates a level effect: the risk free rate is reduced and the risk prices are increased. These features of my model allow to improve on the calibration of the standard recursive utility model and are brought to the data in section 3.1.
Figure 6: Risk Prices - Shock to consumption

The two graphs display the risk prices for an exposure $[\Delta \ 0 \ 0] W_{t+1}$, for the loss aversion model with loss aversion $\alpha = 0.55$ and the standard recursive utility model with risk aversion $\gamma$ (the plane in the 1st graph and the lower dotted line in the second graph) and $\bar{\gamma}$ (the higher dotted line in the 2nd graph). In the second graph, the three cases $\sigma_t \approx 0$, $\sigma_t = 1$ (mean value) and $\sigma_t = 2$ are displayed.

I use the parameters from Hansen, Lee, Polson and Yae (2011) for the consumption process and $\beta = 0.999$, $\gamma = 10$, $\alpha = 0.55$. 

The two graphs display the risk prices for an exposure $[ 0 \quad \Delta \quad 0 ] W_{t+1}$, for the loss aversion model with loss aversion $\alpha = 0.55$ and the standard recursive utility model with risk aversion $\gamma$ (the plane in the first graph and the lower dotted line in the second graph) and $\bar{\gamma}$ (the higher dotted line in the 2d graph). In the second graph, the three cases $\sigma_t \approx 0$, $\sigma_t = 1$ (mean value) and $\sigma_t = 2$ are displayed.

I use the parameters from Hansen, Lee, Polson and Yae (2011) for the consumption process and $\beta = 0.999$, $\gamma = 10$, $\alpha = 0.55$. 

Figure 7: Risk Prices- Shock to long-term consumption
Figure 8: Risk Prices- Shock to consumption volatility

The two graphs display the absolute value of the risk prices for an exposure $\begin{bmatrix} 0 & 0 & \Delta \end{bmatrix} W_{t+1}$ (or $-RP_t (\begin{bmatrix} 0 & 0 & \Delta \end{bmatrix})$), for the loss aversion model with loss aversion $\alpha = 0.55$ and the standard recursive utility model with risk aversion $\gamma$ (the plane in the 1st graph and the lower dotted line in the second graph) and $\bar{\gamma}$ (the higher dotted line in the 2d graph). In the second graph, the three cases $\sigma_t \approx 0$, $\sigma_t = 1$ (mean value) and $\sigma_t = 2$ are displayed.

I use the parameters from Hansen, Lee, Polson and Yae (2011) for the consumption process and $\beta = 0.999$, $\gamma = 10$, $\alpha = 0.55$. 
3 Empirics

In this section, I bring my model to the data and find some support for the loss aversion specification over the standard recursive utility model. In section 3.1, I evaluate the risk free rate, the equity premium and the value premium using my model with loss aversion. I find that my model improves on the standard recursive utility model and allows to match the market premia for lower values of risk aversion. In section 3.2, I analyze the implications of my model regarding the empirical fit of the CAPM model. My model predicts that the CAPM alphas are higher for small CAPM betas and lower for high CAPM betas: the security market line (the excess returns as a function of beta, the exposure to market risk) is flatter than the CAPM. As pointed out in Black, Jensen and Scholes (1972) and more extensively in Frazzini and Pedersen (2010), this is supported by the data for various classes of assets: U.S. equities, 20 global equity markets, Treasury bonds, corporate bonds, and futures. In section 3.3, I show that my model predicts a negative premium for skewness as evidenced in the data (see Harvey and Siddique (2000)).

3.1 Asset Returns

To quantitatively analyze the asset returns in the model with loss aversion, I consider the specific case $B_{\sigma} = 0$ (constant volatility case) and I use the consumption process of Hansen, Heaton and Li (2008). In Hansen, Heaton and Li (2008), the state variable $\{X_t\}$ is explicitly determined by two macro variables: the consumption growth and the earning-to-consumption ratio where earning and consumption are assumed to be cointegrated. The measure of consumption is the seasonally adjusted aggregate consumption of non-durables and services taken from the National Income and Product Accounts (NIPA). The corporate earnings are also taken from NIPA and converted to real terms using the implicit price deflator for non-durables and services. Both are quarterly databases. The parameters of the consumption process and the shocks $\{W_t\}$ are obtained for $\{X_t\}$ constructed with consumption growth and earning-to-consumption ratios on 5 lagged periods. The loadings on the shocks $\{W_t\}$ of any dividend process is obtained directly from the data. The results thus obtained are independent from the choice for the model of preferences and can therefore be used to contrast the implications for asset returns of the standard recursive utility model and of my model with loss aversion.\textsuperscript{15}

\textsuperscript{15}In contrast, in Bansal and Yaron (2004) and Bansal, Kiku and Yaron (2007, 2009), the state variable $\{X_t\}$ is a hidden variable and its evolution, as well as the loadings of the asset returns on the shocks, are chosen to match moments on both consumption and asset returns. The calibration is thus tailored to fit the standard recursive utility model.
In the first two columns, the risk free rate is derived using the standard recursive utility model with risk aversion $\gamma = 10$ and $\bar{\gamma} = \gamma + \frac{\alpha}{1-\alpha} (\gamma - 1)$ with $\gamma = 10$ and $\alpha = 0.55$. The nominal risk free rate is derived from CRSP 30-day-Treasury-bill returns. I correct for inflation using NIPA price indices for non-durables and services. I use the quarterly parameters from Hansen, Heaton and Li (2008) for the consumption process and $\gamma = 10$. Table 1 displays the risk free rate for various parameters of loss aversion and for the standard recursive utility model with loss aversion $\gamma$ and $\bar{\gamma}$ (calculated for $\alpha = 0.55$). As discussed in section 2.2, the risk free rate is decreasing in the parameter $\beta$ that governs the intertemporal rate of discount and in the risk aversion $\gamma$. My model increases the value of risk aversion from $\gamma$ to $\bar{\gamma}$ on part of the domain of consumption realizations. That my model should lower the value of the risk free rate is therefore to be expected. However, as I have discussed above, and as can be seen in Table 1, my model lowers the risk free rate even below the level obtained in the standard recursive utility model with risk aversion $\bar{\gamma}$ (even though the agent behaves as though risk averse with risk aversion $\gamma \leq \bar{\gamma}$ on the non-disappointing outcomes).

My model improves on the standard recursive utility model by lowering the risk free rate closer to historical levels. This improvement goes beyond the direct effect of the partial increase in risk aversion due to the loss aversion specification.

Let’s now turn to the equity premium. The covariation between the market returns and the shocks to aggregate consumption, both immediate and long-term, is too low in the data to generate the equity premium at reasonable levels of risk aversion (the standard recursive utility model requires $\gamma = 79$, and my model would require $\gamma \approx 35$, in order to match the equity premium). In Malloy et al. (2009), the authors adapt the empirical set-up of Hansen, Heaton and Li (2008) to stockholders’ consumption (in contrast to the aggregate consumption). They argue that the relevant Euler Equation for equity is the one resulting from the optimization problem of the stock market’s participants. Using micro-level data from the Consumer Expenditure Survey...
for the period 1982 to 2004, they find that shareholder’s long-run consumption is correlated to the long-run aggregate consumption and three times more volatile.

Motivated by this empirical result, I consider the consumption process:

\[
\log C_{t+1}^s - \log C_t^s = \mu + 3G X_t + HW_{t+1}
\]

\[X_{t+1} = AX_t + BW_{t+1}\]

where \(C^s\) is the stockholders’ consumption and all the parameters are unchanged from Hansen, Heaton and Li (2008) except for the loading on the state variable \(\{X_t\}\) which is leveraged up to 3 times the aggregate consumption loading. This set up implies a volatility for long-term stockholders’ consumption three times that of the long-term aggregate consumption.

Using the consumption process of Eq. (17), I display in Table 2 the equity premium for various parameters of loss aversion and for the standard recursive utility model with loss aversion \(\gamma\) and \(\bar{\gamma}\) (calculated for \(\alpha = 0.55\)). Equity premia are increasing in risk aversion. In my model risk aversion is increased from \(\gamma\) to \(\bar{\gamma}\) on part of the domain of consumption realizations. That my model should increase the equity premia is therefore to be expected. However, as I have discussed above, and as can be seen in Table 1, for reasonably low levels of risk aversion\(^{16}\), my model increases the equity premium beyond the level obtained in the standard recursive utility model with risk aversion \(\bar{\gamma}\) (even though the agent behaves as though risk averse with risk aversion \(\gamma \leq \bar{\gamma}\) on the non-disappointing outcomes).\(^{17}\)

Using the stockholders’ consumption process of Eq. (17) allows obtain an equity premium and a risk free rate close to historical levels for \(\gamma \approx 10\), in the case of the model with loss aversion \(\alpha = 0.55\). Even for a lower risk aversion of \(\gamma = 5\), the model with loss aversion \(\alpha = 0.55\) explains 40% of the historical equity premium (versus 15% for the standard recursive utility model).

Using the stockholders’ consumption process, which is more volatile in the long-run, rather than the aggregate consumption process, improves on the equity premium calibration. Increasing the frequency of the consumption process would also increase the implied equity premium values, as noted in Benartzi and Thaler (1995) and Bansal, Kiku and Yaron (2009). Because the relevant macro-data is available for quarterly frequency, I limit the analysis to the empirical set up of

\(^{16}\)Microeconomics estimates of risk aversion imply \(\gamma \approx 3\). Asset pricing models typically require higher levels of risk aversion, and \(\gamma \leq 10\) is considered as a reasonable level of risk aversion.

\(^{17}\)For \(\gamma > 10\), the risk prices are no longer in a bell shape: risk prices are decreasing with the exposure \(\Delta\) and are always under the risk price level of the standard recursive utility model with risk aversion \(\bar{\gamma}\).
Table 2: Equity Premium

<table>
<thead>
<tr>
<th></th>
<th>standard model</th>
<th>model with loss aversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>risk aversion</td>
<td>risk aversion</td>
<td>α = 0.25</td>
</tr>
<tr>
<td>γ = 5</td>
<td>0.98%</td>
<td>2.12%</td>
</tr>
<tr>
<td></td>
<td>1.21%</td>
<td>1.57%</td>
</tr>
<tr>
<td></td>
<td>2.52%</td>
<td></td>
</tr>
<tr>
<td>γ = 10</td>
<td>2.15%</td>
<td>4.72%</td>
</tr>
<tr>
<td></td>
<td>2.42%</td>
<td>2.91%</td>
</tr>
<tr>
<td></td>
<td>4.58%</td>
<td></td>
</tr>
<tr>
<td>γ = 15</td>
<td>3.32%</td>
<td>7.32%</td>
</tr>
<tr>
<td></td>
<td>3.67%</td>
<td>4.35%</td>
</tr>
<tr>
<td></td>
<td>7.05%</td>
<td></td>
</tr>
<tr>
<td>Equity Premium</td>
<td>from CRSP (1947-2010) = 6.09%</td>
<td></td>
</tr>
</tbody>
</table>

Annualized market excess returns and risk free rate for various values of risk aversion γ and loss aversion α. In the first columns, the equity premium is derived using the standard recursive utility model with risk aversion γ and \( \tilde{\gamma} = \gamma + \frac{\alpha}{1-\alpha} (\gamma - 1) \) with α = 0.55. The equity premium is derived from CRSP value-weighted portfolio returns minus CRSP 30-day-Treasury-bill returns.

I use the quarterly parameters from Hansen, Heaton and Li (2008) for the aggregate consumption process and β = 0.999. The stockholders’ consumption process of Eq. (17) is used for pricing.

Hansen, Heaton and Li (2006), while keeping in mind that a monthly frequency would improve on the empirical fit of my model.

Let’s now turn to the implication of the model with loss aversion for the value premium. As documented in Bansal, Dittmar and Lundblad (2005) as well as in Hansen, Heaton and Li (2008), the value premium can be explained by the long-run risk models. Indeed value stocks have a higher covariance with long-run consumption than growth stocks, which justifies the higher returns they yield. In Table 3, I display the value premium for the standard recursive utility model with risk aversion γ and \( \tilde{\gamma} \) (calculated for a loss aversion coefficient α = 0.55), and for the model with loss aversion. The value premium is calculated as the difference in returns between the portfolio with highest book-to-market ratio (value portfolio) and the portfolio with lowest book-to-market ratio (growth portfolio) from Fama-French (1992) five portfolios sorted on book-to-market ratios. For coherence, I use the stockholders’ consumption process of Eq. (17) to price the assets.18

Micro-level data on risk aversion suggest that γ ≈ 3. For such a low level of risk aversion, the model with loss aversion α = 0.55 can explain the value premium. In contrast the standard recursive utility model explains only 40% of the value premium when γ = 3. The improvement on the calibration of the value premium goes beyond the direct effect of the partial increase in risk aversion due to the loss aversion specification, as can be seen in Table 3.

