The Cost of Constraints:
Risk Management, Agency Theory and Asset Prices

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Abstract:

Traditional academic literature has relied on so-called “limits to arbitrage” theories to explain why investment managers are unable to eliminate the effects of investor “irrational” preferences (either the asset-pricing anomalies or the behavioral finance literature) on asset pricing. We demonstrate, however, that investment managers may not eliminate the observed asset-pricing anomalies because they may contribute to their existence. We show that if managers face constraints such as a “tracking-error constraint,” coupled with the need to hold liquidity to meet redemptions or to actively-manage investments, they optimally hold higher-volatility securities in their portfolios. Investment constraints, such as tracking-error constraints, however, reduce the principal-agent problems inherent in delegated asset management and serve as effective risk-control tools. Liquidity reserves allow managers to meet redemptions or redeploy risks efficiently. We prove that investment managers will combine a portfolio of active risks (a so-called “alpha portfolio”) for a given level of liquidity with a hedging portfolio designed to control tracking error. As the demand for either liquidity or active management increases presumably because of confidence in alpha, the cost of maintaining the tracking-error constraint increases in that the investment managers must finance these demands by selling more lower-volatility securities and holding more higher volatility securities. With more demand for the “alpha” portfolio, managers are forced to buy more of the tracking-error control portfolio. Investment managers and their investors are willing to hold inefficient portfolios and to give up returns, if necessary, to control the tracking-error of their portfolios. Given the liquidity and tracking-error constraints, investment managers concentrate more of their holdings in higher volatility (higher beta) securities. And, we show that it is optimal for investors to limit their manager’s use of leverage, which implies that leverage has a different cost other than the cost of borrowing exceeding the return from lending. Empirically, we show that active investment managers, such as mutual funds, hold portfolios that concentrate in higher volatility securities. Moreover, when they change their holdings of their “alpha” portfolios (reduce or increase their tracking error by choice), the relative prices of higher volatility stocks change according to the predictions of the model. That is, if investment managers move closer to a market portfolio, the prices of lower-volatility stocks rise more than the prices of higher-volatility stocks given changes in the prices of other market factors.
**Introduction:**

The empirical findings that stocks have been priced to earn returns far different from what the Sharpe-Lintner capital asset pricing model (CAPM) would suggest has led behavioral finance specialists to attack efficient and rational markets using observed discrepancies to support alternative behavioral explanations of these so-called “anomalies.” This in turn, has led rational market proponents to claim that all anomalies result from hidden risk factors. Unfortunately, evidence supporting this view is not persuasive. Particularly difficult to reconcile is how “low volatility stocks” have not only generally outperformed other stocks in crisis but also have produced consistently higher returns, on average, with lower risk than have other stocks. Behaviorists have seized on the “volatility” anomaly – whereby high volatility (beta) stocks dramatically underperform low volatility (beta) stocks – and argue that investors have loss aversion or prefer to gamble and, as a result, bid up the prices of higher-beta assets. Baker, Bradley and Wurgler [2011] have gone as far as to say that “among the many candidates for the greatest anomaly in finance, a particularly compelling one is the long-term success of low-volatility and low-beta stock portfolios.” Although sophisticated investors observe these anomalies, a behaviorist would argue that they persist because for them the costs are larger than the benefits of eliminating the discrepancies, the so-called “limits of arbitrage” arguments.

Our alternative model, however, demonstrates that these same rational sophisticated investors are the proper clients for higher-risk assets. They and not the irrational investors might create the observed mispricing. In our model, active management and liquidity demands lead to tracking error; that is, deviations of the portfolio returns from benchmark market-factor returns. Investors, however, impose tracking-error constraints on their managers as a cost-efficient monitoring device. Although they know that the constraints on the investment activities of their managers may not be costless, imposing investment constraints might be less costly for them than other monitoring methods. If the costs of the constraints are positive, and we show that they are in the theoretical and empirical sections of the paper, this leads to pervasive and rational pricing effects, which others have called anomalies.
Our new framework follows on in the spirit of Roll (1992, 2005) and Brennan (1993). Like them, in our model we impose a tracking-error constraint on investment managers. They used a tracking-error constraint, however, to prevent managers from taking on extra risk to outperform a benchmark and showed that their tracking-error constrained portfolio was still on the efficient set. We show, however, that if we allow for the investment manager to have skills, so-called “alpha” and/or liquidity needs that the tracking-error-constrained portfolio will not lie on the efficient set. In general, this and other constraints lead managers to select different portfolios from those on the conventional Markowitz efficient set. In our equilibrium model, investors bid up the prices of higher-volatility stocks to control tracking error while holding alpha-generating portfolios or holding liquid assets to meet contingencies.

In contrast to the noise traders of the behavioral models, in our model, many sophisticated investors cause the apparent mispricing that would be absent in an unconstrained world. Obviously, with positive shadow costs to a constraint, unconstrained investors can profit and, thereby, reduce the cost of the constraint. And, if the costs of the constraints are too high for certain investors, they will find alternatives to active management. Or if the prices of riskier securities are bid up too high by noise traders, they will also find alternatives to active management. If there is an insufficient supply of capital from unconstrained investors to satisfy the excess demand, that is, they are infra-marginal, the expected and observed returns on higher-risk securities will be lower than would be implied from standard asset-pricing models. And, the price effects might be greater, if, in addition, to these rational but constrained managers, irrational investors buy higher-risk securities as lotteries. We have an identification problem as to which group is setting prices that we will break in our theoretical and empirical sections of the paper.

Our framework is dynamic while that of the behaviorist’s is static. As investors and investment managers have different perceptions of their alpha portfolios’ expected performance and its risks or modify contracts or assets under management, the demand for higher volatility assets changes. These changing demands will affect the prices of lower volatility securities
relative to higher volatility securities. While mutual fund constraints might drive mispricing today, hedge fund constraints might also drive mispricing tomorrow.¹

In the next section, we present a representative-agent equilibrium framework that shows exactly how the joint imposition of a tracking error and liquidity constraint leads to an optimal portfolio that over-weights higher volatility stocks relative to lower volatility stocks. We then incorporate active management into the analysis. We show that the greater the expected abnormal return, alpha, relative to its standard error, the greater is the demand for the alpha portfolio and the larger the need to sell lower volatility securities to finance this demand. The equilibrium portfolio is shown to be mean-variance inefficient and this inefficiency is a direct consequence of the cost of the investment constraints. We show the effects of investment managers being able to use leverage (or to be able to equitize the risk of liquidity) on the equilibrium. In the following section, we discuss the economic reasons for the tracking error and liquidity constraints before turning to the empirical evidence. The empirical evidence marries together the actual holdings of mutual funds and the changes in those holdings with the differential returns on high and low volatility stocks. We follow these discussions with the conclusions.

**Theory – Rational Portfolio Choice**

Roll (1992) analyzed the effect of imposing a tracking-error constraint on mean/variance portfolio optimization. He demonstrated, among his other results, that when the benchmark is efficient, the tracking-error efficient portfolio (those optimal portfolios that are subject to a tracking error constraint) is also efficient and lies on the Markowitz frontier. Our first augmentation is to add a liquidity constraint to the Roll model. While tracking error is one of the most widely imposed investment constraints, another pervasive constraint is liquidity.²

¹ Katie Hall, CEO and CIO of the $22.5B Hall Capital investment advisory firm had noticed an important shift in hedge funds that fits our hypothesized dynamics. While hedge funds used to act as unconstrained managers, they now benchmark off one another. Also, too much capital in similar strategies such as long-short market neutral equity has reduced returns in those strategies to near zero. Personal discussions, October 2012.

² Actually, Roll’s model has an investment manager increasing risk (increase beta) to increase return and beat the benchmark while limiting the tracking error results from that strategy. The model assumes managers do not have skill to produce abnormal returns, or “alpha.” In the next section, we introduce “alpha” into our model.
Managers must maintain a certain amount of liquidity or cash in their portfolios to meet investor withdrawal requests or to act on opportunities quickly as they arise.

Therefore, a manager’s portfolio choice problem is affected jointly by both a tracking error constraint and a liquidity constraint. Below we theoretically assess the impact and cost of these constraints on the manager’s choice of the optimal mean/variance portfolio. (At this point, we assume the manager does not have skill. Subsequently, we will add an alpha-generating portfolio, which like a liquidity or another constraint forces trade-offs and therefore increases the shadow prices of the other constraints.) The portfolio choice problem is one in which the manager behaves rationally to minimize tracking error for a given level of return that meets a liquidity constraint. The liquidity constraint is modeled as a cash level that needs to be maintained (i.e., the \( x'1 = k < 0 \) constraint below).

\[
\begin{align*}
\min \& \quad x'\Omega x \\
\text{subject to:} \\
& x'R = G \\
& x'1 = k < 0
\end{align*}
\]

where \( x \) is a vector that represents the differences between the weights of the managed portfolio and the benchmark portfolio, \( \Omega \) is the covariance matrix of \( N \) assets, \( G \) is the target outperformance and the absolute value of \( k \) represents the percentage of cash that is held as the result of the liquidity constraint.

The solution to the above equation (as described fully in the Appendix) is

\[
x = \frac{G}{R_1 - R_0} (q_1 - q_0) + \frac{k}{R_1 - R_0} (q_0R_1 - q_1R_0) \tag{EQ. 1}
\]

In EQ. 1, \( q_0 \) and \( q_1 \) are special mean-variance efficient portfolios. \( q_0 \) is the minimum variance portfolio and \( q_1 \) is the mean-variance efficient portfolio defined by the intersection of the line connecting the origin (return = 0, variance = 0) to the minimum variance portfolio on the
efficient frontier. As a result, both portfolios have the same expected return to variance ratio. The first right-hand term represents the tracking error efficient portfolio as derived by Roll (1992). The second-right hand term represents the deviation from the tracking-error efficient portfolio that accommodates the liquidity constraint. It represents a portfolio with a zero expected return that the optimal portfolio shorts “k” percent of. The optimal portfolio shorts the minimum variance portfolio and buys a higher variance portfolio to maintain minimum tracking error. Since higher volatility assets play a larger role in dictating the movement of the benchmark, by over weighting these assets relative to lower volatility names, managers are able to increase the risk of the portfolio and accommodate the tracking error stemming from the cash holdings. As a result, the dynamic caused by tracking error and liquidity goals is shown to cause a demand for higher volatility assets.

