We study optimal fiscal and redistributive policy in an open economy without commitment. Due to the government’s redistributive motive, inequality affects government’s incentive to default on its external debt while external default constrains the ability of the government to increase output and inequality. When value of external debt is low, the government is not constrained by future government’s decision, external debt is increased and government policy involves high level of redistribution. When the value of external debt is high, an increase in external debt, decreases its future value accompanied by a reduction in transfers. Thus, the economy endogenously fluctuates between episodes that resemble populism and austerity. This result holds when policies are determined via probabilistic voting or the economy is subject to shocks.
1 Introduction

The fiscal policy experiences in various developing nations feature cyclical patterns induced by political factors. A typical observation is that, in good times, governments borrow externally and implement redistributive programs to curb inequality via increases in social transfers and pensions. Such episodes are often followed by debt crisis, a reversal in redistributive policies and implementation of drastic austerity measures. The experiences of Latin American economies exhibit such cycles while the recent dynamics of the fiscal policies in several economies in Southern Europe also show such cyclical pattern. These episodes are often referred to as populist policy cycles.\footnote{Sachs (1989) coined this term in his study of policy cycles in Latin American economies.} The rationale for such policy cycles and more generally the trade-offs between government incentive to borrow and redistribute are not well-understood.

In this paper, we study governments’ incentives to redistribute and borrow externally in a neoclassical model with political frictions. To this end, we develop a tractable heterogeneous-agent overlapping generations framework where redistributive motives by the government determine policies. Households are heterogeneous with respect to their labor productivity, are alive for two periods and decide how much to work when young and save for retirement. Government can use linear taxes on labor and wealth, together with transfers and borrowing and lending from domestic households and foreign creditors to redistribute resources across different income groups and generations.

The key feature of our framework is the lack of commitment by the government where the government in each period chooses contemporaneous policies but can commit to path of future policies. This lack of commitment leads to a game between the government, households and foreign creditors. We study the sub-game perfect equilibria of this game where the choices of each government should be such that the future government do not want to re-optimize; policies must satisfy the sustainability constraint.

The presence of redistributive motives together with linearity of the income tax function implies that inequality (in income and wealth) is necessary for provision of incentive to work, i.e., our model exhibits a trade-off between equality and efficiency. While inequality is ex-ante necessary to provide incentive, it is ex-post costly. In particular, in our OLG framework inequality in earnings among the current young leads to inequality in wealth among the future old. Due to its redistributive motives, from the future government’s perspective this inequality in wealth is undesirable and creates incentives to re-optimize and increase taxes on wealth. A feature that is novel to our model. Similarly, while external debt is required for consumption smoothing, from the future government’s perspective, it is undesirable. Therefore, it creates incentives to re-optimize by the future government similar to other work on sustainability of external debt a la Eaton and Gerso-
The main insight of our model is the conflict of interest for the composition of inequality and external debt between the current and the future government. Holding total consumption fixed, current government wants higher debt and lower inequality relative to the future government. This is because increase inequality coincide with decreases in leisure which is costly for the current government.

We show that in equilibrium this conflict leads to cyclical dynamics in policies resembling populist cycles. In particular, in states of the world where the sustainability constraint is not binding, the current government implements a high value of external debt together with low labor income taxes, high transfers and pension all of which lead to low value for inequality. In the states of the world where the sustainability constraint for the future government is binding, the mixture of inequality and external debt must be tilted in favor of the future government. This means that in such state of the world, transfers and pensions must be low leading to a high value for inequality while external debt must be reduced. Such changes in policies resemble austerity measures. Furthermore, the larger is the initial external debt in such states of the world, the larger is the implemented austerity measures.

In the simplest version of our analysis, we illustrate our main insight in a deterministic economy where individuals have Balanced Growth Preferences between consumption and leisure while policies are chosen by a government that maximizes without government. In this environment, we show that the state of the economy is represented by the level of external debt that country inherits. Our main result is that, when the sustainability constraint is binding, an increase in the inherited external debt decreases the future value of external debt. Underlying this result is the aforementioned conflict of interest between current government and that of the future government regarding the mixture of inequality and external debt. An implication of this result is that when international interest rates are low relative to government’s patience, the deterministic economy convergence to the steady state value of debt is cyclical. This cyclical behavior holds for the aggregate allocations, i.e., consumption, output, etc. as well as that of policies, taxes, transfers and debt. Such dynamics resembles populist cycles.

Additionally, we consider a version of our economy in which policies are determined in a probabilistic voting model as in Lindbeck and Weibull (1987). Compared to the standard problem, since voters only maximize their utility, government in each period is only concerned about the generations that are alive contemporaneously. We characterize various equilibria of this game: the best subgame perfect equilibrium together with a Markov equilibrium a la Eaton Gersovitz. As in the basic setup, we show that the conflict of interest regarding mixture of inequality and external debt is present in the model with political economy frictions. As a result, policies feature cyclical dynamics.
We further show that the conflict of interest between subsequent governments is also present in a version of our model with shocks to terms of trade or government purchases. In this version of the economy, market equilibrium interest rates are naturally lower than the government’s discount factor – as in Alvarez and Jermann (2000). As a result, in the long-run, the economy fluctuates between two phases: one where the current government is not constrained by reneging by the future government during which the government increases its external debt, and redistribute - **populist phase**; one where the current faces a reneging by the future government and implements austerity measures - **austerity phase**. Transition from populism to consolidation is typically triggered by a negative shock to terms of trade.

Finally, we discuss the historical evidence regarding the cyclical pattern of fiscal policy and external debt. As noted by Dornbusch and Edwards (1990), Dornbusch and Edwards (1991), and Sachs (1989), various Latin American economies go through cycles of populism and austerity. We identify particular historical episodes in Chile (under Allende) and Argentina (under Peron) for which our theory does particularly well in explaining the cyclical behavior of fiscal policy and external debt. Furthermore, as we show, some southern European economies, Portugal and Greece in particular, also show patterns that are consistent with our theory of cycles of populism and austerity.

**Literature Review.** Our paper contributes to several strands of the literature in macroeconomics, public finance and international economics.

Our paper is mostly related to the large literature in macroeconomics and international economics that analyzes government’s choices of fiscal and debt policy over time and business cycles; examples include Lucas and Stokey (1983), Chari and Kehoe (1999), Aiyagari et al. (2002), Arellano (2008), Aguiar and Amador (2014), among many others. This literature is mainly silent on issues pertaining to redistribution and inequality. While more recently inequality and redistributive policies are introduced for the study of macroeconomic policies, examples include Werning (2007), and Bhandari et al. (2013), their open economy consequences are primarily unexplored. This is of particular interest given that a large fraction of government outlays are spent on transfers, e.g., social transfers and public pensions. As we show, redistributive considerations significantly change the implications on optimal policies regarding external debt and can shed light on some of the cyclicality in fiscal and external debt policies in developing economies. In this regard, our paper is closely related to Werning (2007) who considered a closed infinite-horizon economy.

Our paper is closely related to the recent literature on optimal policy and social insurance in heterogeneous agent economies in the absence of commitment. Examples include, Sleet and Yeltekin (2006), Farhi et al. (2012), and Scheuer and Wolitzky (2014). Similar to these papers, we consider an intergenerational setting with overlapping generations.
where policies are chosen by governments that cannot commit to future policies (or via a political economy model of voting). While this literature is mainly interested in the shape of the optimal tax functions and the long-run behavior of inequality, we focus on the dynamics of debt and inequality over time and business cycles. Furthermore, we exogenously impose restrictions on the tax function. This allows us to develop a tractable framework for the analysis of the trade-offs between inequality and external debt and its determinants over time.

Two recent papers address some of the issues related to this trade-off: First, D’Erasmo and Mendoza (2013) consider a closed economy with heterogeneous agents in which government does not have access to asset taxes while it can issue debt and default on it. They show that in order to explain the level of domestic debt observed in Europe, government objective must feature a political bias towards government’s creditors. Second, Ferriere (2014) argues that when the government can change the progressivity of the tax code, progressive taxes can be used to mitigate cost of default and cost of borrowing. As in these papers, our model implies that wealth inequality negatively affects government’s ability to borrow. However, our novel result is that due to the interactions between inequality and external debt, it is optimal to make large downward adjustment in external debt when highly indebted and thus policies resemble austerity.

Since Eaton and Gersovitz (1981), a large literature in international economics has analyzed the determinants of sovereign debt based on theories where the government cannot commit to repay its external creditors. Aguiar and Amador (2014) and Aguiar et al. (2009), for example, using the contracting framework of Thomas and Worrall (1994), consider an economy where external debt accumulated in the past affects the ability of a sovereign to invest due to its incentives for expropriation. Such constraints on investment implies that external debt is gradually adjusted over time while investment is gradually increased. This is in sharp contrast with the implication of our model that external debt adjustment is not gradual, i.e., future external debt is inversely related to current external debt. Our result, thus, provides a rationale for the observed austerity measures that are often used in episodes following debt crises.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 characterizes the optimal policies chosen by a planner without commitment. Section 4 extends the result to a setup with probabilistic voting. Section 5 describes a stochastic version of the basic model and Section 6 provides historical evidence in support for our results.
2 Model Setup

In this section, we describe the basic framework in which we analyze the determinants of policies. We consider an overlapping generations small open economy populated by a continuum of heterogeneous households together with the government.

**Households.** Time, \( t \), is discrete and \( t = 0, 1, \cdots \). At the beginning of each period, a generation of households are born and live for two periods. Each generation born at \( t \) consists of a continuum of households who draw a productivity type \( \theta_t \) when they are born. They work, consume and save when young and consume when old. We assume that \( \theta \in \{ \theta^1 < \cdots < \theta^N \} \) and that \( \Pr(\theta_t = \theta^i) = \mu^i \) where the average value of \( \theta^i \) is normalized to 1, i.e., \( \sum_{i=1}^{N} \mu^i \theta^i = 1 \). We refer to households with productivity type \( \theta^i \), as households of type \( i \). The labor productivity of a household of type \( i \) is given by \( \theta^i \).

Households value consumption and labor supply according to the following preferences:

\[
\mathcal{U}^{i}_{t,0} = \log c^{i}_{t,0} + \psi \log \left( 1 - \frac{y^{i}_t}{\theta^i} \right) + \beta \log \left( c^{i}_{t,1} \right) \tag{1}
\]

where \( c^{i}_{t,0}, c^{i}_{t,1} \) are consumption when young and old, respectively while \( y^{i}_t \) is the effective hours worked when young – \( y^{i}_t \) is income. In the above expression, \( \mathcal{U}^{i}_{t,0} \) represents the utility of a young household born at time \( t \) while the utility of an old household is given by \( \mathcal{U}^{i}_{t-1,1} = \log \left( c^{i}_{t-1,1} \right) \). Our assumption on the form of utility function is purely for simplicity of the analysis and can be extended to more general utility functions at the cost of further complications; The Appendix contains the extension of our results to more general preferences.

A young household of type \( i \), has \( a^{i}_{t} \) in wealth at the end of the period. Finally, there is an initial old generation at \( t = 0 \) whose wealth distribution are given by \( \{ a^{i}_{-1} \}_{i=1,\cdots,N} \) while their consumption is given by \( c^{i}_{-1,1} \).

**Technology.** Labor is the only input for production and total output in the economy is given by \( Y_t = \sum_i \mu^i y^i_t \).

**Government.** Government provides transfers to households when young, \( T_t \), pensions to households when old \( P_t \) and pays for \( G_t \) which is government purchases net of any other source of revenue for the government, i.e., revenue from selling commodities. It finances these expenses with revenues raised from taxing labor income, taxing assets saved for retirement as well as borrowing from the rest of the world. Given the structure of government finances, the government budget constraint is given by

\[
B^{d}_{t} + \delta_t B_t + T_t + P_t + G_t = \tau_{t} Y_t + \tau_{a,t} \sum_i \mu^i a^{i}_{t-1} + q_t B_{t+1} + q^d_t B^{d}_{t+1} \tag{2}
\]
where $B^d_t$, and $B_t$ are, respectively, face value of government’s domestic and external debt, $T_t$ is transfers to the young, $P_t$ is public pensions for the old, $\tau_{l,t}$ is the tax rate on earnings, $\tau_{a,t}$ is the tax rate on assets (private pensions), $q_t$ is the price of external debt while $q_t^d$ is the price of domestic government debt. In addition, $\delta_t \in [0, 1]$ is the fraction of the external debt that is repaid in each period. Note that since the government can impose asset taxes on domestic holdings of its debt, it is without loss of generality to assume that all domestic is repaid.

Given the above government policies, we can write households’ budget constraint as

$$
c^i_{t,0} + q^d_t a^i_t \leq y^i_t (1 - \tau_{l,t}) + T_t
$$

(3)

$$
c^i_{t,1} \leq a^i_t (1 - \tau_{a,t+1}) + P_{t+1}
$$

(4)

In addition, the initial old simply consume their after-tax asset income and pensions, or $c^i_{-1,1} = a^i_{-1}(1 - \tau_{a,0}) + P_0$.

Note that we have assumed that government is the only domestic entity that can borrow and lend in international credit markets. While this is an extreme assumption, when the government has the ability to impose capital controls (taxes on foreign transactions), this assumption is without loss of generality. When the government can impose capital controls (differential taxes between domestic and foreign asset income), taxes on foreign transactions can be imposed in such a way that domestic households have no incentive to trade with the rest of the world and only hold domestic debt.

**Markets.** Feasibility of allocations requires that goods and domestic asset markets clear. That is, if $C^i_{t,0}$, $C^i_{t,1}$ are the aggregate consumption of the young and old in each period, we must have that

$$
C^i_{t,0} + C^i_{t-1,1} = Y^i_t + q_t B_{t+1} - \delta_t B_t,
$$

(5)

$$
\sum_i \mu^i a^i_t = B^d_{t+1}.
$$

(6)

Moreover, the price of external government debt must satisfy the no-arbitrage condition for the international lenders:

$$
q_t = \frac{\delta_t + 1}{1 + r}
$$

(7)

where $r$ is the international risk free interest rate.

**Competitive Equilibrium with Taxes.** As it is standard in the Ramsey approach, the above market structure as well as households optimization put a constraint on the set of allocations that can be achieved by the government. Before describing the determination of these policies, our notion of competitive equilibrium makes precise the type of restriction the above market structure puts on government’s choice:
Definition 1 Given a sequence of government policies \( \{ B_t, B_t^d, \tau_{t, t}, \tau_{a, t}, T_t, P_t, \delta_t \}_{t=0}^{\infty} \) as well as international interest rate, \( r \), a competitive equilibrium with taxes is given by \( \{ c_{t, 0}^i, c_{t, 1}^i, y_t^i, q_t^i \}_{t=0}^{\infty} \) together with prices \( \{ q_t, q_t^d \}_{t=0}^{\infty} \) where (i) \( c_{t, 0}^i, c_{t, 1}^i, y_t^i, q_t^i \) maximizes (1) subject to (3) and (4), (ii) external government debt price, \( q_t \), is given by (7), and (iii) domestic bond price, \( q_t^d \), adjust so that domestic credit markets clear, i.e., (5) holds.

2.1 Characterizing the set of Competitive Equilibria

Our assumption of the utility function allows us to provide an extremely simple characterization of the allocations that can constitute a competitive equilibrium with taxes. The following lemma states this result:

**Lemma 1** Suppose that households’ preferences are given by (1). Given initial asset distribution for the initial old, \( \{ a_{0, i} \}_{i=1, \ldots, N} \) and an initial foreign government debt, \( B_0 \), an allocation can be supported as a competitive equilibrium with taxes if and only if the sequence of aggregate allocations \( \{ C_t, 0, C_{t-1, 1}, Y_t, B_t \}_{t=0}^{\infty} \) satisfies (5) and there exists a vector of market weights \( \{ \Phi_t = (\varphi_t^i)_{i \in \{1, \ldots, N\}} \}_{t=-1}^{\infty} \) where \( \sum_i \mu_i \varphi_t^i = 1 \) such that

\[
\begin{align*}
c_{t, 0}^i &= \varphi_t^i C_{t, 0}, \\
c_{t, 1}^i &= \varphi_t^i C_{t, 1}, \\
y_t^i &= \varphi_t^i (1 - Y_t), \quad \forall t \geq 0 \\
\end{align*}
\]

(8)

while the market weights satisfy

\[
\begin{align*}
\varphi_t^i &= 1 + \frac{\psi}{1 + \psi + \beta} \frac{\theta_t^i - 1}{1 - Y_t}, \quad \forall t \geq 0, \\
\varphi_{t-1}^i &= \frac{q_{t-1, 1}^i}{C_{-1, 1}}. \\
\end{align*}
\]

(10)

In words, the above lemma states that competitive equilibria are characterized by aggregate allocations that need only satisfy feasibility together with a representation of the distribution of individual allocations, or inequality. This distribution is captured by market weights \( \varphi_t^i \). Furthermore, the linearity of the tax function implies that inequality is related to aggregate allocations through equation (10). This relationship captures the equity vs efficiency trade-off that is central to our analysis. As GDP, \( Y_t \), rises, the distribution of \( \varphi_t^i \) becomes more dispersed, i.e., increases according to second order stochastic dominance. This captures the idea that an increase in GDP must be accompanied with a decline in taxes which leads to higher inequality. We thus refer to \( \Phi_t \) as inequality and refer to the right side of (10) as \( \hat{\varphi}^i (Y_t) \).