Most consumption-based asset pricing models fail at capturing the equity premium because of

---

18Even if the aggregate consumption is used instead of the stockholders’ consumption, the model is very successful at capturing the value premium.
standard model | model with loss aversion  
---|---  
| risk aversion | risk aversion | \( \bar{\gamma} \) | \( \alpha = 0.25 \) | \( \alpha = 0.25 \) | \( \alpha = 0.55 \)  
| \( \gamma = 3 \) | 1.87\% | 3.94\% | 2.67\% | 3.89\% | 6.65\%  
| Value Premium from Fama-French (1947-2010) = 4.22\%  

Table 3: Value Premium  

Annualized value premia for various values of loss aversion \( \alpha \). In the first columns, the equity premium is derived using the standard recursive utility model with risk aversion \( \gamma \) and \( \bar{\gamma} = \gamma + \frac{\alpha}{1-\alpha} (\gamma - 1) \) with \( \alpha = 0.55 \). The value premium is derived from Fama-French (1992) five portfolios sorted on book-to-market.

I use the quarterly parameters from Hansen, Heaton and Li (2008) for the aggregate consumption process and \( \beta = 0.999 \). The stockholders’ consumption process of Eq. (17) is used for pricing.

the low correlation between stock returns and consumption. As I have shown in Table 1, Table 2 and Table 3, the model with loss aversion improves on the calibration of the risk free rate, the equity premium and the value premium. This improvement goes beyond the direct effect of the partial increase in risk aversion due to the specification of the model (the standard recursive utility model with risk aversion \( \bar{\gamma} \) yields poorer results than my loss aversion model). My model with loss aversion can match the value premium for \( \gamma = 3 \) and the equity premium for \( \gamma \approx 10 \). This is a clear improvement relative to the standard recursive utility model.

To be consistent with the result on the equity premium obtained in Table 2, I use the stockholders’ consumption process of Eq. (17) and \( \gamma = 10 \), in section 3.2 and section 3.3 below.

### 3.2 Prediction for CAPM Alphas

In this section, I analyze the predictions of my model for the fit of the CAPM. In Black, Jensen, and Scholes (1972), the authors point out that the security market line (the excess returns as a function of beta, the exposure to market risk) for U.S. stocks is too flat relative to the CAPM model. They find that the intercept of the security market line is not zero (as predicted by the CAPM) but a positive return, equal to 25\% of the market excess return. Frazzini and Pedersen (2010) show that this empirical result is valid for a wider class of assets (U.S. equities, 20 global equity markets, Treasury bonds, corporate bonds, and futures) and has been persistent over time. They find a similar intercept for the security market line for US stocks, in a 5-factor model that accounts for market, value, size, momentum and liquidity risk. Consistent with this result, they find empirical evidence that the CAPM alphas are decreasing with the CAPM betas, along with the assets’ Sharpe ratios. I find that my model with loss aversion offers a novel theoretical justification
for this central empirical result.

Let’s consider asset returns with log-normal distributions:

\[ R_{i,t+1} - R_f = \phi \left( A_i \exp \left( \Delta_i W_{t+1}^1 - \frac{1}{2} \Delta_i^2 \right) - R_f \right) \]  

(18)

where \( \phi \neq 1 \) allows for more flexibility (and for leverage) on the returns distribution, and \( \{ W_{t+1} \} \) are the shocks to the consumption process. Let’s write \( \Delta_m \) for the market returns’ loadings on the shocks to the consumption process. Because the empirical set-up of Frazzini and Pedersen (2010) accounts for the value, size, momentum and liquidity risk factors, I consider assets that vary only in their exposures to market risk. I consider assets with exposure to the consumption shocks \( \Delta_i \), given by \( \Delta_i = a_i \Delta_m \) where \( a_i \) is a scalar. For a given \( \phi \), the expected returns \( \bar{R}_i \) and the market beta \( m\beta_i \) (the covariation between \( R_i \) and \( R_m \)) determine the parameters \( A_i \) and \( \Delta_i = a_i \Delta_m \), with \( m\beta_i \approx a_i \), for all \( i \). Using the results of section 2.2, I display in Figure 9 the predictions of my model with loss aversion, and of the standard recursive utility model, regarding the fit of the CAPM model for such assets. Because the standard recursive utility model yields a constant risk price on each shock, it predicts a perfect fit of the CAPM model for assets for which the loadings on only one shock vary. In contrast, my model qualitatively predicts that the security market line is above CAPM for betas less than one and below CAPM for betas above one. It predicts that the CAPM alphas and the Sharpe ratios decrease with the CAPM betas, and that the empirical intercept of the security market line is strictly positive for the assets I consider.

I now turn to the quantitative predictions of my model with loss aversion for the fit of the CAPM. I use the parameters of Hansen, Heaton and Li (2008) for the stockholders’ consumption process of Eq. (17). For a given \( \phi \), \( A_m \) and \( \Delta_m \) are determined using the stock market returns from CRSP value weighted portfolio (1947-2010).

In Table 4, I display the results for the fit of the CAPM when using the model with loss aversion. Remember that in the standard recursive utility model, the intercept of the security market line is exactly zero. In contrast, I find that my model with loss aversion can be calibrated to quantitatively explain the historical intercept of 1.5% annual return (25% of the market excess returns).

My analysis concerns assets with specific returns distributions as in Eq. (18). For these assets, a change in the market beta \( m\beta_i \) corresponds to a change in both the volatility and higher moments. In contrast to the standard recursive utility model, the higher moments impact the
Figure 9: Prediction for the fit of the CAPM

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \alpha = 0.10 )</th>
<th>( \alpha = 0.25 )</th>
<th>( \alpha = 0.55 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.02%</td>
<td>0.04%</td>
<td>0.1%</td>
</tr>
<tr>
<td>0.1</td>
<td>0.4%</td>
<td>0.9%</td>
<td>1.7%</td>
</tr>
<tr>
<td>0.02</td>
<td>5%</td>
<td>12%</td>
<td>21%</td>
</tr>
</tbody>
</table>

US stocks intercept = 25\% of Market Excess Returns

Table 4: CAPM intercept as \% of Market Excess Returns

I use the quarterly parameters from Hansen, Heaton and Li (2008) and the stockholders’ consumption process of Eq. (17) with \( \gamma = 10, \beta = 0.999 \). The intercept is calculated by fitting the security market line between a CAPM beta of 0.5 and a CAPM beta of 4.
\[ \alpha = 0.10 \quad \alpha = 0.25 \quad \alpha = 0.55 \]

| \( \phi = 1 \) | 0.02\% | 0.04\% | 0.1\% |
| \( \phi = 0.1 \) | 0.4\% | 0.9\% | 1.7\% |
| \( \phi = 0.02 \) | 5\% | 12\% | 21\% |

US stocks intercept = 25\% of Market Excess Returns

Table 5: Market Betas and Skewness

I use the quarterly parameters from Hansen, Heaton and Li (2008) and the stockholders’ consumption process of Eq. (17) with \( \gamma = 10, \beta = 0.999 \).

The intercept is calculated by fitting the security market line between a CAPM beta of 0.5 and a CAPM beta of 4.

expected returns, in my model with loss aversion. This effect arises from a stochastic discount factor that is not linear in the consumption shocks, in my model. In particular, as \( m\beta_i \) increases for assets with returns as in Eq. (18), so does the skewness of the distribution. This implication of my model for the distribution of returns is supported in the data. Portfolios sorted on their market betas display a clear monotone relation between skewness and market betas (higher betas coincide with higher skewness), using CRSP data (1926-2011). Further, the portfolio with highest market beta \( (m\beta = 2.2) \) has a positive skewness of 4.35. This level of skewness corresponds to a model of returns as in Eq. (18) with \( \phi = 0.05 \). This model of returns allows to find an intercept close to the one from Frazzini and Pedersen (2010).

My model with loss aversion justifies, both qualitatively and quantitatively, why the security market line is flatter than CAPM. Other models in the literature rely on borrowing constraints and heterogeneous agents to obtain the desired results on the security market line (see Black (1972, 1992), Brennan (1971) and Frazzini-Pedersen (2010) among others). My model offers a novel justification for this central issue in financial economics.

### 3.3 Negative Premium for Skewness

In this section, I analyze the predictions of my model regarding the impact on expected returns of the skewness of the returns distribution. In Harvey and Siddique (2000), the authors analyze the empirical price impact of skewness. They find that two assets with same volatility, but different skewness in their returns distribution yield different expected returns. Assets with low skewness yield higher returns than assets with high skewness: there is a negative premium for skewness. I
find that my model with loss aversion offers a novel theoretic justification for this empirical result.

The asset returns are modelled as in Eq. (18). As the loadings on the shocks $\Delta_i$ increase, both the volatility and the skewness of the returns increase. Following the empirical set-up of Harvey and Siddique (2000), let’s take the market portfolio as the reference asset.\textsuperscript{19} Suppose that assets with higher volatility than the market are de-levered, and assets with higher volatility than the market are levered-up, so that all assets have the same volatility as the market portfolio. Using the results of section 2.2, I find that my model predicts relatively less returns when the loadings on the risk, and thus the skewness of the returns distribution, increase. My model with loss aversion qualitatively predicts a negative premium for skewness. This effect arises from a stochastic discount factor that is not linear in the consumption shocks, in my model.

Harvey and Siddique (2010) estimate the premium that assets with low skewness (relative to the market portfolio) can yield over assets with same volatility but high skewness (relative to the market portfolio) at 3.5% annually (or more than half of the equity premium).

For simplification, let’s consider assets with loadings that vary only on one dimension: $\Delta_i = a_i \Delta_m$ as in section 3.2.\textsuperscript{20} In Figure 10, I display the expected returns in excess of the market that assets with same volatility but different skewness can yield for the case $\phi = 0.05$ (for which $m \beta = 2.2$ yields a skewness of 4.35 as in the data). As predicted, the model with loss aversion generates a negative premium for skewness. I find that the model with loss aversion $\alpha = 0.55$ generates a negative premium for skewness of about 1%, which is in the order of magnitude of the one from Harvey and Siddique (2000).

In Harvey and Siddique (2000), the negative premium for skewness obtains from directly modifying the stochastic discount factor (so that it is linear in both the market returns and the square of the market returns). In contrast, my consumption-based asset pricing model is not tailored to address the skewness premium. My model with loss aversion offers a novel theoretical justification for the negative premium for skewness, and matches it both qualitatively and quantitatively.

\textsuperscript{19} Any asset could be taken as the reference, the only reason to take the market portfolio is to test if my model can match the empirical negative skewness premium of Harvey and Siddique (2010)

\textsuperscript{20} This is a simplification. An analysis with assets with varying loadings on all of the shocks would yield similar results.
I use the quarterly parameters from Hansen, Heaton and Li (2008) and the stockholders’ consumption process of Eq. (17) with $\gamma = 10$, $\beta = 0.999$.

The graph displays the annualized average returns in excess of the market for assets with identical volatility but varying skewness.

Figure 10: Negative Premium for Skewness
Conclusion

In this paper, I incorporate loss aversion features in a recursive model of preferences and find tractable solutions to the consumption-based asset pricing model with homogenous agents. The model with loss aversion generates risk prices that vary with the risk exposure (cross-sectional effect) and that are generally higher than in the standard recursive utility model (level effect). The level effect that my model with loss aversion generates allows to match or improve on calibration exercises that use asset returns moments. Accordingly, I find that my model can explain the risk free rate and equity premium for a risk aversion of $\gamma = 10$, and the value premium for a risk aversion of $\gamma = 3$. More striking, my model’s testable implications for the cross section allow to truly differentiate it from the standard recursive utility model. I find that empirical evidence regarding the security market line relative to the CAPM and regarding the negative premium for skewness provide strong support for my model with loss aversion.
References


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Appendix

A Modeling Loss Aversion

In this Appendix, I discuss the assumptions of my model for loss aversion, and relate them to the existing literature. In a first stage, I show how, in my model, loss aversion modifies the effective value of a realized consumption outcome \( C \). In a second stage, I model how the agent values an uncertain outcome \( C \), which effective value is modified by the loss aversion specification.

In Kahneman-Tversky prospect theory model, the agent values realized outcomes as gains or losses relative to a reference point. I modify this model by defining preferences in which the agent gains utility from realized consumption via two channels: utility from consumption outcomes \( u(C) \) (absolute value), and gain-loss utility that depends on the difference between \( u(C) \) and a reference point \( ref \) (relative value). In effect, these preferences combine a traditional von
Neumann-Morgenstern utility function and prospect theory. I also restrict the impact of the gain-loss utility to the losses only, in the spirit of disappointment aversion. When the outcome \( u(C) \) is above the reference \( \text{ref} \), the utility is given by the traditional von Neumann-Morgenstern utility function. Only when the outcome is disappointing, and \( u(C) \) is below the reference \( \text{ref} \), does the gain-loss utility impact the overall utility. For the reasons discussed above, I model the reference point as the expectation of the consumption outcome and set \( \text{ref} = \mathbb{E}(u(C)) \). The expectation is defined loosely here as either the most current expectation or a combination of current and past expectations.