Because of liquidity needs, investment managers have demand for higher volatility stocks. As a result, this demand imbalance might translate into the prices of higher volatility stocks being bid up (lower expected returns) relative to unconstrained asset pricing models representing the cost to control tracking error while maintaining portfolio liquidity. The premium created by this imbalance can be represented by the return give-up between an unconstrained manager and a constrained manager, when both are targeting the same level of risk. In this model, the lost return arises initially from the opportunity costs of holding an inefficient portfolio. This effect might be amplified by the possible price effects of multiple constrained managers buying higher volatility securities for exactly the same reasons, the need to hold liquidity.

**Managed Portfolio Without Additional Market Pricing Effects:**

The inefficiency of the solution given by EQ. 1 can be illustrated by plotting the unconstrained mean-variance optimal frontier side by side with the mean-variance frontier of the tracking-error managed portfolio that is constrained to hold a minimum amount of liquidity.

The unconstrained efficient frontier is estimated using the long-term historical estimates of the returns and variances of market capitalization-weighted US equities and the US minimum
variance portfolio and the correlation between the two.\textsuperscript{3} Table 1 below contains the summary statistics. Assuming that both portfolios are efficient, we constructed the entire mean-variance efficient frontier via “two fund separation.”

### Table 1: Summary Statistics of the Market Cap Weighted and Minimum Variance US Equity Portfolios from 1979 to 2012.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Capitalization Weighted</th>
<th>Minimum Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Return</td>
<td>10.4%</td>
<td>6.5%</td>
</tr>
<tr>
<td>Volatility</td>
<td>15.5%</td>
<td>12.0%</td>
</tr>
<tr>
<td>Risk-free Rate</td>
<td>5.0%</td>
<td>5.0%</td>
</tr>
<tr>
<td>Correlation</td>
<td>.70</td>
<td></td>
</tr>
</tbody>
</table>

In Figure 1, based on the inputs from Table 1, the red-line represents the efficient frontier. As Roll demonstrated, this frontier also describes the frontier of the managed portfolio that is only constrained by tracking error. The blue line represents the managed portfolio that is constrained by both a tracking error (5.5%) and a liquidity (5%) constraint and designed to outperform the benchmark by 1%. Using each portfolio on the efficient frontier as a benchmark, we created a corresponding managed portfolio. The tracking error of these managed portfolios is (see Appendix for proof):

\[
TEV_c = x' \Omega x
\]

\[
TEV_c = TEV - \frac{k}{(R_1 - R_0)^2} \left[2GR_0(\sigma_1^2 - \sigma_0^2) - k(R_1^2 \sigma_1^2 + R_0^2 \sigma_0^2 - 2R_1 R_0 \sigma_0)\right]
\]

\text{EQ 2}

\textsuperscript{3} Estimates of returns and volatilities are based on data from 1979 to 2012.
where TEV is the tracking error variance of Roll’s (1992) tracking error efficient portfolios.
These managed portfolios then plot out the curve labeled “TEV/Liquidity” and each of these portfolios has a tracking error of 5.5% with their corresponding benchmarks and is expected to outperform the benchmark by 1%. As an illustration, the managed portfolio that corresponds to the choice of the tangent portfolio as its benchmark is shown in Figure 1. This portfolio, however, is inefficient because it is a convex combination of the TEV efficient portfolio (which is an overall efficient portfolio) and cash.

Specifically, the managed portfolio is represented by $x + q_B$. Given $x$ (see EQ. 1) has a net negative position (i.e. the managed portfolio holds cash) and is a combination of $q_1$ and $q_0$ as is $q_B$ (assuming $q_B$ is on the frontier), the managed portfolio is a convex combination of cash and an efficient portfolio, with the weights being the absolute value of $k$ and $1 –$ absolute value of $k$, respectively. The mean-variance frontier of the managed portfolio represents a de-levered transformation of the efficient frontier. As such, the frontier of the liquidity-constrained managed portfolio is inefficient for high volatility benchmarks (as in Figure 1 below).

However, Figure 1 also shows that for lower volatility benchmarks, there is a region in which the constrained manager dominates the efficient frontier. The portfolios that are lower risk than the tangent portfolio and hold cash retain the maximum Sharpe ratio of the tangent portfolio by moving down the capital market line. For example, the constrained portfolio that represents a de-levered version of the maximum Sharpe ratio (tangent) portfolio (i.e. 95% tangent portfolio + 5% cash) will dominate any other portfolio on the unconstrained efficient frontier. This region is defined by the condition:

$$\frac{R_p - R_f}{\sigma_p} \geq \frac{R_{BM} - R_f}{\sigma_{BM}}$$

where

$$\sigma_p^2 = TEV_c + \sigma_B^2 + \frac{2G\sigma_B^2}{R_1 - R_0} \left( \frac{R_B}{R_0} - 1 \right) + \frac{2k\sigma_B^2}{R_1 - R_0} \left( \frac{R_B}{R_0} - 1 \right) , R_p = R_B + G , k = -1\% \text{ and } G = 1\%$$

in this example. It lies in the area between the de-levered Capital Market Line (below the tangent portfolio) and the efficient portfolio. In this area it is optimal to de-lever and hold cash versus being fully invested on the efficient frontier.
Using this framework, one can derive the cost of jointly imposing a tracking error and a liquidity constraint. The cost can be conceptually thought of as the difference in expected returns between an optimal constrained mean-variance portfolio and an unconstrained mean-variance portfolio when both portfolios are operating at the same level of volatility. In figure 1 above, the cost is represented by the vertical distance between the unconstrained efficient frontier (red line) and the constrained frontier (blue line). Extending this logic, the cost of the constraints can be calculated at varying levels of both the target benchmark outperformance and liquidity thresholds to assess their marginal effects on the cost. Economically, a manager finds it much more difficult to control tracking error when the target benchmark outperformance is larger.
Outperformance forces deviations from the benchmark and deviations lead to tracking error so managers need to make corrective and costly adjustments to control this tracking error. Each additional unit of tracking error is more costly for managers who aim for greater benchmark outperformance. For example, managers who offer zero benchmark outperformance will be unaffected by tracking-error constraints. They will simply hold the benchmark and incur zero tracking error and zero cost (assuming that the benchmark is efficient). On the other hand, for managers aiming to outperform the benchmark, the tracking error constraint affects their portfolio choice and has a cost. This cost should increase as the outperformance goal increases. Similarly costs increase as managers are forced to hold more liquidity. In summary, the cost of the liquidity and tracking error constraints increase with greater demands for more outperformance and liquidity.

This is seen in Figure 2, where we plot the cost of imposing tracking error and liquidity constraints (z-axis) as a function of a manager’s liquidity demands (x-axis) and benchmark outperformance target (y-axis). Using the market capitalization portfolio as the benchmark, we compare the performance of the market portfolio with the liquidity-constrained tracking-error managed portfolios at different levels of liquidity and outperformance targets (e.g., managed portfolio identified in Figure 1 at different levels of outperformance (G) and liquidity (cash holdings)). The costs of the constraints are then estimated as we do in Figure 1. Each managed portfolio yields a triplet – (outperformance, liquidity, and cost of the constraint). Figure 2 below plots these triplets. As illustrated, as liquidity demands increase a manager incurs larger hedging costs stemming from the liquidity and tracking error constraints. Likewise, as outperformance demands increase, the manager incurs larger hedging costs.
Previously, extra performance relative to a benchmark (G) came from increasing the risk of the portfolio, that is, alpha did not exist. We now introduce manager skill into the framework. Some equate skill with the manager’s ability to identify “alpha” investments. We model alpha as a portfolio of opportunities that generates a return stream that is uncorrelated with all other assets (zero-beta) with an expected return to variance ratio of $\frac{\alpha}{\sigma^2}$. The solution to the manager’s portfolio choice problem when we add the alpha generating portfolio to the mix is:
where the last row of \( x \) (in the numerator) represents the manager’s allocation to the alpha portfolio. Allocating to the alpha portfolio, like cash, leads to tracking error and, as a result, the manager needs to balance the expected gains of allocating more risk to generate alpha against increases in tracking error.

Recall that without an alpha portfolio, a manager who needs to satisfy a liquidity and tracking-error constraint relies on over-weighting relatively higher volatility stocks, represented by \( q_1 \), to outperform the benchmark and finances this over-weighting by selling lower volatility stocks, represented by \( q_0 \). Upon including alpha, the manager now has a second tool to enhance returns. With a tracking error constraint, the manager must decide what to underweight to finance an allocation to the alpha portfolio.

This decision can be understood by first looking by simplifying the solution by removing the liquidity constraint, which is equivalent to setting \( k = 0 \) in \( EQ. 3 \).

\[
x = \frac{G(q_1 - q_0) + k(q_0R_1 - q_1R_0) + G\left(\frac{q_1\sigma_0^2}{\sigma_a^2} - \frac{q_0\alpha\sigma_0^2}{\sigma_a^2R_0}\right) + k\alpha\left(\frac{q_0\alpha\sigma_0^2}{\sigma_a^2R_0} - \frac{q_1\sigma_0^2}{\sigma_a^2}\right)}{(R_1 - R_0 + \frac{R_1\sigma_0^2}{\sigma_a^2} + \frac{\alpha^2\sigma_0^2}{\sigma_a^2R_1} - \frac{2\alpha\sigma_0^2}{\sigma_a^2})}
\]

\( EQ. 3A \)

First, \( EQ. 3A \) shows that the amount allocated to alpha is \( G\left(\frac{\alpha\sigma_0^2}{\sigma_a^2R_0} - \frac{\sigma_0^2}{\sigma_a^2}\right) \), which as expected is an increasing function of the risk adjusted return of the alpha, \( \frac{\alpha}{\sigma_a^2} \), and more so it is an increasing
function of the relative risk adjusted return of the alpha versus that of \( q_0 \). Second, the position in alpha is funded by selling \( G \frac{\sigma^2}{\sigma^2_R R_0} \) of \( q_0 \) in addition to what is sold to finance Roll’s optimal TEV solution (see EQ. 1). Like EQ. 1, the low volatility assets are once again underweight. In fact, if \( \alpha < R_0 \), meaning the funding asset returns more than the alpha portfolio, the alpha portfolio will be shorted to purchase \( q_1 \). For example, when \( \alpha = 0 \), \( \frac{\sigma^2}{\sigma^2_\alpha} \) of alpha is shorted to purchase \( \frac{\sigma^2}{\sigma^2_\alpha} \) of \( q_1 \). EQ. 3A also shows that an additional amount of \( G \frac{\sigma^2}{\sigma^2_\alpha} \) of \( q_1 \) is bought. This is to reduce the tracking error stemming from holding a zero-beta alpha and underweighting \( q_0 \).