**Welfare.** The novelty of our approach allows us to provide a simple and tractable
framework to study optimal policy in presence of heterogeneity. A stark implication of our formulation is the simplicity of welfare analysis. To see this, consider the welfare associated with an arbitrary point on the Pareto frontier for a particular generation represented by a Pareto weights \( \{\alpha^i\} \). The simplicity of our formulation allows us write welfare as

\[
\log C_{t,0} + \psi \log (1 - Y_t) + \beta \log C_{t,1} + (1 + \psi + \beta) \sum_{i}^{\alpha^i} \mu^i \log \left( \hat{\phi}^i (Y_t) \right).
\] (12)

The first three terms represent the planner’s value from aggregate allocations while \( H(Y_t) \) is a convex and increasing function that represent the disutility of inequality. Preferences for redistribution imply that the Pareto weights must satisfy the following assumption:

**Assumption 1 (Redistributive Motives)** \( \alpha^i \) is weakly decreasing in \( i \).

Assumption 1 implies that \( H(Y) \) is increasing in \( Y \). Intuitively, higher output means that inequality in consumption and income is higher in the sense of mean preserving spread which lower social welfare.

Our approach to study optimal taxation is an adoption of the so-called primal approach, following Lucas and Stokey (1983), to an environment with heterogeneity. We focus on allocations that arise in any competitive equilibrium with taxes. Policies, e.g., taxes, transfers and debt, are then determined in order to support a particular allocation. For example, an increase in output must necessarily be accompanied by a decline in labor income taxes and transfers which in turn leads to an increase in inequality. The disutility function \( H(Y) \) captures this effect.

### 3 A Planning Problem

In this section, we establish the main result of our paper. We show that without commitment, optimal policy implies that an increase in the current value of external debt can lead to a decline in its future value. Hence, allocations exhibit cyclical dynamics, i.e., convergence of allocations to their steady state value is cyclical. Thus under lack of commitment, optimal policies exhibit cycles that involve decline in borrowing together with increase in transfers and pensions followed by downward readjustment of government’s external debt and decline in transfers and pensions.

Our main assumption regarding determination of policies is that policies are chosen at each point in time. Thus, at each point in time, government cannot commit to future paths of choices of taxes, transfers and repayments of debt. In this section, we start with the
simplest form of this problem in the spirit of sustainable plans a la Chari and Kehoe (1990) where at each point in time governments maximize a weighted average of households’ utilities. We assume that within a generation welfare weights are as described before in Assumption 1 while across generation we have geometric discounting at rate $\hat{\beta}$.

Formally, as in Chari and Kehoe (1990), policies are determined in a repeated game between a continuum of households, a long-lived government and a continuum of foreign lenders. In the appendix, we show that in such a game an allocations is a subgame perfect equilibrium if and only if it satisfies the competitive equilibrium conditions described in Lemma 1 where feasibility is given by\footnote{Note that we have imposed $\delta_t = 1$ in our feasibility constraint. This is without loss of generality. As has been noted by many authors (for example by Grossman and Van Huyck (1988), Kehoe and Perri (2004), Kehoe and Levine (2009), and Dovis (2013)), the allocations described above can be decentralized via non-stage contingent debt and default as well as state contingent debt and no default. While our model’s implications about default are worth exploring, our main goal is to characterize the dynamics of aggregate quantities: debt (external and domestic), output, inequality, etc. We leave this for future work.}

$$C_{t,0} + C_{t-1,1} = Y_t + \frac{1}{1+r}B_{t+1} - B_t,$$

and the following sustainability constraint:

$$\frac{\hat{\beta}}{\beta} \sum_{i} \alpha_i u_{t-1,1}^i + \sum_{s=t}^{\infty} \hat{\beta}^{s-t} \sum_{i} \alpha_i u_{s}^i \geq W, \quad (13)$$

In the above constraint, the value of $W$ is the value of the best one-shot deviation followed by switching to the worst subgame perfect equilibrium. In the Appendix, we show that this value is associated with an allocation where the young do not save, the old only consume the pensions received from the government, and the economy is in financial autarky with respect to the rest of the world. The government thus gain from not paying its external debt and equalizing consumption among the old while the cost of this is the disruption in domestic asset markets which in turn reduces the ability of the government to redistribute.

The sustainability constraint points to the two main sources of time-inconsistency in determination of policies: First, if the government is a net external debtor, it has an incentive to default on its external debt payments so that it can increase domestic consumption and leisure. Second, at each period, since there is a non-degenerate distribution of wealth among the old households and the government is inequality-averse, it has an incentive to expropriate all the wealth via a 100% tax on wealth and redistribute it equally among the old. The key trade-off is that the inequality among the old is necessary to provide incentive to produce when young while it is costly ex-post. As it becomes clear, the interaction between these two incentives drives most of our results.
Using the above restrictions, we can thus define the *best sustainable equilibrium* as an allocation that satisfies the conditions in Lemma 1 and (13) and it maximizes social welfare at time 0. In our analysis we assume the following:

**Assumption 2 (Impatience)** \( \hat{\beta} (1 + r) \leq 1. \)

Later in section 5, we show that in an environment with shock and a continuum of countries, market clearing interest rate must satisfy the above with strict inequality.

### 3.1 A Recursive Formulation

The problem associated with the best sustainable equilibrium has a natural two-dimensional state variable: external debt and output in the previous period – since inequality among the current old depends only on output in the previous period. Let \( V(B_t, Y_{t-1}) \) be the social associated with the best sustainable equilibrium at \( t \). In the appendix, we show that due to the additive separability of inequality in the social welfare function \( V(B, Y) = W(B) - \frac{1}{\hat{\beta}} H(Y) \) where \( W(\cdot) \) is given by (P1):

\[
W(B) = \max_{C_0, C_1, Y, B'} \beta \log C_1 + \log C_0 + \psi \log (1 - Y) - (1 + \psi + \beta) H(Y) + \hat{\beta} W(B') \quad (P1)
\]

subject to

\[
\begin{align*}
C_0 + C_1 + B + G & \leq \frac{B'}{1 + r} + Y \quad (14) \\
W(B') - \frac{1}{\hat{\beta}} H(Y) & \geq W \quad (15)
\end{align*}
\]

where in the above formulation we have used the characterization in Lemma 1 and the definition of \( H(Y) \) in (12). In words, \( W(B) \) is the value for the planner excluding the disutility of inequality of the current old. The constraint (14) is the aggregate resource constraint for the economy which enters period \( t \) with external debt \( B \) while (15) is the sustainability constraint at \( t + 1 \).

The recursive problem in (P1) is non-standard in the sense that the value function appears in the constraint set. As a result, (P1) might have multiple solutions. Nevertheless, we are able to show that such solution exists and the solution with the highest value among the solutions to the functional equation (P1) solves the optimal policy problem. We do so by focusing on its dual functional equation (with promised utility as state variable) and use standard dynamic programming techniques. All of our results are stated for this solution to (P1).
The following proposition connects the solution to (P1) to the best sustainable equilibrium:

**Proposition 1**  Consider the best solution to (P1), \( W^* \). If an allocation is associated with the best sustainable equilibrium, then the allocations for \( t \geq 1 \) must be generated from the policy functions associated with \( W^* \).

Conversely, if an allocation is generated from the policy function associated with \( W^* \) and satisfies

\[
\limsup_{t \to \infty} \hat{\beta}^t W^*(B_t) = 0
\]

then it must be associated with the best sustainable equilibrium.

Proposition 1 is similar to the standard principle of optimality (see Theorems 4.2 and 4.3 in Stokey et al. (1989)). We provide the proof in the Appendix.

The following lemma establishes the basic properties of the best value function \( W^* (B) \):

**Lemma 2**  The value function \( W^* (B) \) is strictly decreasing, strictly concave and differentiable.

The proof resorts to the duality property stated above and uses standard dynamic programming techniques and is left for the Appendix.

### 3.2 Cyclicality of External Debt

In this section, we provide our main result by characterizing the dynamics of external debt over time.

The problem in (P1) can be simplified. Note that the ratio between consumption of the old and the young is constant. Using that observation and rearranging the terms in (P1) boils down to the following very simple problem:

\[
\begin{aligned}
\max_{B', Y} & \left( \frac{B'}{1 + r} + Y - B - G, Y \right) + \hat{\beta} V (B', Y) \\
\text{subject to} & \quad V (B', Y) \geq W.
\end{aligned}
\]

where \( U (C, Y) = (1 + \beta / \hat{\beta}) \log C + \psi \log (1 - Y) - (1 + \psi) H (Y) \). Our previous discussion implies that \( V (B, Y_{-}) \) is decreasing in both \( B \) and \( Y_{-} \).

The program in (P2) is the central problem in our paper in that it captures the key trade-off of the model which is the tension between external debt an inequality from the current government’s perspective vs. future government. The function \( U (C, Y) \) represents the social welfare of the individuals that are alive in the current period and includes
the cost of inequality. The function $V(B', Y)$ represents the value from future external debt and inequality for the future government. Due to equity efficiency trade-off, higher output today can only be achieved with higher inequality. Hence, the function $V(B', Y)$ is decreasing in $Y$. Note that standard representative agent models of borrowing and lending without commitment are special case of the problem in (P2) where the value function $V(B, Y)$ is independent of $Y$.

The first order conditions associated with (P2) are given by

$$U_C + (\hat{\beta} + \lambda) V_{B'} = 0 \quad (17)$$
$$U_C + U_Y + (\hat{\beta} + \lambda) V_Y = 0 \quad (18)$$

where $\lambda$ is the Lagrange multiplier on (16). Since an increase in output is costly for the current government, $U_Y < 0$, the above implies that,

$$0 > V_Y > V_{B'} \quad (19)$$

The above inequality captures the key tension between current and future government’s perspective. The future government likes to equate the marginal cost of debt and inequality. However, from current government’s perspective, inequality is costly; due to disutility of effort and inequality. Hence, the above implies that from future government’s perspective, there is too little inequality and too much external debt. That is, a reduction in government’s external debt and an increase in output of the same size leads to an increase in the value of the future government while it keeps current consumption unchanged.

The following theorem describes our main result about the shape of policy functions in (P2):

**Theorem 1** The policy function $B'(B)$ in the programming problem (P2) is hump-shaped: for small values of $B$ the sustainability constraint is slack and $B'(B)$ is increasing; for large values of $B$, the sustainability constraint is binding and $B'(B)$ is decreasing. The policy function $Y(B)$ is increasing.

**Proof.** We provide the idea of the proof here and leave the details to the Appendix:

**Case 1:** Suppose that the sustainability constraint is slack. Then we can ignore the constraint (16) in (P2). Since the objective function is supermodular in $(B, B')$ and concave, then standard monotone comparative statics results imply that the policy function $B'(B)$ must be increasing. Furthermore, due to income effect, $Y(B)$ must be increasing.

**Case 2:** Suppose that the sustainability constraint is binding. This constraint implies a mapping from $B'$ to $Y$. The inequality (19) implies that this mapping has a slope that
is less than $-\frac{1}{1+r}$. When replacing in the objective function in (P2), this implies that the objective function is submodular in $(B, B')$ and hence $B'(B)$ is decreasing while $Y(B)$ is increasing.

Consider the comparative statics with respect to $B$. When $B$ is low, the sustainability constraint (16) is slack. Hence, an increase in current external debt should be accompanied by an increase in future external debt for standard consumption smoothing reasons. Furthermore, an increase in external debt increases the interest payment to the foreign lenders which in turn leads to an increase in output and inequality. In other words, from the current government’s perspective, equality (inverse of inequality) is a normal good.

When $B$ is sufficiently high, the sustainability constraint is binding. Due to the sustainability constraint (16), both $B'$ and $Y$ cannot be increased in response to an increase in $B$. Since there is too little inequality from future government’s perspective, i.e., inequality (19) holds, cost of an increase in inequality is lower. Therefore, in response to an increase in $B$, $Y$ must increase and $B'$ should decline.

Intuitively, an increase in external debt, $B$, increases the effective weight on the utility of future government. Since there is too little inequality from the future government’s perspective, the future government always prefers a decline in external debt and an increase in future inequality. Therefore, when current external debt increases, the allocation of future debt and inequality tilts in favor of the future government which implies that $B'$ must decline and inequality must increase.

Figure 1 depicts the policy functions associated with future debt and current inequality ($H = H(Y)$ is the disutility from inequality) for the case where $\hat{\beta} (1 + r) < 1$. When $B$ is below $B^*$, the sustainability constraint is slack, $B'(B)$ is increasing, and $B'(B) > B$. For values of $B$ higher than $B^*$, the sustainability constraint is binding, and $B'(B)$ is decreasing. The steady state level of external debt is given by $B_s$ and is in the region where $B'(B)$ is decreasing.
When debt is above its steady state value, $B_s$, it is reduced below the steady state level and vice versa. We call this feature of the model *overshooting*. This implies that the convergence to the steady state is cyclical. Therefore, the country goes through cycles with two types of episodes: an episode in which external debt is below its steady state value while inequality is low. Low inequality is generated by high value of transfers and pensions and borrowing from the rest of the world, i.e., future debt is high. This episode is then followed by an episode where external debt is high and inequality is high. High inequality is generated by a low value of transfers and pensions and a reduction of external debt. In this sense, during this episode the country is going through a period that resembles *austerity*.

The following corollary states the cyclical pattern of external debt:

**Corollary 1** Suppose that $\hat{\beta}(1 + r) < 1$. Then, the dynamic system implied by $B'(B)$ eventually exhibits oscillatory dynamics. That is, for all $t$ large enough, if $B_t \neq B_s$, $(B_t - B_s)(B_{t+1} - B_s) < 0$. In addition, optimal allocations, $C_{t,0}, C_{t-1,1}, Y_t$ feature the same dynamics.

As it can be seen in problem (P2), an increase $G$ is equivalent to an increase in $B$. This implies that a high enough unexpected shock to $G$ leads to a reduction in the value of outstanding external debt and thus resembles austerity. This will be followed by a cyclical convergence to the steady state. In section 5, we formally analyze a version of our model with shocks to $G$ and describe how cycles between populism and austerity do not dissipate in the long run. This dynamic stands in contrast with most models of international borrowing and lending (see for example Thomas and Worrall (1994), Aguiar et al. (2009), and Aguiar and Amador (2011)), where future outstanding external debt is reduced in response to an increase in current external debt or government spending.
3.3 Dynamics of Taxes, Transfers and Debt

Our previous analysis has focused mainly on allocations. From Lemma 1, there must exist taxes, transfers, pensions and domestic debt policies that support such allocation as competitive equilibrium. In this section, we describe a competitive equilibrium which supports the optimal allocations and has a recursive structure that follows those of the allocations. Hence, policies depend on external debt. The following proposition describes how one set of such policies depend on the state.

**Proposition 2** The optimal allocations generated from policy functions for problem $(P2)$ are supported by government policies where domestic debt is V-shaped in $B$: $B_d$, is decreasing in $B$ when $B < B^*$ and increasing in $B$ when $B \geq B^*$; labor income tax, $\tau_l$, is increasing in $B$ while transfers and pensions are decreasing in $B$.

![Policy Functions for external debt and inequality](image)

Figure 2: Policy Functions for external debt and inequality

Figure 2 describes the behavior of policies in response to changes in external debt. As external debt increases, interest payments to foreign lenders increase and the government...
finances this increase by a decline in transfers and pensions and an increase in labor income taxes. When the sustainability constraint is binding, the change in transfers and pensions are more pronounced while the changes in taxes are less pronounced since the trade-off between inequality and external debt is tilted towards more inequality. When the sustainability constraint is binding, redaction of external debt is partly financed by issuing domestic debt and hence, domestic debt is a V-shaped in external debt.

3.3.1 Role of Preferences

In our analysis above, the key property that generates a hump-shape in future external debt policy and upward sloping inequality is the fact that equality is a normal good, i.e., as countries become richer their demand for equality increases. Log preferences create a direct relationship between inequality and output and normality of equality implies that output also must increase in response to an increase in external debt. With other preferences, the relationship between inequality and output is not as straightforward. In the appendix, we extend our analysis to GHH (given by $\log(c_0 - v(l)) + \beta \log c_1$) and BGP preferences. For all such preferences equality is a normal good, and as a result the policy functions for future external debt and inequality satisfy the same property as in Figure 1. With these preferences, however, the direct relationship between output and inequality does not hold and output can decline as debt increases – in case of GHH preferences where wealth effect is absent. Note that the insights from section 3.3 go through for these alternative specifications of the utility function.

4 Political Economy Model

So far, we have considered an environment in which policies are chosen by a fictitious planner that lacks commitment and cares about all the future generations. In this section we extend this insight to a model where policies are determined according to a probabilistic model of Lindbeck and Weibull (1987) and show that our main results extend to this environment. We sketch the political game here and leave the formal development to the appendix.

We consider a game where in each period, two political parties propose policy platforms that include all the contemporaneous policy variables introduced in section 3. The young and old individuals evaluate such policies based on preferences in A.1 with the addition of a political bias shock toward each party. When the additive political bias shocks are distributed uniformly, the symmetric equilibrium of the political game between the

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3See also Song et al. (2012), Farhi et al. (2012) and Scheuer and Wolitzky (2014) for similar applications to other dynamic settings.
two parties is equivalent to maximization of

$$\omega \sum_i \alpha_i \mu_i u_{i-1,1} + \sum_i \alpha_i \mu_i u_{i,0}$$

(20)

a weighted sum of the utility of the individuals alive at $t$. As before, we assume that political bias shocks are distributed so $\alpha_i$’s are weakly decreasing in $i$.\(^4\)

Similar to the game underlying the model in section 3, policies are determined in a dynamic game between foreign creditors, households and politicians. Our notion of equilibrium is this dynamic game is the standard subgame perfect equilibrium or SPE.