I take the simplest form of that model. The utility of a realized consumption outcome \( C \) is given by:

\[
U(C) = u(C) + b_0 \min(0, u(C) - \mathbb{E}(u(C)))
\]

where \( b_0 \) represents the degree of loss aversion.

This model for loss aversion on a realized outcome \( C \) is similar in spirit to Koszegi and Rabin (2006). In Koszegi and Rabin (2006), the utility also combines a traditional utility from consumption outcomes and a gain-loss utility. The utility of consumption given a reference point \( r \) is given by:

\[
U(c | r) = u(c) + \mu(u(c) - r)
\]

where \( r \) is the reference point, \( u(c) \) is the direct utility from outcome consumption \( c \) and \( \mu \) is piecewise linear with \( \mu(x) = \eta x \) if \( x \geq 0 \) and \( \mu(x) = \lambda \eta x \) if \( x \leq 0 \), and \( \eta \geq 0, \lambda > 1 \). \( \lambda \) represents the degree of loss aversion (how much of a kink there is) and \( \eta \) represents the degree to which loss aversion matters to the agent relative to the standard model of preferences. As discussed above the reference point \( r \) is a stochastic expectation of the stochastic realization of \( u(C) \). Dividing \( U(c | r) \) by \( (1 + \eta) \) limits the effect of gain-loss utility to only the losses and yields a utility function similar to my model.

I define \( \tilde{C} \), where \( U(C) = u(\tilde{C}) \), as the effective value of realized consumption \( C \). It represents the level of consumption that would yield the same utility to the agent in the traditional model without loss aversion.

I focus on the log-linear case of \( u(C) = \log C \). Then \( \text{ref} = \mathbb{E}(\log C) \) and \( U(C) = \log (\tilde{C}) \).

My model defines the effective value of realized consumption \( C \) as \( \tilde{C} \) where

\[
\log \tilde{C} = \log C + b_0 \min(0, \log C - \mathbb{E}(\log C))
\]

Notice that \( \tilde{C} \) is homogeneous of degree one in \( C \).
The effective value $\tilde{C}$ is displayed in figure 11.

If the outcome is a gain relative to the reference point, the effective value from consumption $C$ is unchanged and $\tilde{C} = C$. However, if the outcome is a loss, the effective value is lowered by the loss aversion specification and $\tilde{C} \leq C$. The effective value from consumption $C$ displays a kink around the reference point and the ratio of the slopes above and below the reference point is determined by $1/(1 + b_0)$. I define the parameter governing loss aversion as $\alpha$ where $\alpha \in [0, 1)$ is such that the ratio of the slopes is equal to $1 - \alpha$. $b_0 \geq 0$ is equal to $\alpha/(1 - \alpha)$. In the limit case $\alpha = 0$, the agent displays no loss aversion and $\tilde{C} = C$ for all outcomes of consumption $C$.

$\tilde{C}$ is the effective value of a realized consumption outcome $C$. Let’s now model how the agent values an uncertain consumption outcome. In the standard CRRA model, the agent’s value at period $t$ of an uncertain outcome of consumption $C_{t+1}$ at period $t + 1$ is given by $h(C_{t+1}) = \left(\mathbb{E}_t \left(C_{t+1}^{1-\gamma}\right)\right)^{\frac{1}{1-\gamma}}$ where $\gamma > 1$ is the coefficient of risk aversion. In my model with loss aversion the effective value of a realized outcome $C_{t+1}$ is $\tilde{C}_{t+1}$ as in Eq. (19). I modify $h$ so that $h(C_{t+1}) \propto \left(\mathbb{E}_t \left(\tilde{C}_{t+1}^{1-\gamma}\right)\right)^{\frac{1}{1-\gamma}} = R_t \left(\tilde{C}_{t+1}\right)$.

The reference point is an expectation of the consumption outcome, and it is thus endogenously determined by the agent’s optimal consumption choice. Because the agent is loss averse for outcomes below the reference point, it could be in her best interest at period $t$ to choose a
consumption path that results in a low reference point rather than a high reference point at period \( t + 1 \), thus decreasing the probability of disappointment. In such a case, the agent would sometimes reject first order dominating outcomes. There is some empirical evidence regarding such behavior. Frederick and Loewenstein (1999) consider cases in which a prisoner is better off not trying for parole in order to avoid being disappointed. Gneezy, List, and Wu (2006) observe cases in which an agent chooses a worst outcome for certain rather than a lottery outcome.\(^{21}\) However, in the context of asset pricing, first order stochastic dominance should be preserved to avoid direct violations of the no-arbitrage condition, and I impose that the valuation function \( h \) be increasing. In that regard, I follow Kahneman and Tversky (1979) in which direct violation of dominance is prevented in the first stage of editing.

One way to impose that \( h \) be increasing is to let part of the valuation at time \( t \) come directly from the reference point’s value.

I choose a functional \( h \) of the form:

\[
\begin{align*}
    h (C_{t+1}) &= \left[ \text{Ref}^{b_1} \mathcal{R}_t \left( \tilde{C}_{t+1} \right) \right]^{\frac{1}{1+b_1}} \\
    \mathcal{R}_t \left( \tilde{C}_{t+1} \right) &= \left( \mathbb{E}_t \left( \tilde{C}_{t+1}^{1-\gamma} \right) \right)^{\frac{1}{1-\gamma}}
\end{align*}
\]

with \( b_1 \geq b_0 \geq 0 \) and \( \text{Ref} \) is the reference point for \( C_{t+1} \). Notice that \( h \) is homogeneous of degree one. Further, \( b_1 \geq b_0 \) is a sufficient condition for first order stochastic dominance.\(^{22}\) In my model, I consider the limit case \( b_0 = b_1 \), for which loss aversion is entirely determined by one parameter \( \alpha \), with \( \alpha = \frac{b_0}{1+b_0} = \frac{b_1}{1+b_1} \).

For \( b_0 = b_1 \), Eq. (19) and Eq. (20) become

\[
\begin{align*}
    h (C_{t+1}) &= \mathbb{E}_t \left( \tilde{C}_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \\
    \log \tilde{C}_{t+1} &= \log C_{t+1} - \alpha \max (0, \log C_{t+1} - \mathbb{E} (\log C_{t+1})) \\
    \alpha &\in [0, 1), \ \tilde{\gamma} = \gamma + \frac{\alpha}{\alpha - 1} (\gamma - 1) \geq \gamma > 1
\end{align*}
\]

This is the model I present in section 1.2.

\(^{21}\)See also Akerlov and Dickens (1982) and Matthey (2010)

\(^{22}\)Proof is provided in the Appendix B.
B Monotonicity and Concavity of the Value Function

The model of preferences is given by

\[ V_t = \left( (1 - \beta) C_t^{1-\rho} + \beta (h(V_{t+1}))^{1-\rho} \right)^{\frac{1}{1-\rho}} \]

\[ h(V_{t+1}) = \left[ \left( \exp \left( \tilde{E}_{\xi,t} (\log V_{t+1}) \right) \right) b_1 E_t \left( \frac{\tilde{V}_{t+1}^{1-\gamma}}{1-\gamma} \right) \right]^{\frac{1}{1+b_1}} \]

\[ \log \tilde{V}_{t+1} = \log V_{t+1} + b_0 \min \left( 0, \log V_{t+1} - \tilde{E}_{\xi,t} (\log V_{t+1}) \right) \]

\[ \tilde{E}_{\xi,t} (\log V_{t+1}) = (1 - \xi) \sum_{n=0}^{+\infty} \xi^n E_{t-n} (\log V_{t+1}) \]

\[ 0 < \beta < 1, \ b_0, b_1 \geq 0, \ \rho > 0, \ \gamma \geq 1, \ \xi \geq 0 \]

Let’s show that \( \{V_t\} \) is time consistent, that is that \( h \) is increasing.

Since \( 1 + b_1 > 0 \), \( h \) is increasing if \( h^{1+b_1} \) is increasing.

Let’s rewrite:

\[ h(V_{t+1})^{1+b_1} = E_t \left( \frac{\hat{V}_{t+1}^{1-\gamma}}{1-\gamma} \right) = g(V_{t+1}) \]

with

\[ \hat{V}_{t+1} = v_{t+1} + b_0 \min \left( 0, v_{t+1} - \tilde{E}_{\xi,t} (v_{t+1}) \right) + b_1 \tilde{E}_{\xi,t} (v_{t+1}) \]

\[ g(V_{t+1} + dx) - g(V_{t+1}) = g(V_{t+1})^\gamma E_t \left( \frac{\hat{V}_{t+1}^{1-\gamma}}{1-\gamma} \left( \hat{V}_{t+1} (V_{t+1} + dx) - V_{t+1} (V_{t+1}) \right) \right) \]

\[ V_{t+1}^\gamma (V_{t+1} + dx) - V_{t+1}^\gamma (V_{t+1}) = \hat{V}_{t+1} \frac{dx}{V_{t+1}} \left( 1 + b_0 1_{v_{t+1} \leq \tilde{E}_{\xi,t}(v_{t+1})} \right) \]

\[ + \hat{V}_{t+1} \left( b_1 - b_0 1_{v_{t+1} \leq \tilde{E}_{\xi,t}(v_{t+1})} \right) (1 - \xi) E_t \left( \frac{1}{V_{t+1}} dx \right) \]

It is sufficient to have \( b_1 \geq b_0 \geq 0 \) to ensure that the value function at time \( t \) be strictly increasing in the value function at time \( t + 1 \) and thus for the model of preferences to be time consistent.

I set \( b_0 = b_1 \). Then the preferences can be rewritten as in Eq. (4).
Let’s show that \( V_t \) is concave in \((C_t, V_{t+1})\):

\[
f(x, y) = \left[(1 - \beta) x^{1-\rho} + \beta g(y)^{1-\rho}\right]^\frac{1}{1-\rho}
\]

is concave if \( g \) is concave.

I just need to prove that \( g(Y) = \mathbb{E} \left[Y^{1-\gamma}\right] \) with \( Y = Y \exp \left[-\alpha \max \left(0, y - \tilde{\mathbb{E}}_{t,y} \right)\right] \) is concave.

I can show with Cauchy-Schwarz inequality that \( g(Y) = \mathbb{E} \left[k \left(Y^{1-\gamma}\right)\right] \) is concave if \( k \) is concave.

I just need to prove that \( k(Y) = Y \exp \left[-\alpha \max \left(0, y - \tilde{\mathbb{E}}_{t,y} \right)\right] \) is concave. This is fairly straightforward.

\section{C Euler Equation}

for all returns \( R_{t+1} \), and \( \delta << 1 \), the first order condition of the optimum consumption path is:

\[
\frac{V_t(C_{t+1} + \delta R_{t+1}) - V_t(C_{t+1})}{V_t(C_t + \delta) - V_t(C_t)} = 1
\]

\[
V_t(C_t + \delta) - V_t(C_t) = \delta (1 - \beta) C_t^{-\rho} V_t^\rho
\]

\[
\frac{V_t(C_{t+1} + \delta R_{t+1}) - V_t(C_{t+1})}{V_t(C_t + \delta) - V_t(C_t)} = \frac{V_{t+1} \delta R_{t+1} dV_{t+1}/dC_{t+1}}{V_{t+1} \delta R_{t+1} dV_{t+1}/dC_{t+1}} - \frac{V_{t+1} \alpha 1_{V_{t+1} \geq \tilde{\mathbb{E}}_{t,v_{t+1}} \times \left(1 - \xi \right) \mathbb{E}_t \left(\frac{1}{V_{t+1} \delta R_{t+1} dV_{t+1}/dC_{t+1}} \right)\right)}
\]

\[
\frac{V_t(C_{t+1} + \delta R_{t+1}) - V_t(C_{t+1})}{V_t(C_t + \delta) - V_t(C_t)} = \beta R_t \left(\frac{V_{t+1}}{V_t}\right)^{\gamma-\rho} \mathbb{E}_t \left(\frac{V_{t+1}^{1-\gamma}}{V_t^{1-\gamma}} \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} V_{t+1}^{-\rho-1} \left(1 - \alpha 1_{V_{t+1} \geq \tilde{\mathbb{E}}_{t,v_{t+1}} \times \left(1 - \xi \right) \mathbb{E}_t \left(\frac{1}{V_{t+1} \delta R_{t+1} dV_{t+1}/dC_{t+1}} \right)\right) R_{t+1}\right) + \beta (1 - \xi) R_t \left(\frac{V_{t+1}}{V_t}\right)^{\gamma-\rho} \mathbb{E}_t \left(\alpha 1_{V_{t+1} \geq \tilde{\mathbb{E}}_{t,v_{t+1}} \times \left(1 - \xi \right) \mathbb{E}_t \left(\frac{1}{V_{t+1} \delta R_{t+1} dV_{t+1}/dC_{t+1}} \right)\right) R_{t+1}\right)
\]

The Stochastic Discount Factor is therefore given by:

\[51\]
\[ S_{t,t+1} = \beta \left( \frac{V_{t+1}}{R_t (V_{t+1})} \right)^{\rho - \gamma} \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{V_{t+1}}{V_{t+1}} \right)^{1-\gamma} \times \left( 1 - \alpha \mathbf{1}_{v_{t+1} \geq \bar{\xi}, t v_{t+1}} \right) + \alpha (1 - \xi) \frac{\mathbb{E}_t \left( \mathbf{1}_{v_{t+1} \geq \bar{\xi}, t v_{t+1}} \frac{V_{t+1}^{1-\gamma}}{V_{t+1}} \right)}{V_{t+1}^{1-\gamma}} \]

The Euler Equation is also given by:

\[
\mathbb{E}_t \left\{ \beta R_{t+1} \left( \frac{V_{t+1}}{R_t (V_{t+1})} \right)^{\rho - \gamma} \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{V_{t+1}}{V_{t+1}} \right)^{1-\gamma} \times \left( 1 - \alpha \mathbf{1}_{v_{t+1} \geq \bar{\xi}, t v_{t+1}} \right) \left( 1 - (1-\xi) \mathbb{E}_t \left( R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-1} \left( \frac{V_{t+1}}{V_{t+1}} \right)^{\rho - 1} \right) \right) \right\} = 1
\]

It is sometimes convenient to use this expression rather than the SDF.

