Aggregate Bank Capital and Credit Dynamics

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Abstract

Central banks need a new type of quantitative models for guiding their financial stability decisions. The aim of this paper is to propose such a model. In our model commercial banks finance their loans by deposits and equity, while facing issuance costs when they raise new equity. Because of this financial friction, banks build equity buffers to absorb negative shocks. Aggregate bank capital determines the dynamics of credit. Notably, the equilibrium loan rate is a decreasing function of aggregate capitalization. The competitive equilibrium is constrained inefficient, because banks do not internalize the effect of their individual lending decisions on the future loss-absorbing capacity of the banking sector. In particular, we find that undercapitalized banks lend too much. We show that introducing a minimum capital ratio helps taming excessive lending and enhances financial stability.

Keywords: macro-model with a banking sector, aggregate bank capital, pecuniary externality, capital requirements

JEL: E21, E32, F44, G21, G28

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1 Introduction

Central banks need quantitative models for guiding their financial stability decisions. Indeed, in the context of their new macro-prudential responsibilities, they have recently been endowed with powerful regulatory tools. These tools include the setting of capital requirements for all banks, determining capital add-ons for systemic institutions, and deciding when to activate counter-cyclical capital buffers. The problem is nobody has the slightest idea of the long term impact of these regulations on growth and financial stability. The only quantitative models that central banks currently have at their disposal, the so-called DSGE models, have been designed for very different purposes, namely assessing the short term impact of monetary policy decisions on inflation and economic activity. Until recently, these DSGE models did not even include banks in their representation of the economy. DSGE models are very complex and use very special assumptions, because they have been specifically calibrated to reproduce the short term reaction of prices and employment to movements in central banks’ policy rates. It seems therefore clear that a very different kind of model is needed for analyzing the long term impact of capital regulations (and other macro-prudential tools) on bank credit, GDP growth and financial stability. The aim of this paper is to propose such a model.

Building on the recent literature on macro models with financial frictions, we develop a tractable dynamic model where aggregate bank capital determines the dynamics of credit. Though highly stylized, the model is able to generate predictions in line with empirical evidence. We consider an economy where firms borrow from banks that are financed by deposits and equity. The aggregate supply of bank loans is confronted with the firms’ demand for credit, which determines the equilibrium loan rate. Aggregate shocks impact the firms’ default probability, which ultimately translates into profits or losses for banks. Banks can continuously adjust their volumes of lending to firms. They also decide when to distribute dividends and when to issue new equity. Equity issuance is subject to deadweight costs, which constitutes the main financial friction in our economy and creates room for the loss-absorbing role of bank capital.¹

In a set-up without financial frictions (i.e., no issuance costs for bank equity) and i.i.d. aggregate shocks, the equilibrium volume of lending and the nominal loan rate would be constant. Furthermore, dividend payment and equity issuance policies would be trivial in this case: Banks would immediately distribute all profits as dividends and would issue new shares to offset losses and honor obligations to depositors. This implies that, in a frictionless world, there would be no need to build up capital buffers and all loans would be entirely financed by deposits.

When financial frictions are taken into account, banks’ dividend and equity issuance strategies become less trivial. We show that there is a unique competitive equilibrium, where all variables of

¹Empirical studies report sizable costs of seasoned equity offerings (see e.g. Lee, Lochhead, Ritter, and Zhao (1996), Hennessey and Whited (2007)). Here we follow the literature (see e.g. Décamps et al. (2011) or Bolton et al. (2011)) by assuming that issuing new equity entails a deadweight cost proportional to the size of the issuance.
interest are deterministic functions of the total book value of bank equity, which follows a Markov process reflected at two boundaries. Banks issue new shares at the lower boundary, where total book equity of the banking sector is depleted. When total bank equity reaches its upper boundary, any further earnings are paid out to shareholders as dividends. Between these boundaries, the changes in banks’ equity are only due to their profits and losses. Banks retain earnings in order to increase their loss-absorbing equity buffer and thereby reduce the frequency of costly recapitalizations. This buffer is needed to guarantee the safety of deposits.\footnote{The idea that bank equity is needed to guarantee the safety of deposits is explored in several recent papers. Stein (2012) shows its implication for the design of monetary policy. Hellwig (2015) develops a static general equilibrium model where bank equity is necessary to support the provision of safe and liquid investments to consumers. DeAngelo and Stulz (2014) and Gornall and Strebulaev (2015) argue that, due to the banks’ ability to diversify risk, the actual size of this equity buffer may be very small.}

We start by exploring the properties of the competitive equilibrium in the “laissez-faire” environment, in which banks face no regulation. We first set up the model in discrete time to provide the main economic intuitions. We then present a continuous time version that leads to quasi explicit solutions. This turns out to be helpful for illustrating the features of the equilibrium, notably the long run dynamics. Even though all agents are risk neutral, our model generates a positive spread for bank loans. This spread is decreasing in the level of total bank equity. To get an intuition for this result, note that bank equity is more valuable when it is scarce because profit margins are a decreasing function of total bank equity. Therefore, the marginal (or market-to-book) value of equity is higher when total bank equity is lower. Moreover, profits and losses are positively correlated across banks. Thus each bank anticipates that total bank equity will be lower (higher) in the states of the world where it makes losses (profits). Individual losses are thus amplified by a simultaneous increase in the market-to-book value, whereas individual profits are moderated by a simultaneous decrease in the market-to-book value. As a result, banks only lend to firms when the loan rate incorporates an appropriate premium.

In the continuous-time set-up, the equilibrium dynamics of the loan rate can be obtained in closed form, which enables us to study the long-run behavior of the economy by looking at the properties of the ergodic density function of total bank equity. Our analysis shows that the long-run behavior of the economy is mainly driven by the (endogenous) volatility of total bank equity. In particular, the economy spends most of the time in states with low endogenous volatility. For high recapitalization costs and a low elasticity of demand for bank loans, this can induce long periods of persistently low volumes of credit and low levels of bank equity.

We show that the competitive equilibrium is constrained inefficient. The reason is that competitive banks do not internalize the impact of their individual lending decisions on i) the banking system’s exposure to aggregate shocks and ii) the profit margin on credit. As a result, banks typically lend too much as compared to the socially optimal level, creating inefficiently high exposure to macroeconomic shocks when the banking system is poorly capitalized. Furthermore, inefficiently low profit margins undermine the banking system’s ability to accumulate loss absorbing capital.
through retained earnings.

We use our model to explore the impact of bank capital requirements. A standard argument against high capital requirements is that they would reduce lending and growth. By contrast, the proponents of higher capital requirements put emphasis on their positive impact on financial stability. Solving for the competitive equilibrium under imposed minimum capital requirements enables us to consider the interplay between the aforementioned effects and get some insight into the long run consequences of a substantial increase in minimum capital requirements.

In our framework, imposing a higher capital ratio indeed translates into a higher loan rate and thus reduces lending for any given level of bank capitalization. Importantly, this effect is present even when the regulatory constraint is not binding, because banks anticipate that capital requirements might be binding in the future and require a higher lending premium for precautionary motives. However, reduction in lending also reduces the banks’ exposure to aggregate shocks, while higher loan rates fosters a quicker accumulation of earnings and thus allows for a quicker recovery after the negative shocks, which ultimately makes the banking sector more stable. As a result, in the long run, the economy spends more time in the states of the world with abundant bank capital and cheap credit.

Related literature. From a technical perspective, our paper follows the approach of the new generation of the continuous-time macroeconomic models with financial frictions (see e.g. Brunnermeier and Sannikov (2014, 2015), Di Tella (2015), He and Krishnamurthy (2012, 2013)). Seeking for a better understanding of the transmission mechanisms of monetary policy and the consequences of financial instability, all these papers point out to the key role that balance-sheet constraints and net-worth of financial intermediaries may play in (de)stabilizing the economy in the presence of financing frictions and aggregate shocks. We extend this literature by modeling the banking sector explicitly and relating total bank equity to credit dynamics and financial stability. A closely related paper is Phelan (2015) who explicitly introduces a banking sector in a continuous-time general equilibrium model. In Phelan’s paper also, banks invest in productive capital (land) and depositors obtain utility from holding safe deposits. The focus of our analysis is different: we explicitly study the impact of bank capitalization on the dynamics of the loan rate.

A common feature of the above-mentioned papers is the existence of fire-sale externalities in the spirit of Kiyotaki and Moore (1997) and Lorenzoni (2008) that arise when forced asset sales depress prices, which in turn creates amplifying feedback effects. In our model, amplification and persistent investment distortions emerge even in the absence of fire sales. The main externality imposed by individual banks’ lending decisions is that each bank fails to internalize the impact of its lending choice on the banking system’s exposure to aggregate risk, which increases endogenous volatility. Furthermore, the idea that fierce competition can have a destabilizing effect as it erodes banks’ profit margins and, thus, their ability to accumulate loss absorbing capital is related to Martinez-Miera and Repullo (2010). A similar externality is present in the overlapping generations model
considered by Malherbe (2015): Individual banks neglect the fact that an expansion of lending, due to diminishing returns to productive capital, leads to a deterioration of the marginal loan and, thus, higher bankruptcy costs for all banks in the economy. As a result, banks lend in excessively in booms, which calls for counter-cyclical capital requirements.

The extended version of our model featuring the regulatory leverage constraint contributes to the ongoing investigations of the welfare effects of capital regulation. Most of the literature dealing with this issue is focused on the trade-off between the welfare gains from the mitigation of risk-taking incentives on the one hand\(^3\) and welfare losses caused by lower liquidity provision (e.g., Begenau (2015), Van den Heuvel (2008)), lower lending and output (e.g., Nguyen (2014), Martinez-Miera and Suarez (2014)) on the other hand.\(^4\) In contrast to the above-mentioned studies, the focus of our model is entirely shifted from the incentive effect of bank capital towards its role of a loss absorbing buffer - the concept that is often put forward by bank regulators. Moreover, the main contribution of our paper is qualitative: we seek to identify the long run effects of capital regulation rather than provide a quantitative guidance on the optimal level of a minimum capital ratio.

More broadly, this paper relates to the literature on credit cycles that has brought forward a number of alternative explanations for their occurrence. Fisher (1933) identified the famous debt deflation mechanism, that has been further formalized by Bernanke et al. (1996) and Kiyotaki and Moore (1997). It attributes the origin of credit cycles to the fluctuations of the prices of collateral. Several studies also place emphasis on the role of financial intermediaries, by pointing out the fact that credit expansion is often accompanied by a loosening of lending standards and "systemic" risk-taking, whereas materialization of risk accumulated on the balance sheets of financial intermediaries leads to the contraction of credit (see e.g. Aikman et al. (2014), Dell'Ariccia and Marquez (2006), Jimenez and Saurina (2006)). In our model, quasi cyclical lending patterns emerge due to the reflection property of aggregate bank capital that follows from the optimality of "barrier" recapitalization and dividend strategies.

The rest of the paper is organized as follows. Section 2 presents a simple one-period model that conveys the main intuitions concerning the economic forces working in the full-fledged dynamic setup. In Section 3 we characterize the competitive equilibrium in the discrete-time infinite-horizon dynamic setup. Section 4 presents the continuous-time version of the model, illustrating its application for studying the long-run macrodynamics and discussing inefficiencies inherent in the competitive equilibrium framework. In Section 5 we introduce capital regulation, analysing its implications for bank policies and the lending-stability trade-off. Section 6 concludes. All proofs, model extensions and computational details are gathered in Appendices A-F. Appendix G reports

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\(^3\) Bank capital is often viewed as ‘skin in the game’ needed to prevent the opportunistic behaviours of banks' insiders.

\(^4\) The only exception is the work by De Nicolò et al. (2014) that conducts the analysis of bank risk choices under capital and liquidity regulation in a fully dynamic model, where capital plays the role of a shock absorber, and dividends, retained earnings and equity issuance are modeled under a financial friction captured by a constraint on collateralized debt.
the results of the empirical analysis supporting the key model predictions.

2 One-period model

Before setting-up the fully fledged dynamic model, it is useful to convey some key intuitions in a static benchmark. We start by describing the intermediation and production technologies and then characterize the competitive equilibrium.

2.1 The model set-up

The static model has only two dates $t = 0$ and $t = 1$. There is one physical good, taken as a numeraire, which can be consumed at $t = 0$ or invested to produce consumption at $t = 1$. Banks are needed to channel savings of households to the productive sector.\(^5\) Savings take two possible forms: riskless deposits and risky equity. Banks’ assets consist of loans to the productive sector (see Figure 1) and reserves at the central banks. Banks are competitive: they take both the deposit interest rate $r_D \geq 0$ and loan rate $R$ as given. The central bank sets the rate $r$ at which banks can deposit reserves or refinance. Households cannot directly invest in the productive sector. Instead, at $t = 0$, they invest in deposit and equity claims issued by banks, who then grant loans to firms. Firms are run by penniless entrepreneurs that immediately consume all output net of the loan repayments to banks. Thus, the entire volume of productive investment in the economy is determined by the volume of bank credit.\(^6\) Apart from channelling funding to the productive sector, the major reason why banks matter in our framework is that bank capital allows to buffer losses on loans. As will be shown further, only the aggregate loss absorbing capacity of the banking sector matters for our analysis, whereas the number of banks and their individual sizes do not play any role.

The main financial friction in our model is that issuing new bank equity entails a proportional cost $\gamma$.

Preferences. Households have identical quasi-linear preferences with a discount rate $\rho$. At $t = 0$ they receive an endowment $w_0$ of the good that they allocate between consumption $C_0$, deposits $D$ and investment $I$ in bank equity:

$$w_0 = C_0 + D + (1 + \gamma)I.$$  

Following Stein (2012), we assume that households derive utility both from consumption and from payment services provided by bank deposits, \textit{as long as banks can guarantee perfect safety of }

\(^5\)This standard assumption is usually justified by technological and informational reasons (see e.g. Freixas and Rochet(2008), Chapter 2).

\(^6\)In our model, firms should be thought of as small and medium-sized enterprises (SMEs), which typically rely on bank financing. As is well known, the importance of bank financing varies across countries. For example, according to the TheCityUK research report (October 2013), in EU area, bank loans account for 81% of the long term debt in the real sector, whereas in the U.S. the same ratio amounts to 19%.
the latter. In particular, a representative household has utility:\footnote{As in Brunnermeier and Sannikov (2014), households’ consumption can be both positive and negative.}

\[ U = C_0 + \frac{\mathbb{E}[\tilde{C}_1] + \lambda(D)}{1 + \rho}, \]

where \( \tilde{C}_1 \) denotes consumption at \( t = 1 \),\footnote{Throughout the paper, the symbol \( \sim \) is used to indicate the random character of a variable.} \( D \) denotes aggregate deposits and \( \lambda(D) \) is the utility obtained from holding deposits. It is a concave non-decreasing function of \( D \).

Households’ consumption at date 1 is the sum of profits of banks, \( \tilde{\pi}_B \), and the deposits including interests paid by banks:

\[ \tilde{C}_1 = \tilde{\pi}_B + (1 + r_D)D. \]

Given the linearity of households’ preferences, the following result is immediate (see proof in Appendix A):

**Lemma 1** The aggregate volume of deposits in the economy is implicitly given by \( \lambda'(D) = \rho - r_D \).

The above lemma states that the aggregate demand for deposits, consistent with the individual optimal consumption-portfolio choices, is uniquely determined by the liquidity preference function \( \lambda(.) \), as well as the wedge between the discount factor and the deposit interest rate.

**Productive sector.** The productive sector consists of a continuum of penniless firms endowed with investment projects that are parametrized by a productivity parameter \( x \). The productivity parameter \( x \) is distributed according to a continuous distribution with density function \( f(x) \) defined on a bounded support \([0, \overline{R}]\). Parameter \( \overline{R} \) can be thought of as the maximum productivity among all the firms. This is also the maximum loan rate: when \( R > \overline{R} \), the demand for loans is nil.
Firms’ projects require each an investment of one unit of good at \( t = 0 \). A typical project yields \( x \) units of good at \( t = 1 \) with certainty. For the sake of simplicity, we assume that the firm always repays the interest \( R \) on the loan. However, with some probability, productive capital can be destroyed and the bank only gets \( R \), whereas the firm always gets \( (x - R) \). Firms are protected by limited liability and default when their projects are not successful. Given a nominal loan rate \( R \), only the projects such that \( x > R \) will demand financing. Thus, the total demand for bank loans in the economy is:

\[
L(R) = \int_{R}^{\bar{R}} f(x) dx.
\]

We focus on the simple case where all projects have the same default probability \( \bar{p} \) that depends on the realization of aggregate shocks. For simplicity, we assume that it can take only two values:\(^9\)

\[
\bar{p} = \begin{cases} 
    p, & \text{with probability } 1 - q \text{ (positive shock)}, \\
    \bar{p}, & \text{with probability } q \text{ (negative shock)},
\end{cases}
\]

where \( 1 > p > \bar{p} > 0 \).

Therefore, the net return per loan for a bank at date \( t = 1 \) is

\[
R - r_D - \bar{p},
\]

and the net aggregate output per period in the economy is

\[
F[L(R)] - \bar{p}L(R),
\]

where \( F[L(R)] \) is the aggregate production function:

\[
F[L(R)] = \int_{R}^{\bar{R}} xf(x) dx.
\]

Note that \( F'[L(R)] \equiv R \),\(^{10}\) so that the total expected surplus per unit of time

\[
F[L(R)] - L(R)\left(r_D + E[\bar{p}]\right)
\]

is maximized for \( R_{fb} = r_D + E[\bar{p}] \). Thus, in the first best allocation, the loan rate is the sum of two components: the riskless rate and the expected probability of default. This implies that, in the first-best allocation, banks would make zero expected profit, and the total volume of credit in the

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\(^9\)The extension to heterogeneous default probability is straightforward and would not change our qualitative results.

\(^{10}\)Differentiating \( F[L(R)] = \int_{R}^{\bar{R}} xf(x) dx \) with respect to \( R \) yields \( F'[L(R)]L'(R) = -Rf(R) \). Since \( L'(R) = -f(R) \), this implies \( F'[L(R)] \equiv R \).
economy would be given by \( L(R_{fb}) \).

**Banking sector.** At \( t = 0 \) a typical bank is endowed with \( e \) units of equity (\( E \) for the whole banking sector).\(^{11}\) Before lending to firms, it can distribute dividends \( \delta \geq 0 \) (\( \Delta \) on aggregate) or issue new equity \( i \geq 0 \) (\( I \) on aggregate). Having collected deposits \( d \) (\( D \) on aggregate), a bank deposits (borrows, if negative) \( m \) (\( M \) on aggregate) with the central bank and lends

\[
k = d - m + e - \delta + i
\]

(1)
to firms (\( K \) on aggregate).

At date \( t = 1 \), a bank generates profit

\[
\tilde{\pi}_B = k(R + 1 - \bar{p}) - (1 + r_D)d + (1 + r)m,
\]

which, using (1), can be rewritten as

\[
\tilde{\pi}_B = (1 + r_D)(e - \delta + i) + k(R - \bar{p} - r_D) + m(r - r_D).
\]

(2)

A first simplification comes from the fact that banks are free to choose \( m \) arbitrarily (positive or negative). Thus, the only possible deposit rate \( r_D \) is equal to the central bank policy rate \( r \). Thus, for the rest of the paper, \( r_D \) is set equal to \( r \) and \( \tilde{\pi}_B \) simplifies to:

\[
\tilde{\pi}_B = (1 + r)(e - \delta + i) + k(R - \bar{p} - r).
\]

To guarantee that the bank is able to repay its depositors in full, it must dispose (after dividend payment or recapitalization) of the equity cushion sufficient to absorb the worst possible loss,\(^{12}\) i.e.,

\[
e + i - \delta \geq \frac{k(r + \bar{p} - R)}{1 + r},
\]

(3)

which can be viewed as a market-based leverage constraint. Thus, in our model bank equity plays a role of an absorbing buffer supporting the provision of riskless liquid claims (deposits).\(^{13}\)

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\(^{11}\)Throughout the paper, we use lower case letters for individual bank variables and upper case letters for aggregate variables.