As in section 3, we can show that an allocation is the symmetric SPE of the political game if it satisfies the characterization in Lemma 1 and if the following inequality is satisfied:

$$\omega \sum_i \alpha_i \mu_i u_{i-1,1} + \sum_i \alpha_i \mu_i u_{i,0} \geq V$$

(21)

where $V$ is the value of the worst equilibrium defined in a way analogous to the value of $W$ in section 3. We refer to the above constraint as the political sustainability constraint. A formal characterization of the entire set of SPE outcomes can be found in the appendix.

To derive the implications for policies and allocations associated with the political economy game we need to specify a criterion to select among the set of equilibrium outcomes. We consider two alternatives. First, we consider the SPE outcome that maximizes the same objective function introduced in section 3. We refer to such outcome as best SPE outcome. Second, we consider a selection criterion in the spirit of the canonical Eaton and Gersovitz (1981) model of sovereign default. We refer to this particular SPE as the Eaton-Gersovitz (EG) political equilibrium. We will show that in both cases - with some minor qualifications - the main conclusions derived in section 3 remain valid. In particular, the economy exhibits cyclical dynamics over time.

### 4.1 Best SPE

In this section, we briefly describe the behavior of the best SPE. We rank subgame perfect equilibria according to the welfare function in section 3 that attaches a Pareto weight of $\beta^t \alpha^i$ to an agent of type $i$ born at $t \geq 0$. As before, we assume that Assumption 2 holds. The best SPE is, therefore, the solution of a planning problem that maximizes the social welfare as defined in section 3 subject to the constraints imposed by Lemma 1 and the political sustainability constraint (21).

We use a dual approach for the characterization of the above planning problem. The

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\(^4\)In particular, when political bias shocks have the same uniform distribution for all individuals, this assumption is automatically satisfied.
dual version of the planning problem described above is one where payoff to the foreign lenders is maximized subject to a given level of utility to current generation and a value of social welfare for all generations – together with the constraints imposed by Lemma 1 and (21). This dual problem can be written recursively where the state is given by the average utility for a generation together with time:

\[ v_t = \log C_{t,0} + \psi \log (1 - Y_t) + \beta \log C_{t,1} - (1 + \psi + \beta) H(Y_t). \]

Under this formulation, the political sustainability constraint becomes

\[ v_{t+1} + \omega \log C_{t,1} - \omega H(Y_{t-1}) \geq V. \]

We defer the formal definition of the recursive dual planning problem to the appendix. The main result is that the policy function \( v'(t, v) \), the social welfare associated with future generations, is decreasing in \( v \) and thus we have the following proposition:

**Proposition 3** When the value of initial external debt or inequality is high, the dynamics of \( v_t \) in the best SPE is cyclical, i.e., \( v_s \) exists such that \( \forall t, (v_t - v_s)(v_{t+1} - v_s) < 0. \)

The proof can be found in the appendix.

The dynamics of the best SPE outcome is then consistent with the cycles we described in section 3. When total indebtedness is low and utility for current generation is high, we observe an increase in government debt, especially external debt, and income inequality is low. This increase in indebtedness translates into low utility for future generations and high income inequality. As before, the tradeoff between average value for the next generation and inequality among the future old is tilted towards inequality. Therefore an increase in \( v \) must be accompanied by a change in favor of the future government towards higher inequality among the old and as a result, a lower value for the next generation.

### 4.2 Eaton-Gersovitz Political Equilibrium

We turn to analyze the EG political equilibrium. Such equilibrium satisfies the following two properties. First, strategies are the same after two histories with no default if the inherited external debt, the distribution of domestic assets and pension payments promised by the previous government are the same. Second, a re-optimization by the government triggers reversion to worst equilibrium. In particular, we say that the current government defaults on promises made by previous governments if it does not repay inherited external debt, it taxes assets of the initial old above a certain level exogenously specified or it

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5When \( \hat{\beta}(1 + r) < 1 \) the problem is non-stationary and thus time must be part of the state. When \( \hat{\beta}(1 + r) = 1 \), the problem is stationary and the state is represented only by \( v_t \). See Appendix.
lowers pension payments below what was set by the previous government. Without loss of generality, we set the maximal tax on assets to zero.

In the appendix we show that (other than for period zero), we can characterize the EG political equilibrium outcome recursively using a single state variable: the total government debt given by \( \hat{B}_t = B_t + B^d_t + P_t = B_t + C_{t-1,1} \). In the appendix, we show that the optimal policy problem in this equilibrium is equivalent to maximizing the value of one generation that inherits total debt \( \hat{B} \) and faces a political sustainability constraint given by

\[
w(\hat{B}) = \max_{C_0, Y, C_1, \hat{B}'} \log C_0 + \psi \log (1 - Y) + \beta \log (C_1) - (1 + \psi + \beta) H(Y)
\]  

subject to

\[
\hat{B} + C_0 + \frac{C_1}{1 + \tau} + G \leq Y + \frac{\hat{B}'}{1 + \tau}
\]

\[
\omega [-H(Y) + \log (C_1)] + w(\hat{B}') \geq Y
\]

The main result for this section is that the allocations imposed by the above optimization problem features cycles in total government debt, \( \hat{B}_t \). As before, this is a result of the fact that total debt issued today is decreasing in inherited total debt.

**Proposition 4** The policy rules associated with the problem in (22) are such that: \( Y(\hat{B}) \) is increasing and \( \hat{B}'(\hat{B}) \) is decreasing.

As before, this proposition implies that the equilibrium outcome path exhibits cyclical behavior in total debt. When inherited obligations, \( \hat{B}_t \), are above their steady state value, \( \hat{B}_s \), next period are going to be below \( \hat{B}_s \). The opposite happens when \( \hat{B}_t < \hat{B}_s \). The intuition for this dynamic is similar to the one in section 3. An increase in total debt leads to a tilt in the trade-off between inequality and total debt in favor of the future government which together with the binding sustainability constraint implies that inequality must increase while total debt must decline.

While we are able to show that there are cycles in total government debt (measure of government debt includes external domestic and pensions), the dynamics of the domestic and foreign components depend on parameters. Solving the model numerically, we find that external government debt decreases in total debt if \( \omega \) is sufficiently large but external debt may increase if \( \omega \) is small enough. See Figure 4.2 for an illustration. Intuitively, when \( \omega \) is high, the future government cares a lot about average consumption of the future old and thus the decline in future total debt coincides with an increase in \( C_1 \). This implies that external debt \( B' = \hat{B} - C_1 \) must decline as current total debt increases. Therefore, the dynamics of external and total government debt are aligned. When \( \omega \) is small, this could be reversed and hence external debt is not a monotone function of total debt.
Figure 3: Policy Functions EG Political Equilibrium: High and Low ω

Parameters other than ω used in the simulation are: β = .9625, ˆβ = 1/(1 + r) = .9925, α^i = 1 for all i, ψ = 1, Θ = {1, 1.1, 5} with (μ^i) = {1, .8, .1}.

\section{Stochastic Economy}

In this section, we extend the model to allow for shocks to terms of trade. This formulation allows to draw parallels with the experiences of Latin American economies where the price of a commodity is extremely important for the country’s trade balance and public finances. We show that all of our results in the deterministic model hold in this extension.

Much of the model setup is similar to that presented in section 2. Therefore, we skip many of the details as they remain unchanged and describe the changes made to the model. In each period \( t \geq 0 \), there is a realization of a stochastic event \( s_t \in S = \{s(1) < \cdots < s(K)\} \subset \mathbb{R} \) which is independently and identically distributed according to the distribution \( \pi(s) \). History of shocks are denoted by \( s^t = (s_0, \ldots, s_t) \in S^{t+1} \). With a slight abuse of notation, we represent the probability distribution of \( s^t \) by \( \pi(s^t) = \prod_{\tau=0}^{t} \pi(s_\tau) \). The random variable \( s_t \) directly affects net government purchases given by \( G(s_t) \) where \( G \) is an decreasing function of \( s_t \); that is, \( s(1) \) is the state with highest government spending while \( s(K) \) is the state with lowest value of net government spending. We let average value of net government spending to be \( \bar{G} \). Given this representation of histories, allocations are represented by functions of the history of shocks, \( s^t \).

In line with Lucas and Stokey (1983) and Werning (2007), we assume that asset markets are complete and young households as well as the government can trade a complete set of arrow securities in each state \( s^t \) at prices given by \( \{q_{t+1}^{d}(s^{t+1})\}_{s^{t+1} \succeq s^t} \).
government can issue full state contingent debt in international markets at prices given
by \{q_{t+1}(s^{t+1})\}_{s^{t+1}\geq s^t}.

We make the following assumptions about price of government debt in international
markets:

**Assumption 3** Prices \{q_{t+1}(s^{t+1})\}_{s^{t+1}\geq s^t} satisfy the following properties:

1. **(Stationarity)** There exists an interest rate \( r \) such that
   \[ q_{t+1}(s^{t+1}) = \pi_{t+1}(s^{t+1} | s^t) \frac{1}{1+r} = \pi(s_t) \frac{1}{1+r}, \]
   where \( \pi_{t+1}(s^{t+1} | s^t) \) is the probability of \( s^{t+1} \) conditional on \( s^t \).

2. **(Impatience)** \( \hat{\beta} (1+r) < 1 \).

In the appendix B.1.3, we show that when the world economy consists of identical
countries subject to government spending shocks as described and in an economy À la
Kehoe and Levine (1993) and Alvarez and Jermann (2000), stationarity implies that there
must exist a constant interest rate. Furthermore, since an unconstrained country prices
a state contingent security, the first relationship must be satisfied. Finally, since countries
would like to avoid binding sustainability constraints (borrowing constraints) in the fu-
ture, supply of saving is higher than that of unconstrained steady state and as a result
interest rates must be lower than the discount rate. We can then think of this stochastic
economy as providing a justification for our relative impatience assumption in section 3.

As in section 3, we focus on a planning problem without commitment. An allocation
is, then, said to be sustainable if it satisfies

\[
\frac{\hat{\beta}}{\beta} \sum_i \alpha_i \mu_i u_{-1,1}(s^t) + \sum_{\tau=1}^{\infty} \hat{\beta}^{\tau-1} \sum_{s^{\tau} \geq s^t} \pi_\tau(s^{\tau} | s^t) \sum_i \alpha_i \mu_i u_i(s^\tau) \geq W(s_t). \tag{25}
\]

where as before, \( W(s_t) \), is the highest value to the government when government
and households are in financial autarky defined in a way analogous to the deterministic
model in section 3. As before, the same two stage procedure can be used in order to find the best
sustainable allocations. The associated Bellman equation, associated with the problem for
period 1 and onward can be written as

\[
W(\hat{B}) = \max_{C_0, C_1, Y, B'} \frac{\beta}{\hat{\beta}} \log C_1 + \log C_0 + \psi \log (1-Y) - (1+\psi+\beta) H(Y) + \hat{\beta} \sum \pi(s') W(\hat{B}'(s')) \tag{26}
\]

subject to

\[
C_0 + C_1 + (1+r) \hat{B} = \sum \pi(s') \hat{B}'(s') - \frac{G}{1+r} + ZY \tag{27}
\]

\[
-\frac{\beta}{\hat{\beta}} H(Y) + W(\hat{B}'(s')) \geq W(s') \tag{28}
\]
where \( \hat{B} \) is the value of government’s external debt plus the value of net government purchases next period. In other words, \( \hat{B}_t = B_t + \frac{G_t}{1+r} \). It does represent government’s obligation excluding those to domestic households. Focusing on this greatly simplifies the recursive formulation of the problem.

The decision rules have the same properties as in the deterministic case as shown in the following proposition:

**Proposition 5** Let \( W^* \) be the best solution to the Bellman Equation in 26. Then its associated policy functions satisfy the following properties:

1. \( Y(\hat{B}) \) is decreasing in \( \hat{B} \), \( C_0(\hat{B}) \) and \( C_1(\hat{B}) \) are increasing in \( \hat{B} \),

2. \( \hat{B}'(\hat{B}, s') \) is hump-shaped; there exists \( B^*(s') \), such that \( \hat{B}'(\hat{B}, s') \) is increasing for all \( \hat{B} < B^*(s') \) while it is decreasing for all \( \hat{B} > B^*(s') \);

3. \( \bar{B}^*(s') \) is increasing in \( s' \).

The main result here is that the policy function for debt is hump-shaped in inherited debt. The intuition is the same as in the deterministic case: When the sustainability constraint is binding, an increase in future debt must be accompanied by a fall in inequality and as a result a fall in output. At the optimum, this fall in output has to be larger than the increase in debt since output is a costlier instrument to raise funds than debt. For high values of current external debt, borrowing further cannot increase current consumption. Hence when the government is highly indebted and it values current consumption more, it is willing to make a drastic downward adjustment, sacrifice equality to increase current aggregate consumption.

Figure 4 depicts the policy functions implied by (39) for an example with two values of shocks \( (K = 2) \). In line with proposition 5, policy functions are inverted V-shaped in external debt and decreasing with respect to future government spending, \( G(s') \). The dynamics of debt in the long-run, as implied by policy functions depicted in Figure 4, points toward a stark notion of large debt repayment: A country that has the highest value of external debt sustained in the long-run, upon recovery, must adjust to its highest long-run level. The following proposition summarizes this discussion:

**Proposition 6** Consider a stationary distribution \( \Psi \) implied by the policy functions in 26 and with support \( B \). Let \( \hat{B}_{\min} = \inf B \) and \( \hat{B}_{\max} = \sup B \). Then for every \( \epsilon > 0 \), there exists \( \epsilon' > 0 \) such that \( \Pr (\hat{B}_{t+1} < \hat{B}_{\min} + \epsilon'|\hat{B}_t > \hat{B}_{\max} - \epsilon) \geq \pi(s(K)) \).

Proof can be found in the Appendix.

At the heart of the above result is the hump-shape of the newly issued debt and inequality overhang. As the country experiences a sequence of high government spending...
shocks and increases its foreign borrowing, its debt capacity upon recovery declines. This is because as debt increases, inequality among the young increases and as a result, upon recovery (experiencing a high value of government spending) the country cannot borrow as much.

The dynamics of external debt in this stochastic economy signifies the cyclical pattern of fiscal policy as it fluctuates between populism and austerity. An economy with a history (which can be quite short according to proposition 6) of good shocks, is able to take advantage of low international interest rates and implement relatively large redistributive policies. This coincides with an increase in government’s external debt. This episode can be thought of as the populist phase. When external debt rises above a particular threshold, depending on the form of the shock, external debt is adjusted downward, either in a steep fashion when $G$ is low or slowly when $G$ is high which coincides by a reduction in transfer programs. This episode can be thought of as the consolidation phase.

It is important to notice a difference between our model and that of a standard representative agent model with lack of commitment that have been used to analyze international capital flows. See for instance Aguiar et al. (2009). In a representative agent framework, downward adjustment in external debt happens only when the economy draws a low government spending shocks. In contrast, in our model downward adjustments together with a positive current account also occurs in bad fiscal times.
6 Historical Evidence

In this section, we confront the implications of our model with the accounts of various historical episodes for evolution of fiscal and external debt policies in developing economies.

Cyclicality of fiscal policy and austerity. The equilibrium dynamics of our model is consistent with the populist cycles in macroeconomic policies first identified by Sachs (1989) and Dornbusch and Edwards (1991). In particular, Dornbusch and Edwards (1991) is their study of economic policies in Latin American countries argue that dynamics of fiscal policy involves two main phases. A populist phase in which redistributive policies supporting low and middle-income workers are implemented. In this phase government finances worsen as the government runs large primary deficits. Also the the country’s net foreign asset position deteriorates (typically involving reduction of foreign exchange reserves). A common starting point of the populist phase is the presence of a previous stabilization program that would have generally improved the fiscal and external budget of the country but at the expenses of a highly uneven income distribution usually presents a serious political and economic problem, providing the appeal for a radically different economic program (see Dornbusch and Edwards (1991)).

A consolidation (austerity) phase follows the populist phase attempting to address the fiscal imbalances. In this phase austerity measures are implemented, the government reduces its primary deficits and its external debt position. This phase also typically involves the intervention by foreign organization such as the IMF.

These two phases neatly map into the dynamics predicted by our model. The populist phase then corresponds to instances in our model when external debt values are low and the government borrows heavily externally and expands its redistributive programs. This coincides with a primary deficit for the government in combination with trade inflows and current account deficit. For high enough values of inherited external debt, transfers are reduced and external debt is repaid in an effort to minimize the output distortion in the economy to maximize the amount of resources available. As a result the government runs a primary surplus while the economy runs a current account surplus. Such behavior closely resembles the consolidation phase.