For cases \( v_{t+1} \geq \bar{\xi}, t (v_{t+1}) \), this can be rewritten as:

\[
\beta \left( \frac{\exp \bar{\xi}_{\xi,t} (v_{t+1})}{R_t (V_{t+1})} \right)^{\gamma - \eta} \times \left( 1 - \alpha R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{V_{t+1}}{R_t (V_{t+1})} \right)^{\rho - \gamma} \right) + \alpha \mathbb{E}_t \left( R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{V_{t+1}}{R_t (V_{t+1})} \right)^{\rho - 1} \right) \left( \frac{V_{t+1}}{R_t (V_{t+1})} \right)^{1-\gamma} \]

The Jensen inequality gives us \( \exp \bar{\xi}_{\xi,t} (v_{t+1}) \geq \bar{\xi}_{\xi,t} (V_{t+1}) \geq \bar{\xi}_{\xi,t} (V_{t+1}) \) and \( \mathbb{E}_t \left( V_{t+1}^{1-\gamma} \right) \leq \mathbb{E}_t (V_{t+1}) \). Therefore, when \( \xi = 0 \), \( \left( \frac{\exp \bar{\xi}_{\xi,t} (v_{t+1})}{R_t (V_{t+1})} \right)^{\gamma - \eta} \leq 1 \).

**D Taylor Expansion around \( \rho = 1 \)**

I follow Hansen-Heaton-Lee-Roussanov (2007) and obtain:
\[ v_t = v_t^1 + (\rho - 1)DV_t^1 \]
\[ v_t^1 = (1 - \beta) c_t + \beta Q_t \left( \frac{v_{t+1}}{\alpha} \right) \]
\[ Q_t \left( \frac{\bar{v}_{t+1}}{\alpha} \right) = \frac{1}{1 - \gamma} \log E_t \exp (1 - \gamma) \left( v_{t+1} - \alpha \max \left( 0, v_{t+1} - \bar{E}_{\xi,t}v_{t+1} \right) \right) \]
\[ DV_t^1 = -\frac{1 - \beta}{2\beta} \left( v_t^1 - c_t \right)^2 + \beta \bar{E}_t \left( \frac{DV_{t+1}^1}{\alpha} \right) \]
\[ \bar{DV}_{t+1} = DV_{t+1}^1 - \alpha 1v_{t+1}^1 \bar{E}_{\xi,t}v_{t+1}^1 \left( DV_{t+1}^1 - \bar{E}_{\xi,t}DV_{t+1}^1 \right) \]

where \( \bar{E}_t \) is the distorted expectation operators associated with the Radon-Nicodym derivative

\[ \bar{M}_{t,t+1} = \frac{\exp (1 - \gamma) v_{t+1}^1}{E_t \exp (1 - \gamma) v_{t+1}^1} \]

Approximating the SDF around \( \rho = 1 \), I have

\[ s_{t,t+1} = s_{t,t+1}^1 + (\rho - 1)DS_{t,t+1}^1 \]
\[ s_{t,t+1}^1 = \log \beta + (1 - \gamma) \left( v_{t+1}^1 - Q_t \left( \frac{v_{t+1}}{\alpha} \right) \right) - (c_{t+1} - c_t) \]
\[ - \left( 1 - \gamma \right) \alpha \max \left( 0, v_{t+1}^1 - \bar{E}_{\xi,t}v_{t+1}^1 \right) \]
\[ + \log \left( 1 - \alpha 1v_{t+1}^1 \bar{E}_{\xi,t}v_{t+1}^1 \right) + \alpha (1 - \xi) \frac{E_t \left( 1v_{t+1}^1 \bar{E}_{\xi,t}v_{t+1}^1 \frac{V_{t+1}^1}{\gamma - 1} \right)}{V_{t+1}^1} \]
\[ DS_{t,t+1}^1 = v_{t+1}^1 - Q_t \left( \frac{v_{t+1}}{\alpha} \right) - (c_{t+1} - c_t) + (1 - \gamma) \left( DV_{t+1}^1 - \bar{E}_t \left( \frac{DV_{t+1}^1}{\alpha} \right) \right) \]
\[ - (1 - \gamma) \alpha 1v_{t+1}^1 \bar{E}_{\xi,t}v_{t+1}^1 \left( DV_{t+1}^1 - \bar{E}_{\xi,t}DV_{t+1}^1 \right) \]
\[ - \alpha (1 - \gamma) \frac{DV_{t+1}^1 \bar{E}_t \left( 1v_{t+1}^1 \bar{E}_{\xi,t}v_{t+1}^1 \frac{V_{t+1}^1}{\gamma - 1} \right) - E_t \left( 1v_{t+1}^1 \bar{E}_{\xi,t}v_{t+1}^1 \frac{DV_{t+1}^1 V_{t+1}^1}{\gamma - 1} \right)}{V_{t+1}^1} + \alpha (1 - \xi) \frac{E_t \left( 1v_{t+1}^1 \bar{E}_{\xi,t}v_{t+1}^1 \frac{V_{t+1}^1}{\gamma - 1} \right)}{V_{t+1}^1} \]

Approximating the Euler Equation around \( \rho = 1 \), I have
\[
\begin{align*}
e e_{t+1} &= e e_{t+1} + (\rho - 1) \, D e e_{t+1} \\
\end{align*}
\]

\[
\begin{align*}
ee_{t+1} &= r_{t+1} + \log \beta + (1 - \bar{\gamma}) \left( v_{t+1} - Q_t \left( v_{t+1} \right) \right) - (c_{t+1} - c_t) \\
&\quad - (1 - \bar{\gamma}) \max \left( 0, v_{t+1} - \bar{E}_{t+1} v_{t+1} \right) \\
&\quad + 1_{v_{t+1} \geq \bar{E}_{t+1} v_{t+1}} \log \left( (1 - \alpha) + \alpha (1 - \xi) \frac{E_t \left( R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-1} \right)}{R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-1}} \right)
\end{align*}
\]

\[
\begin{align*}
D e e_{t+1} &= v_{t+1} - Q_t \left( v_{t+1} \right) - (c_{t+1} - c_t) + (1 - \bar{\gamma}) \left( D v_{t+1} - \bar{E}_{t+1} D v_{t+1} \right) \\
&\quad - (1 - \bar{\gamma}) \alpha 1_{v_{t+1} \geq \bar{E}_{t+1} v_{t+1}} \left( D v_{t+1} - \bar{E}_{t+1} D v_{t+1} \right) \\
&\quad - \alpha (1 - \xi) 1_{v_{t+1} \geq \bar{E}_{t+1} v_{t+1}} \frac{E_t \left( R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-1} \right) - E_t \left( v_{t+1} - c_{t+1} \right) R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-1}}{E_t \left( \frac{C_{t+1}}{C_t} \right)^{-1}} \left( 1 - \alpha \right) R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-1} + \alpha (1 - \xi) E_t \left( R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-1} \right)
\end{align*}
\]

### E  Constant Volatility

In all the calculations that follow, I assume \( \xi = 0 \).

The consumption process is given by:

\[
c_{t+1} - c_t = \mu + G X_t + H W_{t+1}
\]

\[
X_{t+1} = A X_t + B W_{t+1}
\]

\( W_t \) iid \( \mathcal{N}(0, I) \).

**Case \( \rho = 1 \)**

I assume \( v_t - c_t = p + Q X_t \)

we want to solve

\[
p + Q X_t = \frac{\beta}{1 - \bar{\gamma}} \log E_t \exp \left[ \left( 1 - \bar{\gamma} \right) \left( p + Q X_{t+1} - \alpha 1_{(H + QB) W_{t+1} \geq 0} (H + QB) W_{t+1} + \mu + G X_t + H W_{t+1} \right) \right]
\]

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Grouping the terms in $X_t$, we find that:

$$Q = \beta G \ (I - \beta A)^{-1}$$

The relevant shock for loss aversion is $LW_t$ with

$$L = H + \beta G \ (I - \beta A)^{-1} B$$

and

$$\frac{1 - \beta}{\beta} p = \mu + \frac{1}{2} \ (1 - \gamma) \ |H + QB|^2 +$$

$$\frac{1}{1 - \gamma} \log \left[ \Phi ((1 - \gamma) \ |H + QB|) + \Phi ((1 - \gamma) \ |H + QB|) \exp \left( \frac{(1 - \gamma)^2} {2} \right) \right]$$

**Approximation around $\rho = 1$**

**Distorted operator**  let’s solve:

$$M_{t,t+1} = \frac{\exp (1 - \gamma) \ v^1_{t+1}}{\mathbb{E}_t \exp (1 - \gamma) \ v^1_{t+1}}$$

$$= \frac{\exp (1 - \gamma) \ ((H + QB) W_{t+1} - \alpha 1 \ (H + QB) W_{t+1} \geq 0 \ (H + QB) W_{t+1})} {\mathbb{E}_t \exp (1 - \gamma) \ ((H + QB) W_{t+1} - \alpha 1 \ (H + QB) W_{t+1} \geq 0 \ (H + QB) W_{t+1})}$$

Let’s define $w_{t+1} = \frac{(H+QB)}{|H+QB|} W_{t+1} = \frac{E}{|E|} W_{t+1}$

for any $y_{t+1}$ independent from $w_{t+1}$

$$\mathbb{E}_t (y_{t+1}) = \mathbb{E}_t (y_{t+1})$$

$$\mathbb{E}_t (1_{w_{t+1} \geq 0}) = \frac{\Phi ((1 - \gamma) \ |H + QB|)}{\Phi (- (1 - \gamma) \ |H + QB|) \exp \left( \frac{(1 - \gamma)^2 (2 - \alpha) |H + QB|^2} {2} \right) + \Phi ((1 - \gamma) \ |H + QB|)}$$

$$\mathbb{E}_t (w_{t+1}) = (1 - \gamma) \ |H + QB| \times$$

$$\left( 1 - \alpha \frac{\Phi ((1 - \gamma) \ |H + QB|)}{\Phi (- (1 - \gamma) \ |H + QB|) \exp \left( \frac{(1 - \gamma)^2 (2 - \alpha) |H + QB|^2} {2} \right) + \Phi ((1 - \gamma) \ |H + QB|)} \right)$$
\[
\mathbb{E}_t \left( w_{t+1}^2 \right) = 1 + (1 - \gamma)^2 |H + QB|^2 - (1 - \gamma) \alpha |H + QB| \times \\
\left( (1 - \gamma) (2 - \alpha) |H + QB| \Phi ((1 - \gamma) |H + QB|) + \phi ((1 - \gamma) |H + QB|) \right) \\
\Phi (- (1 - \gamma) |H + QB|) \exp \left( \frac{(1 - \gamma)^2 \alpha (2 - \alpha) |H + QB|^2}{2} \right) + \Phi ((1 - \gamma) |H + QB|)
\]

\[
\overline{\mathbb{E}}_t \left( 1_{w_{t+1} \geq 0} w_{t+1}^2 \right) = \frac{(1 - \gamma) |H + QB| \Phi ((1 - \gamma) |H + QB|) + \phi ((1 - \gamma) |H + QB|)}{\Phi (- (1 - \gamma) |H + QB|) \exp \left( \frac{(1 - \gamma)^2 \alpha (2 - \alpha) |H + QB|^2}{2} \right) + \Phi ((1 - \gamma) |H + QB|)}
\]

\[
\mathbb{E}_t \left( 1_{w_{t+1} \geq 0} w_{t+1}^2 \right) = \frac{\left( 1 + \left( (1 - \gamma) |L| \right)^2 \right) \Phi ((1 - \gamma) |L|) + (1 - \gamma) |L| \phi ((1 - \gamma) |L|)}{\Phi (- (1 - \gamma) |H + QB|) \exp \left( \frac{(1 - \gamma)^2 \alpha (2 - \alpha) |H + QB|^2}{2} \right) + \Phi ((1 - \gamma) |H + QB|)}
\]

**Solution for \( Dv_t^1 \)**

\[
v_t^1 - \alpha_t = p + QX_t
\]

\[
Dv_t^1 = - \frac{(1 - \beta) (v_t^1 - \alpha_t)^2}{2 \beta} + \beta \mathbb{E}_t \left( Dv_{t+1}^1 \right)
\]

Let’s suppose

\[
Dv_t^1 = q + RX_t + X_t' SX_t
\]

with \( S \) symmetric matrix.