Returning to the setting of a liquidity constraint, the effect of the cash constraint, \( k < 0 \), when alpha is present, compounds the underweight in low volatility stocks and the overweight in higher volatility stocks. The impact is multiplicative in \( k^*\alpha \), as shown in the rightmost term of the top row of EQ. 3. Further, the zero-beta alpha cannot hedge the tracking error of holding cash because cash is also zero beta. As a result, the liquidity constraint, which requires that managers hold cash, leads to a smaller allocation to alpha vs. no requirement to hold cash as long as \( \alpha \) is sufficiently small. That is, if \( \alpha < R_1 \), the position in alpha is reduced by \( \frac{k\sigma^2}{\sigma^2_\alpha} (R_1 - \alpha) \). Otherwise the position in alpha is increased because it is a better vehicle to meet the outperformance return target since it offers a higher expected return than the alternative \( q_1 \) portfolio despite its inability to hedge the tracking error associated with holding cash. Hence as expected, there is an interaction between the imposition of both a cash constraint and the zero-beta alpha portfolio on the manager’s optimal portfolio choices given the tracking-error constraint.

In summary upon equipping managers with the ability to identify alpha, the theoretical findings indicate that low-volatility assets are underweighted and higher-volatility assets are overweighted by managers needing to control tracking error. And these findings hold even upon removing the liquidity constraint. That is, with or without the cash constraint, the manager still is faced with the problem of what names to sell to finance the acquisition of the alpha portfolio to
limit the portfolio’s tracking error. As argued above, low volatility stocks are less “tracking error expensive,” leading to the underweight of low volatility securities to finance the alpha portfolio.

With this formal model, we have the toolset to assess the portfolio managers’ supply and demand for high volatility securities under a variety of scenarios. A question of particular interest to answer is how their ability to use leverage affects their portfolio allocations? Starting with Black (1972), many have pointed to leverage constraints as the source of the low-versus high-volatility premium anomaly. They argue that higher volatility names are bid up to increase risk because lower volatility names cannot be levered as cost-effectively to achieve the same level of risk. We do not believe, however, that leverage constraints are the source of the low volatility anomaly because access to leverage has been available for many decades yet the low volatility “anomaly” persists. For example, many investment vehicles, such as futures, options and more recently levered ETFs, provide easy access to leverage to a very broad set of investors, from retail to institutional. Furthermore, financial institutions have always been able to access direct borrowing.

In addition, and most important, our model indicates that the tracking-error constraint dominates the leverage constraint. That is, with the ability to use leverage, we find that investment managers will use limited amounts of leverage if allowed to do so. In addition, we show (see contour plots below) that although the optimal solution often does employ leverage, the optimal amount of leverage is not a perfect substitute for the overweighting of higher-volatility securities. Specifically at the optimal leverage ratio, managers still underweight lower-volatility securities and overweight higher-volatility securities.

Below is the contour plot for the minimum tracking-error (“TE”) at different pairs of (leverage, k, and outperformance, G). There is an optimal amount of leverage that minimizes tracking error for a given G, i.e. maximizes G/tracking error (information ratio). The contour lines are concave. Minimum tracking-error first falls with the use of more leverage and then increases. So leverage is not a binding constraint in the sense that managers will not use as much
leverage as is allowed. They will limit the use of leverage and still over-bid high volatility stocks and supply low-volatility stocks.

In the contour plot below, for example, if $G = 3\%$ and managers are allowed to use up to 75% leverage the optimal solution indicates that they would only use 41% leverage. This is illustrated by the black dot below. On the black dot, there is a 20% more demand for high-volatility stocks than for low-volatility stocks. At the optimal solution, the low volatility effect is still alive!

An alternative way to display the excess demand for higher volatility securities, the contour lines below represent the excess demand for higher volatility stocks minus the demand for lower volatility stocks. For example, 0.5 means that the relative demand of high volatility stocks versus low volatility stocks is 50%. The red dots represent the optimal solution that minimizes TE for outperformance of level $G$ and leverage $k < \infty$. What we notice from the red optimal solutions is that infinite leverage is not used and the optimal solutions all result in larger demand for high volatility stocks versus low volatility stocks, i.e. they lie above the zero contour line. In the base extreme case of a zero outperformance target ($G = 0$), the manager reduces tracking error to zero and holds the benchmark. Here the red would lie on the zero-contour line, which implies a zero excess demand of both high and low volatility stocks.
EQ. 3 can be decomposed into the following two informative pieces:

1. The outperformance target, $G$, is achieved making adjustments to the portfolio without using leverage, $k = 0$. This results in tracking error. These are represented by the components of EQ. 3 that are a function of $G$, which when combined has an expected return of $G$.

2. If leverage can be used, it will be used to reduce the tracking error from achieving the outperformance target. The tracking-error reducing portfolio that is formed has an expected return of zero. These are represented by the components of EQ. 3 that are a function of $k$.

The components that are a function of $G$ are long $q_1$, short $q_0$ and long $\alpha$. The components that are a function of k act only to control tracking error and do not affect return. Therefore, the best way to control tracking error is to do the opposite of the components of $G$. 
And the opposite is to short $q_1$, go long $q_0$ and to short alpha. Now, shorting alpha only makes sense if in generating returns with no leverage, we had more than enough alpha exposure to meet the $G$ target. And with alpha greater than $R_1$, which is the highest non-competing alpha return, we have more than enough. Hence with $\alpha > R_1$, the optimal portfolio to meet the $G$ target and to minimize tracking error is to use leverage, $k > 0$, and to short $q_1$, long $q_0$ and short $\alpha$, using weights such that the return on this tracking-error control portfolio is zero. But, investors would not want managers to short $\alpha$, which is in scarce supply. Moreover, investors need to worry that managers claim that $\alpha > R_1$ while instead using the Roll model to generate “excess” returns by shorting more $q_1$ and $\alpha$, and using leverage to go long $q_0$ to generate returns. That is, investors are willing to pay managers for generating alpha but not to pay them from holding a higher risk portfolio and levering up beta to “beat” the benchmark. As a result, they would want to restrict the use of leverage to reduce tracking error. If $\alpha > R_1$, they would tolerate the extra tracking error as an alternative to direct monitoring costs.  

If $\alpha < R_1$, however, leverage can be used to generate more alpha (since there is not an excess amount of it) and the optimal tracking-error control portfolio simply is to short $q_1$ and buy $q_0$ and this portfolio has a positive cost proportional to $\alpha - R_1$. This cost is offset, however, by the return generated by the extra alpha as a result of the use of leverage. While, there is no explicit value to leverage because the expected return of the additional holdings stemming from the use of leverage is zero, there might be value in the sense that the marginal tracking error control portfolio has more return coming from alpha and not beta via buying $q_0$. Whether or not it does, depends on the additional return that comes from obtaining more alpha as a result of the use of leverage instead of buying more $q_0$ with leverage. This is so when

$$\alpha < \frac{R_1}{2} \left( 1 - \frac{R_1}{\sigma_1^2} \right) = R^*$$

---

4 Actually, if managers reduced tracking error to zero and leveraged the benchmark portfolio they would expect to beat the benchmark but achieve no value for investors. $k=0$, is a substitute then for adding an additional constraint that the factor exposures of the portfolio remain equal to those in the benchmark.
In other words, the expected return of alpha needs to be less than one half $R_1$ times the expected return to variance improvement of alpha over $q_1$. Otherwise, it is $q_0$. If, $\frac{\alpha}{\sigma^2_\alpha} \gg \frac{R_1}{\sigma^2_1}$, then $\alpha < \frac{R_1}{2}$ and if $\frac{\alpha}{\sigma^2_\alpha} \leq \frac{R_1}{\sigma^2_1}$, then the condition never holds for $\alpha > 0$.

Although there might be a weak case to use leverage if $\alpha < R^*$, the “cheating” costs of managers claiming that to be true and attempting to generate returns by “levering beta” by taking on more $q_1$ or $q_0$ might outweigh the benefits of reducing the tracking error. This indicates that with a tracking-error constraint model investors might also restrict the use of leverage in this case.\(^5\)

The model’s implications further hold up when introducing a short-selling constraint with or without a liquidity constraint. While restricting short-selling does change the mean/variance frontier to be interior to the frontier with short-selling, with respect to a manager’s optimal portfolio choice of high vs. low volatility stocks the largest implication is on the zero-beta alpha. Without short-selling, a zero-beta alpha likely does not exist as a zero-beta asset is typically a long-short portfolio. As a result, the alpha takes the form of a positive-beta alpha asset. In this setting, the manager’s first best choices of assets to sell to finance the investment in the positive-beta alpha portfolio are those assets that have the highest correlation to the positive beta alpha portfolio. These names naturally will have the lowest idiosyncratic volatility. Further, above we showed that maintaining liquidity by holding cash has value because of its positive shadow price. Among the highest correlated assets to the positive-beta alpha portfolio, cash can be generated by choosing to underweight the lowest volatility names and not the highest volatility names. As an illustrative example, and assuming a CAPM world, assume there are two names that belong to the benchmark and have equal correlation to the positive beta alpha. One has a beta of 1 and the other 0.5 with half the idiosyncratic volatility as well. Selling $2$ of the beta 0.5 asset will result in the same tracking error as selling $1$ of the beta 1 asset. However the former yields $1$ of cash holdings. This suggests that given the value of holding cash and maintaining liquidity, active

\(^5\) This indicates that likely candidate portfolios for leverage are low alpha, high information ratio portfolios such as fixed income managers including fixed-income hedge funds, banks, insurance companies, etc. They would use leverage to magnify their alpha signals to meet performance targets.
managers supply (sell) low idiosyncratic and total volatility assets. Empirically this has found to be the case by others, who find that both low idiosyncratic and low total volatility stocks outperform (e.g., Lakonishok and Shapiro (1986) and Ang, Hodrick, Xing and Zhang (2009)).