\(^{12}\)This condition easily extends to the case when depositors accept some probability of default (Value-at-Risk constraint similar to Adrian and Shin (2010)).

\(^{13}\)An equivalent interpretation is that banks finance themselves by repos, and the lender applies a hair cut equal to the maximum possible value of the asset (the loan portfolio) that is used as collateral. The justification of the leverage constraint in our framework differs from the other ones that are put forward by academics. One of them relates to the limitation on pledgeable income by bank insiders (Holmstrom and Tirole (1997)) and backs the concept of "inside" equity that plays the role of "the skin in the game" for banks. The other academic justification for leverage constraints stems from the limited resalability of collateral (Kiyotaki and Moore (1997)), thereby, placing emphasis on the asset side of borrowers’ balance sheets, whereas we focus here on the banks’ liabilities. These three justifications for leverage are different, but not independent.
2.2 Competitive equilibrium in the one-period model

Our objective now is to solve for the competitive equilibrium in this static framework. The competitive equilibrium is characterized by a loan rate $R$, a lending volume $K$ and aggregate reserves $M$ that are compatible with the equilibrium conditions on credit, equity and deposit markets, as well as profit maximization by individual banks.

Consider first the maximization problem of a typical bank. Each bank takes the loan rate $R$ as given and chooses dividend policy $\delta \geq 0$, recapitalization policy $i \geq 0$ and the volume of lending $k \geq 0$ so as to maximize shareholder value, while insuring compliance with the market-based leverage constraint:

$$v = \max_{\delta \geq 0, i \geq 0, k > 0} \left\{ \delta \right. \left[ 1 - (1 + r)(1 + \theta) \right] + \max_{i \geq 0} i \left[ - (1 + \gamma) + \frac{(1 + r)(1 + \theta)}{1 + \rho} \right] + \max_{k > 0} k \left[ R \right. \left. - E[\tilde{p}] - r - \theta(r + \tilde{p} - R) \right] \right\} + \frac{(1 - q)\pi_B(p) + q\pi_B(\bar{p})}{1 + \rho} + \frac{\theta\pi_B(\bar{p})}{1 + \rho},$$

where $\pi_B(p)$ is the profit realization under the positive aggregate shock, $\pi_B(\bar{p})$ is the profit realization after the negative aggregate shock and $\theta$ denotes the Lagrange multiplier associated with the market-based leverage constraint.

Note that the above maximization problem is separable, i.e.,

$$v = \frac{e(1 + r)(1 + \theta)}{1 + \rho} + \max_{\delta \geq 0} \delta \left[ 1 - \frac{(1 + r)(1 + \theta)}{1 + \rho} \right] + \max_{i \geq 0} i \left[ - \frac{(1 + \gamma)(1 + \theta)}{1 + \rho} \right] + \max_{k > 0} k \left[ \frac{R - E[\tilde{p}] - r - \theta(r + \tilde{p} - R)}{1 + \rho} \right].$$

(4)

Optimizing with respect to the bank’s policies yields the following conditions:

$$1 - \frac{(1 + r)(1 + \theta)}{1 + \rho} \leq 0 \quad (= \text{if } \delta > 0), \quad (5)$$

$$- (1 + \gamma) + \frac{(1 + r)(1 + \theta)}{1 + \rho} \leq 0 \quad (= \text{if } i > 0), \quad (6)$$

$$R - E[\tilde{p}] - r = \theta(r + \tilde{p} - R). \quad (7)$$

Condition (5) shows that $\theta > 0$, which implies that the market-based leverage constraint is always binding at both the individual and aggregate levels. This determines completely the loan rate as a function of aggregate bank capitalization, i.e., $R \equiv R(E)$. More specifically, when $\delta = 0$ and $i = 0$, loan rate $R \equiv R(E)$ is the unique solution of the equation:

$$L(R)(R - \bar{p} - r) + (1 + r)E = 0.$$

Equation (7) then determines the value of $\theta$, which also depends on $E$. Finally, the shareholder
The value of each bank is proportional to its book equity, namely:

\[ v \equiv v(e, E) = e \frac{(1 + r)(1 + \theta(E))}{1 + \rho} = e \frac{(1 + r)(\bar{p} - \mathbb{E}[\tilde{p}])}{(1 + \rho)(r + \bar{p} - R(E))} \equiv eu(E), \]

where \( u(E) \) can be interpreted as the market-to-book ratio of equity. Note that \( u(E) \) is a decreasing function of \( E \): bank capital becomes more valuable when it is getting scarce.

It is easy to see from conditions (5) and (6), that the optimal dividend and recapitalization policies are driven by the market-to-book ratio of bank equity. In particular, condition (5) transforms to

\[ u(E) \geq 1, \]

and condition (6) can be rewritten as

\[ u(E) \leq 1 + \gamma. \]

Let \( E_{\text{min}} \) denote the unique level of aggregate equity such that \( u(E_{\text{min}}) = 1 + \gamma \) and \( E_{\text{max}} \) be such that \( u(E_{\text{max}}) = 1 \). Then, when \( E < E_{\text{min}} \), banks will recapitalize by raising in aggregate \( E_{\text{min}} - E \).

Similarly, when \( E > E_{\text{max}} \), aggregate dividends \( E - E_{\text{max}} \) are distributed to shareholders. As a result, the market-to-book ratio of the banking sector always remains within the range \([1, 1 + \gamma]\).

As will be shown further, this feature is preserved in both the discrete-time and continuous-time dynamic versions of our model. The following proposition summarizes our results for the static set-up:

**Proposition 1** The static model has a unique competitive equilibrium. It has the following properties:

a) The loan rate \( R \equiv R(E) \) is a decreasing function of aggregate capital \( E \) and is implicitly given by

\[ L[R(E)][r + \bar{p} - R(E)] = E(1 + r). \]

b) All the banks have the same market-to-book ratio of equity that is a decreasing function of \( E \):

\[ u(E) = \frac{(1 + r)(\bar{p} - \mathbb{E}[\tilde{p}])}{(1 + \rho)(r + \bar{p} - R(E))}. \]

c) Banks pay dividends when \( E \geq E_{\text{max}} \equiv u^{-1}(1) \) and recapitalize when \( E \leq E_{\text{min}} \equiv u^{-1}(1 + \gamma) \).

d) The leverage constraint is always binding for all the banks:

\[ \frac{k}{e} = \frac{L[R(E)]}{E} = \frac{1 + r}{r + \bar{p} - R(E)}. \]

Our parsimonious static set up emphasizes the important idea that the equilibrium loan rate
$R(E)$ is driven by aggregate bank capitalization $E$. Namely, the loan rate is lower when the banking sector is better capitalized. Another key testable prediction generated by this simplest form of our theoretical model is that the banks’ market-to-book ratio of equity should be a decreasing function of aggregate bank capital. Before developing the dynamic version of our theoretical model, it is useful to examine whether these predictions are rejected by the data or not.

In Appendix G, we conduct statistical tests on a data set covering a large panel of publicly traded banks in 43 advanced and emerging market economies for the period 1982-2013. We find that our predictions fit the data extremely well. Table 3 in Appendix G shows the results of simple regressions of loan rates and market-to-book ratios of banks equity in three sub-panels: U.S. banks, Advanced countries’ (excluding U.S.) banks and Emerging countries’ banks. In all cases, the coefficients of Total Bank Equity are negative with $p$-values indistinguishable from zero.

3 Discrete-time model with infinite horizon

We now turn to a stationary version of the static model, which has an infinite number of periods, $t = 0, ..., \infty$. Our objective is to characterize Markovian competitive equilibria, where all aggregate variables are deterministic functions of a single state variable, namely, aggregate bank equity $E_t$. For the sake of tractability, for the rest of the paper we assume that $r = 0$. The case $r > 0$ is studied in Appendix D in the continuous-time framework.

**Definition 1** A (stationary) Markovian competitive equilibrium consists of an aggregate bank capital process $E_t$, loan rate $R(E)$ and credit volume $K(E)$ functions that are compatible with individual banks’ profit maximization and the credit market clearing condition $K(E) = L[R(E)]$.

Given the volume of lending $k_t$, book equity of any individual bank evolves according to

$$\tilde{e}_{t+1} - e_t = k_t(R_t - \tilde{p}_t) - \delta_t + i_t,$$

where $\delta_t \geq 0$ and $i_t \geq 0$ are, respectively, individual dividend payments and recapitalizations at date $t$. Similarly, aggregate capitalization of the banking sector evolves according to

$$\tilde{E}_{t+1} - E_t = L(R_t)(R_t - \tilde{p}_t) - \Delta_t + I_t,$$

where $\Delta_t \geq 0$ and $I_t \geq 0$ are, respectively, aggregate dividend payments and recapitalizations at date $t$.

3.1 Markovian competitive equilibrium

To characterize the competitive equilibrium, one needs to determine the optimal recapitalization and financing decisions of individual banks as well as the functional relation between the aggregate
level of bank equity $E_t$ and the loan rate $R_t$. To this end, consider the optimal decision problem of an individual bank that takes the loan rate function $R_t = R(E_t)$ as given and makes its decisions based on the level of its own equity $e_t$ and aggregate equity $E_t$.

Like in the static model, a fundamental property of the optimization problem of an individual bank is that bank policies are homogeneous of degree one in the individual equity level $e_t$ and thus $v(e,E) \equiv eu(E)$.14 Then, the maximization problem of an individual bank can be stated as follows:

$$e_t u(E_t) = \max_{\delta_t \geq 0, i_t \geq 0, k_t} \delta_t - (1 + \gamma)i_t + E \left[ \tilde{e}_{t+1}u(\tilde{E}_{t+1}) \right], \quad t = 0, ..., \infty,$$

s.t. $k_t[p - R_t] - e_t - i_t + \delta_t \leq 0$,

where $\tilde{e}_{t+1}$ and $\tilde{E}_{t+1}$ are given by Equations (8) and (9) respectively.

Proceeding in the same way as in the static set up, it is easy to show that the optimal dividend and recapitalization policies are of the "barrier" (or $(s,S)$) type.15 Namely, provided that $u(.)$ is a decreasing function of $E$ (this has to be verified ex-post), dividends are distributed only when $E_t = E_{\text{max}}$, where the critical threshold $E_{\text{max}}$ is such that $u(E_{\text{max}}) = 1$. In other words, distribution of dividends only takes place when the marginal value of equity capital equals the shareholders' marginal value of consumption. At the aggregate level, dividend payments made at date $t$ consist of all equity in excess of $E_{\text{max}}$. Similarly, recapitalizations occur only when $E_t$ falls below the critical threshold $E_{\text{min}}$ at which the marginal value of equity equals the total marginal cost of equity issuance $u(E_{\text{min}}) = 1 + \gamma$. Overall, due to the properties of the optimal recapitalization and dividend policies, aggregate bank equity fluctuates in the range $[E_{\text{min}}, E_{\text{max}}]$ (where $E_{\text{min}}$ and $E_{\text{max}}$ are reflecting barriers), the market-to-book ratio of banks never leaves the range $[1, 1 + \gamma]$ and recapitalization/dividend distributions in all banks are perfectly synchronized in time.16

Since for $E \in (E_{\text{min}}, E_{\text{max}})$ the dynamics of $E_t$ (as well as the dynamics of $e_t$) is driven exclusively by retained earnings or absorbed losses, the maximization problem of bank shareholders can be rewritten as follows:

$$(1 + \rho)eu(E) = \max_{k \geq 0} E \left[ \{e + k(R(E) - \bar{p})\} u(\bar{E}) \right] - \theta(E)[k(\bar{p} - R(E)) - e], \quad (10)$$

14. This useful property of the value function is a natural consequence of the scale invariance property of our model.
15. The barrier-type recapitalization and payout policies have been extensively studied by the corporate liquidity management literature (see e.g. Jeanblanc and Shiryaev (1996), Milne and Robertson (1996), Décamps et al. (2011), Bolton et al. (2011, 2013), Hugonnier et Morellec (2015) among others) that places emphasis on the loss-absorbing role of corporate liquid reserves in the presence of financial frictions. In our model, the role of book equity is very similar to the role of liquidity buffers in those models. However, we differentiate from this literature by allowing for the feedback loop between the individual decisions and the dynamics of individual book equity via the general equilibrium mechanism that determines the loan rate and thus affects the expected earnings of banks.
16. Note that the difference $E_{\text{max}} - E_{\text{min}}$ can be interpreted as a target capital buffer. It is then immediate to see that, in the absence of financial frictions (i.e., $\gamma = 0$), there would be no role for capital buffers, as banks could costlessly tap equity markets at any moment in order to guarantee the safety of deposits.
where $\theta(E)$ is the Lagrange multiplier associated with the market-based leverage constraint.

Combined with the equilibrium condition, $K(E_t) = L(R_t)$, the above maximization problem yields the following set of necessary conditions for equilibrium:

$$E \left[ (R - \bar{p})u(\hat{E}) \right] = \theta(E)(\bar{p} - \bar{R}), \quad (11)$$

$$(1 + \rho)u(E) = E \left[ u(\hat{E}) \right] + \theta(E), \quad (12)$$

$$\theta(E) \left\{ L(R)(\bar{p} - \bar{R}) - E \right\} = 0. \quad (13)$$

Here, Equation (11) comes from the First-Order condition of the problem (10) with respect to $k$. Equation (12) is obtained by substituting Equation (11) into the Bellman equation (10). Finally, Equation (13) reflects the fact that the market-based leverage constraint is binding at the individual level if and only if it is binding at the aggregate level.

Given that $u(\hat{E}) \leq u(\min{E}) = 1 + \gamma$, it is easy to see from Equation (12) that $\theta(\min{E}) > 0$ and thus the market-based leverage constraint is binding in the vicinity of the recapitalization threshold. However, in contrast to the static case, $\theta(E)$ can be zero when $E$ is larger than some threshold $\hat{E}$. In general, the competitive equilibrium in the dynamic set-up has two regions: a “constrained” region $[\min{E}, \hat{E})$, in which the market-based leverage constraint is binding, and an “unconstrained” region $(\hat{E}, \max{E})$, in which the market-based leverage constraint is slack.

“Constrained” region. Like in the static version of the model, in the constrained region, the loan rate $R(E)$ is implicitly given by the binding leverage constraint and is a decreasing function of aggregate equity:

$$L[R(E)](\bar{p} - R(E)) = E. \quad (14)$$

Furthermore, the above equality implies that, in the region $[\min{E}, \hat{E})$, any negative shock completely depletes equity, triggering an immediate recapitalization up to $\min{E}$.

Solving the system of equations (11) and (12) for $E \in [\min{E}, \hat{E})$, and using the fact that $u(0) = u(\min{E}) = 1 + \gamma$, yields:

$$u(E) = \left( \frac{1 - q}{1 + \rho} \right) \left[ \frac{\bar{p} - \bar{p}}{\bar{p} - R(E)} \right] u(E^+), \quad (15)$$

$$\theta(E) = \left( 1 - q \right) \left[ \frac{R(E) - \bar{p}}{\bar{p} - R(E)} \right] u(E^+) - q(1 + \gamma), \quad (16)$$

where $E^+ \equiv L[R(E)](\bar{p} - \bar{p})$ and the critical threshold $\hat{E}$ is implicitly defined by equation $\theta(\hat{E}) = 0$.

It is important to emphasize that recapitalizations occur at a strictly positive level of aggregate equity, i.e., $\min{E} > 0$. To see this, assume by way of contradiction that $\min{E} = 0$. Then, from the binding leverage constraint it follows that $R(0) = \bar{p}$ and thus the right-hand side of expression (15)
goes to infinity. Yet, this contradicts the condition \( u(E_{\min}) = 1 + \gamma \) that is implied by the optimal recapitalization policy. Thus one must have \( E_{\min} > 0 \).

**“Unconstrained” region.** For \( E \in [\hat{E}, E_{\max}] \), the equilibrium loan rate \( R(E) \) and the market-to-book value \( u(E) \) are jointly determined by the system:

\[
\begin{align*}
E \left[ (R - \tilde{p})u(\tilde{E}) \right] &= 0, \quad (17) \\
(1 + \rho)u(E) &= E \left[ u(\tilde{E}) \right]. \quad (18)
\end{align*}
\]

Rewriting equation (17), one can express the equilibrium loan rate as follows:

\[
R(E) = \frac{E \left[ \tilde{p}u(\tilde{E}) \right]}{E \left[ u(\tilde{E}) \right]} \geq E [\tilde{p}], \tag{19}
\]

which represents the risk-adjusted expectation of firms’ default probability, using the stochastic (private) discount factor \( \frac{u(\tilde{E})}{(1 + \rho)u(E)} \). The inequality in the right-hand side of the above expression comes from the fact that \( \tilde{p} \) and \( \tilde{E} \) are negatively correlated, whereas \( u(.) \) is a decreasing function.

### 3.2 Constrained inefficiency of the competitive equilibrium

We are now going to show that, because of a pecuniary externality, the competitive equilibrium is never constrained efficient in the region where the market-based leverage constraint is slack.\(^\text{18}\)

To see this, we need to characterize the welfare function \( W(E) \) that corresponds to the competitive lending function \( K(E) \). In the retention region \( (E_{\min}, E_{\max}) \), this welfare function satisfies the following Bellman equation:\(^\text{19}\)

\[
W(E) = \left( F(K) - K F'(K) \right) + \left( \lambda(D) - D \lambda'(D) \right) \left( 1 - q \right) W(E_+) + q W(E_-) \left( 1 + \rho \right),
\]

where \( E_+ \equiv E + K[F'(K) - \tilde{p}] \) and \( E_- \equiv E + K[F'(K) - \tilde{p}] \) are the values of aggregate capitalization in the next period, conditional on the realized aggregate shock. The first term in the above expression captures the firm’s profits that are immediately consumed by householders and the second term is the surplus generated by deposits.\(^\text{20}\)

To understand whether competitive banks lend too much or too little (from a social welfare perspective), one can compute the derivative of the right-hand side of the above equation with

---

\(^\text{18}\)When the market-based leverage constraint is binding, both the competitive equilibrium and the constrained optimum coincide. In particular, the competitive equilibrium is constrained efficient in the static case in Section 2, because in that case the market-based leverage constraint is always binding.

\(^\text{19}\)For clarity of our formulas, throughout this subsection we omit the argument of \( K(E) \).

\(^\text{20}\)Note that the value of the surplus generated by deposits is independent on aggregate capitalization as \( D \) is given by \( \lambda'(D) = \rho - r \).
respect to aggregate lending $K$:

$$
\frac{\partial W(E)}{\partial K} = -F''(K)K + \frac{(1-q)W'(E_+)[KF''(K) + F'(K) - \bar{p}] + qW'(E_-)[KF''(K) + F'(K) - \bar{p}]}{1 + \rho}.
$$

(20)

Using the fact that $F'(K) = R$ and $(1+\rho)W'(E) = (1-q)W'(E_+) + qW'(E_-)$, we can rewrite the above expression as follows:

$$
\frac{\partial W(E)}{\partial K} = F''(K)K \left[ W'(E) - 1 \right] + W'(E) \left[ R(E) - \mathbb{E}^W(\bar{p}) \right],
$$

(21)

where

$$
\mathbb{E}^W(\bar{p}) = \frac{(1-q)W'(E_+)\bar{p} + qW'(E_-)\bar{p}}{(1+\rho)W'(E)}
$$

is the risk-adjusted expectation of firms’ default obtained using the stochastic (social) discount factor $\frac{W'(E)}{(1+\rho)W'(E)}$.