Then

\[
\overline{Dv}_{t+1}^1 = Dv_{t+1}^1 - \alpha_1 v_{t+1}^1 \geq \mathbb{E}_{t+1} \left( Dv_{t+1}^1 - \mathbb{E}_t \overline{Dv}_{t+1}^1 \right)
\]

\[
= q + RX_{t+1} + X_{t+1}' SX_{t+1} - \alpha_1 w_{t+1} \geq 0 \left( (R + 2X_t' A')B W_{t+1} + W_{t+1} B' SB W_{t+1} - \text{Tr} \left( B' SB \right) \right)
\]

and

\[
q + RX_t + X_t' SX_t = - \frac{(1 - \beta)}{2 \beta} \left( p^2 + (Q X_t)^2 + 2pQX_t \right)
\]

\[
+ \beta \mathbb{E}_t \left( q + RX_{t+1} + X_{t+1}' SX_{t+1} - \alpha_1 w_{t+1} \geq 0 \left( (R + 2X_t' A')B W_{t+1} + W_{t+1} B' SB W_{t+1} - \text{Tr} \left( B' SB \right) \right) \right)
\]

Let’s group the quadratic terms:

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\[ S = -\frac{(1 - \beta)}{2\beta}Q'Q + \beta A'SA \]

this is a Sylvester equation, for which we can find the solution \( S \). The solution for \( S \) is the same as in the standard recursive model:

\[ S = -\frac{(1 - \beta)}{2\beta} \sum_{t=0}^{+\infty} \beta^t A''Q'A' \]

\( S \) is negative semi-definite and can be written, using Cholesky decomposition, as \( S = -T'T \) with \( T \) triangular.

I decompose:

\[ W_{t+1} = \frac{(QB + H)'}{|QB + H|}w_{t+1} + Y_{t+1} \]

with \( Y_{t+1} \), vector of variables independent from \( w_{t+1} \)

\[ \mathbb{E}_t (Y_{t+1}) = \mathbb{E}_t (Y_{t+1}) = 0 \]

I group the terms in \( X_t \):

\[ RX_t = -\frac{(1 - \beta)}{\beta}pQX_t + \beta RAX_t + 2\beta X_tA'SB\mathbb{E}_t (W_{t+1} - \alpha 1_{w_{t+1} \geq 0} W_{t+1}) \]

\[ R = \left[ -\frac{(1 - \beta)}{\beta}pQ + 2\beta (1 - \gamma) (H + QB) B'SA \right] (I - \beta A)^{-1} \]

\[ + \left\{ \frac{2\beta \alpha (H + QB) B'SA (I - \beta A)^{-1} \times}{\frac{1 - \gamma}{1 - \alpha} - \Phi((1-\gamma)|H+QB|) \exp\left(\frac{(1-\gamma)^2\alpha(2-\alpha)|H+QB|^2}{2}\right) + \Phi((1-\gamma)|H+QB|)} \right\} \]

finally the constant terms are:

\[ q (1 - \beta) = -\frac{(1 - \beta)}{2\beta}p^2 + \beta R\mathbb{E}_t (W_{t+1} - \alpha 1_{w_{t+1} \geq 0} W_{t+1}) - \beta \mathbb{E}_t \left( |TBW_{t+1}|^2 \right) \]

\[ + \beta \alpha \mathbb{E}_t \left( 1_{w_{t+1} \geq 0} \left( |TBW_{t+1}|^2 - \text{Tr} (TB)'TB \right) \right) \]
\begin{align*}
q &= -\frac{1}{2\beta}p^2 + \frac{\beta}{1-\beta} (1-\gamma) RB (QB + H)' - \frac{\beta}{1-\beta} (1-\gamma)^2 |TB (QB + H)'|^2 + Tr (TB)'TB \\
+ \frac{\beta}{1-\beta} & \left\{ \begin{array}{l}
\frac{RB(QB+H)'}{(1-\gamma)(1-\gamma-\alpha)} \left[ (1-\gamma) (2-\alpha) \Phi ((1-\gamma) |H + QB|) + \right. \\
\left. \frac{1-\gamma}{\Phi(-(1-\gamma)|H+QB|)} exp \left( \frac{(1-\gamma)^2 (2-\alpha) |H + QB|}{2} \right) \Phi((1-\gamma)|H+QB|) \right] \\
+ \frac{1-\gamma}{\Phi(-(1-\gamma)|H+QB|)} exp \left( \frac{(1-\gamma)^2 (2-\alpha) |H + QB|}{2} \right) + \Phi((1-\gamma)|H+QB|) \right\} \\
\end{array} \right.
\end{align*}

Results in the case of constant volatility

SDF

\begin{align*}
s_{t,t+1} &= s_{t,t+1}^1 + (\rho - 1) Ds_{t,t+1}^1 \\
s_{t,t+1}^1 &= \log \beta + (1-\gamma) \left( -\frac{1-\beta}{\beta} p - GX_t + QBW_{t+1} \right) - \gamma (\mu + GX_t + HW_{t+1}) \\
&\quad - (1-\gamma) \alpha 1_{(QB+H)W_{t+1} \geq 0} (QB + H) W_{t+1} \\
&\quad + \log \left\{ \begin{array}{l}
(1-\alpha 1_{(QB+H)W_{t+1} \geq 0}) + \\
\alpha \exp \left( (1-\gamma) (1-\alpha 1_{(QB+H)W_{t+1} \geq 0}) (QB + H) W_{t+1} \right) \times \\
\Phi ((1-\gamma) |QB + H|) \exp \left( \frac{1}{2} (1-\gamma) |QB + H|^2 \right) \\
\end{array} \right. \\
Ds_{t,t+1}^1 &= \left( -\frac{1-\beta}{\beta} p - GX_t + QBW_{t+1} \right) \\
&\quad + (1-\gamma) \left( -\frac{1-\beta}{\beta} q + \frac{p^2}{2\beta} - \frac{1-\beta}{\beta} R (I - \beta A) + \frac{1-\beta}{\beta} pQ \right) X_t \\
&\quad + (1-\gamma) \left( (R + 2X_t'A'S) BW_{t+1} + W_{t+1}'B'SBW_{t+1} \right) \\
&\quad - (1-\gamma) \alpha 1_{(QB+H)W_{t+1} \geq 0} \left( (R + 2X_t'A'S) BW_{t+1} + W_{t+1}'B'SBW_{t+1} - Tr (B'SB) \right) \\
&\quad + \left\{ \begin{array}{l}
Dv_{t+1}^1 \Phi ((1-\gamma) |QB + H|) \exp \left( \frac{1}{2} (1-\gamma) |QB + H|^2 \right) - \\
E_t \left( 1_{(QB+H)W_{t+1} \geq 0} Dv_{t+1}^1 \exp \left( (1-\gamma) (QB + H) W_{t+1} \right) \right) \\
\end{array} \right. \\
&\quad - \alpha (1-\gamma) \left\{ \begin{array}{l}
(1-\alpha 1_{(QB+H)W_{t+1} \geq 0}) \exp \left( (1-\gamma) (1-\alpha 1_{(QB+H)W_{t+1} \geq 0}) (QB + H) W_{t+1} \right) + \\
\alpha \Phi ((1-\gamma) |QB + H|) \exp \left( \frac{1}{2} (1-\gamma)^2 |QB + H|^2 \right) \\
\end{array} \right. \\
\end{align*}

where
\[
\hat{D}_{t,t+1} = \left( (R + 2X_t' A'S) BW_{t+1} + W_{t+1} B'SBW_{t+1} \right) (1 - \alpha \mathbf{1}_{(QB+H)W_{t+1} \geq 0}) \\
+ \alpha \mathbf{1}_{(QB+H)W_{t+1} \geq 0} \text{Tr} (B'SB)
\]

Euler Equation:

\[
e_{t,t+1} = e_{t,t+1}^1 + (\rho - 1) \hat{D}_{t,t+1}
\]

\[
e_{t,t+1}^1 = r_{t+1} + \log \beta + (1 - \bar{\gamma}) \left( -\frac{1 - \beta}{\beta} p - GX_t + QBW_t \right) - \bar{\gamma} (\mu + GX_t + HW_t) \\
- (1 - \bar{\gamma}) \mathbf{1}_{(QB+H)W_{t+1} \geq 0} (QB + H) W_{t+1} \\
+ \mathbf{1}_{(QB+H)W_{t+1} \geq 0} \log \left( (1 - \alpha) + \alpha \frac{\mathbb{E}_t (R_{t+1} \exp(-HW_{t+1}))}{R_{t+1} \exp(-HW_{t+1})} \right)
\]

\[
\hat{D}_{t,t+1} = \left( -\frac{1 - \beta}{\beta} p - GX_t + QBW_t \right) \\
+ (1 - \bar{\gamma}) \left( -\frac{1 - \beta}{\beta} \left( q + \frac{p^2}{2\beta} \right) - \frac{1}{\beta} \left( R (I - \beta A) + \frac{1 - \beta}{\beta} pQ \right) X_t \right) \\
+ (1 - \bar{\gamma}) \left( (R + 2X_t' A'S) BW_{t+1} + W_{t+1} B'SBW_{t+1} \right) \\
- (1 - \bar{\gamma}) \mathbf{1}_{(QB+H)W_{t+1} \geq 0} \left( (R + 2X_t' A'S) BW_{t+1} + W_{t+1} B'SBW_{t+1} - \text{Tr} (B'SB) \right) \\
- \mathbf{1}_{(QB+H)W_{t+1} \geq 0} \frac{QBW_{t+1} \mathbb{E}_t (R_{t+1} \exp(-HW_{t+1})) - \mathbb{E}_t (QBW_{t+1}R_{t+1} \exp(-HW_{t+1}))}{(1 - \alpha) R_{t+1} \exp(-HW_{t+1}) + \alpha \mathbb{E}_t (R_{t+1} \exp(-HW_{t+1}))}
\]

Risk Free Rate when \( \rho = 1 \): The risk free rate is given by

\[
r_{f1}^1 = - \log \mathbb{E}_t (\exp e_{t,t+1}^1 (1))
\]

\[
e_{t,t+1}^1 (1) = \log \beta + (1 - \bar{\gamma}) \left( -\frac{1 - \beta}{\beta} p - GX_t + QBW_t \right) - \bar{\gamma} (\mu + GX_t + HW_t) \\
- (1 - \bar{\gamma}) \mathbf{1}_{(QB+H)W_{t+1} \geq 0} (QB + H) W_{t+1} \\
+ \mathbf{1}_{(QB+H)W_{t+1} \geq 0} \log \left( (1 - \alpha) + \alpha \exp \left( HW_{t+1} + \frac{1}{2} |H|^2 \right) \right)
\]

\[
r_{f1}^1 = - \log \beta + \mu + GX_t + \log \mathbb{E}_t \exp \left[ (1 - \bar{\gamma}) \left( (QB + H) W_{t+1} - \alpha \mathbf{1}_{(H+QB)W_{t+1} \geq 0} (H + QB) W_{t+1} \right) \right] \\
- \log \mathbb{E}_t \exp \left[ \left( (1 - \bar{\gamma}) \left( QB + \frac{\mu}{\bar{\gamma}} H \right) W_{t+1} - \mathbf{1}_{(QB+H)W_{t+1} \geq 0} \right) \left( 1 - \bar{\gamma} \right) \alpha (QB + H) W_{t+1} - \log \left( (1 - \alpha) + \alpha \exp \left( HW_{t+1} + \frac{1}{2} |H|^2 \right) \right) \right]
\]

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\[ rf_t^1 = -\log \beta + \mu + GX_t - \frac{1}{2} |H|^2 + (1 - \gamma) (H + QB) H' \]
\[ + \log \left[ \Phi \left( -(1-\gamma) |L| \right) \exp \left( \frac{(1-\gamma)^2 \alpha (2-\alpha) |L|^2}{2} \right) + \Phi \left( (1-\gamma) |L| \right) \right] \]
\[ - \log \left[ \Phi \left( -(1-\gamma) |L| + \frac{HL'}{|L|} \right) \exp \left( \frac{(1-\gamma)^2 \alpha (2-\alpha) |L|^2}{2} \right) \exp (-\alpha (1-\gamma) LH') + \right] \]
\[ (1-\alpha) \Phi \left( (1-\gamma) |L| - \frac{HL'}{|L|} \right) + \alpha \Phi \left( (1-\gamma) |L| \right) \exp ((1-\gamma) LH') \]

as a second order approximation around \( L = QB + H = 0 \), and \( H = 0 \), I obtain:

\[ rf_t^1 \approx -\log \beta + \mu + GX_t - \frac{1}{2} |H|^2 + (1 - \gamma) (H + QB) H' \]
\[ + \alpha \left\{ -\frac{1}{\sqrt{2\pi}} \frac{H(H+QB)'}{|H+QB|} + \frac{1}{\sqrt{2\pi}} \alpha \left( 1 - \frac{1}{\pi} \right) (1 - \gamma) (H + QB) H' + \right\} \]

the dominant term due to loss aversion is \(-\frac{1}{\sqrt{2\pi}} \alpha \frac{H(H+QB)'}{|H+QB|}\). The net effect of loss aversion decreases the risk free rate: precautionary savings is the dominant effect.