Although the optimization structure underlying the equilibrium model of EQ. 1 is a one-period model, sensitivity of the allocations to changes in the inputs provides evidence as to the dynamics of the system. Based on the sensitivity analysis, as managers expect more alpha or their estimate of the risk of generating alpha falls, (i.e., the alpha-to-risk ratio increases), the manager wants more of the alpha portfolio but is forced to incur higher cost to adhere to liquidity and tracking-error constraints. Further, costs also increase with increases in the amount of liquidity that a manager holds.

Based on this thinking, we now have the logic in place to understand how the introduction of “alpha” to the system affects the costs of the constraints. Active investment managers will underweight less volatile stocks relative to more volatile stocks the more confident they are in their alpha portfolios and the more binding are the tracking error and liquidity constraints. If they become less certain about their abilities to forecast abnormal returns, they will reduce their holdings in higher risk securities and buy lower risk holdings; that is, reduce their tracking-error hedging positions. In the extreme, they could revert back to holding the benchmark or the liquidity only constrained portfolio. If the collective actions of all active managers affect the prices of higher volatility (and lower volatility) securities, as active managers change their views as to their ability to generate alpha either the prices of higher volatility stocks fall relative to lower volatility stocks if they have less confidence in their alpha predictions or the reverse occurs when they have more confidence. Demonstrating this empirically would be strong evidence in support of the clientele hypothesis that argues that since active managers are constrained to hold a disproportionate amount of higher risk securities they

---

6 If managers were allowed to use leverage and had no short-selling constraint, they would finance their positive alpha-zero-beta portfolios by using the proceeds of their short positions to buy their long positions. They would then buy their benchmark portfolio and use leverage to use up the tracking-error budget. Allowed to use leverage, but with short-selling constraints, active managers would first hold the optimal portfolio as if leverage were not allowed, and, finance the alpha using lower volatility names. They then would lever the positive-beta alpha portfolio to use up any remaining tracking-error budget.
cannot be the marginal investors in the market. We summarize the expected dynamics as follows in Table 2.

Table 2: Expected Comparative Statics of Returns to High Volatility minus Low Volatility Securities as Alpha, Volatility of Alpha and Liquidity Requirements Change

<table>
<thead>
<tr>
<th>Parameter Change</th>
<th>Expected Performance of High Volatility minus Low Volatility Securities</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha increases</td>
<td>Positive</td>
<td>Move away from the benchmark to capture alpha</td>
</tr>
<tr>
<td>Alpha decreases</td>
<td>Negative</td>
<td>Move toward the benchmark as reward to tracking error decreases</td>
</tr>
<tr>
<td>Uncertainty increases (volatility of alpha)</td>
<td>Negative</td>
<td>Move towards the benchmark as less reward to risk for tracking error</td>
</tr>
<tr>
<td>Uncertainty decreases (volatility of alpha)</td>
<td>Positive</td>
<td>Move away from the benchmark as more reward to risk for tracking error</td>
</tr>
<tr>
<td>Portfolio Liquidity Increases (k increases)</td>
<td>Negative</td>
<td>Increased demand for liquidity requires low volatility security sales</td>
</tr>
<tr>
<td>Portfolio Liquidity Decreases (k decreases)</td>
<td>Positive</td>
<td>Decreased demand for liquidity requires low volatility security purchases</td>
</tr>
</tbody>
</table>

Behaviorists, on the other hand, would predict the exact opposite, as “gamblers” should increase their demand for volatile securities when uncertainty increases. In the behavioral parlance, the “lotteries” become more “lucrative” when volatility increases (as lotteries can be thought of as call options) increasing demand. Therefore, the pricing dynamics during periods of increasing uncertainty serve as a good test of the competing theories. We investigate these exact dynamics empirically later.

In the next section, we will discuss pricing and why constraints exist and in the empirical section that follows we will demonstrate the relation between the tracking error of the portfolios of active managers and the returns differences on high and low risk securities.
Theory of Investment Constraints and Understanding the Model

Investors impose explicit tracking error constraints on their investment managers to achieve at least two goals: to manage the inherent principal-agent conflicts inherent in delegated (investment) management and to control the risks of their multi-asset/multi-asset externally-managed portfolios. The principal-agent conflicts of generalized delegated management have been widely discussed starting with Jensen and Meckling (1976) classic treatment. We expand on these ideas as they relate to investment management, and more importantly, to asset prices.

Investment managers often find that high returns lead to more assets under management and greater management fees, while lower performance results in slower growth or slow loss of assets over an extended period of time. Incentives are not aligned. Further, it is difficult if not impossible for investors or surrogates to ascertain whether gains or losses were due to skill or luck. The data and the time frame are too short to distinguish among the alternatives. As a result, following the classic solution to any “lemon problem,” investors preemptively act to find the lowest common solution by constraining all managers.7

Constraints are presumably a lower-cost alternative to expending resources (an explicit cost) to monitor the skills and investments (related to risk management) of their investment managers. Investors can measure tracking error, beta and liquidity and use these statistics to keep themselves apprised of the investment activities of the manager at lower costs than alternatives.8 If the investment manager is not adhering to the agreed constraints, investors can withdraw funds or ask the investment manager to explain the discrepancy.9 Moreover, investment managers adhere to the constraints if they know or even suspect that their investors will withdraw funds if deviations occur from the set of agreed to constraints. In addition,

7 Since investors hire multiple investment manager specialists in different categories, they might constrain them to manage the risk of their aggregate portfolio. Moreover, the tracking-error constraint allows investment managers to specialize without having to ascertain where to move the risks of the funds under their individual mandates.
8 This is in contrast to an investor spending time to educate themselves on manager activities and corresponding with managers regularly which is very time consuming and a high cost alternative to keeping appraised of manager activities.
investment managers might act on their own to constrain tracking error fearing that too large a loss relative to the benchmark might lead to investor withdrawals. The most extreme investment constraint is to require that the investment manager tracks an index each day. Since many investors, however, believe that their managers are able to add value to their portfolios; they allow them to deviate from the benchmark with a “promise” to manage with, for example, up to a 5% tracking error.

There is an unusual aspect to constraints in the investment management context. Rating agencies, such as Morningstar, ask investment managers to select a benchmark. These agencies rate managers relative to their selected benchmark. Therefore, to maximize their profits, theoretically, unconstrained investment managers might manage to a tracking error constraint to achieve a higher rating. Investors can more readily separate performance generated from skill, alpha, from risk, if managers select a benchmark comparison. These business dynamics encourage managers to be cognizant of their performance relative to a chosen benchmark.

**Constraints generate hedging demand**

Constraints cause investors to hold inefficient portfolios in the standard mean-variance unconstrained framework. In a world of second best, in a world of information costs (either monitoring or risk management costs), the most efficient equilibrium might be the constrained equilibrium. If investors find methods to reduce the shadow costs of the constraints that they impose on their investment managers the constrained equilibrium will move close to the unconstrained equilibrium. Crucially, these “inefficient” holdings or hedging demands are completely rational: investors know their investment managers will constrain portfolio holdings and are willing to give up returns (pay limited implicit costs) for the value of the constraints. We term the performance drag introduced by the hedging sub-portfolio “the cost of constraints.” It is an implicit cost that manifests itself through lower realized-rates-of-return than those of the unconstrained world. In addition, to the beta and alpha premiums, we conjecture that these “costs of constraints” represent a third premium that collectively we call omega.
In addition, if the demand for higher risk securities influences prices lowering expected returns, the shadow costs of the constraints for investors and their investment managers would be higher than just holding inefficient portfolios. Since most investors who delegate investment management to investment managers have similar monitoring costs, they force their investment managers into similar hedging portfolios, which, when aggregated, generate large hedging demands. We surmise that these hedging demands are large enough to affect the asset prices of higher volatility and higher beta stocks and that these securities are priced to return sizeable expected negative omega premiums.

**Speculators: The supply side of the hedging market**

If active managers are the proper clientele for more volatile securities, who supplies these securities to them or who are short these securities to profit from the shadow cost of the constraints? Active managers rely on speculators to provide this capital. Speculators will hold more of the lower risk securities and hold less of the more risky securities (or short them).

The speculators provide risk-transfer services by buying lower-risk stocks and shorting higher-risk stocks and require compensation for the services that they provide; compensation that comes in the form of earning positive hedging premiums, which we term “omega.” Speculators require adequate compensation for their risk capital and the alternative uses of their time. Therefore, the expected omega premium is positive. Constraints are rational; resulting hedging demands are rational; omega premiums are, therefore, rational.

We expect to observe persistent but dynamic omega premiums in asset prices. This is exactly what we find in our empirical work – the low volatility stock premium is persistent but variable. We demonstrate a link between the level of the expected volatility premium and the tracking error of active managers. Under behavioral-finance assumptions, investors have preferences for higher-volatility stocks, which, unlike our model, are either constant or change randomly or increase with increases in volatility. One possible dynamic under behavior assumptions is that demand for volatile securities increases when volatility increases – “gambler” investors prefer volatile investing landscapes to quiescent ones.
Under our model, investment managers have preferences for higher-volatility stocks. Therefore, we can’t rely on these professional investment managers to supply the capital needed by the behavioral models. Moreover, there is an identification problem. For example, the observed lower returns on high beta stocks might either be caused by “behavioral” models and “irrational” investors or be rational hedging demands of sophisticated investors. These sophisticated investors are willing to give up returns to speculators, who earn an omega premium.

We have already discussed the expected dynamics of the demand side of the omega premiums. The suppliers of risk capital have their own dynamics that are similar to those of the classic speculator. Classical speculators deploy risk capital when they expect to earn at least their required risk-adjusted return on capital. If the opportunities improve, speculators generally deploy more risk capital.