Two particular examples of these populist cycles are worth further consideration: Chile under Allende and Argentina under Perón. In Chile, prior to Allende, in the three years leading to 1970, the conditions of Chilean public finances were favorable, mainly thanks to an increase in price of copper (its main export). Upon winning the election in 1971, the Allende government implemented various populist policies such as redistribu-

\[^6\]Although part of their focus is on the real appreciation of the currency and its effect on import prices and wages in the non-tradable sector, many of the episodes identified also involve labor policies involving low and middle-income workers.
tion of land, nationalization of industries, and increasing minimum wages – real minimum wages for blue collar workers were increased by 56% in the first quarter of 1971. These policies had a positive effect on the income distribution at the cost of reducing the efficiency in the economy and deteriorating the public finance conditions. For instance, Ffrench-Davis (2002) writes that the period in which Allende was in power “shows a distributive improvement but within a deteriorated economy: a less concentrated distribution of a smaller cake.” Moreover, the average change in public external debt minus international reserves was 14.2% of GDP and the primary deficit averaged 13.4% of GDP, reaching a maximal level of 22.5% of GDP in 1973. This populist phase was followed by the advent of Pinochet and a reversal of the social and economic reforms enacted by Allende and sizable primary surplus (3.74% from 1973 to 1982). This resulted in a reduction in external net government debt between 1975 and 1981. In particular, net external government debt as a fraction GDP fell from approximately 52% to 2.2% over a span of five years. Not surprisingly, this coincided with a primary surplus of around 5% of GDP.

Another emblematic case of populist policies that turned into a large fiscal consolidation is the experience of Argentina in 1945-1948, what Mallon and Sourrouille (1975) call “Perón’s period of assault.” As secretary of labor, Perón introduced various pro-labor policies including an increase in minimum wages, substantial increase pension benefits, as well as establishing a close relationship with the labor unions with these policies continuing when he came to power in 1946. During the same episode Argentina experienced a decline in current account (from 1.99 billion Pesos in 1946 to -610 millions Pesos in 1949; see Taylor (1998)) as well as a decline in foreign reserve holdings (from 1.1 billion dollars in 1946 to 258 millions dollars in 1948). By the early 1950’s there are signs of moves towards austerity measures. In a speech in 1952 Perón stated – see Mallon and Sourrouille (1975):

> The justicialista [i.e. Peronist] economy asserts that the production of the economy should first satisfy the needs of its inhabitants and only export the surplus; the surplus, nothing more. With this theory the boys here, of course, eat more every day and consume more, so that the surplus is smaller. But these poor guys have been submerged for fifty years; for this reason I have let them spend and eat and waste everything they wanted to for five years . . . but now we undoubtedly must begin to reorder things so as not to waste any more.

In 1952, wage freezes were introduced and exports were subsidized and import restrictions were implemented. As a result current account increased to 7.761 billion pesos in 1953 (2.1 billions pesos in 1954).

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7The numbers are based on the authors’ calculations on data provided by the Central Bank of Chile.
The predictions of our model are also consistent with the experience of several Southern European countries in the current crisis. In fact, in the aftermath of favorable economic climate following the establishment of euro, the governments of Southern European countries significantly increased public debt (particularly external debt in case of Portugal and Greece) and government expenditures (with particular increase in transfers in the form of pensions, social transfers and compensation of government employees) before 2007. Following the financial crisis of 2007, those countries had to implement drastic austerity measures in an effort to reduce their debt that reduced transfers - especially pensions - and compensation of government employees among other measures.

For example, consider the case of Portugal. The run-up in government debt from 1999 to the current crisis is accompanied by an increasing trend in the share of total government expenditures associated with social protection from 12% of GDP in 1999 to about 18% in 2009. The advent of the crisis is associated with a dramatic changes in public finances and the trade balance. The primary surplus relative to GDP improved from -8.2% in 2010 to .5% in 2014 and the trade balance relative to GDP went from a deficit of 7.1% to a 1.1% surplus in the same period. Over the same period of time the total government expenditures associated with social protection declined by about 73 billion euro from 2010 to 2012. See Reis (2015) for further discussion on the Portuguese government’s finances over this episodes.

Similarly, in Greece the remuneration of public sector employees, a form of transfer, grew by 25% from 2000 to 2009 while this figure is less than 7% for the the Euro-area on average (see OECD (2013) Figure 6). These policies reversed after the crisis with reduction in wages of government employees and pensions being key ingredients of the austerity measures. See Ioannides and Pissarides (2015) for a discussion of the austerity measures.

**Inequality and Incentive to Default.** Cycles in our model arise because countries with high wealth inequality (or equivalently countries that have high income inequality in the past) can support a lower amount of external public debt. Berg and Sachs (1988) find support for this mechanism. They document that higher income inequality is a significant predictor of the probability that a country will reschedule its external debt and of the interest rate spreads of the bond price in secondary market. They consider a sample of emerging economies and find that income distribution - measured as the ratio of household income of top 20 percentiles over bottom 20 percentiles - is a significant predictor of the probability of rescheduling once controlling for other variables such as outward orientation of trade policies, share of agriculture in GNP, level of per capita GDP, changes in

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8Source: Eurostat.
9For instance, “a single salary grid was introduced and then extended to the whole civil service in November 2012, which rationalized the pay structure and reduced average pay levels by nearly 20%” (OECD (2013))
More recently, Aizenman and Jinjarak (2012) find a positive association between sovereign default risk (measured by the CDS spread) and inequality.

Relatedly, Tomz (2004) analyzes a public opinion survey administered in Argentina during the debt crisis of 2001-2002. He finds that on average poor, unemployed and public sector employees are more favorable to a sovereign default consistently with our model predictions. See the survey in Tomz and Wright (2013) for references to other works with similar findings.

7 Conclusion

The political economy and unequal effect of consolidations are important features of sovereign debt crises. In this paper we develop a framework to analyze the interactions of inequality and external debt in an economy without commitment. As we have shown, our model is able to explain the observation that various developing countries go through low frequency cycles in external debt. In explaining these cycles, our theory points toward weak institutions as the main culprit behind this seemingly puzzling observation. That is, according to our theory, the key factor that makes such economies different than the more stable and developed economies is the inability of the institutions in tying the government’s hand in changing policies, i.e. lack of commitment.

We believe that our tractable framework is rich enough for more general analysis of government policy in open economies when the government is concerned about redistribution. Some examples are worth discussing: First, in our analysis, we have focused on an overlapping generations economy without capital. This is partly to separately identify the conflict between current and future governments regarding the trade-off between inequality and external debt as well as the tractability of the repeated game between the government, households, and foreign creditors. With capital, we conjecture that the conflict that we identify is present but the analysis becomes much more complicated since the value associated with the worst equilibrium depends on the value of capital and is harder to characterize. Second, we have stayed silent on default and have focused solely on allocations. One can of course, decentralize the allocations discussed above in a setup where government default on its debt. These are natural extensions of our framework and are worth exploring in future research.

---

10Since most of the countries that rescheduled and have high spreads are in Latin America and since such economies are characterized by extreme income inequality, one may think that high income inequality is just picking up a “Latin American” fixed effect. Interestingly enough, Berg and Sachs (1988) show that introducing a dummy variable for Latin America does not change the result: income inequality is still an important predictor of rescheduling. They find similar result when considering the effect of income inequality on the spreads.
References


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A Omitted Proofs for the Deterministic Log-Log Economy

A.1 Proof of Lemma 1

Note that with log-log preferences, we must have

\[ q^d_t \frac{1}{c^i_{t,0}} = \beta (1 - \tau_{a,t+1}) \frac{1}{c^i_{t,1}} \]

\[ Z \frac{1 - \tau_{l,t}}{c^i_{t,0}} = \psi \frac{1}{\theta^i - y^i_t} \]

Therefore

\[ \frac{c^i_{t,0}}{c^j_{t,0}} = \frac{c^i_{t,1}}{c^j_{t,1}} = \frac{\theta^i - y^i_t}{\theta^j - y^j_t} \quad (29) \]

If we let \( C_{t,0} \) be aggregate consumption for the young at \( t \), and define \( \varphi^i_t = c^i_{t,0}/C_{t,0} \), then the above equations imply that

\[ c^i_{0,t} = \varphi^i_tC_{0,t} \quad (30) \]

\[ c^i_{1,t} = \varphi^i_tC_{1,t} \quad (31) \]

\[ \theta^i - y^i_t = \varphi^i_t (1 - Y_t) \quad (32) \]

where we have used the fact that \( \sum \mu^i \theta^i = 1 \) and assumed that all allocations are interior.\(^{11}\) Note that the above together with (29) imply that

\[ Z \frac{1 - \tau_{l,t}}{C_{t,0}} = \psi \frac{1}{1 - Y_t} \quad q^d_t \frac{1}{C_{t,0}} = \beta (1 - \tau_{a,t+1}) \frac{1}{C_{t,1}} \quad (33) \]

We can write the budget constraint (3) and (4) as

\[ c^i_{t,0} + \frac{q^d_t}{(1 - \tau_{a,t+1})} c^i_{t,1} = Z y^i_t (1 - \tau_{l,t}) + T_t + \frac{q^d_t}{(1 - \tau_{a,t+1})} P_{t+1} \]

Using (33) we have

\[ c^i_{t,0} + \frac{\beta C_{t,0}}{C_{t,1}} c^i_{t,1} = y^i_t \psi C_{t,0} + T_t + \frac{q^d_t}{(1 - \tau_{a,t+1})} P_{t+1} \]

\(^{11}\)Since our preferences do not satisfy Inada condition for hours worked at 0 hours, we need to make assumption about the dispersion of productivities so that everyone works positive hours. When we allow for hours worked to be zero, all of our results will go through while they involve more extensive and cumbersome algebra and are available upon request.
This can be written as
\[ \frac{1}{C_{t,0}} c_{t,0} - \frac{\psi}{1 - Y_t} y_t + \beta \frac{1}{C_{t,1}} c_{t,1} = \hat{T}_t. \]
for some constant \( \hat{T}_t \). Replacing from (30)–(32), we have
\[ \frac{1}{C_{t,0}} \varphi_i^i c_{t,0} - \frac{\psi}{1 - Y_t} \left[ \theta_i - \varphi_i^i (1 - Y_t) \right] + \beta \frac{1}{C_{t,1}} \varphi_i^i c_{t,1} = \hat{T}_t. \]
\[ (1 + \psi + \beta) \varphi_i^i - \frac{\psi \theta_i}{1 - Y_t} = \hat{T}_t. \]

Taking averages across \( i \)'s, we have
\[ (1 + \psi + \beta) \sum_i \mu^i \varphi_i^i - \sum_i \frac{\psi \mu^i \theta_i}{1 - Y_t} = \hat{T}_t \]
\[ (1 + \psi + \beta) - \frac{\psi}{1 - Y_t} = \hat{T}_t. \]

Hence,
\[ (1 + \psi + \beta) \varphi_i^i - \frac{\psi \theta_i}{1 - Y_t} = (1 + \psi + \beta) - \frac{\psi}{1 - Y_t} \]
or
\[ \varphi_i^i = 1 + \frac{\psi \theta_i - 1}{1 + \psi + \beta} \frac{1}{1 - Y_t} \]

This completes the proof. Q.E.D.

A.2 Policy with Commitment

The optimal policy with commitment (the Ramsey outcome) is the competitive equilibrium with taxes that attains the higher value for the planner. We characterize the optimal policy using the primal approach. Using the characterization of competitive equilibrium outcomes in terms of aggregates, it solves the following programming problem:

\[ \max \sum_{\{C_{0,t}, C_{1,t}, Y_t\}} \frac{\beta}{\beta} \sum_i \mu^i \alpha^i \log \left( a^i_{t+1} + P_0 \right) \]
\[ + \sum_{t=0}^{\infty} \beta^t \left[ \log C_{0t} + \psi \log (1 - Y_t) + \beta \log C_{1,t+1} + (1 + \psi + \beta) H(Y_t) \right] \]
subject to

\[ \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t [ZY_t - C_{0,t} - C_{1,t} - G_t] \leq B_0 \]

\[ \left\{ a_{i-1}^i \right\}_{i=1}^{N} B : \text{ given} \]

The first order conditions for the problem for \( t \geq 0 \) are

\[ \hat{\beta}^t \frac{1}{C_{0,t}} = \lambda \left( \frac{1}{1+r} \right)^t \]

\[ \hat{\beta}^t \frac{\beta}{C_{0,t+1}} = \lambda \left( \frac{1}{1+r} \right)^{t+1} \]

\[ \hat{\beta}^t \left( \frac{\psi}{1-Y_t} - (1+\psi+\beta) H'(Y_t) \right) = \lambda \left( \frac{1}{1+r} \right)^{t+1} Z \]

where \( \lambda \) is the multiplier on the consolidated budget constraint for the country. Or

\[ C_{0,t} = \frac{\hat{\beta}^t (1+r)^t}{\lambda}, \quad C_{0,t+1} = \beta \frac{\hat{\beta}^t (1+r)^t}{\lambda}, \quad 1-Y_t = \frac{\hat{\beta}^t (1+r)^t}{\lambda} \left[ \psi - (1+\psi+\beta) H'(Y_t) (1-Y_t) \right]. \]

It is then clear that if \( \hat{\beta}^t (1+r) < 1 \) then \( \{C_{0,t}, C_{1,t+1}, 1-Y_t\}_{t=0}^{\infty} \) have a downward trend converging to zero. Instead if \( \hat{\beta}^t (1+r) = 1 \) then all the allocations are constant over time.

### A.3 Policy Game

The state variable is the distribution of government debt \( B_t, \{a_i^0\}_{i=1}^{N} \) where \( B_t \) is external government debt and \( a_i^0 \) is the government debt held by old agents of type \( i \) (note we are considering symmetric equilibria where all agents of the same type choose the same action). Informally, the sequence of events in any period \( t \) is the following:

- The government chooses its policy, \( \pi_t = (B_{t+1}, B_{t+1}^d, \tau_{i,t}, \tau_{a,t}, T_t, P_t, d_t) \in \Pi \). The government policy \( \pi_t \) must be consistent with the government budget constraint.

- After the government chooses its policy, private agents act. Old agents consume their savings and pension payments, young agents choose their consumption, labor supply and savings, \( c_{0,t}, l_{t}, a_{t+1}^i \), subject to their budget constraint, and foreign lenders choose their holdings of government bonds \( B_{t+1} \) facing prices \( q_{t+1} \).

More formally, let

\[ h^t = \left( h^{t-1}, \pi_t, \{c_{1,t}^i\}_{i=1}^{N}, \{c_{0,t}, l_{t}, a_{t+1}^i\}_{i=1}^{N}, q_t, q_{t}^d \right) \]
be the public history at the end of period $t$. Also define the interim history $h_t = (h_t^{t-1}, \pi_t)$ when private agents act. Let $H^t$ and $H^t_p$ be the space of such histories. Government strategy is $\sigma_t : H^{t-1} \rightarrow \Pi$, young agents strategy is $\sigma^y_t : H^t \rightarrow \mathbb{R}_+^3$, old agents strategy $\sigma^o_t : H^t \rightarrow \mathbb{R}_+$, foreign agents strategy is $\sigma^* : H^t \rightarrow \mathbb{R}_+$, and the pricing rule for government debt are $q_{t+1}, q^d_{t+1} : H^t \rightarrow \mathbb{R}_+$.

We can then define a sustainable equilibrium. A set of strategies $(\sigma, \sigma^y, \sigma^o, \sigma^*)$ is a sustainable equilibrium if: i) For all $h^{t-1}$ the government strategy maximizes (??), and ii) for all $h^t$ the equilibrium strategies induce government policies $\{B_t, B^d_t, \tau_t, \tau_{at}, T_t, P_t, \delta_t\}_{t=0}^\infty$, allocations $\{c^i_{t,0}, c^i_{t,1}, y^i_{t, a_i} \}_{t=0}^\infty$ and prices $\{q_t, q^d_t\}_{t=0}^\infty$ that are competitive equilibrium with taxes.

Before we show the best sustainable allocations can be implemented as a sustainable equilibrium of the game we first characterize the worst equilibrium. It is immediate that the value of the worst equilibrium for the government is given by $W$ in (??).

**Lemma 3** Given initial asset distribution for the initial old, $\{a^i_0\}_{i=1,...,N'}$, and an initial foreign government debt, $B_0$, an aggregate allocation $\{C_0_t, C_1_t, Y_t\}_{t=0}^\infty$ and market weights $\{\Phi_t\}_{t=0}^\infty$ are part of a sustainable equilibrium outcome iff they satisfy (10), (??) and the sustainability constraint

$$
\beta \beta \left\{ \log (C_{1,t}) + H (Y_{t-1}) \right\} + \sum_{s \geq t} \hat{\beta}^{s-t} \left\{ \log C_{0,s} + \psi \log (1 - Y_s) + \beta \log C_{1,s+1} + (1 + \psi + \beta) H (Y_s) \right\} \geq W.
$$

The proof for this lemma follows closely Chari and Kehoe (1990) and it is therefore omitted. It follows as a corollary of the previous lemma that any best sustainable equilibrium outcome solves (??).