**One-period Risk Prices when \( \rho = 1 \):** Take an asset with cash-flows \( \exp(\Delta W_{t+1} - \frac{1}{2} |\Delta|^2) \).

It’s log expected return is

\[ r_t(\Delta) = -\log \mathbb{E}_t \left( \exp ee_{t,t+1}^1 \left( \exp \left( \Delta W_{t+1} - \frac{1}{2} |\Delta|^2 \right) \right) \right) \]

The risk prices are given by

\[ -\frac{\partial}{\partial \Delta} \log \mathbb{E}_t \left( \exp ee_{t,t+1}^1 \left( \exp \left( \Delta W_{t+1} - \frac{1}{2} |\Delta|^2 \right) \right) \right) \]

\[ ee_{t,t+1}^1 \left( \exp \left( \Delta W_{t+1} - \frac{1}{2} |\Delta|^2 \right) \right) = \Delta W_{t+1} - \frac{1}{2} |\Delta|^2 + \log \beta - \mu - GX_t \]
\[ -\log \mathbb{E}_t \exp \left[ (1-\gamma) \left( LW_{t+1} - \alpha 1_{LW_{t+1} \geq 0} LW_{t+1} \right) \right] + ((1-\gamma) (1 - \alpha 1_{LW_{t+1} \geq 0}) LW_{t+1} - H) W_{t+1} \]
\[ + 1_{LW_{t+1} \geq 0} \log \left( (1-\alpha) + \alpha \exp \left( (H - \Delta) W_{t+1} + \frac{1}{2} |H - \Delta|^2 \right) \right) \]

The solution for the excess returns is:
\[ r_t^1(\Delta) - rf_t^1 = (H + (\gamma - 1) L) \Delta' \]

\[ -\log \begin{cases} \Phi \left( - (1 - \gamma) |L| + \frac{(H-\Delta)L'}{|L|} \right) \exp \left( \frac{(1-\gamma)^2 \alpha(2-\alpha)|L|^2}{2} \right) \exp ( - \alpha (1 - \gamma) L (H - \Delta') ) + \\ \left(1 - \alpha\right) \Phi \left( (1 - \gamma) |L| - \frac{(H-\Delta)L'}{|L|} \right) + \alpha \Phi ((1 - \gamma) |L|) \exp ((1 - \gamma) L (H - \Delta')) \end{cases} \]

\[ + \log \begin{cases} \Phi \left( - (1 - \gamma) |L| + \frac{HL'}{|L|} \right) \exp \left( \frac{(1-\gamma)^2 \alpha(2-\alpha)|L|^2}{2} \right) \exp ( - \alpha (1 - \gamma) LH' ) + \\ \left(1 - \alpha\right) \Phi \left( (1 - \gamma) |L| - \frac{HL'}{|L|} \right) + \alpha \Phi ((1 - \gamma) |L|) \exp ((1 - \gamma) LH') \end{cases} \]

As a second order approximation around \( H = 0, L = 0 \) and \( \Delta = 0 \), we have:

\[ r_t^1(\Delta) - rf_t^1 \approx (H + (\gamma - 1) L) \Delta' \]

\[ + \alpha \begin{cases} \frac{1}{\sqrt{2\pi |L|}} \left( 1 - \frac{\alpha}{\sqrt{2\pi}} \frac{HL'}{|L|} \right) \\ \frac{1}{2} \alpha (\bar{\gamma} - 1) \left( 1 - \frac{1}{\pi} \right) \Delta' + \frac{1}{4\pi} \alpha \left( \frac{\Delta L'}{|L|} \right)^2 \end{cases} \]

Let’s define:

\[ f(\Delta) = \exp \frac{1}{2} \left( |(1 - \gamma) L - H + \Delta|^2 - |\Delta|^2 \right) \times \]

\[ \Phi \left( - (1 - \gamma) |L| + \frac{(H-\Delta)L'}{|L|} \right) \exp \left( \frac{(1-\gamma)^2 \alpha(2-\alpha)|L|^2}{2} \right) \exp ( - \alpha (1 - \gamma) L (H - \Delta') ) + \\
\left(1 - \alpha\right) \Phi \left( (1 - \gamma) |L| - \frac{(H-\Delta)L'}{|L|} \right) + \alpha \Phi ((1 - \gamma) |L|) \exp ((1 - \gamma) L (H - \Delta')) \]

The risk prices are given by \( -\frac{f'(\Delta)}{f(\Delta)} \):

\[ -\frac{f'(\Delta)}{f(\Delta)} = H + (\gamma - 1) L + \\
\frac{L}{\sqrt{2\pi |L|}} \exp \frac{1}{2} \left( |H - \Delta|^2 - |\Delta|^2 - \frac{(H-\Delta)L'}{|L|} \right) \frac{f(\Delta)}{f(\Delta)} + \\
+ \alpha (\bar{\gamma} - 1) \left( 1 - \frac{1}{\pi} \right) \frac{f(\Delta)}{f(\Delta)} - \alpha (\bar{\gamma} - 1) \frac{f(\Delta)}{f(\Delta)} \exp \frac{1}{2} \left( |(1 - \gamma) L - H + \Delta|^2 - |\Delta|^2 \right) \]

There are three additional terms due to loss aversion. For \(|L|\) small and \(|\Delta|\) small, the dominating term is \( \alpha \frac{L}{\sqrt{2\pi |L|}} \frac{\exp \frac{1}{2} \left( |H - \Delta|^2 - |\Delta|^2 - \frac{(H-\Delta)L'}{|L|} \right)}{f(\Delta)} \). This term is first order relative to the others and thus increases the risk prices substantially.
Let’s analyze the behavior of $f$:
for $\Delta L' \to \infty$, then $f(\Delta) \sim \exp \left( -H + (\gamma - 1) L \right) \Delta' \times (\alpha \Phi \left( (1 - \gamma) |L| \right) \exp (\gamma - 1) L \Delta') \propto \exp -H \Delta'$
for $\Delta L' \to -\infty$, then $f(\Delta) \sim \exp \left( -H + (\gamma - 1) L \right) \Delta' \times \exp (\alpha (1 - \gamma) L \Delta') \propto \exp -(H + (\gamma - 1) L) \Delta'$
also notice that for $\Delta L'$ positive and “sufficiently” high, we have
\[
f(\Delta) \approx \exp \left\{ \frac{1}{2} \left[ (1 - \gamma) L - H + |\Delta'|^2 - |\Delta|^2 \right] \times \left( 1 + \alpha \left[ \Phi \left( (1 - \gamma) |L| \right) \exp ((\gamma - 1) L \Delta') - \Phi \left( \frac{\Delta'}{|L|} \right) \right] \right\}
\]
and $g(\Delta) = \Phi \left( (1 - \gamma) |L| \right) \exp ((\gamma - 1) L \Delta') - \Phi \left( \frac{\Delta'}{|L|} \right)$
notice that $g$ does not depend on $\alpha$.
\[
g'(\Delta) = (\gamma - 1) L \Phi \left( (1 - \gamma) |L| \right) \exp ((\gamma - 1) L \Delta') - \frac{L}{|L|} \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2} \left( \frac{\Delta'}{|L|} \right)^2
\]
g' is increasing in $\Delta L'$ and has negative values for $\Delta$ small. There is a unique value of $\Delta$, independent from $\alpha$ for which $g' (\Delta^*) = 0$.

Therefore, the risk prices in the model with loss aversion are above the reference model risk prices for $\Delta < \Delta^*$ and below the reference model risk prices for $\Delta > \Delta^*$ where $\Delta^*$ is roughly the same whichever the degree of loss aversion $\alpha$.

F Stochastic volatility in consumption

In all the calculations that follow, I assume $\xi = 0$.

The consumption process is given by:
\[
c_{t+1} - c_t = \mu + GX_t + \sigma_t HW_{t+1}
\]
\[
X_{t+1} = AX_t + \sigma_t BW_{t+1}
\]
\[
\sigma_{t+1} = (1 - a) + a \sigma_t + B_{\sigma} W_{t+1}
\]
$W_t$ iid $\mathcal{N}(0, I)$, $\sigma_t$ scalar.

For simplification, I assume that the shocks to volatility are independent from the shocks to the immediate and future consumption: $B_{\sigma} B' = 0$ and $B_{\sigma} H' = 0$.
case $\rho = 1$

I assume $v_t - c_t = f(X_t, \sigma_t)$

The recursive problem is:

$$f(X_t, \sigma_t) = \frac{\beta}{1 - \gamma} \log \mathbb{E}_t \exp [(1 - \gamma) [f(X_{t+1}, \sigma_{t+1}) + c_{t+1} - c_t - \alpha \max (0, HW_{t+1} + (1 - \mathbb{E}_t) f(X_{t+1}, \sigma_{t+1}))]]$$

This recursive problem satisfies Blackwell conditions for bounded functions. While $f(X_t)$ is not necessarily bounded, it can be approximated to a bounded function (by truncating the domain of $(X_t, \sigma_t)$). In that case, there is a unique solution which can be obtained by iterating the process starting from any initial function.

Suppose that

$$v_t - c_t = p + QX_t + q_1 \sigma_t + q_2 \sigma_t^2$$

then

$$v_{t+1} - \mathbb{E}_t v_{t+1} = (\sigma_t (H + QB + 2q_2 a B_{\sigma}) + (q_1 + 2q_2 (1 - a)) B_{\sigma}) W_{t+1} + q_2 \left( (B_{\sigma} W_{t+1})^2 - B_{\sigma} B_{\sigma}' \right)$$

because $B_{\sigma}$ is small, I ignore the second order term and simplify to:

$$v_{t+1} - \mathbb{E}_t v_{t+1} \approx L(\sigma_t) W_{t+1}$$

with

$$L(\sigma_t) = \sigma_t (H + QB + 2q_2 a B_{\sigma}) + (q_1 + 2q_2 (1 - a)) B_{\sigma}$$

The recursive problem is:

$$p + QX_t + q_1 \sigma_t + q_2 \sigma_t^2 = \beta \left( p + \mu + q_1 (1 - a) + (G + QA) X_t + q_1 a \sigma_t + q_2 (1 - a + a \sigma_t)^2 \right) + \beta \frac{\beta}{1 - \gamma} \log \mathbb{E}_t \exp (1 - \gamma) \left[ L(\sigma_t) W_{t+1} + q_2 (B_{\sigma} W_{t+1})^2 - \alpha \max (0, L(\sigma_t) W_{t+1}) \right]$$

Grouping the $X_t$ terms, I find, as in the standard case:
\[ Q = \beta G (I - \beta A)^{-1} \]

Let’s define \( w_{t+1} = \frac{L(\sigma_t)}{|L(\sigma_t)|} W_{t+1} \)

Then

\[ B_\sigma W_{t+1} = \delta_1(\sigma_t) w_{t+1} + \delta_2(\sigma_t) z_{t+1} \]

with

\[ \mathbb{E}(w_{t+1}z_{t+1}) = 0 \]

and \( z_{t+1} \sim \mathcal{N}(0, 1) \)

\[
\mathbb{E}_t \exp (1 - \gamma) \left[ L(\sigma_t) W_{t+1} + q_2(B_\sigma W_{t+1})^2 - \alpha 1_{L(\sigma_t)w_{t+1} \geq 0} L(\sigma_t) W_{t+1} \right] = \\
\mathbb{E}_t \exp (1 - \gamma) \left[ |L| w_{t+1} + q_2(\delta_1^2 w_{t+1}^2 + \delta_2^2 z_{t+1}^2 + 2\delta_1 w_{t+1} \delta_2 z_{t+1}) - \alpha 1_{w_{t+1} \geq 0} |L| w_{t+1} \right]
\]

If \( 1 - 2q_2(1 - \gamma) \delta_2^2 > 0 \)

\[
\frac{1}{\sqrt{1 - 2(1 - \gamma) q_2 \delta_2^2}} \mathbb{E}_t \exp \left[ (1 - \gamma) |L| w_{t+1} (1 - \alpha 1_{w_{t+1} \geq 0}) + \frac{(1 - \gamma) q_2 \delta_1^2 w_{t+1}^2}{2 - 2(1 - \gamma) q_2 \delta_2^2} \right]
\]

This expectation is defined for \( 1 - 2q_2(1 - \gamma) (\delta_2^2 + \delta_1^2) > 0 \) (since I already assume \( 1 - 2q_2(1 - \gamma) \delta_2^2 > 0 \)). If it is verified, then I automatically have \( 1 - 2q_2(1 - \gamma) \delta_2^2 > 0 \) as well.