At times of crisis or shock, however, the speculators might make profits or suffer losses depending on the dynamics of the omega premiums. For example, in 2008, speculators, holding low volatility securities versus high volatility securities made money as omega premiums fell and lost money as other risk premiums widened. Active managers might not have confidence in their alphas and abandoned active management and bought lower risk stocks from the speculators by using the proceeds from liquidating their higher risk holdings. Moreover, At times of shock, speculators might be unable to understand how to make profits. As a result, if their intermediation services are in demand, they will withdraw risk capital and the omega premium, the cost of risk-transfer services will tend to increase, increase. Furthermore, to the extent speculators in one omega market speculate in other omega markets, there may be contagion among markets if they need to liquidate positions across multiple holdings.

**Empirical Work: Mutual Fund Tracking Error Estimation Procedures**

There is extensive academic literature documenting the outperformance of lower volatility assets relative to higher volatility assets across both systematic measures of volatility (beta) and idiosyncratic measures (residual volatility). Starting as early as the 1970’s, Black,
Jensen and Scholes (1972), Fama and MacBeth (1973), Haugen and Heins (1975) and others have discovered that lower beta stocks offered a higher risk-adjusted return than higher beta stocks, implying the capital asset market line was much flatter than the Sharpe (1964) capital asset pricing model would theoretically predict. Lakonishok and Shapiro (1986) showed that not only do traditional measures of risk, beta, fail to explain the cross-section of US stock returns but non-conventional measures of risk, total volatility and idiosyncratic volatility, also fail to explain the cross-section of returns. In the 1990’s, Fama and French (1992) confirmed the “low volatility anomaly” and showed that beta does not explain the cross-section of expected returns once accounting for size, book to market, earnings yield and leverage. The universality of the anomaly was studied by Ang, Hodrick, Xing and Zhang (2009) whose results showed that the underperformance of stocks with high idiosyncratic volatility extended to international markets. Frazzini and Pedersen (2013) broadened past empirical studies on the beta-return relationship to global fixed income, commodities and currency markets and found that the high beta low return phenomena intact across multiple asset classes and geographies. These rich and robust empirical findings have made the “low volatility anomaly” one of the most well-known puzzles in finance.

Our theoretical model, however, leads active managers to demand higher volatility stocks and supply lower volatility stocks due to a combination of tracking error and liquidity constraints, both with and without the presence of alpha, that constrain their ability to make active investments in stocks that they think will outperform. A key conclusion of the model is that active managers with constraints will hold a larger proportion of their investments in higher volatility assets than the composition of the fund’s benchmark, which implies that they hold fewer lower volatility assets than their benchmark. Unlike previous empirical work, that focuses on unconditionally higher and lower risk stocks, in the empirical work that follows, we condition our analysis by controlling for the volatility of the stocks in the benchmark to assess whether our theoretical framework is consistent with practice. Further, we test the dynamics implied by the theory. Our model predicts that when mutual fund managers believe that the significance of their expected abnormal returns is high, they will take on larger tracking-error and make investment choices to allow them to buy more of the stocks that they want to hold. At these times, tracking error constraints become binding and, as explained above; managers will have more of a demand for high volatility stocks and supply more lower volatility stocks. When the significance of the
expected abnormal returns is low, however, they will reduce tracking error towards zero and the supply of lower volatility stocks will fall as will the demand for high volatility stocks.

We find these dynamics to hold in practice. At times of shock such as in 2008, when, most likely, active managers had difficulty estimating expected returns and their risks, we would expect them to reduce their active tracking errors, leading the tracking-error hedging demands to disappear. They would reduce their holding of higher volatility stocks and increase their holdings of lower volatility stocks. Post the 2008 crisis, as governments around the world communicated their commitments to support economies and uncertainty fell, the model predicts that tracking error would increase with a commensurate increased demand for volatile stocks.

We identify a sample of 95 US mutual funds in the CRSP Mutual Fund database that have 1) realized tracking error between 3% and 10%, 2) an estimated beta to the S&P500 between 0.93 and 1.07, 3) assets under management greater than $500m (at least for 90% of the time period) and 4) returns in the database for over 90% of the available sample period.\(^\text{10}\) Given these characteristics, these funds likely have meaningful (i.e. potentially binding) tracking-error constraints. We anticipate that these funds would have disproportionate holdings of higher volatility securities than their benchmark’s holdings, and, that these funds reduce their tracking error by reducing their holdings of higher volatility securities when confidence in their ability to earn abnormal returns wanes, such as at the time of market shocks (e.g. during the 2008 financial crisis.) We use return data on each of the funds from 1999Q3-2013Q2.\(^\text{11}\)

To show that the tracking error of the funds is not constant but varies over the time period and is related to the uncertainty in the market, we needed (a) to construct an estimated time series

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\(^{10}\) We use the 10% upper bound on tracking error to eliminate funds with potentially different mandates. We use a 3% lower bound to eliminate funds with strict investment guidelines – e.g., index funds. We were left with 95 funds in our sample. Funds in our sample set include the T. Rowe Price Growth Stock Fund (ticker PRGFX, recent AUM of $24B) and the Fidelity Magellan fund (ticker FMAGX, recent AUM of $19B).

\(^{11}\) We apply the first Lipper fund asset code backwards and forwards and then separate out equity funds (code EQ). We do not view this assumption as aggressive because few funds change asset classes. Lipper classifies most mutual funds as either “Growth” or “Growth and Income”. The CRSP data base retains all mutual funds that existed and, therefore, has no survivorship bias.
of their realized tracking error, and (b) to normalize the estimates for changes in the level of market correlation over the sample period to eliminate heteroscedasticity. That is, we normalized the tracking error to separate tracking error due to exogenous increases in residual volatility (decreased stock correlation) that would naturally increase tracking error from increases in tracking error that results from the active decisions of the managers of the funds. For, it is the investment decisions of the active managers that are tied to the theory in the paper.

First, to estimate the non-normalized tracking error, we compute the residual, the absolute difference in daily returns of each mutual fund from the returns on the S&P500, for each day in the sample. We then compute the daily median absolute residual for the 95 funds. We call this MF_ERR and it is a measure of tracking error. That is, the daily MF_ERR_t is

\[
MF_{ERR_t} = \text{median} \left\{ |R_{MF_1,t} - R_{SPX,t}|, \ldots, |R_{MF_N,t} - R_{SPX,t}| \right\}
\]

where

\[
\begin{align*}
\text{Beta} &< 0.93, \\
T_E &< 1.07, \\
2\% &< T_E, \\
AUM_{MF_i} &> \$500M \text{ for } 90\% \text{ of } \text{Obs} \\
\text{Obs}_{MF_i} &> 90\% \times \text{Obs}_{Sample}
\end{align*}
\]

Tracking error (TE) is the annualized daily tracking error (daily average for each year times the number of trading days in the year).

We plot the 30-day moving average of the median mutual fund absolute tracking error in Figure 4.
We observe that realized tracking error spiked during the “internet bubble” of the early 2000s and during the “credit crisis” 2008-09. We also observe that the moving average changes quickly, evidenced by the high number of “spikes” shown in the graph.

Second, we attempt to decompose the above tracking error measure into that caused by changes in correlation (i.e. changes in idiosyncratic volatility) and that caused by active managerial decisions to deviate from the benchmark. To do so, we first measure the average daily absolute beta-adjusted return deviation of each stock contained in the S&P 500 from the S&P 500 return. We call this variable SPX_ABS_IDIO. For each day, it is defined as

\[
SPX_{ABS\_IDIO_t} = \text{average}\left\{ \text{abs}\left( R_{i,t} - B_{i,t} \times R_{SP500,t} \right) \mid \text{Stock } i \in \text{S\&P500 on day } t \right\}
\]

where beta for stock i is the CRSP computed beta as of the most recent previous year end and 1 if unreported. The moving average of this time series is shown in Figure 5.
Not unexpectedly, 2008 and 1999-2003 generated by far the period of highest idiosyncratic deviations as measured by SPX_ABS_IDIO.

While MF_MED_ABS_ERR reflects the tracking error caused by the combination of changes in correlations and changes in manager holdings, SPX_ABS_IDIO reflects tracking error caused by changes in inter-stock correlations (idiosyncratic volatility) only. The part of MF_MED_ABS_ERR unexplained by SPX_ABS_IDIO represents tacking error stemming from changes in active holdings. We compute the residuals from a regression of MF_MED_ABS_ERR onto SPX_ABS_IDIO. The summary regression statistics are shown in Table 3. The regression indicates that 60% of the variability of mutual fund tracking error is caused by changes in inter-stock correlation, with the remainder explained by changes in active holdings.

Regression 3 - Tracking Error
R Square 0.61
Observations 3104

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>SE</th>
<th>t Stat</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-0.001</td>
<td>0.000</td>
<td>-12.53</td>
</tr>
<tr>
<td>SPX_ABS_IDIO</td>
<td>0.189</td>
<td>0.003</td>
<td>70.29</td>
</tr>
</tbody>
</table>

The 30 day moving average of the daily residuals from this regression are shown in Figure 6. These residuals estimate the discretionary average tracking errors of the mutual fund managers (their “target”) in our sample. We call this series “active” tracking error.

Figure 6: 30 day moving average of residuals of regression above

12 Results using non-beta adjusted stock returns are similar to the measure used.
The “active” tracking-error series fits our qualitative expectations remarkably well. Target tracking error was highest during the internet “bubble,” with a spike in late 1999 into early 2000. Investment managers did not “hug” their benchmarks during this period and were more aggressive in taking tracking error. From the theory, it seems reasonable to assume they did so because the expected abnormal returns were greater for the level of idiosyncratic risk assumed during this period. Active tracking error collapsed at the end of the internet “bubble.” It collapsed again during the financial crisis of 2008. In these periods, it is most likely that investment managers lost confidence in their ability to outperform the benchmark and, therefore, increasingly “hugged” the index. That investment managers would lose confidence in 2008 and return their investments towards their benchmarks is economically defensible given their inability to model expected abnormal returns given the tremendous uncertainty in the markets, uncertainties as to the direction of the economy and the fear of their investors that the economy was on the edge of a “depression.” And, many managers reduced risk and increased liquidity in anticipation of increased investor redemption demands. Similarly, it makes qualitative sense that manager confidence was highest in 1999 – 2000 as equities were doing exceptionally well and deviations from the benchmark often were handsomely rewarded by market price changes. The period, 2004 – 2007 were characterized by “normal investing” and was representative of a more typical tracking-error period. In sum, the active tracking-error index fits with our qualitative expectations as to whether managers would have more or less confidence in their ability to outperform their benchmark.