If the competitive equilibrium were constrained efficient, the right-hand side of (21) would be identically zero. Here it generally differs from zero for two reasons. On the one hand, for lower levels of aggregate capitalization, the first term in (21) is negative ($W'(E) > 1$ and $F''(K) < 0$), because profits have more social value when retained in the (undercapitalized) banks rather than immediately consumed by households. On the other hand, the second term does not vanish because the competitive loan rate $R(E)$ is the risk-adjusted expectation of firms’ default using $u(E)$ as the stochastic discount factor (i.e., private discount factor) instead of the social discount factor $W'(E)$.

As we will see from our numerical analysis in the continuous-time version of the model, the sum of these two terms can sometimes be positive, which means that competitive banks can sometimes lend too little. However, in the typical situation, this sum is negative, which means that competitive banks typically lend too much. The reason is that competitive banks do not internalize the fact that an increase in their individual lending volume puts downward pressure on the equilibrium loan rate (pecuniary externality). This pecuniary externality is welfare reducing because it slows down the accumulation of capital within banks, which reduces banks’ lending capacity and ultimately hurts future output. Thus, our model shows that welfare-reducing pecuniary externalities can appear in credit markets even if there are no costly fire sales.\footnote{Examples of macro models with costly fire sales are e.g. Brunnermeier and Sannikov (2015) and Stein (2012).}

4 Continuous-time model

While the discrete time version of our model has more economic appeal than the continuous-time version, the latter turns out to be much more tractable from an analytical perspective and thus is very helpful for illustrating the main properties of the competitive equilibrium. Furthermore, as we demonstrate in this section, it allows an easy analysis of the long-run macroeconomic dynamics.
4.1 The competitive equilibrium in continuous time

In the continuous time set-up, the return on assets for bank follows the process:

\[(R_t - p)dt - \sigma_0 dZ_t,\]  \hspace{1cm} (22)

where \(p\) denotes the unconditional default probability of firms, \(\sigma_0\) reflects the exposure to aggregate shocks and \(\{Z_t, t \geq 0\}\) is a standard Brownian motion.\(^{22}\)

For an individual bank, let \(k_t, \delta_t, i_t\) denote, respectively, the volume of lending, cumulative dividends and cumulative equity injections at time \(t\). Thus, book equity of an individual bank evolves according to

\[de_t = k_t[(R(E_t) - p)dt - \sigma_0 dZ_t] - d\delta_t + di_t.\]  \hspace{1cm} (23)

Similarly, aggregate equity \(E_t\) evolves according to

\[dE_t = K(E_t)[(R(E_t) - p)dt - \sigma_0 dZ_t] - d\Delta_t + dI_t,\]  \hspace{1cm} (24)

where \(K(E_t), \Delta_t\) and \(I_t\) denote, respectively, the aggregate volumes of lending, cumulative dividends and cumulative equity injections at time \(t\).\(^{23}\) Note that Equations (23) and (24) are the exact (continuous-time) analogues of the dynamic Equations (8) and (9) formulated in discrete time.

To characterize the competitive equilibrium in continuous time, consider the optimal decision problem of an individual bank. Again, bank shareholders choose lending \(k_t \geq 0\), dividend \(d\delta_t \geq 0\) and recapitalization \(di_t \geq 0\) policies so as to maximize the market value of equity:

\[v(e, E) = \max_{k_t, \delta_t, i_t} \mathbb{E} \left[ \int_0^\tau e^{-\rho t} (d\delta_t - (1 + \gamma)di_t)|e_0 = e, E_0 = E \right] \equiv eu(E),\]  \hspace{1cm} (25)

where \(\tau := \inf\{t : e_t < 0\}\),\(^{24}\) and \(e_t\) and \(E_t\) evolve according to (23) and (24), respectively. Note that, in continuous time, the market-based leverage constraint boils down to the requirement of non-negative book equity \(e_t \geq 0\) \((E_t \geq 0\) at the aggregate level).

As before, we use the homotheticity property of the shareholder value function and work directly with the market-to-book ratio \(u(E)\). By the standard dynamic programming arguments, \(u(E)\)

\(^{22}\)In Appendix B we demonstrate that the results of the discrete-time version of the model converge to the ones of the continuous-time version studied in this section.

\(^{23}\)The aggregate volume of reserves (or borrowing from the central bank, if negative) is given by the aggregate balance sheet constraint \(M(E) = D - K[R(E)] + E\).

\(^{24}\)If bank recapitalizations are optimal, default never occurs and \(\tau \equiv \infty\).
satisfies the Bellman equation:

\[
\rho u(E) = \max_{k \geq 0, d\delta \geq 0, di \geq 0} \left\{ \frac{d\delta}{e} [1 - u(E)] - \frac{di}{e} [1 + \gamma - u(E)] + \frac{k}{e} [(R(E) - p)u(E) + \sigma_0^2 K(E)u'(E)] \right\}
+ K(E)(R(E) - p)u'(E) + \frac{\sigma_0^2 K^2(E)}{2} u''(E).
\]

(26)

Maximization with respect to the volume of lending \(k\) shows that the optimal lending policy of the bank is indeterminate, i.e., bank shareholders are indifferent with respect to the volume of lending. Instead, the latter is entirely determined by the firms’ demand for credit, so that at the aggregate level we have \(K(E) = L[R(E)]\).\(^{25}\) However, for \(k > 0\), it must hold that

\[
\frac{u'(E)}{u(E)} = -\frac{R(E) - p}{\sigma_0^2 L[R(E)]}.
\]

(27)

Assuming that \(R(E) \geq p\) (which will be verified ex-post), it follows from the above expression that \(u(E)\) is a decreasing function of \(E\). Then, the optimal payout policy maximizing the right-hand side of (26) is characterized by a critical barrier \(E_{\text{max}}\) satisfying

\[
u(E_{\text{max}}) = 1,
\]

(28)

and the optimal recapitalization policy is characterized by a barrier \(E_{\text{min}}\) such that

\[
v(E_{\text{min}}) = 1 + \gamma.
\]

(29)

Thus, the optimal dividend and recapitalization policies in continuous time are very similar to the ones defined in the discrete-time set-up. Namely, banks distribute any excess profits as dividends so as to maintain aggregate equity at or below \(E_{\text{max}}\). Similarly, recapitalizations are undertaken so as to offset losses and to maintain aggregate equity at or above \(E_{\text{min}}\).

Under the optimal bank policies summarized in Equations (27), (28) and (29), in the region \(E \in (E_{\text{min}}, E_{\text{max}})\), the market-to-book value \(u(E)\) satisfies:

\[
\rho u(E) = L[R(E)](R(E) - p)u'(E) + \frac{\sigma_0^2 (L[R(E)])^2}{2} u''(E).
\]

(30)

We demonstrate in Appendix B that Equations (27) and (30) are the continuous-time limits of Equations (17) and (18) that we have encountered in the unconstrained regime of the discrete-time version of the model. Combining (27) and (30) shows that the equilibrium loan rate \(R(E)\) satisfies

---

\(^{25}\)This situation is analogous to the case of an economy with constant returns to scale, in which the equilibrium price of any output is only determined by technology (constant marginal cost), whereas the volume of activity is determined by the demand side.
a first-order differential equation:

$$R'(E) = \frac{1}{H[R(E)]}, \quad \text{with} \quad H(R) \equiv \frac{\sigma_0^2[L(R) - (R - p)L'(R)]}{2\rho \sigma_0^2 + (R - p)^2}. \quad (31)$$

Moreover, considering the limit of the discrete-time version of the model (see Appendix B), it can be shown that $R(E_{\text{max}}) = p$, which provides a necessary condition to solve for the equilibrium loan rate. Given that $L'(R) < 0$, it is easy to see that $R'(E) < 0$ and thus $R(E) > p$ for any $E \in [E_{\text{min}}, E_{\text{max}})$. Thus, the loan rate carries a positive lending premium. Indeed, rewriting (27) yields:

$$R(E) - p = \sigma_0^2 K(E) \left[ - \frac{u'(E)}{u(E)} \right]. \quad (32)$$

The “raison d’être” for this premium is rooted in the joint impact of aggregate shocks and financing frictions. To see this, consider the impact of the marginal unit of lending on shareholder value $v(e, E) \equiv eu(E)$. A marginal increase in the volume of lending increases the bank’s exposure to aggregate shocks. However, note that the aggregate shock not only affects the individual bank’s equity $e_t$ but also aggregate equity $E_t$ and thus the market-to-book ratio $u(E)$. In the presence of financing frictions, i.e., when $\gamma > 0$, the latter is monotonically decreasing in $E$.\(^{26}\) Thus, if there is a negative aggregate shock that depletes the individual bank’s equity, the effect of the loss on shareholder value gets amplified via the market-to-book ratio. Symmetrically, a positive aggregate shock, while increasing book equity, translates into a reduction of the market-to-book ratio, which reduces the impact of positive profits on shareholder value. This mechanism gives rise to **effective risk aversion** with respect to variation in aggregate capital, which explains why risk-neutral bankers require a positive spread for accepting to lend.

Once the equilibrium loan rate $R(E)$ has been specified, the market-to-book ratio can easily be computed by solving equation (27) with the boundary condition $u(E_{\text{max}}) = 1$. This yields:

$$u(E) = \exp \left( \int_{E}^{E_{\text{max}}} \frac{R(s) - p}{\sigma_0^2 L[R(s)]} ds \right). \quad (33)$$

The typical patterns of the loan rate $R(E)$ and the market-to-book ratio $u(E)$ that emerge in the competitive equilibrium are illustrated in Figure 2.

The following proposition summarizes the characterization of the continuous-time version of the competitive equilibrium.

**Proposition 2** There exists a unique Markovian equilibrium in continuous time, in which aggregate

\(^{26}\)Intuitively, having an additional unit of equity reduces the probability of facing costly recapitalizations in the short-run, so that the marginal value of equity, $u(E)$, is decreasing with bank capitalization.
Figure 2: Loan rate and market-to-book ratio in the competitive equilibrium

Notes: this figure reports the typical patterns of the loan rate $R(E)$ (left panel) and market-to-book ratio of equity $u(E)$ (right panel) in the competitive equilibrium.

Bank capital evolves according to:

$$dE_t = L[R(E_t)][(R(E_t) - p)dt - \sigma_0 dZ_t].$$

(34)

The loan rate function $R(E)$ is implicitly given by the equation

$$E(R) = \int_{R}^{R_{max}} \frac{\rho^2 \sigma_0^2 [L(s) - (s - p)L'(s)]}{2\rho^2 \sigma_0^2 + (s - p)^2} ds,$$

(35)

where $R_{max}$ is the unique solution of

$$\int_{p}^{R_{max}} \frac{(s - p)[1 - (s - p)L'(s)]}{2\rho^2 \sigma_0^2 + (s - p)^2} ds = \ln(1 + \gamma).$$

(36)

Banks distribute dividends when $E_t$ reaches the threshold $E_{max} = E(p)$ and recapitalize when $E_t$ reaches 0.

In the competitive equilibrium, the loan rate $R_t$ fluctuates between its first-best level $p$ and $R_{max}$. Note that the lending pattern emerging in our model is procyclical which matches empirical evidence (see e.g. Becker and Ivashina (2014)). Moreover, it follows immediately from Expression (36) that the maximum lending premium, $R_{max} - p$, is increasing with the magnitude of financial frictions, $\gamma$. Thus, our model predicts that loan rates, lending and, thereby, output should exhibit more dispersion in the economies with stronger financial frictions. At the same time, Expression (35) shows that the target level of bank capitalization, $E_{max}$, is increasing with $R_{max}$. Thus, the loss absorbing capacity of equity becomes more important under stronger financing frictions. By contrast, in the absence of financial frictions, i.e., when $\gamma \equiv 0$, one would have $R_{max} = p$ and $E \equiv 0$, so that there would be no role for bank equity and no fluctuations of credit.

A prominent feature of the competitive equilibrium in continuous time is that banks always
postpone recapitalizations until \( E_{\text{min}} = 0 \). This marks a sharp difference with respect to the discrete-time version of the model, in which recapitalizations occur at a strictly positive level of aggregate equity. The explanation for this feature is closely linked to the fact that, when banks have a possibility to \textit{instantaneously} adjust lending and aggregate losses are small (which is implied by the diffusion process), the constrained region \([E_{\text{min}}, \hat{E}]\) identified in the discrete-time set-up shrinks to a single point, \( \{E_{\text{min}}\} \) (this is proven in Appendix B). However, since the benefit of accelerating recapitalization disappears in this case (i.e., shareholders no longer bear the “shadow costs” associated with the market leverage constraint), it also becomes optimal to postpone recapitalizations until the last moment.

4.2 Long run behavior of the economy

We now exploit the continuous-time version of the model to study the long-run behavior of the economy in the competitive equilibrium. To this end, we look at the long-run behavior of the loan rate. Note that the loan rate function \( R(E) \) cannot generally be obtained in closed form. However, it turns out that the dynamics of the loan rate \( R_t = R(E_t) \) is explicit. Indeed, applying Itô’s lemma to \( R_t = R(E_t) \) yields:

\[
dR_t = L[R(E_t)] \left( (R(E_t) - p)R'(E_t) + \frac{\sigma_0^2 L[R(E_t)]}{2} R''(E_t) \right) \frac{\mu(R_t)}{\sigma(R_t)} dt - \sigma_0 L[R(E_t)]R'(E_t) dZ_t. \tag{37}
\]

After some computations involving the use of Expression (31), one can obtain the drift and the volatility of \( R_t = R(E_t) \) in closed form. This yields the following proposition:

---

27 However, the property \( E_{\text{min}} = 0 \) is not a general feature of the continuous-time set-up. In the (unreported) version of our model in which the dynamics of equity follows a jump process, the constrained region (on which the market-based leverage constraint binds) does not vanish and, as a result, banks recapitalize at a strictly positive level of aggregate capital. Moreover, even in the diffusion set-up, the property \( E_{\text{min}} = 0 \) may not hold if one allows for a time-varying financing cost. In Appendix F we allow \( \gamma \) to switch between two values \( \gamma_G \) (Good state) and \( \gamma_B \) (Bad state) such that \( \gamma_B > \gamma_G \). Our numerical analysis reveals that, when \( \gamma_B \) is very high, in the Good state banks will raise new equity at a strictly positive level of \( E \) in order to reduce the probability of incurring larger financing costs conditional on a sudden transition to the Bad state. This result is in line with the “market timing” effect that was first identified in the dynamic setting by Bolton et al. (2013) in the context of a partial-equilibrium liquidity-management model with the time-varying fixed cost of recapitalization.

28 This feature echoes the result obtained in the model by Biais et al. (2007) in the context of a principal-agent model. In particular, they show that the region on which the project has to be downsized in discrete time shrinks to a single point in the continuous-time limit.

29 One could equivalently look at the dynamics of the state variable \( E_t \). However, in this simple set-up with no regulation, working with \( R_t \) instead of \( E_t \) enables us to provide an analytic characterization of the system’s behavior, because the drift and volatility of \( R_t \) have closed-form expressions. By contrast, the drift and volatility of the process \( E_t \) cannot in general be obtained in closed form, since \( R(E) \) has an explicit expression only for particular specifications of the credit demand function.
Proposition 3 The loan rate $R_t = R(E_t)$ has explicit dynamics

$$dR_t = \mu(R_t)dt + \sigma(R_t)dZ_t, \quad p \leq R_t \leq R_{\text{max}}, \quad (38)$$

with reflections at both ends of the support. The volatility function is given by

$$\sigma(R) = \frac{2\rho\sigma_0^2 + (R - p)^2}{\sigma_0 \left(1 - (R - p)^2 L(R)\right)}. \quad (39)$$

The drift function is

$$\mu(R) = \frac{\sigma(R) h(R)}{2}, \quad (40)$$

where

$$h(R) = \frac{\sigma(p) - \sigma(R)}{R - p} - \frac{R - p}{\sigma_0} + \sigma'(R). \quad (41)$$

The impulse response methodology that is usually employed to study the long-run dynamics in the traditional macro models consists in simulating the dynamics of a model after a single unanticipated shock at $t = 0$. In our model, this would correspond to the particular trajectory of the Brownian motion $dZ_t \equiv 0$ for $t > 0$, which yields $dR_t = \mu(R_t)dt$. It can be shown that $\mu(p) = 0$, so that the first best rate $R_{fb} = p$ is a steady state of this deterministic system. In Appendix E we study the properties of the deterministic steady-state and demonstrate that it is generally not stable. Moreover, we show below that this "deterministic" steady state does not provide a correct picture of the asymptotic behavior of the economy, as the latter is mainly driven by the endogenous risk $\sigma(R)$ neglected by the impulse response analysis.

Given the loan rate dynamics defined in Proposition 3, the asymptotic behavior of the economy can be described by the ergodic density function which measures the average time spent in the neighborhood of each possible loan rate $R$: the states with low $R$ (equivalently, high aggregate capital $E$) can be interpreted as "boom" states and the states with high $R$ (equivalently, low aggregate capital $E$) can be thought of as "bust" states. This ergodic density function can be computed by solving the Kolmogorov forward equation (see Appendix A for details).

Proposition 4 The competitive loan rate process $R_t$ is ergodic. Its asymptotic distribution is characterized by the probability density function

$$g(R) = \frac{C_0}{\sigma^2(R)} \exp\left(\int_p^R 2\mu(s) \sigma^2(s) ds\right), \quad (42)$$

where the constant $C_0$ is such that $\int_p^{R_{\text{max}}} g(R)dR = 1.$
By differentiating the logarithm of the ergodic density defined in (42), we obtain:

\[
\frac{g'(R)}{g(R)} = \frac{2\mu(R)}{\sigma^2(R)} - \frac{2\sigma'(R)}{\sigma(R)}. \tag{43}
\]

Using the general formulas for \(\sigma(R)\) and \(\mu(R)\), it can be shown that \(\sigma(p) = 2\rho\sigma_0\), \(\sigma'(p) = 2\rho\sigma_0L'(p) < 0\) and \(\mu(p) = 0\). Hence, \(g'(p) > 0\), which means that the state \(R = p\) that would correspond to the "deterministic" steady state is definitely not the one at which the economy spends most of the time in the stochastic setup. To get a deeper understanding of the determinants of the system behavior in the long run, we resort to the particular demand specification:

\[
L(R) = \alpha(R - R)^{\beta}, \tag{44}
\]

where \(\beta \geq 0\), \(p < R\) and the constant \(\alpha \equiv (R - p)^{-\beta}\) is chosen so as to normalize the maximum feasible volume of lending (it is attained for \(R = p\)) to 1.

Figure 3: Volatility and ergodic density of \(R\)

Notes: this figure reports the typical patterns of the loan rate volatility (left panel) and the ergodic density (right panel). Parameter values: \(\rho = 0.04\), \(\sigma_0 = 0.05\), \(p = 0.02\), \(\beta = 0.5\) and \(R = 0.06\).