Observe that \( (\delta_2^2 + \delta_1^2) = B_\sigma B'_\sigma \). A solution for \( q_2 \) exists if and only if \( 1 - 2q_2(1 - \gamma) B_\sigma B'_\sigma > 0 \).

Let’s define \( \Sigma = \sqrt{\frac{1 - 2(1 - \gamma) q_2 \delta_2^2}{1 - 2q_2(1 - \gamma) B_\sigma B'_\sigma}} \)

then:

\[
p + q_1 \sigma_t + q_2 \sigma_t^2 = \beta \left( p + \mu + q_1(1 - a) + q_1 a \sigma_t + q_2 (1 - a + a \sigma_t)^2 \right)
\]

\[
-\frac{1}{2} \frac{\beta}{1 - \gamma} \log \left( 1 - 2q_2(1 - \gamma) B_\sigma B'_\sigma \right) + \frac{\beta (1 - \gamma) |L(\sigma_t) \Sigma(\sigma_t)|^2}{2} + \\
\frac{\beta}{1 - \gamma} \log \left[ \Phi \left( (1 - \gamma) |L(\sigma_t) \Sigma(\sigma_t)| \right) + \Phi \left( (1 - \gamma) |L(\sigma_t) \Sigma(\sigma_t)| \right) \exp \left( -\frac{(1 - \gamma)^2 \alpha (2 - \alpha) |L(\sigma_t) \Sigma(\sigma_t)|^2}{2} \right) \right]
\]

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and

\[ |L(\sigma_t) \Sigma(\sigma_t)| = \sqrt{\sigma_t^2 |H + QB|^2 + \frac{B_\sigma B'_\sigma (q_1 + 2q_2 (1-a + a\sigma_t))^2}{1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma}} \]

I linearize around \( |L(\sigma_t) \Sigma(\sigma_t)| = 0 \) up to the second order:

\[
p + q_1 \sigma_t + q_2 \sigma_t^2 = \beta \left( p + \mu + q_1 (1-a) + q_1 a \sigma_t + q_2 (1-a + a \sigma_t)^2 \right) - \frac{1}{2} \frac{\beta}{1-\gamma} \log (1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma) + \frac{1}{2} \beta (1-\gamma) \left( \sigma_t^2 |H + QB|^2 + \frac{B_\sigma B'_\sigma (q_1 + 2q_2 (1-a + a\sigma_t))^2}{1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma} \right) \left( 1 - \alpha + \frac{\alpha^2}{2} \left( 1 - \frac{1}{\pi} \right) \right)
\]

Let’s group the terms in \( \sigma_t^2 \):

\[
0 = 2q_2^2 |B_\sigma|^2 \left( \frac{1}{\beta} - a^2 \alpha \left( 1 - \frac{\alpha}{2} \left( 1 - \frac{1}{\pi} \right) \right) \right) + q_2 \left( \frac{a^2}{1-\gamma} - (1-\gamma) |H + QB|^2 |B_\sigma|^2 \left( 1 - \alpha + \frac{\alpha^2}{2} \left( 1 - \frac{1}{\pi} \right) \right) \right) + \frac{1}{2} |H + QB|^2 \left( 1 - \alpha + \frac{\alpha^2}{2} \left( 1 - \frac{1}{\pi} \right) \right)
\]

This is a second order equation in \( q_2 \) where \( q_2 \) is the only unknown. The solution has to satisfy \( 1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma > 0 \). There is a unique choice of \( q_2 \) such that \( (X_t, \sigma_t) \) is stable under the distribution.

I suppose that \( \sigma_t^2 \left( |H + QB|^2 + \frac{4q_2^2 a^2 |B_\sigma|^2}{1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma} \right) \gg \frac{B_\sigma B'_\sigma (q_1 + 2q_2 (1-a + a\sigma_t))^2}{1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma} \) and use the simplification:

\[
\sqrt{\sigma_t^2 |H + QB|^2 + \frac{B_\sigma B'_\sigma (q_1 + 2q_2 (1-a + a\sigma_t))^2}{1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma}} = \sigma_t \sqrt{\left( |H + QB|^2 + \frac{4q_2^2 a^2 |B_\sigma|^2}{1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma} \right)}
\]

Let’s group the terms in \( \sigma_t \):

\[
q_1 = \left( -\frac{\alpha}{a \sqrt{\pi}} \sqrt{\left( |H + QB|^2 + \frac{4q_2^2 a^2 |B_\sigma|^2}{1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma} \right)} + \frac{2q_2 (1-a) (1-2q_2 (1-\gamma)) B_\sigma^2 (1-\frac{\alpha}{2} (1 - \frac{1}{\pi}))}{1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma} \right)
\]

\[
q_2 = \left( \frac{1}{\beta a} - \frac{1-\alpha 2q_2 (1-\gamma) |B_\sigma|^2 (1-\frac{\alpha}{2} (1 - \frac{1}{\pi}))}{1 - 2q_2 (1-\gamma) B_\sigma B'_\sigma} \right)
\]

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The changes due to the loss aversion specification enters both through the modified value for \( q_2 \) and through the extra terms in \( q_1 \).

Let’s group the constant terms:

\[
\begin{align*}
\frac{1 - \beta}{\beta} p &= \mu + q_1 (1 - a) + q_2 (1 - a)^2 - \frac{1}{2} \frac{1}{1 - \bar{\gamma}} \log (1 - 2q_2 (1 - \bar{\gamma}) B_\sigma B'_\sigma) \\
&- \frac{1 - \beta}{\sqrt{2\pi}} \frac{\alpha}{\sqrt{1 - 2q_2 (1 - \bar{\gamma}) B_\sigma B'_\sigma}} \\
&+ \frac{1}{2} (1 - \bar{\gamma}) \frac{|B_\sigma|}{1 - 2q_2 (1 - \bar{\gamma}) B_\sigma B'_\sigma} \left[ 1 - \alpha + \frac{\alpha^2}{2} \left( \frac{1}{2} \pi \right) \right]
\end{align*}
\]

The changes due to the loss aversion specification enters through changes in the values of \( q_2 \) and \( q_1 \) and through the extra terms in \( p \).

Since it was obtained through several steps of approximation, this linear solution is not the exact solution to the recursive problem

\[
f (\sigma_t) = \frac{\beta}{1 - \bar{\gamma}} \log \mathbb{E}_t \exp \left[ (1 - \bar{\gamma}) \left\{ f (\sigma_{t+1}) + \mu + \sigma_t (QB + H) W_{t+1} - \alpha \max (0, \sigma_t (QB + H) W_{t+1} + (1 - \mathbb{E}_t) f (\sigma_{t+1})) \right\} \right]
\]

where \( v_t - c_t = QX_t + f (\sigma_t) \)

To check the robustness of my approximate solution, I numerically derive the unique solution to the recursive problem. In Figure 2 I plot the exact solution and the approximate solution. The approximate solution is very close to the exact solution, which justifies using the approximate solution to derive asset prices.

Results in the case of stochastic volatility, \( \rho = 1 \)

SDF

\[
\begin{align*}
s_{1,t+1}^1 &= \log \beta + (1 - \bar{\gamma}) \left( v_{t+1}^1 - Q_t \left( \frac{v_{t+1}^1}{v_{t+1}} \right) \right) - (c_{t+1} - c_t) \\
&- (1 - \bar{\gamma}) \alpha \max (0, v_{t+1}^1 - \mathbb{E}_t v_{t+1}^1) \\
&+ \log \left( \left( 1 - \alpha 1_{v_{t+1}^1 \geq \mathbb{E}_t v_{t+1}^1} \right) + \frac{\mathbb{E}_t \left( 1_{v_{t+1}^1 \geq \mathbb{E}_t v_{t+1}^1} V_{t+1}^{1-\gamma} \right)}{V_{t+1}^{1-\gamma}} \right)
\end{align*}
\]

becomes
\[ s_{t,t+1}^1 = \log \beta + (1 - \gamma) \left( \frac{1 - \beta}{\beta} p - \frac{\gamma}{(1 - \gamma)} \mu + q_1 (1 - a) + q_2 (1 - a)^2 \right) \]
\[ + (1 - \gamma) \left( q_1 \left( a - \frac{1}{\beta} \right) + 2aq_2 (1 - a) \right) \sigma_t + q_2 \left( a^2 - \frac{1}{\beta} \right) \sigma_t^2 \] - \sigma_t HW_{t+1} + (1 - \gamma) \left( L (\sigma_t) W_{t+1} + q_2 (B_\sigma W_{t+1})^2 \right) - (1 - \gamma) \alpha 1_{L(\sigma_t)W_{t+1} \geq 0} L (\sigma_t) W_{t+1} \]
\[ + \log \left\{ \left( 1 - \alpha 1_{L(\sigma_t)W_{t+1} > 0} \right) + \alpha \exp \left( - (1 - \gamma) \left( 1 - \alpha 1_{L(\sigma_t)W_{t+1} \geq 0} \right) L (\sigma_t) W_{t+1} + q_2 (B_\sigma W_{t+1})^2 \right) \right\} \times \frac{\exp \left( (1 - \gamma)^2 L^2 \Sigma^2 \right)}{\sqrt{1 - 2q_2 (1 - \gamma) B_\sigma B_\sigma'}} \]

**Euler Equation:**

\[ ee_{t,t+1}^1 = r_{t+1} + \log \beta + (1 - \gamma) \left( v_{t+1}^1 - Q_t \left( \frac{1}{v_{t+1}^1} \right) \right) - (c_{t+1} - c_t) \]
\[ - (1 - \gamma) \alpha \max (0, v_{t+1}^1 - E_t v_{t+1}^1) \]
\[ + 1_{v_{t+1}^1 \geq E_t v_{t+1}^1} \log \left( (1 - \alpha) + \alpha \frac{E_t \left( R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-1} \right)}{R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{-1}} \right) \]

becomes

\[ ee_{t,t+1}^1 = r_{t+1} + \log \beta + (1 - \gamma) \left( \frac{1 - \beta}{\beta} p - \frac{\gamma}{(1 - \gamma)} \mu + q_1 (1 - a) + q_2 (1 - a)^2 \right) \]
\[ + (1 - \gamma) \left( q_1 \left( a - \frac{1}{\beta} \right) + 2aq_2 (1 - a) \right) \sigma_t + q_2 \left( a^2 - \frac{1}{\beta} \right) \sigma_t^2 \] - \sigma_t HW_{t+1} + (1 - \gamma) \left( L (\sigma_t) W_{t+1} + q_2 (B_\sigma W_{t+1})^2 \right) - (1 - \gamma) \alpha 1_{L(\sigma_t)W_{t+1} \geq 0} L (\sigma_t) W_{t+1} \]
\[ + 1_{L(\sigma_t)W_{t+1} \geq 0} \log \left( (1 - \alpha) + \alpha \frac{E_t \left( R_{t+1} \exp (-\sigma_t HW_{t+1}) \right)}{R_{t+1} \exp (-\sigma_t HW_{t+1})} \right) \]