We could improve these results with additional tests to measure the manager’s reward-to-risk of their active investments. If these measures were available, we would expect that we would observe larger tracking errors with greater expected reward-to-risk ratios. We expect that changes in the active tracking error index would correlate highly with changes in their certainty of their reward to risk ratios of their active portfolio.
Linking Changes in Active Tracking Error to Changes in Asset Prices:

We now turn to link changes in active target tracking error to changes in asset prices. Investment managers optimize their portfolios using a hedging portfolio that we believe is overweight volatile stocks, on average, and the proportion invested in that portfolio changes with changes in target tracking error. We expect that low volatility stocks will outperform high volatility stocks when active tracking error falls. We used BARRA factors to estimate the returns on volatility factors. More information on BARRA factors can be found on their website.\(^\text{13}\)

For estimates of changes in returns associated with changes in volatility, we use the returns on two sets of long-short, volatility-sorted portfolios. First, we compute the returns on each of four long-short portfolios constructed by sorting on the factor loadings of four separate BARRA volatility-related factors: (1) Beta, (2) Volatility, (3) Total Risk and (4) Specific Risk (that is, BBeta, BVol, BTRisk and BSRisk) by buying the stocks with the lowest factor loadings for each of these “volatility” measures (the less volatile stocks) and selling the stocks with the highest factor loading for each of these “volatility” measures.\(^\text{14}\) Our volatility sorted portfolios are rebalanced once a month, are equally weighted and are industry neutral. Second, we use the BARRA volatility factor returns.\(^\text{15}\) For ease of comparison with our long-short portfolios, we consider the negative of the BARRA volatility factor returns (nVolatility) for the BARRA volatility factor is effectively long more volatile stocks while our long-short portfolios are long less volatile stocks so taking the negative return makes nVolatility investing in lower volatility stocks and shorting higher volatility stocks.

There are several advantages for us to use the BARRA model risk estimates. First, BARRA is a third party model that is constructed for risk management uses – exactly the activity we want to measure. Second, the BARRA factor returns are particularly useful for asset pricing

\(^\text{14}\) BARRA is an MSCI division that provides equity risk models and factors correlated with the volatility and risk of single stocks.
\(^\text{15}\) According to BARRA, referencing their USE3 model, “Volatility — captures relative volatility using measures of both long-term historical volatility (such as historical residual standard deviation) and near-term historical volatility (such as high-low price ratio, daily standard deviation, and the cumulative range over the last 12 months). Other proxies for volatility (volume beta) are also included.”
tests as they are independent of all other BARRA factors including industry and beta. These are “clean” long-short returns versus those of, for example, book-to-market sorted factor returns that may have industry exposures that could cause spurious asset-pricing relations. We believe this is a novel approach.

As shown in Table 3, we identify the five major changes in our target tracking error series by finding local maximums and minimums of our multiyear tracking error series:

Table 3: Major changes in target tracking error index and expected asset price performance

<table>
<thead>
<tr>
<th>Major Changes in Target Tracking Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Date</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Sep-98</td>
</tr>
<tr>
<td>Apr-00</td>
</tr>
<tr>
<td>Oct-02</td>
</tr>
<tr>
<td>Mar-08</td>
</tr>
<tr>
<td>Nov-08</td>
</tr>
</tbody>
</table>

Again, for each shift, we predict the cumulative return to our volatility factors: when target tracking error increases (decreases), we expect volatile stocks to outperform (underperform).

In line with our predictions, in Table 4, we find strong evidence that links changes in tracking error to changes in asset prices. When our target tracking error measure increases, our volatility factors have large positive returns. Conversely, when our target tracking error measure decreases, the volatility factors have large negative returns as seen below.
Table 4: Major changes in target tracking error index and realized asset price performance

<table>
<thead>
<tr>
<th>Date</th>
<th>Start</th>
<th>End</th>
<th>Low Volatility - High Volatility Expected Performance</th>
<th>Realized Factor Performance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Average</td>
</tr>
<tr>
<td>Sep-98</td>
<td>Apr-00</td>
<td></td>
<td>Underperform</td>
<td>(2.6%)</td>
</tr>
<tr>
<td>Apr-00</td>
<td>Oct-02</td>
<td></td>
<td>Outperform</td>
<td>37.7%</td>
</tr>
<tr>
<td>Oct-02</td>
<td>Nov-03</td>
<td></td>
<td>Underperform</td>
<td>(16.2%)</td>
</tr>
<tr>
<td>Mar-08</td>
<td>Nov-08</td>
<td></td>
<td>Outperform</td>
<td>13.2%</td>
</tr>
<tr>
<td>Nov-08</td>
<td>May-10</td>
<td></td>
<td>Underperform</td>
<td>(39.5%)</td>
</tr>
</tbody>
</table>

In each of the major instances of tracking error changes, the average realized factor performance is as predicted. These results strongly indicate that the hedging activities of active managers drive demand for and changes in demand for high volatility stocks versus low volatility stocks. And the results as economically significant: for example, from April 2000 to October 2002, when our target tracking error measure decreased sharply, nVolatility returned 9.0%, BBeta returned 41.3%, BVol returned 26.3%, BTRisk returned 57.1% and BSRisk returned 54.6%

Remember, that the BBeta, BVol, BTRisk and BSRisk are long-short dollar neutral returns neutral to other stock factors, including industry.\(^{16}\)

Changing weights of the Aggregated Mutual Fund Portfolio Relative to the S&P 500 benchmark weights:

As further tests, we sorted all of the stocks in the S&P500 (using the Fidelity S&P500 index mutual fund composition to proxy for the S&P500) from lowest-to-highest volatility as of December 31, 1999. We estimated the volatility of each stock in the index using daily return data for the previous quarter (October to December 1999). We then grouped the stocks into deciles based on this ranking with the lowest volatility stocks assigned to group 1, the highest to group 10. For each group we computed the percentage market weight of the S&P500 as of the end of December 1999. At the end of 1999, the lowest and highest volatility groups comprised far less than 10% of the market value of the stocks within the S&P500.

\(^{16}\) The BARRA US equity model consists of 13 systematic stock factors.
Using the 95 active mutual funds in our sample of active-constrained equity managers, we aggregated their portfolio holdings to construct a grand “mutual fund.” As with the stocks in the S&P500, we repeated the grouping procedure based on ranking the stocks in the aggregated mutual fund portfolio on their quarterly estimated volatility (October to December 1999), but used the volatility boundaries of the 10 S&P500 decile portfolios to designate the boundaries for the ten groups for the grand mutual fund. We computed the market value weights of the stocks in each of the ten groups in the grand mutual fund portfolio as a fraction of the total market value of stocks in fund for December 1999.17

We repeated the process of ranking on volatility at the end of each subsequent quarter; that is, dividing the stocks into 10 groups based on the previous quarter’s volatility estimates and computing the S&P portfolio group weights for each quarter through 6/2013, 56 quarters in all. As we did for December 1999, we computed the quarterly market-value weights for the grand mutual fund portfolio for each quarter, until 6/2013, 56 quarters in all.

Generally, for the S&P500 benchmark, groups 1 through 5, the lower volatility stocks, contained 60% to 72% of the market value of the stocks in the index over the 56 quarters. The mean and standard deviations (of groups 1 through 5) were 66.04% and 6.01%, respectively, with a median of 68.09%. For the grand mutual fund portfolio, however, for the same groups, the mean was 56.17% and the standard deviation was 6.04%. The median stock holdings of the aggregated mutual fund lower volatility groups were 56.33%. The weights in the lower volatility groups 1 through 5 were more variable but entirely below those of the S&P 500 for each quarter. The mean difference was -9.87% with a standard deviation of 4.52%.

These test procedures control for both changes in volatility and the market value of stocks in the benchmark over the 56 quarters from December 1999 through June 2013. The mutual funds were holding higher volatility stocks over this period than those contained in the benchmark portfolio. The t-statistic of the mean difference of -9.87% in their respective

---

17 If a stock in the aggregated portfolio had an estimated volatility in excess of the largest estimated volatility of the S&P500 stocks that quarter, it was placed in group 10.
portfolio weights was -16.31. The active manager that is subject to a tracking-error constraint holds less low volatility stocks.

There are many market forces that affect the returns on higher volatility stocks differently from the returns on lower volatility stocks. One might be that the active managers change their portfolio compositions based on their ability to forecast abnormal returns. The greater their expected ability to forecast returns; that is, the greater the reward-to-error of their forecasts, the more tracking error they would entertain for their portfolios. Our theory, however predicts that the greater their expected ability, the greater the proportion of their portfolio that they would hold in higher volatility stocks. The higher volatility stock holdings mitigate the costs of tracking error constraints. Higher expected ability compensates, in part, for the costs of additional tracking error. We presume, however, that at times of shock, the ability to forecast returns falls and those active managers reduce their active portfolios and reduce their holdings of higher-volatility stocks as they move closer to their benchmarks.

Figure 7 is a plot of the differences in the weights of the aggregate mutual fund from the S&P500 weights for three groups. In the MF High minus S&P High (diamond line), we plot the difference between the portfolio market weights of the top three groups ranked on volatility of the aggregate mutual fund portfolio from the top three group weights of the S&P 500. Similarly, we plot the same statistics for the Mid Diff (difference in weights for the middle four deciles) and the Low Diff (difference in weights for the lowest three deciles in volatility.) We also show both the plus and minus one standard deviation of the MF High - S&P 500 High.

The mutual funds began decreasing their exposures to higher volatility stocks starting in 2008 and increased their holdings of the middle volatility stocks relative to the S&P 500. The same is true after the “dot.com bubble” in 2000. The reduction in higher volatility holdings and the increase in the holdings of lower volatility stocks is exactly as the theory predicts. This combination brings the mutual fund holdings closer to market weights.