Figure 3 reports the typical patterns of the endogenous volatility \(\sigma(R)\) (the left-hand side panel) and the ergodic density \(g(R)\) (the right-hand side panel) for the above loan demand specification. It shows that the extrema of the ergodic density almost coincide with those of the volatility function, i.e., the economy spends most of the time in the states with the lowest loan rate volatility. Intuitively, the economy can get "trapped" in the states with low loan rate volatility because the endogenous drift is generally too small to move it away from these states. In fact, \(\sigma(R)\) turns out to be much larger than \(\mu(R)\) for any level of \(R\), so that the volatility impact always dominates the drift impact.\(^{30}\) In this light, relying on the results of the impulse response analysis that is typically used in the standard macromodels in order to infer the long-run behavior of the economy would be misleading.

Note that functions \(\sigma(.)\) and \(g(.)\) must be truncated (and, in the case of the ergodic density,\(^{30}\) The reason is that the factor \(h(R)\) in the expression of \(\mu(R)\) is very small.

\[30\]
rescaled) on \([p, R_{\text{max}}]\), where \(R_{\text{max}}\) depends on the magnitude of the maximum level of issuing costs \(\gamma\). For the specification of the loan demand function stated in (44) we always have \(R_{\text{max}} < \overline{R}\). However, \(R_{\text{max}}\) can be arbitrary close to \(\overline{R}\), which typically happens with very strong financial frictions and low elasticity of credit demand. In that case the economy will spend quite some time in the region where the loan rate is close to \(R_{\text{max}}\). We interpret this situation as a persistent "credit crunch": it manifests itself via scarce bank equity capital, high loan rates, low volumes of lending and output.

This "credit crunch" scenario is reminiscent to the "net worth trap" documented by Brunnermeier and Sannikov (2014). In their model, the economy may fall into a recession because of the inefficient allocation of productive capital between more and less productive agents, which they call "experts" and "households" respectively. This allocation is driven by the dynamics of the equilibrium price of capital, which depends on the fraction of the total net worth in the economy that is held by experts. After experiencing a series of negative shocks on their net worth, experts have to sell capital to less productive households, so that the average productivity in the economy declines. Under a reduced scale of operation, experts may struggle for a long time to rebuild net worth, so that the economy may be stuck in a low output region. In our model, the output in the economy is driven by the volume of credit that entrepreneurs can get from banks, whereas the cost of credit depends on the level of aggregate bank capitalization. When the banking sector suffers from a series of adverse aggregate shocks, its loss absorbing capacity deteriorates. As a result, the amplification mechanism working via the market-to-book value becomes more pronounced and bankers thus require a larger lending premium. The productive sector reacts by reducing its demand for credit and the banks have to shrink their scale of operations, which makes it even more difficult to rebuild equity capital.

4.3 Inefficiencies

In this section we show that the competitive equilibrium is not constrained efficient due to a pecuniary externality that each bank imposes on its competitors. More precisely, competitive banks do not take into account the effect of their individual lending decisions on the dynamics of aggregate bank equity. This leads to excessive lending when banks are poorly capitalized, implying overexposure of the banking sector to aggregate shocks and the erosion of its ability to accumulate earnings.

We organize our discussion as follows. First we evaluate social welfare in the competitive equilibrium. Then we analyze the second best allocation in the set-up with completely inelastic demand for loans, which allows us to isolate the inefficiencies caused by excessive exposure of the banking sector to aggregate shocks. Finally, we turn to the more general case with downward sloping loan demand where, in addition to overexposure of the banking sector, a second, distributive inefficiency

\[\int \frac{(R-\rho)H(R)}{\sigma^2L(R)} dR \text{ diverges.}\]

\[\text{In a partial equilibrium set-up, a similar result is found by Isohätälä, Mihle and Robertson (2014).}\]
comes into play: Allowing banks to make more profits can be socially desirable even though this reduces firm profits, in particular, if the banking sector is poorly capitalized.

4.3.1 Welfare in the competitive equilibrium

Social welfare in our framework can be computed as the expected value of households’ total discounted consumption, i.e., discounted firm profits, which are immediately distributed to households, plus the present value of banks’ expected dividend payments net of expected capital injections:

\[
W(E) = E \left[ \int_0^{+\infty} e^{-\rho t} (\pi_F(K) + d\Delta t - (1 + \gamma) dI) | E_0 = E \right],
\]

(45)

where \( \pi_F(K) \equiv F(K) - KF'(K) \) denotes the aggregate instantaneous production of firms net of credit costs.

By using aggregate bank equity \( E \) as a state variable, we can apply standard methods to compute the social welfare function. As long as banks neither distribute dividends nor recapitalize, households’ consumption consists only of firm profits. Therefore, in the region \( E \in (0, E_{\text{max}}) \), the social welfare function, \( W(E) \), must satisfy the following differential equation:

\[
\rho W(E) = F(K) - K(E)F'(K) + K \left( F'(K) - p \right) W'(E) + \frac{\sigma^2}{2} K^2 W''(E),
\]

(46)

subject to the boundary conditions:

\[
W'(0) = 1 + \gamma,
\]

(47)

\[
W'(E_{\text{max}}) = 1.
\]

(48)

The above boundary conditions reflect the fact that dividend distribution and bank recapitalization only affect the market value of banks, without producing any immediate impact on the firms’ profit.

Before solving for the constrained welfare optimum in the next subsection, we first evaluate the social welfare function at the competitive equilibrium in order to get a first grasp of the prevailing distortions. To this end, we take the first derivative of the right-hand side of equation (46) with respect to \( K \). Since we are interested in the impact of a marginal increase in lending on welfare at the competitive equilibrium, we substitute the competitive loan rate from equilibrium condition

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33 In our welfare analysis, we neglect the value generated by deposits as it does not depend on aggregate bank capitalization. It can easily be accounted for by adding a constant term \( \frac{\Delta(D) - d_X(D)}{\rho} \) to \( W(E) \).

34 As will be shown below, also when bank policies are dictated by the social planner, the optimal dividend and recapitalization policies are of the barrier type.

35 For the sake of space, we omit the argument of \( K(E) \).
(27) into the derivative to get:

$$F''(K)K \left[ W'(E) - 1 \right] + W'(E) \left[ \frac{W''(E)}{W'(E)} - \frac{u'(E)}{u(E)} \right] \sigma^2 K. \quad (49)$$

Note that expression (49) is equivalent to expression (21) that we have encountered in the discrete-time version of the model. The first term in (49) sheds light on the distributive inefficiency: An increase in aggregate lending drives down the loan rate ($F''(K) = \partial R/\partial K < 0$). This increases firms’ profits but bites into the profits of banks and, thus, impairs the banks’ ability to accumulate loss absorbing capital in the form of retained earnings.\(^{36}\) As long as the (social) marginal value of aggregate equity, $W'(E)$, is larger than one, welfare could be improved by restricting the total volume of lending below its competitive level. This is true at least when aggregate equity is low and thus its marginal (social) value is high (recall that $W'(0) = 1 + \gamma$).

The second term in (49) captures the inefficiency stemming from the fact that competing banks do not take into account how their individual lending decisions affect the exposure of the banking sector to aggregate shocks. The term in square brackets captures the difference between the social planners and individual banks’ effective risk aversion with respect to variations in aggregate capital.\(^{37}\)

Yet, without making further assumptions, both terms in (49) cannot be signed globally. We now turn to the case with inelastic demand ($F''(K) = 0$) in order to isolate the impact of the second term in (49), thereby, focusing on the inefficient exposure of the banking sector to aggregate shocks.

**4.3.2 Inelastic demand**

This subsection assumes that the firms’ demand for loans is constant and equal to 1 as long as the loan rate does not exceed some maximum rate $\bar{R}$:\(^{38}\)

$$L(R) = \begin{cases} 1 & \text{for } R \leq \bar{R}, \\ 0 & \text{for } R > \bar{R}. \end{cases} \quad (50)$$

This specification leads to closed-form solutions for the value functions, loan volumes and loan rates for both the competitive and the second-best allocations.

In the competitive equilibrium, banks always lend at the maximum feasible scale $L(R) = 1$ as

---

\(^{36}\)This "margin effect" of competition is also acknowledged in Martinez-Miera and Repullo (2010) albeit in an entirely different application.

\(^{37}\)Recall from (27) that banks' risk-aversion determines the size of "lending premium" for an aggregate level of lending $K$.

\(^{38}\)This loan demand specification can be obtained from the demand specification (44) by taking the limit case $\beta \equiv 0$. 

long as \( R(E) \) in condition (27) does not exceed \( \overline{R} \) and, thus,\(^{30}\)
\[
K^{CE}(E) = 1.
\]

Then, by using the results of Proposition 2, one can obtain the explicit characterization of the competitive equilibrium. In particular, the equilibrium loan rate can be found in closed-form:
\[
R^{CE}(E) = p + \sqrt{2}\rho\sigma_0 \tan \left[ \frac{\sqrt{2}\rho}{\sigma_0} (E_{\text{max}} - E) \right].
\]

Moreover, solving Equation (36) yields a closed form for the maximum interest rate that prevails upon recapitalization,
\[
R_{\text{max}}^{CE} = p + \sigma_0 \sqrt{2\rho(2\gamma + \gamma^2)},
\]
and the target level of aggregate equity is given by
\[
E_{\text{max}}^{CE} = \frac{\sigma_0}{\sqrt{2}\rho} \arctan \sqrt{2\gamma + \gamma^2}.
\]

We now analyze the second best allocation, that is, the welfare maximizing policies subject to the same constraint as in the competitive equilibrium, i.e., that issuing new capital is costly.

\[
W(\tilde{E}) = \max_{R,K,d\Delta,dI} \mathbb{E} \left[ \int_{0}^{+\infty} e^{-\rho t} \left( \pi_F(K_t, R_t) + d\Delta_t - (1 + \gamma) dI_t \right) | E_0 = \tilde{E} \right],
\] (51)

where the instantaneous firms’ profit is given by \( \pi_F(K_t, R_t) = K_t(\overline{R} - R_t) \). The single state variable in the social planner’s problem is aggregate bank equity \( E \) and the market for bank credit must clear, i.e., \( K_t = L(R_t) \). Assuming that the welfare function is concave (this is verified ex-post), it is immediate that optimal dividend and recapitalization policies are of the “barrier type” and thus aggregate equity fluctuates within a bounded support \([E_{\text{SB}}^{\text{min}}, E_{\text{SB}}^{\text{max}}]\), over which the social welfare function satisfies the ODE
\[
\rho W(E) = \max_{R \leq \overline{R}, K \leq 1} \left( K(\overline{R} - R) + K(R - p)W'(E) + \sigma_0^2 K^2 W''(E) \right).
\] (52)

It can be established that \( E_{\text{SB}}^{\text{min}} = 0 \) and that the optimal dividend barrier \( E_{\text{SB}}^{\text{max}} \) is given by the super-contact condition
\[
W''(E_{\text{SB}}^{\text{max}}) = 0.
\] (53)

As a consequence, it holds that \( W'(E) > 1 \) for \( E \in [0, E_{\text{SB}}^{\text{max}}] \), implying that social welfare is maximized at the highest possible (“reservation”) loan rate \( R_{\text{SB}}^{CE}(E) = \overline{R} \). Therefore, firms’ profits will be equal to zero and the social welfare function coincides with the value function of a

\(^{30}\)To ensure that a competitive equilibrium exists we make the implicit assumption that \( \overline{R} \geq R_{\text{max}}^{CE} \).
monopolistic bank, satisfying the ODE\footnote{In the setting with inelastic demand social welfare coincides with the market value of a monopolistic bank. However, this result does not extend to the more realistic cases where demand for loans is elastic.}

\[ \rho W(E) = \max_{K \leq 1} K (R - p) W'(E) + \frac{\sigma^2_0 K^2}{2} W''(E). \quad (54) \]

Maximizing the right-hand side of the above equation with respect to the aggregate lending volume \( K \) yields:

\[ K^{SB}(E) = \min \left\{ 1, \frac{(R - p)}{\sigma^2_0} \left( -\frac{W''(E)}{W'(E)} \right)^{-1} \right\}. \quad (55) \]

Note that this condition shares some similarities with condition (27) in the competitive equilibrium, with the notable difference that loan supply depends on the social implied risk aversion with respect to variation in aggregate equity, \(-W''(E)/W'(E)\), whereas in the competitive equilibrium, banks’ inverse supply \( R \) is driven by the private implied risk aversion \(-u'(E)/u(E)\). Substituting \( K^{SB} \) back into (54) implies that for interior loan volumes (\( K^{SB} < 1 \)), the social welfare function has an explicit solution

\[ W_1(E) = c_2 \left( \frac{E}{a} - c_1 \right)^{\eta}, \quad (56) \]

where \( \eta \equiv \frac{2\rho \sigma^2_0}{(R - p)^2 + 2\rho \sigma^2_0} \) and \( \{c_1, c_2\} \) are constants to be determined.\footnote{We indicate the solution by \( W_1 \) in order to differentiate it from its counterpart \( W_2 \) in the region where the corner solution \( R = 1 \) applies.}

Substituting the general solution of the welfare function (56) into Equation (55) shows that the optimal volume of lending, if interior, linearly increases with aggregate capital \( E \):

\[ K^{SB}(E) = \min \left\{ 1, \frac{2\rho}{R - p} \left( \frac{E}{a} - c_1 \right) \right\}. \quad (57) \]

Let \( \hat{E}_K \) denote the critical level of aggregate capital that satisfies

\[ K^{SB}(\hat{E}_K) = 1, \quad (58) \]

such that the welfare maximizing loan volume satisfies \( K^{SB}(E) < 1 \) for \( E < \hat{E}_K \) and \( K^{SB}(E) = 1 \) for \( E \geq \hat{E}_K \).\footnote{Note that \( \hat{E}_K \) does not need to be positive, in which case \( K^{SB}(E) = 1 \) for all \( E \in \left[ E^{SB}_{min}, E^{SB}_{max} \right] \). This scenario might materialize for relatively low values of \( \gamma \) and \( \sigma_0 \).} That is, for low levels of aggregate bank capital, the social planner restricts aggregate lending below the level that is attained in the competitive equilibrium. The reason for this is that, in the competitive equilibrium, each bank maximizes its profits, taking the aggregate capitalization of the banking sector as given. In particular, banks do not take into account how their individual lending decisions affect the banking sector’s exposure to common productivity shocks (i.e., the endogenous volatility of aggregate bank capital).
Consider now the region \((\hat{E}_K, E_{SB}^{max})\), in which the social planner would allow the banking sector to operate at the maximum feasible scale, i.e., \(K^{SB}(E) = 1\). Substituting \(K^{SB}(E) = 1\) into (54) and solving the obtained ODE under the boundary conditions (48) and (53), yields the explicit solution of the welfare function for \(E \in (\hat{E}_K, E_{SB}^{max})\),

\[
W_2(E) = \frac{1}{\beta_1 - \beta_2} \left[ \frac{\beta_2 e^{\beta_1(E-E_{SB}^{max})}}{\beta_1} + \frac{\beta_1 e^{\beta_2(E-E_{SB}^{max})}}{\beta_2} \right],
\]

where \(\beta_1 > 0\) and \(\beta_2 < 0\) are the roots of the characteristic equation \(\rho = (\bar{R} - p)\beta + \frac{\sigma_0^2}{2} \beta^2\).\(^{43}\)

Piecing two regions together, we require the continuity of the welfare function and its first derivative at the critical threshold \(\hat{E}_K\), i.e.,

\[
W_1(\hat{E}_K) = W_2(\hat{E}_K),
\]

\[
W_1'(\hat{E}_K) = W_2'(\hat{E}_K).
\]

We are thus left with four conditions (47), (58), (60), and (61), which allows us to determine the four remaining unknowns \(c_1, c_2, \hat{E}_K, \) and \(E_{SB}^{max}\).

Figure 4 illustrates the optimal loan rate (the left-hand side panel) and lending (the right-hand side panel) in the competitive equilibrium (CE) and in the second best (SB). In the second best allocation, banks lend (weakly) less than in the competitive equilibrium, which reflects the pecuniary externality inflicted by each bank on its competitors. The social planner internalizes this externality and reduces the exposure to macro shocks \((\sigma_0 K(E))\), in the states of the world where the implied risk aversion with respect to fluctuations in \(E\) is most severe (i.e., close to the recapitalization barrier \(E_{min} = 0\)). Moreover, note that, while in the second best, the optimal loan rate is always equal to the maximum rate \(\bar{R}\), in the competitive setting, it is strictly decreasing in aggregate equity. As a result, the instantaneous profit of the banking sector in the second best allocation is larger than in the competitive equilibrium, which makes it easier for banks to rebuild equity buffers after negative shocks. Thus, the second best allocation implies lower risk exposure and higher margins. This implies a somewhat counterintuitive result: the optimal target equity buffers in the second best allocation are smaller than in the competitive equilibrium, i.e., \(E_{SB}^{max} < E_{CE}^{max}\).

### 4.3.3 Linear demand

We now turn to the example of a linear demand for loans. To this end, we consider the loan demand specification introduced in (44) with \(\beta = 1\), i.e.,

\[
L(R) = \frac{\bar{R} - R}{R - p}.
\]

\(^{43}\)In particular, we have \(\beta_{1,2} = -\frac{R - p \pm \sqrt{(\bar{R} - p)^2 + 2p\sigma_0^2}}{\sigma_0^2}\).
Figure 4: Loan rate and credit volume under *inelastic* loan demand: CE vs SB

Notes: this figure reports the typical patterns of the loan rates (the left-hand side panel) and credit volumes (the right-hand side panel) in the competitive equilibrium (dash-dotted line) and the second best allocation (solid line). Parameter values: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $\gamma = 2$, $R = 0.1$, $\beta = 0$.

The second best lending strategy can be defined as the solution to the following Bellman equation

$$
\rho W(E) = \max_{K \geq 0} (R - p) \frac{K^2}{2} + K (1 - K) (R - p) W'(E) + \frac{\sigma_0^2}{2} K^2 W''(E),
$$

subject to the boundary conditions $W'(0) = 1 + \gamma$ and $W'(E_{SB_{max}}) = 1$. The optimal dividend barrier $E_{SB_{max}}$ satisfies $W''(E_{SB_{max}}) = 0$. The first-order condition of the above maximization problem determines the optimal level of lending:

$$
K_{SB}(E) = \frac{(R - p) W'(E)}{(R - p)[2W'(E) - 1] - \sigma_0^2 W''(E)}.
$$

With elastic demand, social welfare does no longer coincide with the value of a monopolistic bank. Therefore, let us consider the latter as another benchmark. The value function of a monopolistic bank, $V(E)$, satisfies the Bellman equation

$$
\rho V(E) = \max_{K \geq 0} K (1 - K) (R - p) V'(E) + \frac{\sigma_0^2}{2} K^2 V''(E),
$$

which differs from (62) only by the fact that the firms’ profits do not show up in the Bellman equation.

The optimal level of lending chosen by a monopolistic bank is given by

$$
K_{Monop}(E) = \frac{(R - p) V'(E)}{2(R - p) V'(E) - \sigma_0^2 V''(E)}.
$$

\footnote{It is easy to see that, at the target level of aggregate bank equity, $E_{SB_{max}}$, that satisfies equations $W'(E_{SB_{max}}) = 1$ and $W''(E_{SB_{max}}) = 0$, we have $R_{SB}(E_{max}) = p$.}
and the value function $V(E)$ is a solution of the ODE:

$$V''(E) = \frac{(\bar{R} - p)V'(E)\left[4\rho V(E) - (\bar{R} - p)V'(E)\right]}{2\rho \sigma^2_0 V(E)},$$

subject to the usual conditions $V'(0) = 1 + \gamma$ and $V'(E_{\text{max}}^{\text{Monop}}) = 1$, with $E_{\text{max}}^{\text{Monop}}$ satisfying $V''(E_{\text{max}}^{\text{Monop}}) = 0$.