**Risk Free Rate when \( \rho = 1 \):** The risk free rate is given by

\[ r_f = - \log E_t \left( \exp ee_{t,t+1}^1 (1) \right) \]
\[ r_f = -\log \beta + (1 - \gamma) \left( \frac{1 - \beta}{\beta} p + \frac{\gamma}{1 - \gamma} \mu - q_1 (1 - a) - q_2 (1 - a)^2 \right) + G \tilde{X}_t \\
- (1 - \gamma) \left( q_1 \left( a - \frac{1}{\beta} \right) + 2a q_2 (1 - a) \right) \sigma_t + q_2 \left( a^2 - \frac{1}{\beta} \right) \sigma_t^2 \\
- \log E_t \exp \left\{ (1 - \gamma) \left( L(\sigma_t) W_{t+1} + q_2 (B_\sigma W_{t+1})^2 \right) - \sigma_t H W_{t+1} \right\} \\
+ 1_{L(\sigma_t) W_{t+1} \geq 0} \left[ - (1 - \gamma) \alpha L(\sigma_t) W_{t+1} + \log \left( (1 - \alpha) + \alpha \exp \left( \sigma_t H W_{t+1} + \frac{1}{2} \sigma_t^2 |H|^2 \right) \right) \right]\]

remember

\[ p + q_1 \sigma_t + q_2 \sigma_t^2 = \beta \left( p + \mu + q_1 (1 - a) + q_1 a \sigma_t + q_2 (1 - a + a \sigma_t)^2 \right) + \frac{\beta}{1 - \gamma} \log E_t \exp (1 - \gamma) \left[ L(\sigma_t) W_{t+1} + q_2 (B_\sigma W_{t+1})^2 - \alpha 1_{L(\sigma_t) W_{t+1} \geq 0} L(\sigma_t) W_{t+1} \right] \]

and so

\[ r_f = -\log \beta + \mu + G \tilde{X}_t \\
+ \log E_t \exp (1 - \gamma) \left[ L(\sigma_t) W_{t+1} + q_2 (B_\sigma W_{t+1})^2 - \alpha 1_{L(\sigma_t) W_{t+1} \geq 0} L(\sigma_t) W_{t+1} \right] \\
- \log E_t \exp \left\{ (1 - \gamma) \left( L(\sigma_t) W_{t+1} + q_2 (B_\sigma W_{t+1})^2 \right) - \sigma_t H W_{t+1} \right\} \\
- 1_{L(\sigma_t) W_{t+1} \geq 0} \left[ (1 - \gamma) \alpha L(\sigma_t) W_{t+1} - \log \left( (1 - \alpha) + \alpha \exp \left( \sigma_t H W_{t+1} + \frac{1}{2} \sigma_t^2 |H|^2 \right) \right) \right]\]

and so

\[ r_f = -\log \beta + \mu + G \tilde{X}_t - \frac{1}{2} \sigma_t^2 |H|^2 + \frac{1}{2} \sigma_t^2 \left| \frac{HL(\sigma_t)'}{L(\sigma_t)} \right|^2 \left( 1 - (\Sigma(\sigma_t))^2 \right) \\
+ (1 - \gamma) \left( \Sigma(\sigma_t) \right)^2 \sigma_t L(\sigma_t) H' \\
+ \log \left\{ \Phi \left( - (1 - \gamma) |L(\sigma_t) \Sigma(\sigma_t)| \right) \exp \left( \frac{(1 - \gamma)^2}{2} |L(\sigma_t) \Sigma(\sigma_t)|^2 \right) \right\} \\
+ \Phi \left( \left( - (1 - \gamma) |L(\sigma_t) \Sigma(\sigma_t)| \right) + \sigma_t \frac{HL(\sigma_t)'}{L(\sigma_t)} \Sigma(\sigma_t) \right) \times \exp \left( \frac{(1 - \gamma)^2}{2} |L(\sigma_t) \Sigma(\sigma_t)|^2 - (\Sigma(\sigma_t))^2 \sigma_t L(\sigma_t) H' \right) \\
- \log \left\{ \Phi \left( \left( (1 - \gamma) |L(\sigma_t) \Sigma(\sigma_t)| - \sigma_t \frac{HL(\sigma_t)'}{L(\sigma_t)} \Sigma(\sigma_t) \right) \times \exp \left( \left( (\Sigma(\sigma_t))^2 \sigma_t L(\sigma_t) H' \right) \times \exp \left( \frac{1}{2} \sigma_t^2 \left| \frac{HL(\sigma_t)'}{L(\sigma_t)} \right|^2 \left( 1 - (\Sigma(\sigma_t))^2 \right) \right) \right) \right\} \]

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as a second order approximation around \( L(\sigma_t) \Sigma(\sigma_t) = 0 \), and \( H = 0 \):

\[
\begin{align*}
    r^1_t & \approx -\log \beta + \mu + GX_t - \frac{1}{2} \sigma_t^2 |H|^2 + \frac{1}{2} \sigma_t^2 \left| \frac{HL(\sigma_t)'}{L(\sigma_t)} \right|^2 (1 - (\Sigma(\sigma_t))^2) \\
    & \quad + (1 - \gamma) \left( \Sigma(\sigma_t) \right)^2 \sigma_t L(\sigma_t) H' + \\
    & \quad + \alpha \left\{-\frac{1}{\sqrt{2\pi}} \sigma_t \frac{HL(\sigma_t)'}{L(\sigma_t)} \Sigma(\sigma_t) + \frac{1}{2} \alpha (\Sigma(\sigma_t))^2 \left( 1 - \frac{1}{\pi} \right) (1 - \gamma) \sigma_t L(\sigma_t) H' + \\
    & \quad - \frac{1}{4} \frac{\sigma_t^2}{\sigma_t^2} \left| \frac{HL(\sigma_t)'}{L(\sigma_t)} \right|^2 (1 - (\Sigma(\sigma_t))^2 \left( 1 + \frac{\alpha}{\sigma_t^2} \right) \right\} \\
    & \quad \text{the dominant term due to loss aversion is} \quad -\frac{1}{\sqrt{2\pi}} \alpha \sigma_t \frac{HL(\sigma_t)'}{L(\sigma_t)} \Sigma(\sigma_t). \quad \text{The net effect of loss aversion decreases the risk free rate: precautionary savings is the dominant effect.}
\end{align*}
\]

**One-period Risk Prices when \( \rho = 1 \):** Let’s now turn to risk prices

Take an asset with cash-flows \( \exp \left( \Delta W_{t+1} - \frac{1}{2} |\Delta|^2 \right) \). Its log expected return is

\[
    r_t(\Delta) = -\log \mathbb{E}_t \left( \exp e e_{t,t+1}^1 \left( \exp \left( \Delta W_{t+1} - \frac{1}{2} |\Delta|^2 \right) \right) \right)
\]

The risk prices are given by

\[
    -\frac{\partial}{\partial \Delta} \log \mathbb{E}_t \left( \exp e e_{t,t+1}^1 \left( \exp \left( \Delta W_{t+1} - \frac{1}{2} |\Delta|^2 \right) \right) \right)
\]

The expected excess returns are thus:

\[
    r^1_t(\Delta) - rf^1_t =
\]

\[
    -\log \mathbb{E}_t \exp \left[ \begin{cases} 
    \Delta W_{t+1} - \frac{1}{2} |\Delta|^2 + (1 - \bar{\gamma}) \left( L(\sigma_t) W_{t+1} + q_2 (B_\sigma W_{t+1})^2 \right) - \sigma_t HW_{t+1} \\
    \quad + 1_{L(\sigma_t)W_{t+1} \geq 0} \times \\
    \quad \left[ - (1 - \bar{\gamma}) \alpha L(\sigma_t) W_{t+1} + \log \left( (1 - \alpha) + \alpha \exp \left( (\sigma_t H - \Delta) W_{t+1} + \frac{1}{2} |\sigma_t H - \Delta|^2 \right) \right) \right] \\
    \end{cases} \right]
\]

\[
    + \log \mathbb{E}_t \exp \left[ \begin{cases} 
    (1 - \bar{\gamma}) \left( L(\sigma_t) W_{t+1} + q_2 (B_\sigma W_{t+1})^2 \right) - \sigma_t HW_{t+1} \\
    \quad + 1_{L(\sigma_t)W_{t+1} \geq 0} \times \\
    \quad \left[ - (1 - \bar{\gamma}) \alpha L(\sigma_t) W_{t+1} + \log \left( (1 - \alpha) + \alpha \exp \left( \sigma_t HW_{t+1} + \frac{1}{2} \sigma_t^2 |H|^2 \right) \right) \right] \\
    \end{cases} \right]
\]

and

\[
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\]
Let's define

\[
\Phi \left( (1 - \gamma) |L| + \frac{\sigma_t H L'}{|L|} \right) \Sigma \exp \left( \frac{(1-\gamma)^2 \alpha (2-\alpha) |L| \Sigma^2}{2} - \Sigma^2 \alpha (1 - \gamma) (\sigma_t H L') \right) \\
+ (1 - \alpha) \Phi \left( (1 - \gamma) |L| - \frac{\sigma_t H L'}{|L|} \right) \Sigma \exp \left( \frac{1}{2} \sigma_t^2 \left| \frac{H L'}{|L|} \right|^2 (1 - \Sigma^2) \right) \\
+ \alpha \Phi \left( (1 - \gamma) |L| \right) \exp \left( \Sigma^2 \alpha (1 - \gamma) (\sigma_t H - \Delta) L' \right) \exp \left( \frac{1}{2} \left| \frac{\sigma_t (H - \Delta) L'}{|L|} \right|^2 (1 - \Sigma^2) \right)
\]

The risk prices are given by:

\[
\frac{\partial}{\partial \Delta} \log E_t \left( \exp \left( \frac{1}{2} \left| \Delta \right|^2 \right) \right)
\]

\[
= - \frac{\partial}{\partial \Delta} \log E_t \left[ \Delta W_t + \frac{1}{2} |\Delta|^2 + (1 - \gamma) \left( \frac{\gamma}{L} (\sigma_t |W_t + q_L B |) - \sigma_t H W_t \right) \right] \\
\left[ - (1 - \gamma) \alpha L (\sigma_t) W_t + \log \left( 1 - \alpha + \alpha \exp \left( (\sigma_t H - \Delta) W_t + \frac{1}{2} |\sigma_t H - \Delta|^2 \right) \right) \right]
\]

Let's define

\[
f(\Delta) = \Phi \left( (1 - \gamma) |L| + \frac{\sigma_t H - \Delta |L|}{|L|} \right) \Sigma \exp \left( \frac{1}{2} \Sigma^2 \left( \left| (1 - \gamma) |L| - \frac{\sigma_t H - \Delta |L|}{|L|} \right|^2 - \left| \frac{\Delta |L|}{|L|} \right|^2 \right) \right) + \]

\[
(1 - \alpha) \Phi \left( (1 - \gamma) |L| - \frac{\sigma_t H - \Delta |L|}{|L|} \right) \Sigma \exp \left( \frac{1}{2} \Sigma^2 \left( \left| (1 - \gamma) |L| - \frac{\sigma_t H - \Delta |L|}{|L|} \right|^2 - \left| \frac{\Delta |L|}{|L|} \right|^2 \right) \right) + \]

\[
\alpha \Phi \left( (1 - \gamma) |L| \right) \exp \left( \frac{1}{2} \left| \Delta L' \right|^2 \Sigma^2 + \left| \frac{\sigma_t H - \Delta |L|}{|L|} \right|^2 \right)
\]

the risk prices are given by

\[
\frac{\Delta L' |L|}{|L|^2} (1 - \Sigma^2) + \sigma_t H \left( I - \frac{L' |L|}{|L|^2} \right) - \frac{f'(\Delta)}{f(\Delta)}.
\]

The risk prices are given by:
\[
\sigma_t H + \Sigma^2 (\gamma - 1) L + \left( \frac{\Delta L' L}{|L|^2} - \frac{\sigma_t H L' L}{|L|^2} \right) (1 - \Sigma^2) + \\
\left\{ \begin{array}{l}
\frac{L}{\sqrt{2\pi|L|}} \Sigma^2 \exp \frac{1}{2} \Sigma^2 \left| \frac{\Delta L'}{|L|} \right|^2 \exp \left( \frac{1}{2} \Sigma^2 \left| \frac{(\gamma - 1)|L| + (\sigma_t H - \Delta)L'}{|L|} \right|^2 \right) \times \\
\Phi \left( \left( (1 - \gamma)|L| + (\sigma_t H - \Delta)L'/|L| \right) \Sigma^2 \exp \left( \frac{1}{2} \Sigma^2 \left| \frac{|(1 - \gamma)|L| - (\sigma_t H - \Delta)L'}{|L|} \right|^2 \right) \right) \\
\Phi \left( (1 - \gamma)|L| \Sigma \exp \frac{1}{2} \left( (1 - \gamma)|L| \Sigma^2 + \left| \frac{(\sigma_t H - \Delta)L'}{|L|} \right|^2 \right) \right) \times \\
\frac{1}{f(\Delta)} \\
\end{array} \right\}
\]

There are three additional terms due to loss aversion. For $|L|$ small and $|\Delta|$ small, the dominating term is $\alpha \Sigma^2 \frac{L}{\sqrt{2\pi|L|}} \exp \frac{1}{2} \Sigma^2 \left| \frac{\Delta L'}{|L|} \right|^2 \exp \left( \frac{1}{2} \Sigma^2 \left| \frac{|(1 - \gamma)|L| - (\sigma_t H - \Delta)L'}{|L|} \right|^2 \right) \times \Phi \left( (1 - \gamma)|L| \Sigma \exp \frac{1}{2} \left( (1 - \gamma)|L| \Sigma^2 + \left| \frac{(\sigma_t H - \Delta)L'}{|L|} \right|^2 \right) \right) \times \frac{1}{f(\Delta)}$. This term is first order relative to the others and thus increases the risk prices substantially.