Using the difference in the holdings of the top five groups (highest volatility of the aggregate portfolio from the similar groups of the S&P500, the mutual fund managers held
Figure 7: Plot of the Differences in Holdings of the Aggregate Mutual Fund and the Benchmark for High, Median, and Low Volatility Portfolios from end-1999 to June 2013.
These results are similar to those of our previous tests. Here, however, we controlled for the changes in volatility of the S&P500 benchmark portfolio. As the theory suggests, the active managers reduce their relative holdings of the higher volatility stocks with increases in market uncertainty. [ASH: Should we take the below sections out? I don’t completely understand the differences in return differences and differences in holding differences…hard to get my hands around second derivatives. Changing to just changes might be useful. We can then show that the demanders (in this case mutual funds) do affect prices. When mutual funds reduce their holdings of high vol stocks, if we can show high vol stocks underperform that would be great. And vice versa for low vol stocks.] We plotted the quarterly changes in the return differences of the low volatility grand mutual funds minus the low volatility S&P500 benchmark stocks against the quarterly changes in Low Diff.\textsuperscript{18} The quarterly returns versus portfolio changes are shown in Figure 8. As the active managers move toward index weights, the change in returns tends to be positive. As they move away from index weights, the change in returns tends to be negative. This is what the theory suggests. In times of shock or increased uncertainty when managers move towards the index, the low volatility stocks should outperform more than previously. This is what we find.

\textsuperscript{18} The portfolio returns each quarter are similar to a strategy of buying the active manager portfolio of low volatility stocks (first three active groups, value-weighted) and selling the first three groups of the S&P 500 and computing the return differences of this portfolio. We compare these return differences quarter by quarter to the changes in holdings of the similar portfolio of active managers. The theory suggests that the changes in returns should be associated with changes in their holdings.
We repeated these tests for the MF-High minus SP500 high group. Those results are plotted in Figure 9. As the active managers move towards index weights the higher volatility stocks do not perform as well as when they move away from active weights. Including the big quarterly changes in weights, the $R^2$ of the relation is .08. Ignoring the two large changes in weights, we can still observe a relation between return changes and weight changes. At times of shock or increased uncertainty, when managers move towards the index, the high volatility stocks should under perform more so than previously. This is what we find.
To estimate the “costs of investment constraints” that mutual funds face, we decompose the relative performance difference between the mutual funds and their benchmark, which is the S&P500 for all of the selected mutual funds. At the end of each quarter, from December 1999 through March, 2013, we computed the returns on the S&P500 and the grand mutual fund portfolio, “aggregate portfolio,” for the subsequent quarter. That is, we used each stock’s weight of the S&P500 at the end of each quarter and we held the stock for the entire next quarter ignoring changes to the composition of the index during that quarter. Similarly, we used the weights of the aggregate portfolio at the end of each quarter to compute its returns for the subsequent quarter.

We regress the quarterly returns on the aggregate portfolio on the quarterly returns on the S&P500 portfolio. These results are shown in Table 5.

<table>
<thead>
<tr>
<th></th>
<th>Coefficient</th>
<th>SE</th>
<th>t Stat</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.365</td>
<td>0.313</td>
<td>1.17</td>
<td>0.249</td>
</tr>
<tr>
<td>S&amp;P</td>
<td>1.073</td>
<td>0.037</td>
<td>29.04</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Notice from these results that the intercept (“alpha”) is positive but insignificant and the regression coefficient (“beta”) is significant and greater than 1. We did not include cash holdings, in the return calculations for the aggregate portfolio, which, if included would have moved the “beta” closer to one.\(^{19}\) Moreover, we did not consider changes in the aggregate portfolio during the quarter that would have also increased cash holdings and reduced beta. The aggregate managers appear not to exhibit stock picking skills.

In Table 6, we display the summary statistics of the regression of the quarterly returns of the aggregate portfolio on the quarterly returns on the S&P 500 and the differential holdings (in percent) of the aggregate mutual funds from the S&P500 composition for the high volatility groups 8 through 10.

\(^{19}\) In constructing the aggregate portfolio we used the stock holdings of the 95 mutual funds. We noticed that the cash holding of the mutual funds fell over time from approximately 5% in the earlier years to 3% in the later years. We did notice, (Figure 7) that the funds added more middle volatility, likely more liquid stocks, to their portfolios in recent years. With changes in transaction technologies, these stocks might be used to reduce the need to hold cash.
Notice now that the mutual funds do appear to have skills at predicting “alpha,” abnormal returns. The intercept is 1.29% per quarter with a t-statistic of 2.15. However, as they overweight the higher volatility stocks to control tracking error, the coefficient on the differential holdings is -.122 (with a t-statistic of -1.79). The higher the differential holdings of higher volatile stocks, the lower are the return on the aggregate portfolio. These results support the notion that the stock-holding “beta” is greater than zero because of cash holdings, stock picking skills are significant (“positive alphas”) but the benefits of alpha are mainly offset by the need to overweight higher volatility stocks, which underperform, potentially as a result of the demand for institutional investors to hold these stocks. If the average overweight was 7.6% in high volatility stocks, the point estimate of lost return using these statistics is about 3.7% a year (i.e., -.12 x 7.6 = 0.93% per quarter or 3.7% per year).

In these empirical results, we find evidence consistent with the predictions of the model. Active managers hold a disproportionate amount of high volatility stocks relative to their benchmark and do so consistently over time. With tracking error and liquidity constraints, active managers hold higher volatility stocks relative to their benchmark. The greater their significance of expectations of excess returns, the greater their tracking error and, relative to the benchmark, they acquire additional higher volatility stocks. With greater uncertainty, managers have less confidence in their ability to predict abnormal returns. Consistent with the theory, as uncertainty increases they reduce their holdings of higher volatility stocks and reduce their tracking error.
We also find that managers do have skill in identifying alpha. Upon taking into account the costs of constraints those managers pay by demanding higher volatility names to control tracking error and maintain liquidity, historical performance of the 95 mutual funds in our sample, on average, display no alpha. In fact, for the riskier stocks (8-10 groups) the aggregate portfolio returned $.546% per quarter more than the S&P500 riskier stocks (8-10 groups). When the differential returns on the aggregate portfolio from the S&P500 returns for the respective 8-10 groups were regressed on the returns on the S&P500 the slope was insignificantly different from zero and the intercept was $.546% with a t-statistic of 1.39. Although the performance was not significantly different from zero, the higher risk stocks that the active managers did select, did outperform the higher risk stocks in the S&P500, but not enough to overcome the loss of returns of higher weighting in the higher risk stocks, in general, to control tracking error.\(^{20}\)

The Roll model indicates that managers buy high beta (high volatility) stocks to outperform their benchmark. This arises because managers cannot use leverage to enhance returns. Although we show similar results, managers change their holdings as a result of uncertainty with regard to their confidence in their active investments. In addition, unlike the Roll model, we demonstrate that managers can buy lower beta (lower volatility stocks) and finance the purchase by selling other low volatility stocks and hold additional higher volatility stocks.

In addition, these results do not support the behavioral models. The active mutual fund managers tend to be the natural clientele for higher volatility stocks. Contrary to active institutional investors being the natural arbitrageurs or speculators, they tend to move in the same direction as the “gamblers.” We find, however, that when active managers change their holdings (in the direction that the model predicts) they move back to the index weights and returns move in the same direction. Although as volatility increases and active managers reduce their holdings of volatile stocks, the gamblers don’t seem to be adding to their investments in volatile names, or they are insufficient in economic import to buy up the stocks vacated by active managers as they

\(^{20}\) We found similar results in the other grouped portfolios. When we took the difference in return between the low risk group (groups 1-3) in the aggregate portfolio and the low risk group in the S&P portfolio, the average difference was 0.542% per quarter with a t-stat of 1.53. For the medium risk group, the average difference was 0.437% with a t-stat of 0.83. While these results are not statistically significant, they suggest economically significant manager skill.
reduce their tracking error. Gamblers would find higher risk stocks that resemble “lottery tickets” most attractive when volatility increases because a “lottery ticket” is nothing more than a call option and the value of an option increases when volatility increases.

**Conclusion: Asset Pricing Implication of Delegated Investment Management**

We present a model with supporting empirical evidence that shows that active investment managers, subject to common investment constraints of tracking error and/or liquidity, are natural clients for higher volatility assets. The constrained asset-pricing model is an alternative explanation of the empirically observed “low volatility” or “low beta” puzzle. With these constraints broadly imposed across active management in many asset classes and regions, from US equities to international fixed income, empirical findings of the “low volatility” anomaly in many asset classes are a consistent implication of our framework.

These professional investment managers rationally chose to hold inefficient portfolios to meet investment constraints. If multiple asset managers face similar constraints, their common demands for higher volatility assets might cause their prices to be bid up such that their expected returns are lower than those predicted by the standard unconstrained asset-pricing models. When we add an ability to predict stock returns into the model, an alpha, we show that the greater the expected abnormal returns to risk ratio, the more managers want to deviate from their benchmark; that is, the more tracking error they want to take. Moreover, they finance their active bets, either in low or higher risk assets, by selling other lower risk assets while retaining their higher risk assets. In addition, the dynamics suggest that as either their ability to forecast alpha levels falls or the uncertainty of their forecasts increases, they will reverse their holdings and reduce their tracking error. We demonstrate that at time of abnormal volatility in the market that is exactly what occurs.

Managers hold inefficient portfolios and higher risk assets than they would otherwise hold in an unconstrained setting. In doing so, they incur an implicit cost in that they and their investors are willing to give up returns to satisfy constraints. We argue that without constraints, explicit monitoring and risk management costs would increase. Investors trade off these costs in
selecting active managers. The constrained equilibrium asset-pricing model incorporates these costs through tracking-error and/or liquidity constraints and is a more realistic model than the no-cost alternatives. Moreover, arbitrageurs who might benefit from any miss-pricing by buying lower-volatility assets and selling higher-volatility assets might have similar constraints as do active managers. Or, if unconstrained, they need to earn a return on the capital that they employ, and would not eliminate all return differences between low and high volatility assets, whether measured by beta or sigma.