Figure 5 contrasts the loan rate and lending functions emerging in the competitive, second best and monopolistic settings. The qualitative insights from the inelastic case framework are still true, albeit less clear, since now one the social planner takes into account the price depressing effect of an increase in lending and the associated reallocation of profits. This interference of the effects no longer allows us to clearly sign the lending distortions in the competitive equilibrium. However, the social planner still wants to restrict lending below the competitive level when $E$ is low, even though this reduces firm’s profits. The reason is that when aggregate capital is low, the social value of bank profits (which are used to build equity buffers) is larger than that of the value of the firms’ profits (which are consumed immediately) and also the implied risk-aversion is high. Yet, the loan rate (lending) corresponding to the second best allocation is always lower (higher) as compared to the one chosen in the monopolistic setting, which is a natural consequence of the fact that a monopolistic bank does not internalize the impact of its lending decisions on the firms’ profits. Moreover, in the second best allocation, a poorly capitalized banking sector is less (more) exposed to aggregate shocks which leads to higher (smaller) instantaneous bank profits than in the competitive (monopolistic) setting. This implies that the optimal dividend barrier under the second best allocation is smaller (larger) than in the competitive (monopolistic) setting, i.e., $E_{\text{max}}^{\text{Monop}} < E_{\text{max}}^{\text{SB}} < E_{\text{max}}^{\text{CE}}$.

Figure 5: Loan rate and credit volume under elastic loan demand: CE vs SB

Notes: this figure reports the typical patterns of the loan rates (the left-hand side panel) and credit volumes (the right-hand side panel) in the competitive equilibrium (dash-dotted lines), the second best allocation (solid lines) and the monopolistic setting (dashed lines). Parameter values: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $\gamma = 0.2$, $\bar{R} = 0.15$, $\beta = 1$. 

30
5 Impact of capital regulation

So far our analysis has been focused on the "laissez-faire" environment in which banks face no regulation. Our objective in this section is to understand the impact of capital regulation on bank policies. Let us assume that public authorities enforce a minimum capital requirement, under which each bank must maintain equity capital above a certain fraction of loans, i.e.,

\[ e_t \geq \Lambda k_t, \]

where \( \Lambda \in (0, 1] \) is the minimum capital ratio.\(^{45} \)

Note that, under such a formulation, banks have two options to comply with minimum capital requirements. The first option is to immediately recapitalize as soon as the regulatory constraint starts binding. The second option consists in cutting on lending and simultaneously reducing deposit taking. We show below that, in our model, banks use the first option when \( E_t \) is small and the second one when \( E_t \) is large. In other words, a capital ratio does two things: it forces banks to recapitalize earlier (i.e., \( E_{\text{min}} > 0 \)) and to reduce lending as compared to the unregulated case.

To solve for the regulated equilibrium, we again start by looking at the maximization problem of an individual bank. As in the unregulated set-up, bank shareholders maximize the market value of their claim by choosing their lending, recapitalization and dividend policies subject to the regulatory restriction on the volume of lending:

\[
v_{\Lambda}(e, E) \equiv e u_{\Lambda}(E) = \max_{k_t \leq \frac{e}{\Lambda}, d\delta_t, d\gamma_t} \mathbb{E} \left[ \int_0^{+\infty} e^{-\rho t} (d\delta_t - (1 + \gamma) d\gamma_t) | e_0 = e, E_0 = E \right]. \tag{64} \]

To have the intuition of the solution to the above problem, recall that, in the unregulated case, bank recapitalizations take place only when equity is completely depleted. Thus, it is natural to expect that the regulatory constraint will be binding for relatively low levels of equity. Indeed, in the general case, the bank may find itself in one of two cases: (i) when its level of equity is relatively high, the regulatory constraint is not binding and the volume of lending is still determined by the firms’ demand for credit; (ii) in the states with low equity, the regulatory constraint binds and the volume of lending is determined by \( k_t = e_t / \Lambda \). Due to the homotheticity property, at each point in time, all banks have the same leverage ratio. Thus, it is legitimate to anticipate the existence of the critical level of bank capital \( E_{\text{c}}^\Lambda \), such that the regulatory constraint binds (for all banks) for any \( E \in [E_{\text{min}}^\Lambda, E_{\text{c}}^\Lambda] \) and is slack for any \( E \in (E_{\text{c}}^\Lambda, E_{\text{max}}^\Lambda] \). This critical threshold \( E_{\text{c}}^\Lambda \) must satisfy

\[
\frac{K(E_{\text{c}}^\Lambda)}{E_{\text{c}}^\Lambda} = \frac{1}{\Lambda}.
\]

\(^{45}\)Since our model only considers one type of bank assets (loans), we cannot discuss the issue of risk weights or distinguish a leverage ratio from a risk-weighted capital ratio.
For Λ high enough, $E_t^\Lambda$ tends to $E_{max}^\Lambda$ and the unconstrained region disappears entirely. For future reference, we denote the critical leverage ratio above which the unconstrained region disappears by $\Lambda^*$.  

**Proposition 5** For all $\Lambda \in (0, 1]$, there exists a unique regulated equilibrium, where the support of $E_t$ is $[E_{min}^\Lambda, E_{max}^\Lambda]$. This equilibrium is characterized by one of two regimes:

a) for $\Lambda \geq \Lambda^*$, the regulatory constraint binds over the entire interval $[E_{min}^\Lambda, E_{max}^\Lambda]$. The loan rate is explicitly given by

$$R(E_t) = L^{-1}[E_t/\Lambda],$$

where $L^{-1}$ is the inverse function of the loan demand. The dynamics of aggregate bank capital is given by:

$$\frac{dE_t}{E_t} = \frac{1}{\Lambda} \left( (L^{-1}[E_t/\Lambda] - p)dt - \sigma_0 dZ_t \right), \quad E \in (E_{min}^\Lambda, E_{max}^\Lambda).$$

b) for $\Lambda < \Lambda^*$, the capital constraint only binds for $E \in [E_{min}^\Lambda, E_c^\Lambda]$ and is slack for $E \in (E_c^\Lambda, E_{max}^\Lambda]$, where $E_c^\Lambda$ is a critical capitalization level. When $E \in (E_{min}^\Lambda, E_c^\Lambda]$, the dynamics of aggregate equity and the loan rate function are defined as in the regime a). When $E \in (E_c^\Lambda, E_{max}^\Lambda]$, the loan rate satisfies the first-order differential equation

$$R'(E) = -\frac{1}{H[R(E)]}$$

with the boundary condition $R(E_c^\Lambda) = L^{-1}[E_c^\Lambda/\Lambda]$.

In either regime, banks distribute dividends when $E_t = E_{max}^\Lambda$ and recapitalize when $E_t = E_{min}^\Lambda$.

We show in Appendix A that, in the unconstrained region $(E_c^\Lambda, E_{max}^\Lambda)$, the market-to-book value still simultaneously satisfies Equations (27) and (30), whereas in the constrained region $(E_{min}^\Lambda, E_c^\Lambda)$ it satisfies instead

$$\rho = \frac{E(L^{-1}[E/\Lambda] - p) u_\Lambda'(E)}{\lambda u_\Lambda(E)} + \frac{\sigma_0^2 E^2}{2\Lambda^2} \frac{u_\Lambda''(E)}{u_\Lambda(E)},$$

under the condition

$$\frac{u_\Lambda'(E)}{u_\Lambda(E)} \geq -\frac{L^{-1}[E/\Lambda] - p}{\sigma_0^2 E/\Lambda},$$

with equality at $E = E_c^\Lambda$.

The optimal recapitalization and payout decisions are characterized by two boundaries, $E_{min}^\Lambda$ (recapitalizations) and $E_{max}^\Lambda$ (dividend payments), such that $u_\Lambda(E_{min}^\Lambda) = 1 + \gamma$ and $u_\Lambda(E_{max}^\Lambda) = 1$. In Appendix C.1 we provide the detailed description of the computational procedure that enables us to numerically solve for the regulated equilibrium.

To get general insight into the impact of capital regulation on the cost of credit and the bank’s policies, we perform a comparative static analysis by computing the equilibrium characteristics of...
bank policies for a wide range of minimum capital ratios. The left panel of Figure 6 reports the values of $E_{\min}^\Lambda$ and $E_{\max}^\Lambda$ (solid lines), contrasting them to the values $E_{\min}$ and $E_{\max}$ computed in the unregulated setting (dashed lines). The boundaries of the corresponding loan rates are reported in the right panel of Figure 6. As long as the capital ratio is not too high, the bank may find itself in either constrained or unconstrained region, but above some critical level $\Lambda^*$ of a capital ratio (about 62% in this numerical example), the regulatory constraint is always binding. In contrast with the unregulated set-up, shareholders always recapitalize the bank before completely exhausting bank capital.47 The minimum loan rate $R_{\min}^\Lambda$ is still equal $p$ as long as the unconstrained region exists ($\Lambda < \Lambda^*$). However, for $\Lambda \geq \Lambda^*$, the increase in $\Lambda$ entails the upward shift in the entire support of $R$, as the loan rate becomes entirely determined by the binding regulatory constraint.

Figure 6: Minimum capital requirements and bank policies

Notes: this figure illustrates the effect of minimum capital requirements on banks’ policies. Solid lines in the left panel depict the recapitalization ($E_{\min}^\Lambda$) and payout ($E_{\max}^\Lambda$) barriers, as well as the critical threshold $E_{\min}^\Lambda$ such that the regulatory constraint is binding for $E \leq E_{\min}^\Lambda$. Solid lines in the right panel depict the minimum ($R_{\min}^\Lambda \equiv R(E_{\min}^\Lambda)$) and maximum ($R_{\max}^\Lambda \equiv R(E_{\max}^\Lambda)$) boundaries for the loan rate, as well as the critical loan rate $R_{\min}^\Lambda$ such that the regulatory constraint is binding for any $R \geq R_{\min}^\Lambda$. Dashed lines in both panels represent the outcomes of the competitive equilibrium in the unregulated set-up. Dot-dashed lines represent the temporal average values of aggregate capitalization and loan rate. For $\Lambda > \Lambda^*$, a critical level $E_{\min}^\Lambda$ does not exist, i.e., the regulatory constraint is binding for any $E \in [E_{\min}^\Lambda, E_{\max}^\Lambda]$. Parameter values: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $\gamma = 0.2$, $\beta = 1$ and $\bar{R} = 0.15$.

Figure 7 illustrates the impact of capital regulation on loan rates (left panel) and lending (right panel). It shows that increasing the minimum capital ratio leads to an increase of the loan rate for any level of bank capitalization, which ultimately translates into the reduction of lending. Interestingly, this happens even when the capital constraint is not binding, i.e., when $E > E_{\min}^\Lambda$. The reason is that banks anticipate that the regulatory constraint will bind in the future and require higher lending premium by precautionary reasons. Overall, these results suggest that, even in the absence of bank defaults, capital regulation can be useful to address the problem of excessive lending revealed by our welfare analysis in the previous section.

47In this regard, a parallel can be made with the discrete-time version of our model in which the presence of shadow costs associated with the market-based leverage constraint induced banks to recapitalize at a strictly positive level of
Figure 7: Impact of capital regulation on loan rates and credit volume

Notes: this figure illustrates the impact of a constant minimum capital ratio on loan rates (the left hand side panel) and lending (the right hand side panel). Solid curves correspond to the unregulated set-up where $\Lambda = 0\%$. Under the low levels of minimum capital ratio (5%, 20%, 35% in this numerical example), the regulatory constraint binds only for the lower levels of bank capitalization. For the high levels of capital ratio (65% in our example), the slope of the lending curve is entirely determined by the binding regulatory constraint. Parameter values: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $\gamma = 0.2$, $\beta = 1$ and $\overline{R} = 0.15$.

It is also worthwhile to note that the immediate implications of an unanticipated increase of the minimum capital ratio would differ depending on the current level of aggregate bank capitalization $E_t$. To see this, let us consider the following thought experiment: assume that 10% capital requirements are imposed on the previously-unregulated banking sector and there is a credible commitment for no further changes in the capital regulation framework. In the new regulated equilibrium, aggregate bank capital will fluctuate between $E_{0.1}^{\min}$ and $E_{0.1}^{\max}$. If current aggregate capitalization of the banking sector is low (i.e., $E_t < E_{0.1}^{\min}$), banks will have to raise $\Delta E \equiv E_{0.1}^{\min} - E_t$ of new equity, which would be followed then by a substantial reduction of lending. By contrast, for highly-capitalized banks with aggregate equity $E_t \geq E_{0.1}^{\min}$ there is no need to raise new equity. Moreover, in this case banks will generally implement less dramatic reductions in lending as compared to undercapitalized banks.

Understanding the long run effects of capital requirements, however, requires a more subtle approach than a simple comparison of the lending patterns that arise under different minimum capital ratios. In particular, the probabilistic behavior of the economy should be taken into account. We therefore analyze the long-run impact of capital regulation on lending by looking at the temporal average loan rate $\overline{R}^\Lambda$ that is computed according to the following formula:

$$\overline{R}^\Lambda = \lim_{T \to \infty} \frac{1}{T} \int_0^T R_t dt = \int_{E_{0.1}^{\min}}^{E_{0.1}^{\max}} R(E) g_\Lambda(E) dE,$$

where $g_\Lambda(E)$ is the ergodic density function of $E$ in the regulated competitive equilibrium (see aggregate capitalization.

---

48 All further temporal average values we use in our analysis are computed according to the general formula: $\overline{x} = \int_{E_{0.1}^{\min}}^{E_{0.1}^{\max}} x(E) g_\Lambda(E) dE$, where $x(E)$ is any function of $E$. 34
The dash-dotted line in the right hand side panel of Figure 6 corresponds to the typical pattern of the temporal average loan rate $\overline{R}$. It shows that, as $\Lambda$ goes up, $\overline{R}$ increases but tends to the lower boundary of feasible loan rates, $R_{\min}$. In other words, even though a higher minimum capital ratio induces a substantial shift of the feasible range of the loan rates, its long-run impact on lending is rather moderate. To understand this result, it is instructive to consider the effect of higher minimum capital ratios on the temporal average level of bank capitalization, $E$, that is depicted by the dash-dotted line in the left-hand side panel of Figure 6. Its pattern (converging to the dividend barrier $E_{\max}$ when $\Lambda$ goes up) suggests that, under tighter capital requirements, the banking sector becomes more stable so as in the long run banks spend more times in the states with abundant capital.

To properly measure the effect of capital requirements on the stability of the banking sector, we compute the expected time to reach the recapitalization barrier starting from the temporal average level of aggregate capitalization $\overline{E}$. Figure 8 depicts the typical pattern of this stability measure as the function of $\Lambda$, suggesting that increasing the minimum capital ratio enhances the stability of the banking sector.

A substantial body of the existing literature attributes the stability effect of higher capital requirements to the reduction in the moral hazard incentives on the banks’ side (see e.g., Martinez-Miera and Suarez (2014), Nguyen (2014)). To understand why higher minimum capital ratios help promote financial stability in our setting (which completely abstracts from a moral hazard problem), it is important to emphasize that the primary effect of the enhanced capital requirements on banks’ policies in the presence of financing frictions manifests itself via the reduction in lending. In this regard, minimum capital regulation can be seen as a tool addressing the problem of excessive lending by competitive banks that was discussed in Section 4.3 (i.e., by using the minimum capital regulation, the social planner can implicitly control the endogenous volatility of aggregate capital). Under higher minimum capital requirements, banks have lower exposure to aggregate shocks and, at the same time, charge higher loan rates, which makes it easier for them to recover after the series of negative shocks. This enables banks to spend more time in the states with higher aggregate capitalization and thus relatively lower loan rates (within the feasible corridor). By reducing the temporal average loan rate in the long run, the positive effect from enhanced financial stability ultimately mitigates the direct negative impact inflicted by higher capital requirements on lending.

### 6 Conclusion

This paper develops a general equilibrium model of commercial banking, in which banks satisfy households’ needs for safe deposits and channel funds to the productive sector. Bank capital plays
the role of a loss-absorbing buffer that insulates banks from the need to undertake costly recapitalizations too often. In our model, the aggregate level of bank capitalization drives the cost and the volume of lending. Specifically, we establish a negative relation between the equilibrium loan rate and the level of aggregate bank capital that is supported by data.

In the continuous-time version of the model with no capital regulation, the equilibrium dynamics of loan rates can be obtained in closed form, which enables us to study analytically the long-run behavior of the economy. We show that this behavior is ergodic and is essentially determined by the volatility of the loan rate and the magnitude of financing frictions. The competitive equilibrium is not constrained efficient because of a pecuniary externality which leads to excessive risk exposure of the banking sector (via excessive lending).

A natural application of our model is the analysis of the impact of minimum capital requirements on lending. We find that implementing a higher capital ratio produces two intrinsically related effects. The immediate effect is an increase of loan rates and reduction in lending, which implies a reduction of banks’ exposure to aggregate shocks. As a result, the banking sector becomes more stable. Namely, banks spend more time in the states with abundant capitalization and thus relatively low loan rates, which ultimately leads to a moderate increase in average loan rate in the long run.

It should be acknowledged that our model suffers from several limitations. First, it only considers commercial banking activities (deposit taking and lending), while neglecting market activities such as investment in securities and derivatives trading. Second, it only considers diffusion risks that do not lead to actual bank defaults, but merely fluctuations in the size of the banking sector. A consequence of these limitations is that we do not address the important questions of banks’ excessive risk-taking and the role of capital regulation in the mitigation of this behavior. However, these issues have already been the subject of a large academic literature. Finally, in this paper we have focused on the scenario in which private bank recapitalizations prevent systemic crises.
from happening. A potential direction of further investigations would be to explore the alternative scenario, where shareholders do not spontaneously inject new capital, and public authorities are forced to intervene.
Appendix A. Proofs

Proof of Lemma 1. The utility of the representative consumer is

\[ U = C_0 + \frac{E[\tilde{C}_1] + \lambda(D)}{1 + \rho}, \]

where the consumptions at \( t = 0 \) and \( t = 1 \) are given by aggregate budget constraints:

\[ C_0 = w_0 - D - (1 + \gamma)I, \]
\[ \tilde{C}_1 = (1 + r_D)D + \tilde{\pi}_B, \]

where \( \tilde{\pi}_B \) denotes the aggregate profits of banks. Inserting the expressions of \( C_0 \) and \( \tilde{C}_1 \) into \( U \) yields

\[ U = w_0 - (1 + \gamma)I + \frac{E[\tilde{\pi}_B]}{1 + \rho} + \frac{(r_D - \rho)D + \lambda(D)}{1 + \rho}. \]

The first-order condition with respect to \( D \) immediately yields \( \lambda' = \rho - r_D \). Q.E.D.

Proof of Lemma 2. Omitted.

Proof of Proposition 1. Omitted.

Proof of Proposition 2. By the standard dynamic programming arguments, shareholder value \( v(e, E) \) must satisfy the Bellman equation:\(^{50}\)

\[ \rho v = \max_{k \geq 0, \delta \geq 0, d \geq 0} \left\{ d\delta(1 - v_e) - di(1 + \gamma - v_e) \right. \]
\[ + k[(R(E) - p)v_e + \sigma_0^2 K(E)v_{ee}] + \frac{k^2 \sigma_0^2}{2}v_{ee} \]
\[ + K(E)(R(E) - p)v_E + \frac{\sigma_0^2 K^2(E)}{2}v_{EE} \right\}. \tag{A1} \]

Using the fact that \( v_e = eu(E) \), one can rewrite the Bellman equation (A1) as follows:

\[ \rho u(E) = \max_{k \geq 0, \delta \geq 0, d \geq 0} \left\{ \frac{d\delta}{e}[1 - u(E)] - \frac{di}{e}[1 + \gamma - u(E)] \right. \]
\[ + \frac{k}{e}[(R(E) - p)u(E) + \sigma_0^2 K(E)u'(E)] \]
\[ + K(E)(R(E) - p)u'(E) + \frac{\sigma_0^2 K^2(E)}{2}u''(E) \right\}. \tag{A2} \]

A solution to the maximization problem in \( k \) only exists when

\[ \frac{u'(E)}{u(E)} \leq -\frac{R(E) - p}{\sigma_0^2 K(E)}, \tag{A3} \]

with equality when \( k > 0 \).