**Remark 1** For convenience, in this Appendix we will derive our results by considering the dual of (??):

$$
B (V) = \max ZY - C_0 - C_1 - G + \frac{B (V')} {1 + r}
$$

subject to

$$
\frac{\beta}{\hat{\beta}} \log C_1 + \left( \log C_0 + \psi \log (1 - Y) + (1 + \psi + \beta) H (Y) \right) + \hat{\beta} V' = V
$$

$$
\frac{\beta}{\hat{\beta}} H (Y) + V' \geq W
$$

\[12\] To be precise, since $\log$ is unbounded, to prove that $W$ is the value of the worst equilibrium we must assume that in any equilibrium there is a minimal level of consumption and leisure that must be delivered. Such minimal value can be made arbitrarily close to zero.
A.4 Proof of Lemma 2

We prove that the value function is concave. The proof of differentiability and monotonicity is standard and follows the same techniques from Stokey et al. (1989). In order to show concavity of the value function, it is sufficient to show that the constraint set is convex and the rest of the proof follows from Stokey et al. (1989). Note that given lemma 1, the constraint set is given by

\[ \log C_0 + \psi \log (1 - Y) + (1 + \psi + \beta) H(Y) + \beta \log C_1 + \hat{\beta} V' = V \]

\[ \frac{\beta}{\hat{\beta}} [\log C_1 + H(Y)] + \beta V' \geq W \]

where

\[ H(Y) = \sum_i \alpha^i \mu^i \log \left( 1 + \frac{\psi}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \right). \] (35)

Since log is a concave function, in order to prove the convexity of the constraint set, it is sufficient to show that \( H(Y) \) is concave. The following lemma, proves this result:

**Lemma 4** Let \( H(Y) \) be the function defined by (35). Then \( H(Y) \) is strictly decreasing and strictly concave.

**Proof.** We have

\[ H(Y) = \sum \mu^i \alpha^i \log \left[ 1 + \frac{\psi}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \right] \]

\[ H'(Y) = \sum \mu^i \alpha^i \frac{\psi}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \]

\[ = \frac{1}{1 - Y} \sum_{i=1}^N \mu^i \frac{\alpha^i}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \]

From assumption ??, \( \alpha^i \) is decreasing in \( i \) and so is \( \frac{1}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \). Hence, \( \frac{\alpha^i}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \) is decreasing in \( i \) while \( \frac{\psi}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \) is increasing in \( i \). Hence from Chebyshev’s sum inequality (see Hardy et al. (1952)), we must have

\[ \sum_{i=1}^N \mu^i \frac{\alpha^i}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \psi \frac{\theta^i - 1}{1 - Y} < \sum_{i=1}^N \mu^i \frac{\alpha^i}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \psi \frac{\theta^i - 1}{1 - Y} \sum_{i=1}^N \mu^i \frac{\alpha^i}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \psi \frac{\theta^i - 1}{1 - Y} \]

\[ = \sum_{i=1}^N \mu^i \frac{\alpha^i}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y} \psi \frac{\theta^i - 1}{1 - Y} \left( \sum_{i=1}^N \mu^i \theta^i - 1 \right) \]

\[ = 0 \]
This establishes that $H(Y)$ is strictly decreasing in $Y$.

Furthermore

$$H'(Y) = \frac{1}{1-Y} \sum \mu^i \alpha^i \frac{\psi (\theta^i - 1)}{1-Y + \frac{\psi (\theta^i - 1)}{1+\psi+\beta}}$$

$$H''(Y) = \frac{1}{(1-Y)^2} \sum \mu^i \alpha^i \frac{\psi (\theta^i - 1)}{1-Y + \frac{\psi (\theta^i - 1)}{1+\psi+\beta}} + \frac{1}{1-Y} \sum \mu^i \alpha^i \frac{\psi (\theta^i - 1)}{(1-Y + \frac{\psi (\theta^i - 1)}{1+\psi+\beta})^2}$$

$$= \frac{H'(Y)}{1-Y} + \frac{1}{1-Y} \sum \mu^i \alpha^i \frac{\psi (\theta^i - 1)}{(1-Y + \frac{\psi (\theta^i - 1)}{1+\psi+\beta})^2}$$

Note that in the final expression above, the first term is negative while a similar application of Chebyshev’s sum inequality implies that the second term is negative as well and hence $H(Y)$ must be strictly concave.

This concludes the proof.

Q.E.D.

### A.5 Proof of Theorem 1

Note that since $U^p$ and $U^p_1$ are both logarithmic in terms of $C_1$ and are separable, the margin between $C_1$ and $V'$ in (P1) is not distorted, i.e., we have

$$B'(V') = -\hat{\beta} C_1$$

Since $B$ is concave, $B'$ is decreasing and therefore the changes in $C_1$ and $V'$ in response to changes in $V$ is in the same direction (both decreasing or both increasing). Therefore, if the sustainability constraint is binding, it is sufficient to show that $Y$ is decreasing in $V$.

Since, we have shown that the value function is differentiable, we can take first order conditions from (P1). Note that there are two possibilities:

1. The sustainability constraint is slack.

In this case, we have the following FOC

$$Z + \lambda \left[ -\frac{\psi}{1-Y} + (1+\psi+\beta) H'(Y) \right] = 0$$

where $\lambda$ is the lagrange multiplier associated with the promise keeping constraint. Fur-
thermore, from envelope condition for (P1), we must have

\[ B'(V) = -\lambda. \]

Since \( B(V) \) is concave, the above condition implies that \( \lambda \) is increasing in \( V \). We can then write the FOC associated with \( Y \) as

\[ \frac{Z}{\lambda} = \frac{\psi}{1 - Y} - (1 + \psi + \beta) H'(Y) \]

Since \( H \) is concave, the right hand side of the above equation is increasing in \( Y \) and the left hand side is decreasing in \( \lambda \), it must be that \( Y \) decreases as \( \lambda \) increases. In other words, \( Y \) is a decreasing function of \( V \).

Furthermore, when the sustainability constraint is not binding, the first order condition associated with \( V' \) together with the envelope condition implies that

\[ \frac{1}{1 + r} B'(V) + \lambda \beta = 0 \rightarrow \frac{1}{1 + r} B'(V) - \hat{\beta} B'(V) = 0 \]

Since \( \hat{B} \) is concave, the above implies that \( V' \) is an increasing function of \( V \). Similarly, the FOC associated with \( C_1 \) is given by

\[ -1 + \frac{\lambda \beta}{C_1} = 0 \rightarrow C_1 = \beta \lambda = -\beta B'(V) \]

Hence \( C_1 \) must be an increasing function of \( V \). Hence, when the constraint is slack, the left hand side of (?) is increasing in \( V \). So there must exists a \( V^* \) so that for values of \( V > V^* \) the sustainability constraint is slack while for values of \( V \leq V^* \), the sustainability constraint must be binding. Hence, we have established that for values of \( V > V^* \), \( Y \) is a decreasing function of \( V \).

2. Sustainability constraint is binding.

As we show above \( V^* \) must exist such that sustainability constraint is binding only if \( V < V^* \). When \( V \) is below \( V^* \), sustainability constraint binds and hence, we must have

\[ V' = \frac{W}{\beta} \left[ \log C_1 + H(Y) \right] \quad \text{(36)} \]

Using this relationship the promise keeping constraint becomes

\[ \log C_0 + \psi \log (1 - Y) + (1 + \psi) H(Y) + \hat{\beta} W = V \]
while the objective becomes

\[ ZY - C_0 + \frac{1}{1+r} \left[ -C_1 + B \left( W - \frac{\beta}{\beta} (\log C_1 + H(Y)) \right) \right] \]

Using this objective and taking first order conditions with respect to \( Y \) and \( C_1 \), we have

\[
Z - \frac{\beta}{\beta} (1 + r) B' \left( V' \right) H' (Y) + \lambda \left[ -\psi \frac{Y}{1-Y} + (1 + \psi) H' (Y) \right] = 0 \quad (37)
\]

\[
1 + \frac{\beta}{\beta C_1} B' \left( V' \right) = 0 \quad (38)
\]

where \( V' \) is given by (33). As before, the envelope condition implies that \( B' \left( V \right) = -\lambda \) and hence, we need to show that \( Y \) is a decreasing function of \( \lambda \). Consider a small change in \( \lambda \), given by \( d\lambda \). Let the change in \( C_1 \) and \( V' \) be defined by \( dC_1 \) and \( dV' \), respectively. Then, we must have the following relationships – based on (33) and (38)

\[
dV' = \frac{1}{B' \left( V' \right)} (dC_1 + H' (Y) dY)
\]

\[
dC_1 = -\frac{\beta}{\beta} B'' \left( V' \right) dV'
\]

which implies that

\[
dV' = \frac{H' (Y)}{B' \left( V' \right) + \frac{\beta}{\beta} B'' \left( V' \right)} dY
\]

Using the above equation and taking a total derivative from (37), we have

\[
- \frac{\beta}{\beta} (1 + r) \left[ B'' \left( V' \right) H' (Y) dV' + B' \left( V' \right) H'' (Y) dY \right] + \lambda \left[ -\frac{\psi}{(1-Y)^2} + (1 + \psi) H'' (Y) \right] dY \]
\]

\[
+ d\lambda \left[ -\psi \frac{Y}{1-Y} + (1 + \psi) H' (Y) \right] = 0
\]

\[
- \frac{\beta}{\beta} (1 + r) \left[ B'' \left( V' \right) \frac{\left( H' (Y) \right)^2}{B' \left( V' \right) + \frac{\beta}{\beta} B'' \left( V' \right)} dY + B' \left( V' \right) H'' (Y) dY \right] + \lambda \left[ -\frac{\psi}{(1-Y)^2} + (1 + \psi) H'' (Y) \right] dY
\]

\[
= d\lambda \left[ \frac{\psi}{1-Y} - (1 + \psi) H' (Y) \right]
\]

Hence,

\[
\frac{dY}{d\lambda} = \frac{\frac{\psi}{1-Y} - (1 + \psi) H' (Y)}{-\frac{\beta}{\beta (1+r)} \left( B'' \left( V' \right) \frac{\left( H'(Y) \right)^2}{B'(V') + \frac{\beta}{\beta} B''(V')} + B' \left( V' \right) H'' (Y) \right) + \lambda \left[ -\frac{\psi}{(1-Y)^2} + (1 + \psi) H'' (Y) \right]}
\]
Note that $H' > 0$ and hence the numerator is a positive number. Furthermore, $B'' < 0$, $B' < 0$, $H'' < 0$ together with $\lambda > 0$ imply that the denominator is negative. Hence, $\frac{dY}{d\lambda} < 0$.

This completes the proof.

Q.E.D.

B Omitted Proofs for the Extensions

B.1 Economy with shocks

We again consider the dual of the problem solved by the government. With shocks this can be written as

$$B(V) = \max_{Y, C_0, C_1, V'(s')} ZY - C_0 - C_1 + \frac{1}{1 + r} \sum_{s'} \pi(s') B(V'(s'))$$

subject to

$$\frac{\beta}{\beta} \log C_1 + \left[ \log C_0 + \psi \log (1 - Y) + (1 + \psi + \beta) H(Y) \right] + \frac{\beta}{\beta} \sum_{s'} \pi(s') V'(s') = V$$

$$\frac{\beta}{\beta} H(Y) + V'(s') \geq W(s')$$

where note that $s$ is not part of the state because the term $G(s)$ is additive. Then foreign debt is $B(V) - \left[ G(s) + \frac{EG(s')}{r} \right]$.

B.1.1 Proof of Proposition 5

We first show the following Lemma:

**Lemma 5** The solution to the functional equation (39) satisfies the following:

1. A unique $B(V)$ exists that satisfies (39). Furthermore, $B(V)$ is strictly decreasing, strictly concave and differentiable.

2. The policy function $V'(V, s)$ is increasing in $s$.

**Proof. Properties of the Value Function.** The proof follows closely that of Lemma 2. Specifically since the function $H(Y)$ is concave in $Y$, the constraint set is convex and as a result the value function is strictly concave. The rest of the claims can be proven using standard techniques from Stokey et al. (1989).
Properties of the Policy Function. Here, we show that \( V' (V, s) \) is increasing in \( s \). Note that for any \( s \), we must have that

\[
\frac{\beta}{\hat{\beta} C_1(s)} = -\frac{1}{B'(V'(s))}
\]

This implies that if \( C_1(s) > C_1(s') \), then \( V'(s) > V'(s') \), since \( B(\cdot) \) is decreasing and concave. Consider the set of states \( S_1 \subset S \) for which the sustainability constraint is slack. Then, the first order conditions associated with (P1) imply that

\[
\forall s, s' \in S_1, C_1(s) = C_1(s'), V'(s) = V'(s').
\]

Consider a state \( s \in S - S_1 \). We show that \( \min S - S_1 > \max S_1 \). Suppose not. Let \( \hat{s} = \max S_1 > \bar{s} = \min S - S_1 \). Then, it must be that

\[
\frac{\beta}{\hat{\beta}} \log C_1(\hat{s}) + V'(\hat{s}) > W(\hat{s}) - \frac{\beta}{\hat{\beta}} H(Y) \quad (40)
\]

Since \( W(s) \) is increasing in \( s \), simply because government spending decreases with \( s \), we must have \( W(\hat{s}) > W(\bar{s}) \). Hence, the above inequality and equality imply that \( V'(\hat{s}) > V'(\bar{s}) \). We show that if this is the case, there is a perturbation that increases the objective and hence we have a contradiction. Consider a small increase in \( V'(\hat{s}) \) by \( \varepsilon > 0 \) and a small decrease in \( V'(\bar{s}) \) by \( \frac{\pi(\hat{s})}{\pi(\bar{s})} \varepsilon \). This perturbation does not change the value that the current government receives from the allocation and hence the promise keeping constraint is satisfied. Furthermore, since \( \varepsilon \) is small and (40) is strict inequality, sustainability constraints are satisfied. However, the difference between the value of the objective for this perturbation to the original objective is given by

\[
\pi(\hat{s}) B(V'(\hat{s}) + \varepsilon) + \pi(\bar{s}) B\left(V'(\bar{s}) - \frac{\pi(\hat{s})}{\pi(\bar{s})} \varepsilon\right) - \pi(\hat{s}) B(V'(\hat{s})) - \pi(\bar{s}) B(V'(\bar{s}))
\]

The derivative of the above expression at \( \varepsilon = 0 \) is given by

\[
-\pi(\hat{s}) B'(V'(\hat{s})) - \pi(\bar{s}) B'(V'(\bar{s})) = \pi(\hat{s}) [B'(V'(\hat{s})) - B'(V'(\bar{s}))] > 0
\]

where the last inequality follows from strict concavity of \( B \). This means that for \( \varepsilon > 0 \) and small enough this perturbation increases the value of the objective in (P1) and hence we have a contradiction. Therefore, it must be that \( \min S - S_1 > \max S_1 \). Hence, if we let
\(s^* = \max S_1\), we must have
\[
\frac{\beta}{\bar{\beta}} \log C_1(s) + V'(s) = \frac{W(s)}{\bar{\beta}} \mathcal{H}(Y), \forall s > s^*,
\]
\[
\frac{\beta}{\bar{\beta}} \log C_1(s) + V'(s) > \frac{W(s)}{\bar{\beta}} \mathcal{H}(Y), \forall s \leq s^*
\]

The first set of equalities above imply that \(V'(s)\) must be increasing in \(s\). Furthermore, a similar argument to the perturbation above can be used to show that \(V'(s') = V'(s^*) < V'(s)\), for all \(s > s^* \geq s'\). This implies that \(V'(s)\) is weakly decreasing.

Next, we prove the claims in Proposition 5.

Proof of 1.

As the proof in B.1.1 shows, for each \(V\) there is a state \(s^*\) so that for all \(s \leq s^*\), the sustainability constraint is slack while for all \(s > s^*\), the sustainability constraint is binding. By continuity of the policy function, it must be that this property holds for an interval including \(V\). We can then use the binding sustainability constraints to solve for \(V'(s)\) and write the promise keeping constraint as
\[
\log C_0 + \psi \log (1 - Y) + (1 + \psi) \mathcal{H}(Y) + \sum_{s \leq s^*} \pi(s) \left[ \beta \mathcal{H}(Y) + \beta \log C_1(s) + \hat{\beta} V'(s) \right] + \hat{\beta} \sum_{s > s^*} \pi(s) W(s) = V
\]
while the value of the objective is given by
\[
ZY - C_0 + \frac{1}{1 + r} \sum_{s \leq s^*} \left[ -C_1(s) + B \left( V'(s) \right) \right]
\]
\[
+ \frac{1}{1 + r} \sum_{s > s^*} \pi(s) \left[ -C_1(s) + B \left( W(s) - \frac{\beta}{\bar{\beta}} \left( \log C_1(s) + \mathcal{H}(Y) \right) \right) \right]
\]
Hence, the FOC associated with \(Y\) as well as that of \(\{C_1(s)\}_{s > s^*}\) are given by
\[
Z - \sum_{s > s^*} \pi(s) \frac{\beta}{\bar{\beta} (1 + r)} B' \left( V'(s) \right) H'(Y) + \lambda \left[ -\frac{\psi}{1 - Y} + \left( 1 + \psi + \beta \sum_{s \leq s^*} \pi(s) \right) H'(Y) \right] \quad \text{(41)}
\]
\[
1 + B' \left( V'(s) \right) \frac{\beta}{\bar{\beta} C_1(s)} = 0
\]
Note that as before, the envelope condition implies that \(B'(V) = -\lambda\) and in order to show that \(Y(V)\) is decreasing in \(V\), it is sufficient to show that as \(\lambda\) rises, \(Y\) decreases. Taking
Taking a total derivative from (41), we have
\[
dC_1(s) = -\frac{\beta}{\hat{\beta}} B''(V'(s)) \, dV'(s)
\]
\[
dV'(s) = -\frac{\beta}{\hat{\beta}} \left[ \frac{1}{C_1(s)} \, dC_1(s) + H'(Y) \, dY \right]
\]
The above imply that
\[
dV'(s) = -\frac{\beta}{\hat{\beta}} \left[ -\frac{\beta}{\hat{\beta}} B'(V'(s)) \, dC_1(s) + H'(Y) \, dY \right]
\]
\[
dV'(s) = -\frac{\beta}{\hat{\beta}} \left[ \frac{\hat{\beta}}{\hat{\beta} B'(V'(s))} B''(V'(s)) \, dV'(s) + H'(Y) \, dY \right]
\]
Taking a total derivative from (41), we have
\[
-\frac{\beta}{\hat{\beta}} \frac{1}{1+r} \sum_{s>s^*} \pi(s) \left[ B''(V'(s)) \, H'(Y) \, dV'(s) + B'(V'(s)) \, H''(Y) \, dY \right]
\]
\[
+ d\lambda \left[ -\frac{\psi}{1-Y} + \left( 1 + \psi + \beta \sum_{s\leq s^*} \pi(s) \right) H'(Y) \right]
\]
\[
+ \lambda \left[ -\frac{\psi}{(1-Y)^2} + \left( 1 + \psi + \beta \sum_{s\leq s^*} \pi(s) \right) H''(Y) \right] \, dY = 0
\]
or
\[
-\frac{\beta}{\hat{\beta}} \frac{1}{1+r} \sum_{s>s^*} \pi(s) \left[ B''(V'(s)) \left( \frac{-\beta}{\hat{\beta}} (H'(Y))^2 \right) \frac{1}{1 + \frac{\beta}{\hat{\beta}} B''(V'(s))} + B'(V'(s)) \, H''(Y) \right]
\]
\[
+ d\lambda \left[ -\frac{\psi}{1-Y} + \left( 1 + \psi + \beta \sum_{s\leq s^*} \pi(s) \right) H'(Y) \right]
\]
\[
+ \lambda \left[ -\frac{\psi}{(1-Y)^2} + \left( 1 + \psi + \beta \sum_{s\leq s^*} \pi(s) \right) H''(Y) \right] \, dY = 0
\]
which with more concise notation can be written as
\[
\frac{dY}{d\lambda} = \frac{-\psi}{(1-Y)^2} \left( 1 + \psi + \beta \sum_{s \leq s^*} \pi(s) \right) H'
\]
\[
\lambda \left[ -\frac{\psi}{(1-Y)^2} - \left( 1 + \psi + \beta \sum_{s \leq s^*} \pi(s) \right) H'' \right] + \frac{\beta}{\hat{\beta}(1+r)} \sum_{s > s^*} \pi(s) \left[ B''(s) \left( \frac{-\beta}{\hat{\beta}} (H'')^2 \right) \frac{1}{1 + \frac{\beta}{\hat{\beta}} B''(s)} + B'(s) \, H'' \right]
\]
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The denominator of the above expression is positive while the denominator is negative. This is because $H' < 0$, $H'' < 0$, $B' < 0$, and $B'' < 0$. Therefore, $\frac{dY}{d\lambda} < 0$. This proves the first claim.