We present evidence that active managers do take tracking error and the level changes, as is expected, are related to the alpha to sigma ratio. We show that the managers do have skill as they select stocks that outperform their benchmark across the volatility spectrum. Managers, however, overweight high volatility stocks, relative to the benchmark, and these stocks underperform. When we perform a simple regression of the returns of the aggregate mutual fund portfolio on the returns of the benchmark, it appears that managers have no skill. Their excess performance is insignificantly different from zero over the 2000-2013 timeframe. However, conditioning on the level of overweighting in higher volatility assets, we show managers have economically significant skill. Our skilled market participants earn alpha from the market in their “alpha portfolio” but pay it back to the market in their “hedging portfolio” – that portfolio which is held to adhere to investment constraints.

Other models that attempt to explain the “volatility puzzle” fall into two camps. First are members of the behavioral economics camp who argue that noise traders typically have behavioral biases the lead them to overpay for higher volatility assets and that professional investors are limited in their ability to supply these securities to the noise traders because of “limits of arbitrage” arguments. Our model is diametrically opposed in that in our model, professional arbitrageurs contribute to the mispricing of highly volatile assets instead of working to correct them. Our skilled market participants earn alpha from the market in their “alpha portfolio” across low risk and high risk securities and they, too, overweight high volatility assets relative to low volatility assets to finance the holdings of this “alpha portfolio.” The expected and demonstrated dynamics of our model also stands in contrast to the behavioral models. In our model, managers dynamically allocate to their alpha and hedging portfolios and asset prices
follow their actions. When the return/risk of deviating from the benchmark falls, typically in crisis and high volatility periods, high volatility stocks underperform. In the behaviorist model, either they are silent on the dynamics or if certain investors enjoyed lotteries, they would drive up the prices of the most “lucrative lotteries” in the market when volatility increases implying that high volatility stocks should outperform. We find the empirical results that support our model.21

The second camp arises from the Black (1972) model, followed by Pederson and Frazzini and Pederson (2013), who argue that a leverage constraint combined with unequal borrowing and lending costs leads to a “kinked” capital markets line and an underperformance of high beta securities. When we allow for leverage in our model, we find that, consistent with the Black (1972), that leverage does dilute the excess demand of high volatility stocks and excess supply of low volatility stocks, but does not cause the imbalances to disappear. Tracking error becomes the binding constraint, yielding an optimal portfolio to control tracking error that is overweight high volatility names and underweight low volatility names. A very interesting implication of our model is that managers use leverage to reduce their exposures to alpha when the expected return to alpha is high and vice versa when alpha is low. Given alpha returns warrant management fees, but beta returns do not, investors may decide to restrict the use of leverage depending on a managers ability to generate alpha. This can shed insight to explain optimal capital structure. Our model is richer than the simple cost of leverage model. First, the cost of leverage has fallen dramatically over time with the advent of and the use of highly liquid futures and options markets. Sophisticated investors can quickly gain access to high leverage levels through broker-dealers, as can individual investors through brokerage accounts and leveraged ETFs. Second, the factor returns and change in the holdings of mutual funds strongly support our model’s predictions in that low volatility stocks outperform in periods of enhanced volatility as managers move towards their benchmarks. In a cost of leverage model, one would expect as volatility increases the cost of using leverage increases leading to additional demand for high volatility stocks and contemporaneous outperformance. Third, in our model investors hire skilled

21 There is another prediction difference worth noting: Behaviorists, if we understand their argument, would argue that certain investors would clamor for the most volatile stocks. We should see empirically that there are poor returns to only the most volatile stocks. Instead, results generally show monotonic results to large sets of stocks sorted by volatility.
investment managers to manage in different markets. In the leverage constraint model, there are no managerial skills and the investors control their own portfolio.\textsuperscript{22}

In summary, moving to understand the import of constraints in delegated asset management and other market frictions might unlock some of the mysteries that we observe when using the unconstrained asset-pricing models as the norm. In our model, we have the freedom to substitute other constraints and costs and test them. This avenue of approach might lead to clearer understanding of the anomalies that we observe in asset prices.

\textsuperscript{22} We wonder why that investor would use implicit leverage in each piece of his portfolio – if given the choice of high beta equities, low beta equity, high duration fixed income and low duration fixed income, we do not believe he would optimally chose high beta equities and high beta fixed income. If he wanted higher returns, would he not shift a higher allocation into equities from fixed income?
Appendix

Optimal tracking error portfolio choice with liquidity constraint:

The solution to the optimal portfolio choice, where the goal is to minimize tracking error for a level of expected return, while maintaining liquidity in the form of cash.

\[
\min_x x' \Omega x
\]

subject to:
\[
x' R = G
\]
\[
x' 1 = k < 0
\]

The optimization problem can be formulated as the solution to the following Lagrangian equation:

\[
\Omega x - \lambda_1 R - \lambda_2 1 = 0
\]

Yielding,

\[
x = \Omega^{-1} [R \hspace{1em} 1] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}
\]

A-3

Multiplying (A-3) by \([R \hspace{1em} 1]'\), gives

\[
\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1} \begin{bmatrix} G \\ k \end{bmatrix}
\]

A-4

where A is a key matrix in mean-variance algebra.

\[
A = \begin{bmatrix} R' \Omega^{-1} R & R' \Omega^{-1} 1 \\ R' \Omega^{-1} 1 & 1' \Omega^{-1} 1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}
\]

The right-hand side matrix relates to the properties of the global minimum variance portfolio, q0, and the portfolio, q1; that is, represented by the intersection of the line connecting q0 with the origin to the efficient frontier.
Table A.1: Properties of Efficient Portfolio \( q_0 \) and \( q_1 \).

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Composition</th>
<th>Mean</th>
<th>Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_0 )</td>
<td>( \Omega^{-1}\frac{1}{c} )</td>
<td>( r_0 = \frac{b}{c} )</td>
<td>( \sigma_0 = \sqrt{\frac{1}{c}} )</td>
</tr>
<tr>
<td>( q_1 )</td>
<td>( \Omega^{-1}\frac{R}{b} )</td>
<td>( r_1 = \frac{a}{b} )</td>
<td>( \sigma_1 = \sqrt{\frac{a}{b^2}} )</td>
</tr>
</tbody>
</table>

Substitution A into (A-4) and then substituting (A-4) into (A-3) yields:

\[
x = \Omega^{-1}[R 1] \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \begin{bmatrix} G \\ k \end{bmatrix} = \frac{1}{ac - b^2} \Omega^{-1}[R 1] \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \begin{bmatrix} G \\ k \end{bmatrix}
\]

Multiplying the above out yields:

\[
x = \frac{1}{ac - b^2} \Omega^{-1}(R(Gc - bk) + 1(ak - Gb))
\]

\[
= \frac{1}{bc} \Omega^{-1}(R(Gc - bk) + 1(ak - Gb))
\]

and with some algebraic manipulation and substitution yields:

\[
x = \frac{G}{R_1 - R_0} (q_1 - q_0) + \frac{k}{R_1 - R_0} (\Omega^{-1} 1 \frac{a}{bc} - \Omega^{-1} R \frac{1}{c}) \tag{A-5}
\]

By substituting the definitions of portfolio \( q_0 \) and \( q_1 \) into (A-5), it can be rewritten as:

\[
x = \frac{G}{R_1 - R_0} (q_1 - q_0) + \frac{k}{R_1 - R_0} (q_0 R_1 - q_1 R_0) \tag{A-6}
\]

Variance of Tracking Error Portfolio with liquidity constraint:

The variance of the tracking error portfolio with liquidity constraint is:

\[
\sigma^2_{T_{EVc}} = x' \Omega x =
\]
\[
\left[ \frac{g}{R_1-R_0} (q_1 - q_0) + \frac{k}{R_1-R_0} (q_0 R_1 - q_1 R_0) \right]' \Omega \left[ \frac{g}{R_1-R_0} (q_1 - q_0) + \frac{k}{R_1-R_0} (q_0 R_1 - q_1 R_0) \right]
\]

\[
\sigma^2_{TEV_c} = \frac{1}{(R_1 - R_0)^2} \left[ q_1 (G - k R_0) + q_0 (k R_1 - G) \right]' \Omega \left[ q_1 (G - k R_0) + q_0 (k R_1 - G) \right]
\]

\[
\sigma^2_{TEV_c} = \frac{1}{(R_1 - R_0)^2} \left[ q_1 (G - k R_0) + q_0 (k R_1 - G) \right]' \Omega \left[ q_1 (G - k R_0) + q_0 (k R_1 - G) \right]
\]

Simplifying using the definitions of \(q_0\) and \(q_1\) gives:

\[
\sigma^2_{TEV_c} = \frac{1}{(R_1 - R_0)^2} \left[ (G - k R_0)^2 \sigma_1^2 + (k R_1 - G)^2 \sigma_0^2 + 2 (k R_1 - G) (G - k R_0) \sigma_0^2 \right]
\]

which after some algebra and substitution of EQ. 3 yields EQ. 4:

\[
TEV_c = TEV - \frac{k}{(R_1 - R_0)^2} \left[ 2 G R_0 (\sigma_1^2 - \sigma_0^2) - k (R_0^2 \sigma_1^2 + R_1^2 \sigma_0^2 - 2 R_1 R_0 \sigma_0^2) \right]
\]

\[A-7\]

Variance of Managed Portfolio:

Variance of the managed portfolio with both a tracking error and liquidity constraint equals:

\[
\sigma^2_p = (x + q_B)' \Omega (x + q_B)
\]

\[
\sigma^2_p = \sigma^2_B + x' \Omega x + 2 x' \Omega q_B
\]

\[A-8\]

Upon substituting A-6 and the definitions of \(q0\) and \(q1\) into A-8, yields:

\[
\sigma^2_p = TEV_c + \sigma^2_B + \frac{2 G \sigma_0^2}{R_1-R_0} \left( \frac{R_B}{R_0} - 1 \right) + \frac{2 k \sigma_0^2 (R_1-R_B)}{R_1-R_0}
\]

\[A-9\]
References