\(^{50}\)For the sake of space, we omit the arguments of function \( v(e, E) \).
Under conjecture that \( R(E) \geq p \) (which will be verified ex-post), it follows from the above expression that \( u(E) \) is a decreasing function of \( E \). Then, the optimal payout policy maximizing the right-hand side of (A2) is characterized by a critical barrier \( E_{\text{max}} \) satisfying

\[
 u(E_{\text{max}}) = 1,
\]  
(A4)

and the optimal recapitalization policy is characterized by a barrier \( E_{\text{min}} \) such that

\[
 u(E_{\text{min}}) = 1 + \gamma. 
\]  
(A5)

In other words, dividends are only distributed when \( E_t \) reaches \( E_{\text{max}} \), whereas recapitalization occurs only when \( E_t \) reaches \( E_{\text{min}} \). Given (A3), (A4), (A5) and \( k > 0 \), it easy to see that, in the region \( E \in (E_{\text{min}}, E_{\text{max}}) \), market-to-book value \( u(E) \) satisfies:

\[
 p u(E) = K(E)(R(E) - p)u'(E) + \frac{\sigma_0^2 K^2(E)}{2} u''(E). 
\]  
(A6)

Note that, at equilibrium, \( K(E) = L[R(E)] \). Taking the first derivative of (A3), we can compute \( u''(E) \). Inserting \( u''(E) \) and \( u'(E) \) into (A6) and rearranging terms yields:

\[
 R'(E) = -\frac{1}{\sigma_0^2} \frac{2\rho \sigma_0^2 + (R(E) - p)^2}{L[R(E)] - [R(E) - p]L'[R(E)]}. 
\]  
(A7)

Since \( L'(R(E)) < 0 \), it is clear that \( R'(E) < 0 \) if \( R(E) > p \). To verify that \( R(E) > p \) for any \( E \in [E_{\text{min}}, E_{\text{max}}] \), it is sufficient to show that \( R_{\text{min}} \equiv R(E_{\text{max}}) \geq p \).

To obtain \( R_{\text{min}} \), let

\[
 V(E) \equiv E u(E)
\]
denote the market value of the entire banking sector. At equilibrium, dividends are distributed when the marginal value of bank capital equals the marginal value of dividends, which implies

\[
 V'(E_{\text{max}}) = u(E_{\text{max}}) + E_{\text{max}} u'(E_{\text{max}}) = 1.
\]

Similarly, recapitalizations take place when the marginal value of bank capital equals the marginal costs of recapitalizing the banks, which implies

\[
 V'(E_{\text{min}}) = u(E_{\text{min}}) + E_{\text{min}} u'(E_{\text{min}}) = 1 + \gamma.
\]

Given (A4) and (A5), it must hold that

\[
 u'(E_{\text{max}}) = 0,
\]

and

\[
 E_{\text{min}} = 0.
\]

Inserting \( u'(E_{\text{max}}) = 0 \) into the binding condition (A3) immediately shows that \( R_{\text{min}} = p \), so that \( R(E) > p \) for any \( E \in [E_{\text{min}}, E_{\text{max}}] \).

\[\text{As we show in Appendix B, these properties can be alternatively established by looking at the limit of the discrete-time equilibrium characterization for } h \to 0.\]
Hence, the loan rate $R(E)$ can be computed as a solution to the differential equation (A7), which yields:

$$\int_0^E R'(s) ds = R(E) - R_{\text{max}},$$

(A8)

where $R_{\text{max}} \equiv R(0)$.

To obtain $E_{\text{max}}$, we use the fact that individual banks’ optimization with respect to the recapitalization policy implies $u(0) = 1 + \gamma$. Integrating equation (A3) in between $E_{\text{min}} = 0$ and $E_{\text{max}}$, while taking into account the condition $u(E_{\text{max}}) = 1$, yields an equation that implicitly determines $E_{\text{max}}$:

$$u(E_{\text{max}}) \exp \left( \int_0^{E_{\text{max}}} \frac{R(E) - p}{\sigma^2 R(E)} dE \right) = 1 + \gamma.$$  

(A9)

Equation (36) immediately follows from the change of variable of integration in equation (A8) and (A9), i.e., $dR = R'(E)dE$. Q.E.D.

Proof of Proposition 3. Omitted.

Proof of Proposition 4. Consider the process $R_t$ that evolves according to

$$dR_t = \mu(R_t) dt + \sigma(R_t) dZ_t, \quad p \leq R_t \leq R_{\text{max}},$$  

(A10)

with reflections at both ends of the support.

Let $g(t, R)$ denote the probability density function of $R_t$. It must satisfy the forward Kolmogorov equation:

$$\frac{\partial g(t, R)}{\partial t} = -\frac{\partial}{\partial R} \left( \mu(R) g(t, R) - \frac{1}{2} \frac{\partial^2}{\partial R^2} \left[ \sigma^2(R) g(t, R) \right] \right).$$  

(A11)

Since the process $R_t$ is stationary, we have $\frac{\partial g(t, R)}{\partial t} = 0$ and thus $g(t, R) \equiv g(R)$. Integrating Equation (A11) over $R$ yields:

$$\mu(R) g(R) = \frac{1}{2} \frac{\partial}{\partial R} \left[ \sigma^2(R) g(R) \right],$$

where the constant of integration is set to zero because of reflection properties of the process. Solving the above equation by using the change of variable $\tilde{g}(R) = \sigma^2(R) g(R)$ ultimately yields:

$$g(R) = \frac{C_0}{\sigma^2(R)} \exp \left( \int_p^{R_{\text{max}}} \frac{2\mu(s)}{\sigma^2(s)} ds \right),$$  

(A12)

where the constant $C_0$ is chosen so as to normalize the solution to 1 over the region $[p, R_{\text{max}}]$, i.e., $\int_p^{R_{\text{max}}} g(R) dR = 1$.

To ensure that the distribution of $R$ is non-degenerate, it is sufficient to check that $\sigma(R) > 0$ for any $R \in [p, R_{\text{max}}]$. From the expression of $\sigma(R)$, it is easy to see that this condition holds for any loan demand specifications such that $L'(R) < 0$ and $L(R) > 0$. Q.E.D.

Proof of Proposition 5. Consider the shareholders’ maximization problem stated in (64). By the standard dynamic programming arguments and the fact that $v_\Lambda(e, E) = eu_\Lambda(E)$, where $u_\Lambda(\dot{E})$
satisfies the following Bellman equation:

\[
\rho u_A(E) = \max_{d\delta \geq 0, d\gamma \geq 0} \left\{ \frac{d\delta}{e} [1 - u_A(E)] - \frac{d\gamma}{e} [1 + \gamma - u_A(E)] \right\} + \\
+ \max_{0 < k \leq \epsilon / \Lambda} \left\{ \frac{k}{e} \left[ (R(E) - p)u_A(E) + \sigma_0^2 K(E) u_A'(E) \right] \right\} + \\
+ K(E) [R(E) - p]u_A'(E) + \frac{\sigma_0^2 K^2(E)}{2} u_A''(E).
\]  

(A13)

A solution to (A13) exists only if \(K(E) \leq E / \Lambda\), and

\[
B(E) := \frac{R(E) - p}{\sigma_0^2 K(E)} \geq -\frac{u_A'(E)}{u_A(E)} \equiv y(E),
\]

with equality when \(K(E) < E / \Lambda\).

The optimal dividend and recapitalization policies are characterized by barriers \(E_{\text{max}}^\Lambda\) and \(E_{\text{min}}^\Lambda\) such that \(u_A(E_{\text{max}}^\Lambda) = 1 \) and \(u_A(E_{\text{min}}^\Lambda) = 1 + \gamma\). Moreover, by the same reason that in the unregulated equilibrium, it must hold that

\[
E_{\text{max}}^\Lambda u_A'(E_{\text{max}}^\Lambda) = 0,
\]

and

\[
E_{\text{min}}^\Lambda u_A'(E_{\text{min}}^\Lambda) = 0.
\]

From the first equation we immediately have \(u_A'(E_{\text{max}}^\Lambda) = 0\). The implication of the second equation is less trivial. Recall that in the unregulated equilibrium, we had \(E_{\text{min}} = 0\). However, under capital regulation, this is no longer possible, as \(K(E) = E / \Lambda\) on the constrained region and thus equation (A13) could not hold when \(E \to 0\). Therefore, it must be that \(u_A'(E_{\text{min}}^\Lambda) = 0\).

With the conjecture that there exists some critical threshold \(E_c^\Lambda\) such that the regulatory constraint is binding for \(E \in [E_{\text{min}}^\Lambda, E_c^\Lambda]\) and is slack for \(E \in [E_c^\Lambda, E_{\text{max}}^\Lambda]\), equation (A13) can be rewritten as follows:

\[
\rho = -\pi_B(E) y(E) + \frac{\sigma_0^2 K^2(E)}{2} [y^2(E) - y'(E)] + \mathbbm{1}_{E \in [E_{\text{min}}^\Lambda, E_c^\Lambda]} \frac{[R(E) - p - \sigma_0^2 K(E) y(E)]}{\Lambda}
\]

(A15)

where \(\mathbbm{1}_{\{\}}\) is the indicator function and \(\pi_B(E)\) denotes the aggregate expected profit of banks:

\[
\pi_B(E) = K(E) [R(E) - p],
\]

the volume of credit \(K(E)\) satisfies

\[
K(E) = \begin{cases} 
E / \Lambda, & E \in [E_{\text{min}}^\Lambda, E_c^\Lambda] \\
\Lambda [R(E)], & E \in (E_c^\Lambda, E_{\text{max}}^\Lambda],
\end{cases}
\]

(A16)

and the loan rate \(R(E)\) is given by

\[
R(E) = \begin{cases} 
L^{-1} [E / \Lambda], & E \in [E_{\text{min}}^\Lambda, E_c^\Lambda] \\
R'(E) = -1 / \Lambda [R(E)], & R(E_c^\Lambda) = L^{-1} [E_c^\Lambda / \Lambda], & E \in (E_c^\Lambda, E_{\text{max}}^\Lambda],
\end{cases}
\]

(A17)
where $L^{-1}$ is the inverse function of the demand for loans and function $H(.)$ is defined in (35). The critical threshold $E^\Lambda_c$ must satisfy equation

$$y(E^\Lambda_c) = B(E^\Lambda_c).$$

If $y(E) < B(E)$ for any $E \in [E^\Lambda_{min}, E^\Lambda_{max}]$, then the regulatory constraint is always binding and $y(E)$ satisfies equation (A15) with $E^\Lambda_c = E^\Lambda_{max}$. The condition $u'_\Lambda(E^\Lambda_{min}) = 0$ yields the boundary condition $y(E_{min}) = 0$. Similarly, the condition $u'_\Lambda(E^\Lambda_{max}) = 0$ translates into the boundary condition $y(E_{max}) = 0$. Q.E.D.

Appendix B. Discrete-time model with infinite horizon: example

B.1. Illustrative example

In this Appendix we provide the discrete-time characterization of the competitive equilibrium for the particular specification of the firms’ default probability which allows for convergence to the diffusion process in the continuous time limit:\footnote{Another interesting example would be the specification of the default probability generating a jump-process in the continuous-time limit:

$$\tilde{p}_t(h) = \begin{cases} p, & \text{with probability } 1 - \phi h \text{ (positive shock)}, \\ p - \frac{l}{h}, & \text{with probability } \phi h \text{ (negative shock)}, \end{cases}$$

where $\phi$ is the intensity of large losses and $l$ is the size of a (proportional) large loss.}

$$\tilde{p}_t(h) = \begin{cases} p - \sigma_0 \sqrt{h}, & \text{with probability } 1/2 \text{ (positive shock)}, \\ p + \sigma_0 \sqrt{h}, & \text{with probability } 1/2 \text{ (negative shock)}. \end{cases}$$

The individual bank’s optimization problem with respect to the volume of lending becomes:

$$(1 + \rho h) e_t u^h(E_t) = \max_{k^h_t} \mathcal{L}(e_t, E_t), \quad t = 0.. + \infty, \quad (B1)$$

with

$$\mathcal{L}(e_t, E_t) = \mathbb{E}\left[ (e_t + k^h_t[R^h(E_t) - \tilde{p}_t(h)]h) u^h(\tilde{E}_{t+h}) \right] - \theta^h(E_t) \left\{ k^h_t \sigma_0 \sqrt{h} - (R^h(E_t) - p)h - e_t \right\},$$

where $E \in [E^h_{min}, E^h_{max}]$, with $E^h_{min} = [u^h]^{-1}(1 + \gamma)$ and $E^h_{max} = [u^h]^{-1}(1)$, and $\theta^h(E_t)$ is the Lagrange multiplier associated with the market leverage constraint.

Conditionally on $E_t = E$, the value of $E_{t+h}$ only depends on the realization of the aggregate risk:

$$E_{t+h} = \begin{cases} E^+ \equiv E + L[R(E)]A^+, & \text{with } A^+ \equiv (R^h(E) - p)h + \sigma_0 \sqrt{h}, \\ E^- \equiv E + L[R(E)]A^-, & \text{with } A^- \equiv (R^h(E) - p)h - \sigma_0 \sqrt{h}, \end{cases}$$

Another interesting example would be the specification of the default probability generating a jump-process in the continuous-time limit:

$$\tilde{p}_t(h) = \begin{cases} p, & \text{with probability } 1 - \phi h \text{ (positive shock)}, \\ p - \frac{l}{h}, & \text{with probability } \phi h \text{ (negative shock)}, \end{cases}$$

where $\phi$ is the intensity of large losses and $l$ is the size of a (proportional) large loss.
Given the above dynamics of aggregate capitalization, necessary conditions for equilibrium are:

\[
\frac{1}{2} \left[ A^+ u^h(E^+) + A^- u^h(E^-) \right] = \theta^h(E) [\sigma_0 \sqrt{h} - (R^h(E) - p) h], \quad (B2)
\]

\[
\frac{1}{2} \left[ u^h(E^+) + u^h(E^-) \right] = (1 + \rho h) u^h(E) - \theta^h(E), \quad (B3)
\]

\[
\theta^h(E_i) \left\{ L[R^h(E)] [\sigma_0 \sqrt{h} - (R^h(E) - p) h] - E \right\} = 0. \quad (B4)
\]

We solve the above problem by conjecturing and verifying the existence of two non-empty regions: \([E^h_{\min}, \hat{E}^h]\), where the market leverage constraint is binding and \(\theta^h(E) > 0\), and \([\hat{E}^h, E^h_{\max}]\) where the market leverage constraint is slack and \(\theta^h(E) = 0\). The critical barrier \(\hat{E}^h\) is implicitly given by \(\theta^h(\hat{E}^h) = 0\).

First, consider the constrained region \([E^h_{\min}, \hat{E}^h]\). The equilibrium interest rate, \(R^h(E)\), is implicitly given by the binding market leverage constraint:

\[
L[R^h(E)] [\sigma_0 \sqrt{h} - (R^h(E) - p) h] = E.
\]

The above equation implies that \(E^+ \equiv 0\) in the region \([E^h_{\min}, \hat{E}^h]\). Furthermore, we have \(E^+ = 2\sigma_0 \sqrt{h} L[R^h(E)]\) and \(u(E^-) = u(E^h_{\min}) = 1 + \gamma\). Solving the system of equations (B3) and (B2) with respect to \(\theta^h(E)\) and \(u^h(E)\) yields:

\[
\theta^h(E) = \frac{1}{2} \left( \frac{1 + \beta^h[R^h(E)]}{1 - \beta^h[R^h(E)]} \right) u(2\sigma_0 \sqrt{h} L[R^h(E)]) - 1 - \gamma, \quad (B5)
\]

\[
u^h(E) = \frac{1}{1 + \rho h} \frac{u(2\sigma_0 \sqrt{h} L[R^h(E)])}{1 - \beta^h[R^h(E)]}, \quad (B6)
\]

where

\[
\beta^h[R^h(E)] = \frac{R^h(E) - p}{\sigma_0 \sqrt{h}}.
\]

In the unconstrained region, \([\hat{E}^h, E^h_{\max}]\), the equilibrium loan rate and the market-to-book value satisfy the system of equations:

\[
\left( 1 + \beta^h[R^h(E)] \right) u^h(E^+) = \left( 1 - \beta^h[R^h(E)] \right) u^h(E^-), \quad (B7)
\]

\[
2(1 + \rho h) u^h(E) = u^h(E^+) + u^h(E^-). \quad (B8)
\]

Rewriting Equation (B7) after taking expectation yields:

\[
R^h(E) = p + \frac{\sigma_0}{\sqrt{h}} \frac{u^h(E^-) - u^h(E^+)}{u^h(E^+) + u^h(E^-)} > p.
\]

It remains to check that \(E^h_{\max} > \hat{E}^h > E^h_{\min} > 0\).

First, let us show that \(E^h_{\min} > 0\). Suppose by way of contradiction that \(E^h_{\min} = 0\). Then, from the binding leverage constraint at the aggregate level it follows that \(R^h(0) = p + \frac{\sigma_0}{\sqrt{h}}\) and thus \(\beta^h[R^h(0)] = 1\). The optimal recapitalization policy implies that at \(E^h_{\min}\) it must hold that
$u^h(E^h_{\text{min}}) = 1 + \gamma$. However, evaluating (B6) at $E^h_{\text{min}} = 0$ yields $u(0) \equiv \infty$, which is incompatible with the previous statement. Therefore, $E^h_{\text{min}} > 0$, as claimed.

Next, we show that $\hat{E}^h > E^h_{\text{min}}$. Combining (B3) and $u^h(E^h_{\text{min}}) = 1 + \gamma$, it is immediate to see that $\theta^h(E^h_{\text{min}}) > 0$. Then, by continuity, it must be that $\theta^h(E) > 0$ in the vicinity of $E^h_{\text{min}}$.

Finally, let us show that $\hat{E}^h > E^h_{\text{max}}$. Assume by way of contradiction that $E^h_{\text{max}} = \hat{E}^h$. Then it must hold that $\theta^h(E^h_{\text{max}}) \geq 0$. The optimal dividend policy implies that $u(E^+) = 1$ for $E = E^h_{\text{max}}$. Note that $\theta^h(E^h_{\text{max}}) \geq 0$ if and only if
\[
\xi(h) := 1 + \beta^h[R^h(E^h_{\text{max}})] - (1 + \gamma)(1 - \beta^h[R^h(E^h_{\text{max}})]) > 0.
\]
Yet, when $h \to 0$, we have $\beta^h[R^h(E^h_{\text{max}})] \to 0$ and thus $\xi(h) < 0$, a contradiction. Hence, $E^h_{\text{max}} > \hat{E}^h$.