**Proof of 2.**

Given the first claim that $Y(V)$ is decreasing, whenever the sustainability constraint is binding, we must have

$$\frac{\beta}{\hat{\beta}} \left[ \log C_1(s) + H(Y) \right] + V'(s) = W(s)$$

Since $Y$ is decreasing in $V$, we must have that

$$\frac{\beta}{\hat{\beta}} \log C_1(s) + V'(s) = W(s) - \frac{\beta}{\hat{\beta}} H(Y)$$

and since $H$ is decreasing, $\frac{\beta}{\hat{\beta}} \log C_1(s) + V'(s)$ must be decreasing in $V$ as well. Since

$$\frac{\beta}{\hat{\beta}} C_1(s) = -\frac{1}{B'(V'(s))}$$

$C_1(s)$ and $V'(s)$ must move together and hence, $C_1(s)$ and $V'(s)$ are decreasing whenever the sustainability constraint is binding. It remains to show that there must exist $V^*(s)$ so that for all values of $V < V^*(s)$, the sustainability constraint for state $s$ is binding and vice versa. Suppose this does not hold. That is, suppose that $V_1$ exists such that for a neighborhood of values of $V$ below $V_1$, the sustainability constraint is slack while for a neighborhood of values of $V$ above $V_1$, the sustainability constraint is binding. Then by continuity of the policy function from theorem of the maximum, it must be that at $V_1$ the sustainability constraint is binding. Note that since for values of $V \in (V_1 - \varepsilon, V_1)$ for a small $\varepsilon > 0$, the sustainability constraint is slack, and we must have

$$\frac{1}{1 + r} B'(V'(V, s)) \beta B'(V) = 0$$

$$-\frac{1}{1 + r} \beta B'(V) \frac{1}{C_1(V, s)} = 0$$

where we have used the FOCs associated with $C_1$ and $V'$ and the envelope condition. From concavity of the value function, the above imply that $V'(V, s)$ and $C_1(V, s)$ are both increasing in $V$. Hence, it must be that

$$V'(V_1, s) \geq V'(V, s), \forall V \in (V_1 - \varepsilon, V_1)$$

$$C_1(V_1, s) \geq C_1(V, s), \forall V \in (V_1 - \varepsilon, V_1)$$
From part 1, we know that $Y(V_1) \leq Y(V), \forall V \in (V_1 - \varepsilon, V_1)$. Therefore, for $V \in (V_1 - \varepsilon, V_1)$

$$\frac{\beta}{\beta} [\log C_1 (V_1, s) + H (Y (V_1))] + V'(V_1, s) \quad \geq \quad \frac{\beta}{\beta} [\log C_1 (V, s) + H (Y (V))] + V'(V, s)$$

$$> W(s)$$

Which is a contradiction. This completes the proof.

*Proof of 3.*

Suppose not. That is, there exists $i, i+1$ such that $V^*(s(i)) > V^*(s(i+1))$. This implies that for values of $V \in (V^*(s(i+1)), V^*(s(i)))$, the sustainability constraint for $s(i)$ is binding but that of $s(i+1)$ is slack. We know from lemma 5 that $V'(s(i+1)) > V'(s)$. Hence a perturbation of the form $\hat{V}'(s(i+1)) = V'(s(i+1)) - \varepsilon$ and $\hat{V}'(s(i)) = V'(s(i)) + \varepsilon \frac{\pi(s(i+1))}{\pi(s(i))}$ increases the objective and satisfies all the constraints. This concludes the proof.

### B.1.2 Proof of Proposition 6

We first show the existence of a stationary distribution. Given a solution to the above functional equation, $B(V)$ and its associated policy function for continuation values, $V'(V, s)$, the ergodic set associated with this dynamic system is given by $V \subset \mathbb{R}$ where

$$V'(V, S) = V. \quad (42)$$

Furthermore, the invariant distribution associated with $V'(V, s)$ is given by $\mu$ which is a probability measure on $(V, \mathcal{V})$ where $\mathcal{V}$ is the set of all Borel subsets of $V$ and must satisfy

$$\mu(A) = \int_{V} \sum_{s \in S} \pi(s) 1[V'(V, s) \in A] d\mu(V) \quad (43)$$

**Lemma 6** The dynamic system implied by the policy function $V'(V, s)$ has a compact ergodic set $V \subset \mathbb{R}$. Furthermore, a probability measure $\mu$ exists that satisfies (43).

**Proof.** We first show that there exists an ergodic set $V$ that is a subset of a bounded interval. Namely, let $\underline{V} = V'(V^*(s(1)), s(1))$ and $\overline{V} = V'(V, s(K))$. We show that for any $V \in [\underline{V}, \overline{V}]$ and $\forall s \in S$, then $V'(V, s) \in [\underline{V}, \overline{V}]$. Note that for any $V > V^*(s(1))$, $V'(V, s(1)) \geq V'(V^*(s(1)), s(1)) = \underline{V}$. Furthermore, from lemma 5, we have that

$$V'(V, s(i)) > V'(V, s(1)) \geq \underline{V}, \forall 1 \leq i \leq K$$

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Now suppose that \( V < V^* (s(1)) \), then from proposition 5 we have that
\[
V'(V, s(i)) \geq V'(V, s(1)) \geq V'(V^* (s(1)), s(1)) = V
\]
Hence, for all \( V \), we must have that \( V'(V, s) \geq V \).

Now, suppose that \( V \in [V, \overline{V}] \). Then either sustainability constraint for \( s(K) \) is binding or slack. If it is slack, then we must have
\[
B'(V'(V, s(K))) = (1 + r) \beta B'(V)
\]
since \( \beta (1 + r) < 1 \) and \( B' < 0 \),
\[
B'(V) < B'(V'(V, s(K)))
\]
and concavity of \( B(\cdot) \) implies that \( V > V'(V, s(K)) \). Hence,
\[
\overline{V} \geq V > V'(V, s(K)) \geq V'(V, s(i)), \forall i
\]
If the sustainability constraint is binding,
\[
V < V \rightarrow V'(V, s(K)) < V'(V, s(K)) = \overline{V}
\]
and furthermore
\[
V'(V, s(i)) < V'(V, s(K)) = \overline{V}.
\]
This establishes that any ergodic set \( V \) must be a subset of \([V, \overline{V}]\).

Next, we show that a stationary distribution \( \mu \) exists. Since \([V, \overline{V}]\) is a compact subset of \( \mathbb{R} \), then by Riesz Representation Theorem (Dunford and Schwartz (1958), IV.6.3), the space of regular measures on \([V, \overline{V}]\) is isomorphic to the space \( C^* ([V, \overline{V}] \), the dual of the space of bounded continuous functions on \([V, \overline{V}]\). Moreover, by Banach-Alaoglu Theorem (Rudin (1991), Theorem 3.15), the set \( \{f \in C^* ([V, \overline{V}] \}; \|f\| \leq k \} \) is a compact set in the weak-* topology for any \( k > 0 \). Equivalently, the set of regular measures, \( \psi \), with \( \|\psi\| \leq 1 \), is compact. Since non-negativity and full measure on \([V, \overline{V}]\) are closed properties, we must have the set
\[
M([V, \overline{V}]) = \{\psi; \psi \text{ a regular measure on } [V, \overline{V}], \psi([V, \overline{V}]) = 1, \psi \geq 0\}
\]
is compact in weak-* topology. Since any ergodic set \( V \) is a subset of \([V, \overline{V}]\), the transfor-
mation $T$ on $M([\mathcal{V}, \overline{\mathcal{V}}])$ given by

$$T(\psi)(A) = \int_{[\mathcal{V}, \overline{\mathcal{V}}]} \sum_{s \in S} \pi(s) 1[V'(V, s) \in A] \, d\psi(V)$$

maps $M([\mathcal{V}, \overline{\mathcal{V}}])$ into itself. Since from theorem of maximum, $V'(V, s)$ is continuous in $V$, the above mapping is continuous. Thus by Schauder-Tychonoff Theorem (Dunford and Schwartz (1958), V.10.5), $T$ has a fixed point $\mu$. This concludes the proof. ■

To prove the claim in proposition 6, it is sufficient to show that $V'(V, s(K)) = V$. Suppose not. That is, suppose that $V'(V, s(K)) < V$. Since $\overline{\mathcal{V}} = \sup \mathcal{V}$, it must be that for some $V < \overline{\mathcal{V}}$, $V'(V, s(K)) = \overline{\mathcal{V}} - \overline{\mathcal{V}}'$ is increasing in $s$. Since $\overline{\mathcal{V}} = \inf \mathcal{V}$, it must be that $V > \overline{\mathcal{V}}$. Now, if the sustainability constraint for $s(K)$ is slack at $V$, it must be that $V > \overline{\mathcal{V}}$ which cannot be. So it must be that the sustainability constraint for $s(K)$ at $V$ is binding. As a result, $V'(V, s(K)) > V'(V, s(K)) = \overline{\mathcal{V}}$ which is a contradiction. Hence, we must have $V'(V, s(K)) = \overline{\mathcal{V}}$. This concludes the proof.

B.1.3 A Model of The World Economy

In this section, we describe a world economy consisted of identical countries who experience i.i.d. shocks to government spending. We show that in the steady state of such economy, the world interest rate is constant and satisfies the relationship $\hat{\beta}(1+r) < 1$.

The world is consisted of a continuum of countries that are as described in section 5. The initial state of each country is defined by its initial inequality $\Phi - 1$, external debt $B$ and government spending represented by $s_0$. We let $x = (\Phi - 1, B, s_0)$ and $\psi_0$ be the initial distribution of $x$ across countries. In particular, let $\mathcal{X}$ be the set of all possible $x$. Then, from market clearing, $\psi_0$ must satisfy

$$\int_{\mathcal{X}} B(x) \, d\psi_0(x) = 0$$

Allocations for each country can be identified by its history of shocks and its initial state $x$. These allocations are given by $\{C_{t,0}(s^t, x), C_{t-1,1}(s^t, x), Y_t(s^t, x), \Phi_t(s^t, x)\}$, where $s^t = (s_1, \cdots, s_t)$.

Hence, the world planning problem that treats each country symmetrically can be written as

$$\max \int_{\mathcal{X}} \left\{ \frac{\hat{\beta}}{\beta} U_1^p(C_{t,0}(s^t, x) ; \Phi_{t-1}(x)) + \sum_{t=0}^{\infty} \sum_{s^t} \hat{\beta}^t \pi(s^t) U_p \left( C_{t,0}(s^t, x), C_{t,1}(s^{t+1}, x), Y_t(s^t, x); \Phi_t(s^t, x) \right) \right\} \, d\psi_0(x)$$

(44)
subject to

$$\int \sum_{s_t} \pi(s^t) \left[ C_{t,0}(s^t, x) + C_{t-1}(s^t, x) + G(s_t) - ZY_t(s^t, x) \right] d\psi_t(x) = 0 \quad (45)$$

$$\frac{\beta}{\beta} U^p_t \left( C_{t-1,1}(s^t, x); \Phi_{t-1}(s^t, x) \right) + \sum_{\tau=t}^{\infty} \sum_{s^\tau \geq s_t} \pi(s^\tau|s^t) \hat{\beta}^{\tau-t} U^p \left( C_{\tau,0}(s^\tau, x), \Phi_{\tau} \left( s^\tau, x \right) \right) \geq W(s_t) \quad (46)$$

Note that at each point in time, the state of the economy can be characterized by the distribution of external debt, inequality and government spending given by $\psi_t$ appropriately defined. Let $\lambda_t$ be the lagrange multiplier associated with aggregate resource constraint (45). Then it must be that in a steady state where $\psi_t = \psi_{t+1}$, $\lambda_{t+1}/\lambda_t = q$ is constant. The ratio $q$ is equal to $\frac{1}{1+r}$ where $r$ is the international interest rate considered in section 5. In particular, it can be simply shown that the state of each country can be represented by promised utility $V_t$ as defined in section 3. Then, it can be shown that each individual country’s problem at each point in time is equivalent to the recursive formulation in (39).

Note that first order conditions of the above problem with respect to consumption imply that

$$\begin{bmatrix} \hat{\beta}^t + \sum_{s^\tau \geq s_t} \hat{\beta}^{t-\tau} \mu_{\tau} (s^\tau, x) \\ \hat{\beta}^{t-1} + \sum_{s^\tau \geq s_t} \hat{\beta}^{t-1-\tau} \mu_{\tau} (s^\tau, x) \end{bmatrix} \frac{1}{C_{t,0}(s^t, x)} = \lambda_t$$

Hence,

$$\frac{C_{t+1,0}(s^{t+1}, x)}{C_{t,0}(s^t, x)} = \frac{\lambda_t \hat{\beta}^{t+1} \sum_{s^\tau \geq s_{t+1}} \hat{\beta}^{t+1-\tau} \mu_{\tau} (s^\tau, x)}{\lambda_{t+1} \hat{\beta}^t \sum_{s^\tau \geq s_t} \hat{\beta}^{t-\tau} \mu_{\tau} (s^\tau, x)} = \frac{\lambda_t}{\lambda_{t+1}} \left( \frac{\hat{\beta} + \sum_{s^\tau \geq s_t} \hat{\beta}^{t-\tau} \mu_{\tau} (s^\tau, x)}{\hat{\beta} + \sum_{s^\tau \geq s_t} \hat{\beta}^{t-\tau} \mu_{\tau} (s^\tau, x)} \right)$$

$$\frac{C_{t,1}(s^{t+1}, x)}{C_{t-1,1}(s^t, x)} = \frac{\lambda_t \hat{\beta}^{t+1} \sum_{s^\tau \geq s_{t+1}} \hat{\beta}^{t+1-\tau} \mu_{\tau} (s^\tau, x)}{\lambda_{t+1} \hat{\beta}^t \sum_{s^\tau \geq s_t} \hat{\beta}^{t-\tau} \mu_{\tau} (s^\tau, x)} = \frac{\lambda_t}{\lambda_{t+1}} \left( \frac{\hat{\beta} + \sum_{s^\tau \geq s_t} \hat{\beta}^{t-\tau} \mu_{\tau} (s^\tau, x)}{\hat{\beta} + \sum_{s^\tau \geq s_t} \hat{\beta}^{t-\tau} \mu_{\tau} (s^\tau, x)} \right)$$

Note that the above equations imply that

$$\frac{C_{t+1,0}(s^{t+1}, x)}{C_{t,0}(s^t, x)} \geq \frac{\hat{\beta}}{q}$$

$$\frac{C_{t,1}(s^{t+1}, x)}{C_{t-1,1}(s^t, x)} \geq \frac{\hat{\beta}}{q}$$
with strict inequality if the sustainability constraint binds for \((s^{t+1}, x)\). This implies that we must have that in a steady state \(\frac{\hat{\beta}}{q} \leq 1\), otherwise aggregate consumption increases without bounds. Additionally, if the sustainability constraint binds with positive probability, we must have that \(\frac{\hat{\beta}}{q} < 1\). Hence, we have the following proposition:

**Proposition 7** Consider a steady state of the planning problem in (44) where \(\lambda_{t+1}/\lambda_t = q\) is the ratio of the multiplier on (45). Then it must be that \(\hat{\beta}/q \leq 1\). Furthermore, if in steady state, the sustainability constraint (46) binds with positive probability, then \(\hat{\beta}/q < 1\).

### B.2 GHH Preferences

Let \(\gamma = 1 + \frac{1}{\varepsilon}\). Then market utility is given by

\[
U^M_0(C, Y; \Phi) = \max \sum \varphi_i \mu_i \log \left(c^i - \psi \left(\theta^i \right)^\gamma / \gamma \right)
\]

subject to

\[
\sum \mu_i c^i = C, \sum \mu_i \theta^i l^i = Y
\]

Then FOCs are given by

\[
\frac{\varphi_i}{c^i - \psi \left(\theta^i \right)^\gamma / \gamma} = \lambda_c \quad \lambda_c \in \mathbb{R}
\]

\[
\frac{\varphi_i}{c^i - \psi \left(\theta^i \right)^\gamma / \gamma} \psi \left(\theta^i \right)^{-1} = \lambda_Y \theta^i
\]

and we must have \(\psi \left(\theta^i \right)^\gamma - 1 = \frac{\theta^i \lambda_c}{\lambda_Y}\). Then

\[
l^i = \left(\frac{\theta^i \lambda_c}{\psi \lambda_Y}\right)^\varepsilon \to \theta^i l^i = \left(\frac{\theta^i}{\theta^i}\right)^{1+\varepsilon} \theta^i l^i \to \theta^i l^i = \frac{(\theta^i)^{1+\varepsilon}}{\sum \mu_j (\theta^j)^{1+\varepsilon} Y}
\]

We define average productivity as

\[
\hat{\theta}^{1+\varepsilon} = \sum \mu_i \left(\theta^i\right)^{1+\varepsilon}
\]

Then,

\[
l^i = \left(\frac{\theta^i}{\hat{\theta}^{1+\varepsilon}}\right) Y
\]
Furthermore

\[
\frac{\varphi_i}{\lambda C} = c^i - \psi \left( \frac{(t^i)^\gamma}{\gamma} \right)
\]

\[
\frac{1}{\lambda C} = \sum \mu^i c^i - \psi \sum \mu^i \left( \frac{(t^i)^\gamma}{\gamma} \right)
\]

\[
= C - \frac{\psi \sum \mu^i (\theta^i)^{1+\epsilon}}{\theta^{(1+\epsilon)} \gamma^\gamma}
\]

\[
= C - \frac{\psi \hat{\theta}^{(1+\epsilon)}}{\theta^{(1+\epsilon)} \gamma^\gamma}
\]

\[
= C - \frac{\psi \hat{\theta}^{(1+\epsilon)(1-\gamma)} \gamma^\gamma}
\]

\[
= C - \frac{\psi \hat{\theta} \gamma Y^\gamma}
\]

Then,

\[
c^i = (C - \nu (Y/\hat{\theta})) \varphi^i + \frac{\psi}{\gamma} \left( \frac{(\theta^i)^{1+\epsilon}}{\hat{\theta}^{(1+\epsilon)}} \right)^\gamma
\]

\[
= \varphi^i C - \varphi^i \nu (Y/\hat{\theta}) + \frac{\psi (\theta^i)^{1+\epsilon}}{\hat{\theta}^{(1+\epsilon)} \gamma} \gamma^\gamma
\]

\[
= \varphi^i C - \varphi^i \nu (Y/\hat{\theta}) + \frac{(\theta^i)^{1+\epsilon}}{\hat{\theta}^{(1+\epsilon)}} \frac{\psi \hat{\theta}^{(1+\epsilon)(1-\gamma)} \gamma^\gamma}
\]

\[
= \varphi^i C - \nu (Y/\hat{\theta}) \left( \varphi^i - \left( \frac{\theta^i}{\hat{\theta}} \right)^{(1+\epsilon)} \right)
\]

then the market utility can be written as

\[
U(C, Y; \Phi) = \sum \mu^i \varphi^i \log \left( (C - \nu (Y/\hat{\theta})) \varphi^i \right)
\]

\[
= \log \left( C - \nu (Y/\hat{\theta}) \right) + \sum \varphi^i \mu^i \log \varphi^i
\]

and so

\[
U_C = \frac{1}{C - \nu (Y/\hat{\theta})}
\]

\[
U_Y = -\frac{1}{C - \nu (Y/\hat{\theta})} \frac{1}{\hat{\theta}^\nu} \left( \frac{Y}{\hat{\theta}} \right)
\]
Hence, the budget constraint can be written as

\[
\frac{1}{C_0 - \nu(Y/\hat{\theta})} \left[ \varphi^i C_0 - \nu(Y/\hat{\theta}) \left( \varphi^i - \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\epsilon} \right) \right] - \frac{1}{C_0 - \nu(Y/\hat{\theta})} \frac{1}{\hat{\theta}} \nu(Y/\hat{\theta}) \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\epsilon} Y + \beta \varphi^i = \hat{T}
\]

\[
(1 + \beta) \varphi^i - \frac{1}{\epsilon} \frac{1}{C_0 - \nu(Y/\hat{\theta})} \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\epsilon} \nu(Y/\hat{\theta}) = \hat{T}
\]

\[
1 + \frac{1}{1 + \beta} \frac{\nu(Y/\hat{\theta})}{C_0 - \nu(Y/\hat{\theta})} \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\epsilon} - \hat{T} = \varphi^i
\]

The objective for the government is

\[
(1 + \beta) \sum \alpha^i \mu^i \log \varphi^i + \log(C_0 - \nu(Y/\hat{\theta})) + \beta \log C_1
\]

So, we can write the recursive problem as

\[
B(V) = \max \{ZY - C_0 - \frac{1}{1+r} C_1 + \frac{1}{1+r} B(V)\}
\]

subject to

\[
(1 + \beta) \sum \alpha^i \mu^i \log \varphi^i + \log(C_0 - \nu(Y/\hat{\theta})) + \beta \log C_1 + \hat{\beta} V' = V
\]

\[
\frac{\beta}{\hat{\beta}} \sum \alpha^i \mu^i \log \varphi^i + \log C_1 + V' \geq V
\]

\[
1 + \frac{1}{1 + \beta} \frac{\nu(Y/\hat{\theta})}{C_0 - \nu(Y/\hat{\theta})} \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\epsilon} - \hat{T} = \varphi^i
\]

Some preliminary results:

Let \( H(Y, C_0) = \sum \alpha^i \mu^i \log \left(1 + \frac{1}{1 + \beta} \frac{\nu(Y/\hat{\theta})}{C_0 - \nu(Y/\hat{\theta})} \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\epsilon} - \hat{T} \right) \) where \( \hat{H}(x) = \sum \alpha^i \mu^i \log \left(1 + \frac{1}{1 + \beta} \frac{\nu(x/\hat{\theta})}{x - \nu(x/\hat{\theta})} \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\epsilon} - \hat{T} \right) \). Then we have:

**Lemma 7** The function \( \hat{H} \) is strictly increasing and concave. Furthermore, the function \( H(Y, C_0) \) is decreasing in \( Y \) and increasing in \( C_0 \) and if \( \gamma \geq 2 \), it is strictly concave in \( (Y, C_0) \).

**Proof.** Some algebra:
\[ \hat{H}'(x) = - \sum \frac{\alpha_i}{\varphi_i \mu_i} \frac{1}{1 + \beta} \frac{v(1/\hat{\theta})}{x - v(1/\hat{\theta})} \frac{(\theta_i)^{1+\varepsilon} - \hat{\theta}^{1+\varepsilon}}{\hat{\theta}^{1+\varepsilon}} > 0 \]

\[ \hat{H}''(x) = 2 \sum \frac{\alpha_i}{\varphi_i \mu_i} \frac{1}{1 + \beta} \frac{v(1/\hat{\theta})}{x - v(1/\hat{\theta})} \frac{(\theta_i)^{1+\varepsilon} - \hat{\theta}^{1+\varepsilon}}{\hat{\theta}^{1+\varepsilon}} \]

\[ (\gamma + 1) \hat{H}'(x) + \gamma x \hat{H}''(x) = - \frac{v(1/\hat{\theta})}{(x - v(1/\hat{\theta}))^3} \frac{1}{(\gamma + 1) (x - v(1/\hat{\theta})) + 2\gamma x} \sum \frac{\alpha_i}{\varphi_i \mu_i} \frac{1}{1 + \beta} \frac{(\theta_i)^{1+\varepsilon} - \hat{\theta}^{1+\varepsilon}}{\hat{\theta}^{1+\varepsilon}} \]

\[ - \sum \frac{\alpha_i}{(\varphi_i)^2 \mu_i} \left( \frac{1}{1 + \beta} \frac{v(1/\hat{\theta})}{x - v(1/\hat{\theta})} \frac{(\theta_i)^{1+\varepsilon} - \hat{\theta}^{1+\varepsilon}}{\hat{\theta}^{1+\varepsilon}} \right)^2 < 0 \]

\[ = \frac{v(1/\hat{\theta})}{(x - v(1/\hat{\theta}))^3} \frac{1}{(\gamma + 1) v(1/\hat{\theta}) + (\gamma - 1) x} \sum \frac{\alpha_i}{\varphi_i \mu_i} \frac{1}{1 + \beta} \frac{(\theta_i)^{1+\varepsilon} - \hat{\theta}^{1+\varepsilon}}{\hat{\theta}^{1+\varepsilon}} \]

\[ - \gamma \hat{H}'(x) - \gamma x \hat{H}''(x) = \frac{v(1/\hat{\theta})}{(x - v(1/\hat{\theta}))^2} \gamma \sum \frac{\alpha_i}{\varphi_i \mu_i} \frac{1}{1 + \beta} \frac{(\theta_i)^{1+\varepsilon} - \hat{\theta}^{1+\varepsilon}}{\hat{\theta}^{1+\varepsilon}} \]

\[ - 2\gamma x \frac{v(1/\hat{\theta})}{(x - v(1/\hat{\theta}))^3} \sum \frac{\alpha_i}{\varphi_i \mu_i} \frac{1}{1 + \beta} \frac{(\theta_i)^{1+\varepsilon} - \hat{\theta}^{1+\varepsilon}}{\hat{\theta}^{1+\varepsilon}} \]

\[ + \gamma x \sum \frac{\alpha_i}{(\varphi_i)^2 \mu_i} \left( \frac{1}{1 + \beta} \frac{v(1/\hat{\theta})}{x - v(1/\hat{\theta})} \frac{(\theta_i)^{1+\varepsilon} - \hat{\theta}^{1+\varepsilon}}{\hat{\theta}^{1+\varepsilon}} \right)^2 > 0 \]
Then

\[ H_C = Y^{-\gamma} \hat{H}' (C_0 Y^{-\gamma}) > 0 \]
\[ H_Y = -\gamma C_0 Y^{-\gamma-1} \hat{H}' (C_0 Y^{-\gamma}) < 0 \]
\[ H_{YC} = Y^{-2\gamma} \hat{H}'' (C_0 Y^{-\gamma}) < 0 \]
\[ H_{YY} = \gamma (\gamma + 1) C_0 Y^{-\gamma-2} \hat{H}' (C_0 Y^{-\gamma}) \]
\[ + \gamma^2 C_0^2 Y^{-2\gamma-2} \hat{H}'' (C_0 Y^{-\gamma}) \]
\[ = \gamma C_0 Y^{-\gamma-2} [(\gamma + 1) \hat{H}' (C_0 Y^{-\gamma}) + \gamma C_0 Y^{-\gamma} \hat{H}'' (C_0 Y^{-\gamma})] < 0 \]
\[ H_{YY} H_{CC} - (H_{YC})^2 = \gamma (\gamma + 1) C_0 Y^{-\gamma-2} \hat{H}' (C_0 Y^{-\gamma}) \]
\[ + \gamma^2 C_0^2 Y^{-2\gamma-2} \hat{H}'' (C_0 Y^{-\gamma}) \]
\[ - \gamma^2 Y^{-2\gamma-2} (\hat{H}' (C_0 Y^{-\gamma}) + C_0 Y^{-\gamma} \hat{H}'' (C_0 Y^{-\gamma}))^2 \]
\[ = \gamma (\gamma + 1) C_0 Y^{-3\gamma-2} \hat{H}' \hat{H}'' + \gamma^2 C_0^2 Y^{-4\gamma-2} (\hat{H}'')^2 \]
\[ - \gamma^2 Y^{-2\gamma-2} ((\hat{H}')^2 + 2 C_0 Y^{-\gamma} \hat{H}' \hat{H}'' + C_0^2 Y^{-2\gamma} (\hat{H}'')^2) \]
\[ = \left( \gamma - \gamma^2 \right) C_0 Y^{-3\gamma-2} \hat{H}' \hat{H}'' - \gamma^2 Y^{-2\gamma-2} (\hat{H}')^2 \]
\[ = -\gamma Y^{-2\gamma-2} \hat{H}' ((\gamma - 1) C_0 Y^{-\gamma} \hat{H}'' + \gamma \hat{H}') \]
\[ = -\gamma Y^{-2\gamma-2} \hat{H}' ((\gamma - 1) x \hat{H}'' + \gamma \hat{H}') \]

The above expression is positive as long as \( \gamma \geq 2 \). This implies that the function \( H \) is concave, dec in \( Y \) and increasing in \( C \).

Q.E.D.

Let \( \tilde{\beta} = \frac{\beta}{\tilde{\beta}} \).
Note that with the above definition, we can write the problem

\[ B(V) = \max ZY - C_0 - \frac{1}{1+r} C_1 + \frac{1}{1+r} B(V') \tag{FE2} \]

subject to

\[
\begin{align*}
\log \left( C_0 - v \left( Y/\tilde{\theta} \right) \right) + (1 + \beta) H(C_0, Y) + \beta \log C_1 + \tilde{\beta} V' &= V \\
\hat{\beta} \log C_1 + \hat{\beta} H(C_0, Y) + V' &\geq V
\end{align*}
\]

We analyze two cases:

**Case 1.** \( V \) is high so that the sustainability constraint is not binding. Then FOCs are given by

\[
\begin{align*}
-1 + \frac{\lambda}{C_0 - v \left( Y/\tilde{\theta} \right)} + (1 + \beta) \lambda H_C(C_0, Y) &= 0 \\
-\frac{1}{1+r} + \frac{\beta \lambda}{C_1} &= 0 \\
Z - \frac{\lambda}{C_0 - v \left( Y/\tilde{\theta} \right)} v' \left( Y/\tilde{\theta} \right) \frac{1}{\tilde{\theta}} + (1 + \beta) \lambda H_Y(C_0, Y) &= 0 \\
\frac{1}{1+r} B'(V') + \beta \lambda &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{1}{C_0 - v \left( Y/\tilde{\theta} \right)} + (1 + \beta) H_C(C_0, Y) &= \frac{1}{\lambda} \\
\frac{1}{C_0 - v \left( Y/\tilde{\theta} \right)} v' \left( Y/\tilde{\theta} \right) \frac{1}{\tilde{\theta}} - (1 + \beta) H_Y(C_0, Y) &= \frac{Z}{\lambda}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{C_0 - v \left( Y/\tilde{\theta} \right)} + (1 + \beta) Y^{-\gamma} \hat{\Theta}'(C_0 Y^{-\gamma}) &= \frac{1}{\lambda} \\
\frac{1}{C_0 - v \left( Y/\tilde{\theta} \right)} v' \left( Y/\tilde{\theta} \right) \frac{1}{\tilde{\theta}} + (1 + \beta) \gamma Y^{-\gamma-1} C_0 \hat{\Theta}'(C_0 Y^{-\gamma}) &= \frac{Z}{\lambda}
\end{align*}
\]

Multiplying the first equation by \( C_0 \) and the second equation by \( \frac{Y}{\gamma} \) and subtracting the second from the first, we have

\[
1 = \frac{C_0 - ZY / \gamma}{\lambda} \rightarrow C_0 = \lambda + \frac{ZY}{\gamma}
\]
So, the equation that governs $Y$ can be written as

$$\frac{1}{\lambda + ZY/Y - Y^{\gamma}v(1/\theta)} + (1 + \beta) Y^{-\gamma} \hat{H}'((\lambda + ZY/\gamma) Y^{-\gamma}) = \frac{1}{\lambda}$$

We can rewrite the above equation as

$$\frac{1}{(\lambda + ZY/\gamma) Y^{-\gamma} - v(1/\theta)} + (1 + \beta) \hat{H}'((\lambda + ZY/\gamma) Y^{-\gamma}) = \frac{Y^\gamma}{\lambda}$$

As we have noted above, inequality is governed by the variable $x = C_0 Y^{-\gamma} = (\lambda + ZY/\gamma) Y^{-\gamma}$ – As $x$ goes up inequality goes down. We want to characterize the behavior of various variables as a function of $V$. Note that since the value function is concave – the constraint set is convex and the objective is linear, $B'(V)$ is a decreasing function of $V$. Since by the envelope condition $B'(V) = -\lambda$ and hence as $V$ increases $\lambda$ increases as well. So the characterization comes down to the behavior of various variables as a function of $\lambda$. We start with inequality $x$. Note that

$$dx = Y^{-\gamma}d\lambda + \left[-\gamma Y^{-\gamma - 1} \lambda + \frac{1}{\gamma} YZ^{-\gamma} \right] dY$$

$$dY = \frac{dx - Y^{-\gamma}d\lambda}{t}$$

where $t = \gamma Y^{-1 - \gamma} \lambda + \frac{Y^{-1}}{\gamma} ZY^{-\gamma} > 0$. Then we can write the above equation as

$$\lambda \left[ \frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}'(x) \right] - Y^\gamma = 0$$

Taking derivatives

$$d\lambda \left[ \frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}'(x) \right] + \lambda \left[ -\frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}''(x) \right] dx - \gamma Y^{\gamma - 1} dY = 0$$

or

$$d\lambda \left[ \frac{\gamma Y^{-1}}{t} \frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}'(x) \right] + \left[ \left( -\frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}''(x) \right) \lambda - \frac{\gamma Y^{\gamma - 1}}{t} \right] dx = 0$$

As a result

$$\frac{dx}{d\lambda} = -\frac{\gamma Y^{-1}}{t} \frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}'(x) \left( -\frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}''(x) \right) \lambda - \frac{\gamma Y^{\gamma - 1}}{t} > 0$$

and this is positive since the numerator is negative and denominator is positive.
implies that $x$ increases with $V$ and as a result inequality decreases as $V$ increases.