B.2. Convergence

We now establish convergence of the discrete-time version of the competitive equilibrium studied above to its continuous-time analogue studied in Section 4. Establishing convergence implies proving the following properties for $h \to 0$:

- $\hat{E}^h \to E^h_{\text{min}}$ and $E^h_{\text{min}} \to 0$;
- $R(E^h_{\text{max}}) \to p$;
- Equations (B7) and (B8) converge to Equations (27) and (30) when $h \to 0$.

a) First, let us show that $\hat{E}^h \to E^h_{\text{min}}$ when $h \to 0$. Recall that $\hat{E}^h$ is implicitly defined by Equation $\theta(\hat{E}^h) = 0$, which holds if and only if
\[
(1 + \beta^h[R^h(\hat{E}^h)])u(2\sigma_0\sqrt{h}L[R^h(\hat{E}^h)]) = (1 + \gamma)(1 - \beta^h[R^h(\hat{E}^h)]).
\]
When $h \to 0$, we have $\lim_{h \to 0} \beta^h[R^h(\hat{E}^h)] \to 0$, so that the above equality transforms to:
\[
u(2\sigma_0\sqrt{h}L[R^h(\hat{E}^h)]) = 1 + \gamma.
\] (B9)

At the same time, $E^h_{\text{min}}$ satisfies $u^h(E^h_{\text{min}}) = 1 + \gamma$, which is equivalent to
\[
\frac{1}{1 + \rho h} \frac{u(2\sigma_0\sqrt{h}L[R^h(E^h_{\text{min}})])}{1 - \beta^h[R^h(E^h_{\text{min}})]} = 1 + \gamma.
\]
When $h \to 0$, the above equation transforms to
\[
u(2\sigma_0\sqrt{h}L[R^h(E^h_{\text{min}})]) = 1 + \gamma.
\] (B10)

Equations (B9) and (B10) can simultaneously hold only when $\hat{E}^h = E^h_{\text{min}}$.

Second, the property $E^h_{\text{min}} \to 0$ for $h \to 0$ immediately follows from the aggregate market leverage constraint that holds with equality at $E = E^h_{\text{min}}$. 

44
b) Next, we demonstrate that \( R(E_{\text{max}}^h) \rightarrow p \) when \( h \rightarrow 0 \). To this end, consider the system of Equations (B7) and (B8) evaluated at \( E_{\text{max}}^h \):

\[
\begin{align*}
(1 + \beta^h[R^h(E_{\text{max}}^h)]) u^h(E) &= 1 - \beta^h[R^h(E_{\text{max}}^h)], \\
2(1 + \rho^h) u^h(E_{\text{max}}^h) &= u^h(E^+) + u^h(E^-).
\end{align*}
\]

(B11) \hspace{1cm} (B12)

The optimal dividend policy implies that \( E^+ \equiv E_{\text{max}}^h \) and \( u(E_{\text{max}}^h) = 1 \). Then, solving the above system one obtains:

\[
u^h(E^-) = \frac{1 - \beta^h[R^h(E_{\text{max}}^h)]}{1 + \beta^h[R^h(E_{\text{max}}^h)]},
\]

and

\[
\beta^h[R^h(E_{\text{max}}^h)] = \frac{\rho^h}{1 + \rho^h}.
\]

Using the definition of \( \beta^h(\cdot) \), one can show that:

\[
R^h(E_{\text{max}}^h) = p + \sigma_0^2 \frac{\rho \sqrt{h}}{1 + \rho^h}.
\]

It is easy to see from the above equation that \( R^h(E_{\text{max}}^h) \rightarrow p \) when \( h \rightarrow 0 \).

c) Finally, let us show that Equations (B7) and (B8) converge to Equations (27) and (30) when \( h \rightarrow 0 \). To see this, consider first a first-order Taylor expansion of Equation (B7). Neglecting the terms of order higher than \( h \), we get:

\[
\begin{align*}
(1 + \beta^h[R^h(E)]) \left[ u^h(E) + [u^h(E)]' L[R^h(E)] \sigma_0 \sqrt{h} \right] &= \left(1 - \beta^h[R^h(E)]\right) \left[ u(E) - [u^h(E)]' L[R^h(E)] \sigma_0 \sqrt{h} \right],
\end{align*}
\]

(B13)

which, after simplification, yields

\[
-u^h(E)(R^h(E) - p)h - \sigma_0^2 [u^h(E)]' L[R^h(E)] h = u^h(E)(R^h(E) - p)h + \sigma_0^2 [u^h(E)]' L[R^h(E)] h.
\]

(B14)

When \( h \rightarrow 0 \), this ultimately leads to

\[
\frac{u'(E)}{u(E)} = \frac{R(E) - p}{\sigma_0^2 L[R(E)]},
\]

which is exactly (27).

Similarly, applying a second-order Taylor expansion to the right-hand side of Equation (B8) and letting \( h \) go to zero yields:

\[
(1 + \rho) u(E) = u(E) + u'(E)(R(E) - p)L[R(E)] + \frac{1}{2} u''(E) \sigma_0^2 (L[R(E)])^2,
\]

(B15)

which is exactly (30).
Appendix C. Competitive equilibrium with capital regulation

C.1. Numerical procedure

This numerical algorithm solving for the competitive equilibrium with minimum capital regulation can be implemented with the Mathematica software:53

- Pick a candidate value $\hat{E}^{\Lambda}_{\min}$.
- Assume that the regulatory constraint always binds. Solve ODE (A15) for $y(E)$ under the boundary condition $y(\hat{E}_{\min}) = 0$.
- Compute a candidate value $\hat{E}^{\Lambda}_{\max}$ such that satisfies equation $y(\hat{E}^{\Lambda}_{\max}) = 0$.
- Check whether $y(\hat{E}^{\Lambda}_{\max}) \leq B(\hat{E}^{\Lambda}_{\max})$.
- Conditional on the results of the previous step, one of the two scenarios is possible:
  a) if $y(\hat{E}^{\Lambda}_{\max}) \leq B(\hat{E}^{\Lambda}_{\max})$, then the regulatory constraint is always binding for a given $\Lambda$, i.e., there is a single “constrained” region. In this case market-to-book value $u_{\Lambda}(\hat{E}^{\Lambda}_{\min})$ can be computed according to

$$u_{\Lambda}(\hat{E}^{\Lambda}_{\min}) = u_{\Lambda}(\hat{E}^{\Lambda}_{max}) \exp\left(\int_{\hat{E}^{\Lambda}_{\min}}^{\hat{E}^{\Lambda}_{\max}} y(E)dE\right) = \exp\left(\int_{\hat{E}^{\Lambda}_{\min}}^{\hat{E}^{\Lambda}_{\max}} y(E)dE\right).$$

  b) observing $y(\hat{E}^{\Lambda}_{\max}) > B(\hat{E}^{\Lambda}_{\max})$ means that, for given $\hat{E}^{\Lambda}_{\min}$, the constrained and unconstrained regions coexist. To find the critical level of aggregate equity $\hat{E}^{\Lambda}_{c}$ above which the regulatory constraint is slack, one needs to solve the following equation:

$$y(\hat{E}^{\Lambda}_{c}) = B(\hat{E}^{\Lambda}_{c}).$$

- using $\hat{E}^{\Lambda}_{c}$ and (A17), define the function $R(E)$ for $E > \hat{E}^{\Lambda}_{c}$;
- compute a new candidate for the dividend boundary, $\hat{E}^{\Lambda}_{\max}$, such that $B(\hat{E}^{\Lambda}_{\max}) = 0$ (note that this is equivalent to solving equation $R(\hat{E}^{\Lambda}_{\max}) = p$);
- compute the market-to-book value $u_{\Lambda}(\hat{E}^{\Lambda}_{\min})$ according to:

$$u_{\Lambda}(\hat{E}^{\Lambda}_{\min}) = \exp\left(\int_{\hat{E}^{\Lambda}_{\min}}^{\hat{E}^{\Lambda}_{c}} y(E)dE\right) \exp\left(\int_{\hat{E}^{\Lambda}_{c}}^{\hat{E}^{\Lambda}_{\max}} B(E)dE\right).$$

- If $u_{\Lambda}(\hat{E}^{\Lambda}_{\min}) = 1 + \gamma$, then $E^{\Lambda}_{\min} = \hat{E}^{\Lambda}_{\min}$, $E^{\Lambda}_{\max} = \hat{E}^{\Lambda}_{\max}$ (or $\hat{E}^{\Lambda}_{\max}$ if 2 regions) and $E^{\Lambda}_{c} = \hat{E}^{\Lambda}_{c}$ (if 2 regions). Otherwise, pick a different $\hat{E}^{\Lambda}_{\min}$, repeat the procedure from the beginning.

C.2. Ergodic density function of $E$

Since all variables of interest are the deterministic functions of $E$, we directly use the ergodic density of aggregate bank capitalization in order to compute their average values. Given the dynamics of

53 In all computations $\Lambda$ is taken as a parameter.
$E$ defined in the Proposition 5, the ergodic density function of $E$ in the regulated competitive equilibrium can be computed according to:

$$g_{\Lambda}(E) = \frac{C_{\Lambda}}{\sigma_0^2 K^2(E)} \exp \left( \int_{E_{\min}^{\Lambda}}^{E_{\max}^{\Lambda}} \frac{2(R(E) - p)}{\sigma_0^2 K(E)} dE \right),$$

where $K(E)$ and $R(E)$ are defined in (A16) and (A17) respectively, and the constant $C_{\Lambda}$ is such that $\int_{E_{\min}^{\Lambda}}^{E_{\max}^{\Lambda}} g_{\Lambda}(E) dE = 1$. Figure 9 illustrates the typical patterns of the ergodic density function of $E$ for different levels of the minimum capital ratio.

**Figure 9: The ergodic density of $E$ for different minimum capital ratios**

Notes: this figure illustrates the typical patterns of the ergodic density function of $E$ for different levels of minimum capital ratios. For lower levels of $\Lambda$ for which the constrained and constrained regimes coexist, ergodic density functions exhibit kinks at $E_{\Lambda}^c$. For $\Lambda = 65\%$ in this numerical example, the regulatory constraint binds for any $E \in [E_{\Lambda}^{\min}, E_{\Lambda}^{\max}]$ and the ergodic density function is of the class $C^1$. Parameter values: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $\gamma = 0.2$, $\beta = 1$, $\bar{R} = 0.15$.

**C.3. Expected time to reach the recapitalization barrier**

Let $T_{\gamma}(E)$ denote the expected time it takes to reach the recapitalization boundary $E_{\min}^{\Lambda}$ starting from any $E \geq E_{\min}^{\Lambda}$. Since $t + T_{\gamma}(E_t)$ is a martingale, function $T_{\gamma}(E)$ satisfies the following differential equation:

$$\frac{[K(E)\sigma_0]^2}{2} T''_{\gamma}(E) + K(E) [R(E) - p] T'_{\gamma}(E) + 1 = 0, \quad (E_{\min}^{\Lambda}, E_{\max}^{\Lambda}),$$

where functions $K(E)$ and $R(E)$ are defined in (A16) and (A17) respectively.

The above equation is subject to the following two boundary conditions: $T_{\gamma}(E_{\min}^{\Lambda}) = 0$ (i.e., it takes no time to reach $E_{\min}^{\Lambda}$ from $E_{\min}^{\Lambda}$), and $T_{\gamma}(E_{\max}^{\Lambda}) = 0$, which emerges due to the reflection property of aggregate equity. To measure the impact of minimum capital requirements on financial stability in Section 5, for each level of $\Lambda$, we compute the expected time to reach the recapitalization boundary $E_{\min}^{\Lambda}$ starting from the long-run average level of aggregate equity, $E_{\Lambda}^{\bar{}}$. 

47
Appendix D. Competitive equilibrium with $r > 0$

In the core of the paper we characterized the competitive equilibrium for the particular case $r = 0$. In this Appendix we solve for the competitive equilibrium for $r > 0$, focusing on the continuous-time framework. When $r > 0$, the dynamics of equity of an individual bank is given by:

$$dE_t = re_t dt + k_t[(R(E_t) - p - r)dt - \sigma_0dZ_t] - d\delta_t + di_t. \quad (D1)$$

The aggregate equity of the banking sector evolves according to:

$$dE_t = [rE_t + K(E_t)(R(E_t) - p - r)]dt - \sigma_0K(E_t)dZ_t - d\Delta_t + dI_t. \quad (D2)$$

Solving the shareholders’ maximization problem in the same way as we did in the proof of Proposition 2 and allowing for $k > 0$ yields us two equations:

$$\frac{u'(E)}{u(E)} = -\frac{R(E) - p - r}{\sigma_0^2K(E)}, \quad (D3)$$

$$(p - r)u(E) = [rE + K(E)(R(E) - p - r)]u'(E) + \frac{\sigma_0^2K^2(E)}{2}u''(E). \quad (D4)$$

Using (D3) to compute $u''(E)$ and substituting it together with $u'(E)$ into (D4) (while taking into account the equilibrium condition $K(E) = L[R(E)]$), enables us to express $R'(E)$:

$$R'(E) = -\frac{1}{\sigma_0^2} \frac{2(p - r)\sigma_0^2 + (R(E) - p - r)^2 + 2(R(E) - p - r)rE/L[R(E)]}{L[R(E)] - (R(E) - p - r)L'[R(E)]}. \quad (D5)$$

Applying the same arguments as in the setting with $r = 0$, we can show that $E_{min} = 0$ and $R_{min} = r + p$. The boundary $R_{max}$ can be computed numerically by solving equation

$$\int_p^{R_{max}} E'(R) \frac{(R - p - r)}{\sigma_0^2L(R)} dR = \log(1 + \gamma), \quad (D6)$$

where $E'(R) = 1/R'(E)$.

Note that the left-hand side of the above expression is increasing in $R_{max}$. Hence, there exists a unique solution to (D6), which guarantees the uniqueness of the equilibrium.

The typical patterns of the equilibrium loan rate $R(E)$ and net deposits $D - M(E)$ emerging for different values of $r$ are reported in Figure 10. The following remarks are in order concerning these numerical results. First, higher $r$ drives up the loan rate, yet, in a non-linear way. Second, as higher $R(E)$ triggers reductions in lending and thus reduces the overall exposure of the banking sector to aggregate risk, the target level of aggregate capitalization ($E_{max}$) declines when $r$ increases. Thus, the interest rate on deposits can be viewed as an additional regulatory instrument of keeping control over aggregate lending and thus financial stability. Finally, under assumption that $\lambda(\cdot)$ is a decreasing function of its argument, condition $\lambda'(D) = \rho - r$ implies that the aggregate volume of deposits increases with $r$. Thus, aggregate reserves given by $M(E) = D - K(E) + E$ also increase with $r.$
Figure 10: The impact of $r > 0$ on the loan rate and net deposits

Notes: this figure illustrates the typical patterns of the equilibrium loan rate (left hand-side panel) and the corresponding levels of net deposits (right hand-side panel) for positive levels of $r$. Parameter values: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $\gamma = 0.2$, $\beta = 1$, $\mathcal{R} = 0.15$. The solid lines in each graph correspond to $r = 0$.

Appendix E. Impulse response analysis

In this Appendix we apply the impulse response methodology to study the stability of the deterministic steady state. Such an exercise shows that the long-run behavior predicted by the impulse response analysis can be misguided because of the neglected impact of endogenous risk.

The usual approach to studying the macrodynamics in a DSGE model requires linearizing this model around the deterministic steady-state and perturbing the system by a single unanticipated shock. The equivalent approach in our framework would be to consider a particular trajectory of realization of aggregate shocks such that $dZ_t = 0$ for $t > 0$. Then, the dynamics of the system can be described by the ordinary differential equation (linearization is not needed here):

$$dR_t = \mu(R_t)dt,$$

where the initial shock determines $R_0 > p$.

By using expression (40), it can be shown that $\mu(p) = 0$. Hence, the frictionless loan rate ($R_t = p$) is an equilibrium of the deterministic system. It is locally stable when $\mu'(p) < 0$ and is globally stable when $\mu(R) < 0$ for all $R$. After some computations, it can be shown that

$$\mu'(p) = 2\rho^2\sigma_0^2 L''(p) L(p).$$

Hence, the steady state is locally stable when $L''(p) < 0$. Moreover, it also follows from (41), that condition $L''(R) < 0$ ensures global stability.

Illustrative example. Under our usual loan demand specification, $L(R) = \left(\frac{\mathcal{R}}{R-p}\right)^\beta$, the drift of
the loan rate is given by\textsuperscript{54}

$$\mu(R) = \sigma(R) \frac{\beta(R - p)Q(R)}{2\sigma_0|\beta - 1)(R - \beta p)|^2},$$  \hspace{1cm} (E1)

where $Q(R)$ is a quadratic polynomial:

$$Q(R) = (1 - \beta)((R - p)^2 - 2\rho\sigma_0^2) - 2(R - p)(\overline{R} - p).$$  \hspace{1cm} (E2)

Given the above specification, it can easily be shown that, when $\beta < 1$ (which is equivalent to $L''(R) < 0$), $\mu'(p) < 0$ and $\mu(R) < 0$ in the entire interval $[p, \overline{R}]$. Thus, the steady state is locally and globally stable. By contrast, when $\beta > 1$ (which is equivalent to $L''(R) > 0$), the steady state is locally unstable, i.e., $\mu'(p) > 0$. Moreover, when the recapitalization cost $\gamma$ is not too small, there exists a unique $R^* \in (p, \overline{R})$ such that $\mu(R)$ is positive in the region $(0, R^*)$ and negative in the region $(R^*, \overline{R})$.

While the outcomes of the impulse response analysis applied to our model might suggest that the economy should remain most of the time at the "deterministic" steady state, the analysis of the long-run behavior of the economy conducted in Section 4.2 shows that it is actually never the case because of the significant impact of endogenous risk.

### Appendix F. Time-varying financing conditions

In the core of the paper we have considered the setting in which the refinancing cost $\gamma$ was constant over time. In this section we extend the basic model to time-varying financing conditions. Assume that the cost of issuing new equity depends on a macroeconomic state that can be Good or Bad, with the respective costs of recapitalization $\gamma_G$ and $\gamma_B$, such that $\gamma_B > \gamma_G$. Let $\psi_B$ denote the intensity of transition from the Good to the Bad state and $\psi_G$ denote the intensity of transition from the Bad to the Good state.

Note that the homogeneity property of the individual decision problem is still preserved, so we can again work directly with the market-to-book value of banks. To ensure that the maximization problem of bank shareholders has a non-degenerate solution, the market-to-book value must satisfy the system of simultaneous equations:

$$\rho u_G(E) = \frac{[K_G(E)\sigma_0]^2}{2} u''_G(E) + K_G(E) [R_G(E) - p] u'_G(E) - \psi_B [u_G(E) - u_B(E)], \ E \in (E^G_{\min}, E^G_{\max})$$  \hspace{1cm} (F1)

$$\rho u_B(E) = \frac{[K_B(E)\sigma_0]^2}{2} u''_B(E) + K_B(E) [R_B(E) - p] u'_B(E) - \psi_G [u_B(E) - u_G(E)], \ E \in (E^B_{\min}, E^B_{\max})$$  \hspace{1cm} (F2)

\textsuperscript{54}Endogenous volatility $\sigma(R)$ is given by:

$$\sigma(R) = \frac{2\rho\sigma_0^2 + (R - p)^2}{\sigma_0[R + (\beta - 1)R - \beta p]}.$$

50
along the system of the FOCs for the individual choices of lending in each state:

\[
\begin{align*}
    u_G(E)[R_G(E) - p] &= -\sigma^2 G^2 K_G(E)u'_G(E), \\
    u_B(E)[R_B(E) - p] &= -\sigma^2 B^2 K_B(E)u'_B(E).
\end{align*}
\]  

(F3)  

(F4)

Thus, the market-to-book value, the loan rate and aggregate lending functions will have different expressions conditional on financing conditions. The differential equations characterizing them take into account the possibility of transitions between the states. Note that equations (F1)-(F2) are similar to equation (30) obtained in the setting with the time-invariant financing cost. However, compared to (30), each equation carries the additional term 

\[
-\psi_j \left[ u_j(E) - u_j(E) \right]
\]

reflecting the possibility that the cost of raising new equity can suddenly change from \(\gamma_j\) to \(\gamma_\bar{j}\) (here \(\bar{j}\) denotes the state complementary to the state \(j\)).

For given recapitalization \((E^G_{\text{min}}, E^G_{\text{max}})\) and dividend \((E^B_{\text{max}}, E^B_{\text{max}})\) boundaries, the boundary conditions can be established by solving for the optimal recapitalization and payout policies, which yields:

\[
\begin{align*}
    u_G(E^G_{\text{max}}) &= u_B(E^B_{\text{max}}) = 1, \\
    u_G(E^G_{\text{min}}) &= 1 + \gamma_G, \quad u_B(E^B_{\text{min}}) = 1 + \gamma_B.
\end{align*}
\]

(F5)  

(F6)

To define the boundaries \(E^G_{\text{min}}, E^G_{\text{max}}, E^B_{\text{min}}, E^B_{\text{max}}\), we follow the same logic that was used in the setting with the time-invariant cost and consider the marginal value of the entire banking sector at the boundaries. The absence of arbitrage opportunities implies that the following condition must hold at \(E^j_{\text{max}}, j \in \{G, B\}\):

\[
V_j'(E^j_{\text{max}}) = u_j(E^j_{\text{max}}) + E^j_{\text{max}} u_j'(E^j_{\text{max}}) = 1.
\]

Combining the above condition with (F5) and taking into account the fact that \(E^j_{\text{max}} > 0\) yields us a couple of equations needed to compute the values of \(E^j_{\text{max}}, j \in \{G, B\}\):

\[
u_j'(E^j_{\text{max}}) = 0, \quad j \in \{G, B\}.
\]

(F7)

Similarly, at refinancing boundaries \(E^j_{\text{min}}, j \in \{G, B\}\), it must hold that

\[
V_j'(E^j_{\text{min}}) = u_j(E^j_{\text{min}}) + E^j_{\text{min}} u_j'(E^j_{\text{min}}) = 1 + \gamma_j,
\]

which implies

\[
E^j_{\text{min}} u_j'(E^j_{\text{min}}) = 0, \quad j \in \{G, B\}.
\]

(F8)

Note that, compared with the setting involving the time-invariant refinancing cost, two polar cases are possible now: a) \(u_j'(E^j_{\text{min}}) > 0\) with \(E^j_{\text{min}} = 0\); or b) \(u_j'(E^j_{\text{min}}) = 0\) with \(E^j_{\text{min}} > 0\). As we discuss below, which of these cases ultimately materializes depends on the magnitude of the refinancing cost in the Bad state, \(\gamma_B\).

Finally, before we turn to the numerical illustration of the equilibrium properties, note that the
loan rate in state \( j \in \{G, B\} \) satisfies the following differential equation:
\[
R_j'(E) = -\frac{1}{\sigma_0^2} \left( 2\rho \sigma_0^2 + (R_j(E) - p)^2 + \psi_j \left( 1 - \frac{u_j'(E)}{u_j(E)} \right) \sigma_0^2 \right).
\]

Compared with equation (31) in Section 4.1, the right-hand side of the above equation contains an additional term in the numerator, \( \psi_j \left( 1 - \frac{u_j'(E)}{u_j(E)} \right) \sigma_0^2 \). This suggests that in the current set-up the loan rate will carry an extra premium/discount.

**Numerical example.** To illustrate the numerical solution and properties of the competitive equilibrium in the setting with time-varying refinancing costs, we resort to the simple (linear) specification of the demand for loans, i.e.,
\[
K_j(E) = \mathcal{R} - R_j(E), \quad j \in \{G, B\}.
\]

We first work with the systems of equations (F1)- (F2) and (F3)- (F4). Replacing \( K_j(E) \) in (F3)- (F4), one can express \( R_j(E) \) as a function of \( u_j(E) \) and \( u_j'(E) \). In our simple linear case, this yields:
\[
R_j(E) = \frac{pu_j(E) - \rho \sigma_0^2 u_j'(E)}{u_j(E) - \sigma_0^2 u_j'(E)}, \quad j \in \{G, B\}.
\]  

Substituting the above expression(s) in the system (F1)- (F2) leaves us with a system of two simultaneous second-order differential equations that can be solved numerically by using the four boundary conditions stated in (F5)-(F6). To obtain a solution to this system, we conjecture that \( E_{\max}^G < E_{\max}^B \) (this is verified ex-post) and use the fact that \( u_G(E) \equiv 1 \) when \( E \in [E_{\max}^G, E_{\max}^B] \).

To solve numerically for the equilibrium, we proceed in three steps: (i) first, we take the boundaries \( \{E_{\max}^G; E_{\max}^B\} \) as given and solve for the optimal \( E_{\min}^G \) and \( E_{\min}^B \) that satisfy conditions (F8); (ii) second, we search for the couple \( \{E_{\max}^G; E_{\max}^B\} \) that satisfies conditions (F7); (iii) finally, we uncover the equilibrium loan rates by substituting the functions \( u_G(E) \) and \( u_B(E) \) into equations (F9).

We now turn to the discussion of two possible cases that may arise depending on the magnitude of the refinancing cost \( \gamma_B \). For this discussion, it is helpful to introduce two benchmarks: in the first benchmark, the economy never leaves the Good state, i.e., \( \psi_B \equiv 0 \); in the second benchmark, the economy always remains in the Bad state, i.e., \( \psi_G \equiv 0 \). The first benchmark gives us the lower bound for \( E_{\max}^G \) that we will further label \( E_{\max}^* \), whereas the second benchmark gives the upper bound for \( E_{\max}^B \) that we will label \( E_{\max}^{**} \). We also denote \( u(E|\psi_B \equiv 0) (R(E|\psi_B \equiv 0)) \) the market-to-book value (loan rate) in the set-up with \( \psi_B \equiv 0 \) and \( u(E|\psi_G \equiv 0) (R(E|\psi_G \equiv 0)) \) the market-to-book value (loan rate) in the set-up with \( \psi_G \equiv 0 \).

**Case 1: \( \gamma_B \) low.** First we consider the case in which the refinancing cost in the Bad state is relatively low. The solid lines in Figure 11 depict the typical patterns of the market-to-book ratios (left panel) and the corresponding loan rates (right panel) computed in each state. These outcomes are contrasted with the patterns emerging in the two benchmarks (dashed lines).

---

55This equation can be obtained by combining equations (F1) and (F3) (or, equivalently, (F2) and (F4)).

56To find a numerical solution, one can start with a guess for \( u_B(E) \) and proceed iteratively until the difference between the solutions on the consequent iterations vanishes. It turns out to be convenient to take as the initial guess for \( u_B(E) \) the solution to the ODE (F2) with \( \psi_G \equiv 0 \).
Several equilibrium features are worth mentioning at this stage. First, one can note that $E_{G_{\text{min}}} = E_{B_{\text{min}}} = 0$, so that banks do not change their recapitalization policies as compared to those implemented in the benchmark settings. By contrast, $E_{G_{\text{max}}} > E_{*_{\text{max}}}$ and $E_{B_{\text{max}}} < E_{**_{\text{max}}}$, which means that banks will delay the distribution of dividends as compared to the setting in which the economy permanently remains in the Good state and will accelerate dividend payments as compared to the setting in which the economy is permanently locked in the Bad state. Second, in the Good state, the loan rate carries a discount (as compared to its benchmark value obtained for $\psi_B \equiv 0$) when $E$ is low and an extra “premium” when $E$ is high enough. By contrast, the loan rate in the Bad state carries an extra premium (as compared to its benchmark value obtained for $\psi_G \equiv 0$) for lower level of $E$ and a “discount” when $E$ is high.

Figure 11: Competitive equilibrium with time-varying financing costs: $\gamma_B$ low

Parameter values: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $\gamma_G = 0.2$, $\gamma_B = 1$, $\psi_G = 0.1$, $\psi_B = 0.05$; $L(R) = \overline{R} - R$ with $\overline{R} = 0.1$. The benchmark payout barriers are: $E_{G_{\text{max}}} \approx 0.007$ and $E_{B_{\text{max}}} \approx 0.012$. The payout barriers in the Good and Bad regimes are: $E_{G_{\text{max}}} \approx 0.008$ and $E_{B_{\text{max}}} \approx 0.011$.

Case 2: $\gamma_B$ high. We now turn to the setting involving very high refinancing costs in the Bad state. The properties of the competitive equilibrium emerging in this case are illustrated in Figure 12. The key departure from the previous case pertains to the fact that, for fear of incurring substantial recapitalization costs if the economy slides in the Bad state, in the Good state banks will raise new equity capital at a strictly positive level of aggregate capitalization, i.e., $E_{G_{\text{max}}} > 0$. In a dynamic setting, this phenomenon of “market timing” was first identified by Bolton et al. (2013) within a partial-equilibrium liquidity-management model with the stochastically changing fixed cost of equity issuance. The main general-equilibrium implication of this feature in our setting is that the loan rate (and, therefore, the volume of lending) converges to its First-Best level when $E \rightarrow E_{G_{\text{min}}}$ and financing conditions are good (i.e., $j = G$).

Appendix G. Empirical analysis

This Appendix presents a simple assessment of the consistency of the key predictions of our model with the data. As stated in Proposition 1, our model delivers two key predictions: bank loan rates and the ratio of market-to-book equity are (weakly) decreasing functions of aggregate bank equity.
Figure 12: Competitive equilibrium with time-varying financing costs: $\gamma_B$ high

Parameter values: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $\gamma_G = 0.2$, $\gamma_B = +\infty$, $\psi_G = 0.1$, $\psi_B = 0.05$; $L(R) = \overline{R} - R$ with $\overline{R} = 0.1$. The benchmark payout barriers are: $E_{max}^G \approx 0.007$ and $E_{max}^{**} \approx 0.017$. The payout barriers in the Good and Bad regimes are: $E_{max}^G \approx 0.01$ and $E_{max}^B \approx 0.014$. The recapitalization barriers are: $E_{min}^G \approx 0.0005$ and $E_{min}^B = 0$.

We assess these predictions by estimating conditional correlations via regression analysis between measures of bank gross returns on earning assets, the ratio of market-to-book equity, and aggregate bank equity. We use a large bank-level panel dataset. Its use, and the attendant heterogeneity of the data for banks belonging to a specific country group, places a strong consistency requirement on the predictions of our model, which is constructed under the simplifying assumption of homogeneous banks.

Our results indicate that the key predictions of our model are not rejected by simple statistical tests, are consistent with a large variety of country circumstances, and are robust to data heterogeneity owing to our use of a firm-level panel dataset. This suggests that our model is useful and relevant.

G.1. Data and statistics

The data consists of consolidated accounts and market data for a panel of publicly traded banks in 43 advanced and emerging market economies for the period 1982-2013 taken from the Wordscope database retrieved from Datastream. The sample is split into three sub-samples: U.S. banks (753 banks with about 10,500 bank-year observations); banks in advanced economies excluding the U.S. (500 banks with about 8,000 bank-year observations); banks in emerging market economies (185 banks with about 2,000 bank-year observations). Table 1 summarizes the definitions of the variables considered.

Note that bank gross return on assets include revenues accruing from investments other than loans; however, in the analysis below we will condition our estimates on asset composition using the % of loans to assets as a bank control. Furthermore, the variable total bank equity is the sum of

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57 The Standard Industrial Classification (SIC) System is used to identify the types of financial institutions included in the sample, which are: National Commercial Banks (6021), State Commercial Banks (6022), Commercial Banks Not Elsewhere Classified (6029), Savings Institution Federally Chartered (6035), Savings Institutions Not Federally Chartered (6036). In essence, the sample includes all publicly quoted depository institutions in the database. This panel dataset is unbalanced due to mergers and acquisitions, but all banks active in each period are included in the sample to avoid survivorship biases.
Table 1: Definition of variables

<table>
<thead>
<tr>
<th>Identifier</th>
<th>Variable</th>
<th>Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>ret</td>
<td>bank gross return on assets</td>
<td>total interest income/earning assets</td>
</tr>
<tr>
<td>mtb</td>
<td>market-to-book equity ratio</td>
<td>market equity/book equity</td>
</tr>
<tr>
<td>logta</td>
<td>bank size</td>
<td>Log(assets)</td>
</tr>
<tr>
<td>loan asset</td>
<td>% of loans to assets</td>
<td>total loans/total assets</td>
</tr>
<tr>
<td>bequity</td>
<td>bank book equity</td>
<td>bank book equity</td>
</tr>
<tr>
<td>mmr</td>
<td>short-term rate</td>
<td>short-term rate</td>
</tr>
<tr>
<td>TBE</td>
<td>total bank equity</td>
<td>sum of bequity</td>
</tr>
</tbody>
</table>

the equity of banks belonging to a particular country: this amounts to assuming that the relevant banking market is the country. All other variables are exact empirical counterparts of the variables defined in the model.

Table 2 reports sample statistics (Panel A) and some (unconditional) correlations (Panel B). Note that the correlation between bank returns, the market-to-book equity ratio, and total bank equity is negative and significant. However, we wish to gauge conditional correlations, to which we now turn.

Table 2: Sample statistics and unconditional correlations

**Panel A: Sample Statistics**

<table>
<thead>
<tr>
<th>Variable</th>
<th>US</th>
<th>Advanced (ex. US)</th>
<th>Emerging</th>
</tr>
</thead>
<tbody>
<tr>
<td>ret (%)</td>
<td>10712</td>
<td>6.68</td>
<td>1.78</td>
</tr>
<tr>
<td>mtb</td>
<td>10094</td>
<td>1.41</td>
<td>0.71</td>
</tr>
<tr>
<td>logta</td>
<td>11630</td>
<td>13.58</td>
<td>1.65</td>
</tr>
<tr>
<td>loan asset</td>
<td>11442</td>
<td>65.6</td>
<td>13.37</td>
</tr>
<tr>
<td>bequity (US$ bln)</td>
<td>11555</td>
<td>943.52</td>
<td>8942.38</td>
</tr>
<tr>
<td>mmr</td>
<td>25874</td>
<td>5.33</td>
<td>3.92</td>
</tr>
<tr>
<td>TBE (US$ bln)</td>
<td>25113</td>
<td>330299.8</td>
<td>363168.8</td>
</tr>
</tbody>
</table>

**Panel B: Correlations**

<table>
<thead>
<tr>
<th></th>
<th>ret</th>
<th>mtb</th>
<th>logta</th>
<th></th>
<th>ret</th>
<th>mtb</th>
<th>logta</th>
<th></th>
<th>ret</th>
<th>mtb</th>
<th>logta</th>
</tr>
</thead>
<tbody>
<tr>
<td>ret</td>
<td>0.1653*</td>
<td>1</td>
<td>0.0471*</td>
<td>1</td>
<td>0.0584*</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mtb</td>
<td>-0.1163*</td>
<td>0.2124*</td>
<td>1</td>
<td>-0.2418*</td>
<td>0.1637*</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>logta</td>
<td>-0.7860*</td>
<td>-0.3016*</td>
<td>0.1747*</td>
<td>-0.5266*</td>
<td>-0.1340*</td>
<td>0.2914*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TBE</td>
<td>-0.3680*</td>
<td>0.0818*</td>
<td>1</td>
<td>-0.1991*</td>
<td>-0.0539*</td>
<td>0.3755*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:** * indicates significance at 5% level.

G.2. Conditional correlations via panel regressions

We test whether there exist a negative conditional correlation between bank returns, market-to-book equity and total bank equity by estimating panel regressions with $Y_{it} \in (ret, mtb)$ as the
dependent variable of the form:

\[ Y_{it} = \alpha + \beta E_{it} + \gamma_1 \text{bequity}_{it} + \gamma_2 \text{Logta}_{it} + \gamma_3 \text{loanasset}_{it} + \gamma_4 \text{mmr}_t + \gamma_5 \text{rgdpg}_t + \gamma_6 \text{infl}_{it} + \epsilon_{it}. \]  

Model (G1) is used for the US sample, while we add to Model (G1) country specific effects in the estimation for the advanced economies and emerging market samples. Our focus is on the coefficient \( \beta \). Bank specific effects are controlled for by the triplet (bequity, logta, loanasset), while country specific time-varying effects are controlled for by the short-term rate, real GDP growth and inflation (mmr, rgdpg, and infl respectively).

Table 3 reports the results. The coefficient \( \beta \) is negative and (strongly) statistically significant in all regressions. The quantitative impact of changes in total bank equity on both bank returns and the market-to-book ratio is substantial as well. Thus, we conclude that the two key predictions of our model are consistent with the data.

### Table 3: Panel regressions

<table>
<thead>
<tr>
<th>Explanatory variables</th>
<th>US (1) ( \text{ret} ) &amp; ( \text{mb} )</th>
<th>Advanced (ex. US) (3) ( \text{ret} ) &amp; ( \text{mb} )</th>
<th>Emerging (5) ( \text{ret} ) &amp; ( \text{mb} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>(-2.17e-06***) &amp; (-6.75e-07***)</td>
<td>(-8.65e-07***) &amp; (-2.52e-06***)</td>
<td>(-1.67e-05**) &amp; (-8.33e-06***)</td>
</tr>
<tr>
<td>\text{bequity}</td>
<td>(-1.33e-06) &amp; (-7.05e-06***)</td>
<td>(-9.42e-06***) &amp; (-1.73e-06***)</td>
<td>(-2.12e-06) &amp; (-5.23e-06)</td>
</tr>
<tr>
<td>\text{logta}</td>
<td>(0.00419) &amp; (0.139***)</td>
<td>(-0.276***) &amp; (0.0762***)</td>
<td>(-0.590***) &amp; (0.107***)</td>
</tr>
<tr>
<td>\text{loanasset}</td>
<td>(0.0183***) &amp; (0.00291***)</td>
<td>(-0.0238***) &amp; (-0.00302***)</td>
<td>(0.0368***) &amp; (-0.00202)</td>
</tr>
<tr>
<td>\text{mmr}</td>
<td>(0.413***) &amp; (-0.0375***)</td>
<td>(0.589***) &amp; (-0.00545)</td>
<td>(0.410***) &amp; (-0.0222***)</td>
</tr>
<tr>
<td>\text{rgdpg}</td>
<td>(-0.110***) &amp; (0.116***)</td>
<td>(-0.0636***) &amp; (0.100***)</td>
<td>(0.0179) &amp; (0.0781***)</td>
</tr>
<tr>
<td>\text{infl}</td>
<td>(-0.0350***) &amp; (-0.0509***)</td>
<td>(0.0657***) &amp; (-0.00966)</td>
<td>(0.148***) &amp; (-0.00444)</td>
</tr>
<tr>
<td>\text{Constant}</td>
<td>(5.689***) &amp; (-0.349***)</td>
<td>(11.71***) &amp; (0.660***)</td>
<td>(18.47***) &amp; (-0.111)</td>
</tr>
</tbody>
</table>

Notes: robust p-values in brackets (*** \( p < 0.01 \), ** \( p < 0.05 \), * \( p < 0.1 \)).
References


Pandit, V., 2010. We must rethink Basel, or growth will suffer. Financial Times, 10 November.


TheCityUK report, October 2013. Alternative finance for SMEs and mid-market companies.