**Case 2.** Suppose that the constraint is binding. Then we can write the program as

$$B(V) = \max ZY - C_0 - \frac{1}{1+r} C_1 + \frac{1}{1+r} B \left( V - \frac{\hat{\beta}}{1-r} \log C_1 - \frac{\hat{\beta}}{1-r} H(C_0, Y) \right)$$

subject to

$$\log \left( C_0 - v \left( \frac{Y}{\hat{\theta}} \right) \right) + H(C_0, Y) + \frac{\hat{\beta}}{1-r} V = V$$

FOCs are

$$-1 - \frac{\lambda}{C_0 - v \left( \frac{Y}{\hat{\theta}} \right)} \frac{1}{\hat{\beta}} (1+r) B' \left( V' \right) Y^{-\gamma} \hat{H}' \left( C_0 Y^{-\gamma} \right) + \lambda Y^{-\gamma} \hat{H}' \left( C_0 Y^{-\gamma} \right) = 0$$

$$-1 - \frac{1}{1+r} - \frac{1}{\hat{\beta}} (1+r) B' \left( V' \right) \frac{1}{C_1} = 0$$

$$Z + \frac{1}{\hat{\beta}} (1+r) Y^{-\gamma-1} \hat{H}' \left( C_0 Y^{-\gamma} \right) B' \left( V' \right) - \lambda \frac{v' \left( \frac{Y}{\hat{\theta}} \right)}{C_0 - v \left( \frac{Y}{\hat{\theta}} \right)} - \lambda Y^{-\gamma-1} \hat{H}' \left( C_0 Y^{-\gamma} \right) = 0$$

Note that as before

$$C_0 = \lambda + \frac{ZY}{\gamma}$$

So similar to what we had before, we can write

$$dx = Y^{-\gamma} d\lambda + \left[ -\gamma Y^{-\gamma-1} \lambda + \frac{1-\gamma}{\gamma} Z Y^{-\gamma} \right] dY$$

$$dY = \frac{dx - Y^{-\gamma} d\lambda}{t}$$

We can write the first FOC as

$$-\gamma Y^{-\gamma} - \frac{\lambda}{C_0 Y^{-\gamma} - v \left( 1/\hat{\theta} \right)} - \frac{1}{\hat{\beta}} (1+r) B' \left( V' \right) \hat{H}' \left( C_0 Y^{-\gamma} \right) + \lambda \hat{H}' \left( C_0 Y^{-\gamma} \right) = 0$$

or

$$-\gamma Y^{-\gamma} - \frac{\lambda}{x - \tilde{v}} - \frac{1}{\hat{\beta}} (1+r) B' \left( V' \right) \hat{H}' \left( x \right) + \lambda \hat{H}' \left( x \right) = 0$$

Taking derivative we can write

$$-\gamma Y^{-\gamma-1} dY + \lambda \left( -\frac{1}{(x - \tilde{v})^2} + \hat{H}'' \left( x \right) \right) dx + \left( \frac{1}{x - \tilde{v}} + \hat{H}' \left( x \right) \right) d\lambda$$

$$-\frac{1}{\hat{\beta} (1+r)} B' \left( V' \right) \hat{H}'' \left( x \right) dx - \frac{1}{\hat{\beta} (1+r)} \hat{H}' \left( x \right) B'' \left( V' \right) dV' = 0$$

56
From the FOCs, we have

\[ C_1 = -\frac{1}{\beta}B'(V') \rightarrow dC_1 = -\frac{1}{\beta}B''(V') \, dV' \]

\[ V' = V - \tilde{\beta}^{-1} \log C_1 - \tilde{\beta}^{-1} \hat{H}(x) \rightarrow dV' = -\frac{1}{\beta} \frac{1}{C_1} dC_1 - \frac{1}{\beta} \hat{H}'(x) \, dx \]

\[ dC_1 = -\frac{1}{\beta} B''(V') \left[ -\frac{1}{\beta} \frac{1}{C_1} dC_1 - \frac{1}{\beta} \hat{H}'(x) \, dx \right] \]

\[ = -\frac{1}{\beta} B''(V') \left[ \frac{1}{B'(V')} dC_1 - \frac{1}{\beta} \hat{H}'(x) \, dx \right] \]

\[ = -\frac{1}{\beta} B'' B' \, dC_1 + \frac{1}{\beta^2} B'' \hat{H}' \, dx \]

\[ dC_1 = \frac{1}{\beta^2} B'' \hat{H}' \, dx \]

\[ dV' = -\frac{1}{\beta} \tilde{\beta} dC_1 \rightarrow dV' = -\frac{1}{\beta} \hat{H}' \, dx \]

\[ \frac{dC_1}{dV'} = -\frac{1}{\beta \tilde{\beta} B'} \frac{1}{\beta} B'' \hat{H}' \, dx \]

Note that \( \frac{B''}{\tilde{\beta} B'} > 0 \). So we can rewrite

\[ \frac{-\gamma Y^{-1}}{t} dY - Y^{-\gamma} d\lambda + \lambda \left( \frac{-1}{(x - \bar{v})^2} + \hat{H}''(x) \right) \, dx + \left( \frac{1}{x - \bar{v}} + \hat{H}'(x) \right) \, d\lambda \]

\[ -\frac{1}{\beta (1 + r) B'(V')} \hat{H}''(x) \, dx + \frac{1}{\beta (1 + r) \hat{H}'(x) B''(V')} \frac{1}{\beta} \frac{1}{\beta^2} \hat{H}' \, dx = 0 \]

and we can write

\[ dx \left[ -\frac{\gamma Y^{-1}}{t} + \lambda \left( -\frac{1}{(x - \bar{v})^2} + \hat{H}''(x) \right) - \frac{1}{\beta (1 + r) B'(V')} \hat{H}''(x) \right. \]

\[ + \frac{1}{\beta (1 + r) \hat{H}'(x) B''(V')} \frac{1}{\beta} \frac{1}{\beta^2} \hat{H}' \] \]

\[ \left. \left[ -\hat{H}' - \frac{1}{x - \bar{v}} - \frac{\gamma Y^{-1}}{t} \right] \, d\lambda \rightarrow \frac{dx}{d\lambda} > 0 \right] \]

\[ \frac{dV'}{d\lambda} = -\frac{1}{\beta} \frac{\hat{H}'}{1 + \frac{1}{\beta^2} \tilde{\beta} B'} \, dx < 0 \]

This completes the proof that \( V' \) is decreasing in \( \lambda \) and as a result in \( V \). Hence, we have the following proposition:

**Proposition 8** The continuation value for the policy function \( V'(V) \) in (FE2) is U-shaped in \( V \).
B.3 BGP Preferences

For generic time separable preferences $U(C_0, Y) + \beta U(C_1, 0)$, we cannot write the planning problem recursively using external government debt as a state variable. However, there is an alternative formulation in which we use total debt $\hat{B}$ as a state variable and solve:

$$V(\hat{B}) = \max \sum_i \mu^i \alpha^i \left[ U\left(c^i_0, y^i\right) + \beta U\left(c^i_1, 0\right) \right] + \hat{\beta} V\left(\frac{c^i_1}{1+r} + B'\right)$$

subject to

$$\sum_i \mu^i \left[c^i_0 + \frac{c^i_1}{1+r}\right] + G + \hat{B} \leq \sum_i \mu^i y^i + \frac{B'}{1+r}$$

$$\frac{\hat{\beta}}{\beta} \sum_i \mu^i \alpha^i U\left(c^i_1, 0\right) + V(\hat{B}') \geq W$$

Let preferences be BGP (Cobb-Douglass):

$$U(c, l) = \frac{(c^{1-\psi}(1-1)^{\psi})^{1-\sigma}}{1-\sigma}$$

If the market weights are $\Phi = \{\phi^i\}$ then

$$c^i_0 = \frac{\mu^i \phi^i_0^{\psi/(\sigma-1)/\sigma}}{\sum_j \mu^j \phi^j_0^{\psi/(\sigma-1)/\sigma}} C_0 = s^i_0(\Phi) C_0 \quad (47)$$

$$1 - l^i = \frac{\phi^i_1^{\psi/(\sigma-1)/\sigma} (1-Y)}{\theta^i \theta^i_1^{\psi/(\sigma-1)/\sigma}} = s^i_1(\Phi) \frac{(1-Y)}{\theta^i_1} \quad (48)$$

$$c^i_1 = \frac{\mu^i \phi^i_1^{-1/[1-(1-\psi)(1-\sigma)-1]} C_1 = s^i_1(\Phi) C_1 \quad (49)}{\sum_j \mu^j \phi^j_1^{-1/[1-(1-\psi)(1-\sigma)-1]}}$$

so that

$$\sum_i \mu^i \alpha^i U\left(c^i_0, y^i\right) = A_0(\Phi) U\left(C_0, Y\right) \quad (50)$$

$$\sum_i \mu^i \alpha^i U\left(c^i_1, 0\right) = A_1(\Phi) U\left(C, 0\right) \quad (51)$$
where

\[ A_0 (\Phi) = \sum_i \mu^i \alpha^i s_0^i (\Phi)^{1-\sigma} \theta_i^{-\psi(1-\sigma)} \]  

\[ A_1 (\Phi) = \sum_i \mu^i \alpha^i s_1^i (\Phi)^{(1-\sigma)(1-\psi)} \]  

Moreover the implementability conditions are for all \( i \)

\[ \hat{T} = c_0^i + \beta \frac{(1-\psi)}{(1-\sigma)(1-\psi)} \frac{U(c_0^i, l_i)}{1-\psi} \left( 1 - \psi l_i \right) c_1^i \]  

for some \( \hat{T} \). Using (47), (48), and (49) in (54) implicitly defines \( \Phi = \Phi (C_0, Y, C_1) \). We can then write the problem (in dual form) using only aggregates as

\[ \hat{B} (V) = \max ZY - C_0 - \frac{C_1}{1+r} - G + \frac{1}{1+r} \hat{B} (V') \]

subject to

\[ A_0 (\Phi (C_0, Y, C_1)) U (C_0, Y) + \beta A_1 (\Phi (C_0, Y, C_1)) U (C_1, 0) + \hat{B} V' = V \]

\[ \frac{\beta}{\hat{B}} A_1 (\Phi (C_0, Y, C_1)) U (C_1, 0) + V' \geq W \]

Figure 5 shows the policy functions associated with this problem. The main thing to observe is that \( V' (V) \) is U-shaped (and so \( \hat{B}' \) and \( B' \) are hump-shaped).

C Omitted Proofs for the Political Economy Model

C.1 Characterization of SPE Outcomes

We start our analysis by characterizing the set of SPE of the political economy game. We focus on the symmetric equilibria and derive necessary and sufficient conditions that allocations resulting from the equilibria of the political economy game must satisfy. The following lemma characterizes the set of allocations that can be implemented as a SPE of the political economy game.

Lemma 8 Given the international interest rate \( r \), initial external debt, \( B_0 \), and an initial distribution of assets for the initial old, \( \{ a_{i-1} \} \), government policies \( \{ B_t, B_t^d, \tau_{l,t}, \tau_{a,t}, T_t, P_t, \delta_t \} \), allocations \( \{ c_{i,0}^1, c_{i,1}^1, y_i^1, a_1^1 \} \), and prices \( \{ q_t, q_t^d \} \), are a SPE outcome if and only if:

1. The allocations, prices and government policies are a competitive equilibrium with taxes;
2. The allocations satisfy the following political sustainability constraint for all $t \geq 0$

$$
\omega \sum_i \alpha_i \mu_i u_{t-1,1}^i + \sum_i \alpha_i \mu_i u_{t,0}^i \geq V,
$$

where $V$ is the value of the worst equilibrium and it is given by the following programming problem:

$$
V = \max \quad \omega \sum_i \alpha_i \mu_i u(c_1) + \sum_i \alpha_i \mu_i u(c_0^i, y^i, \theta^i) \quad (55)
\text{s.t.} \quad \sum_i \mu_i c_0^i + c_1 + G = Z \sum_i \mu_i y^i \\
\theta^i (1 - \tau_1) u_{c_0}^i + u_y^i = 0 \\
c_0^i = (1 - \tau_1) Z y^i + T
$$

Figure 5: Policy functions with BGP preferences
The proof for this lemma is omitted because the argument is essentially the same as in Chari and Kehoe (1990) and Abreu (1988).

C.2 Best SPE Outcome

Under our preference specification (??), the welfare function (??) can be written solely in terms of aggregate allocations and initial pension payments as

$$\frac{\beta}{\hat{\beta}} \sum_i \mu^i \alpha^i \log \left( (1 - \tau_{a,0}) a^i_0 + p_0 \right) + \sum_{t=0}^{\infty} \hat{\beta}^t \log C_{0t} + \Omega (Y_t) + \beta \log C_{1t+1},$$

(56)

where we define $\Omega (Y) = \psi \log (1 - Y) + (1 + \psi + \beta) H (Y)$. The problem for the planner at $t = 0$ is then to choose an aggregate allocation $\{C_{0t}, C_{1t}, Y_t\}_{t=0}^{\infty}$ and time zero policies $\{P_0, \tau_{a,0}, \delta_0\}$ that maximize (56) subject to the present value version of the consolidated budget constraint for the country

$$\sum_t \left( \frac{1}{1 + \tau} \right)^t [ZY_t - C_{0t} - C_{1t} - G] \geq B\delta_0$$

(57)

and the sequence of political sustainability constraints from $t = 1$ and onward,

$$\omega \log C_{1t} + H (Y_{t-1})] + [\log C_{0t} + \Omega (Y_t) + \beta \log C_{1t+1}] \geq V,$$

(58)

and for $t = 0$,

$$\omega \sum_i \mu^i \alpha^i \log \left( (1 - \tau_{a,0}) a^i_0 + p_0 \right) + [\log C_{0t} + \Omega (Y_t) + \beta \log C_{1t+1}] \geq V.$$  

(59)

C.2.1 Proof of Proposition 3

C.3 EG Political Equilibrium

Setup. We make our selection operational by setting up the equilibrium recursively. The state variable is $\langle B, z \rangle$ where $B = (B, \{a^i\}, P^e)$, $P^e$ are pension payments promised by the previous government, and $z \in \{0, 1\}$ is a variable that records if previous governments defaulted in the past. We adopt the convention that $z = 1$ if previous governments defaulted in the past and zero otherwise. Given our selection, if $z = 1$ then value for the current government is $V$ for all $B$, $V (B, 0) = V$. Taking as given the decision of the future governments, denoted by a bar, the value function for the government that inherited a state $(B, 0)$ is given by

$$V (B, 0) = \max \{v (B), V\}$$

(60)
where the value \( v \) is the value associated with honoring all debt obligations and it can be expressed in primal form as

\[
v(B) = \max_{\pi=(P,C_0,Y,B')} \omega \sum_i \mu^i \alpha^i \log \left( a^i + P \right) + \log C_0 + \hat{\Omega}(Y) + \beta \sum_i \mu^i \alpha^i \log \left( a^{i'} + \bar{P}(B') \right)
\]  

subject to

\[
\sum_i \mu^i \left( a^i + P \right) + C_0 + B + G \leq ZY + \frac{1}{1+r}B
\]  

\[
P \geq P^e
\]  

\[
\bar{v}(B') \geq V
\]  

\[
a^{i'} + \bar{P}(B') = \phi^i(Y) \left( \sum_i \mu^i a^{i'} + \bar{P}(B') \right)
\]  

(61)

(62)

(63)

(64)

(65)

where \( \hat{\Omega}(Y) = \Omega(Y) - \beta H(Y) \), \( \bar{P} \) and \( \bar{v} \) are the policy rule and the value function used by future government, and recall that \( \phi^i(Y) \) is the consumption share for type \( i \) household given by (10).

The constraint (62) is the consolidated budget constraint for the country in sequential form. Constraint (63) requires that the pension payments by the current government, \( P \), are at least what promised by the previous government, \( P^e \). The constraint (64), guarantees that next period government has no incentive to default. The current government leaves a state to next period government such that it finds not optimal to default. Finally, constraint (65) ensures that assets holdings \( \{ a^{i'} \} \) are consistent with optimality of the households saving decisions. With log-preferences, this amount to assume that next period consumption share of type \( i \) agent is equal to current consumption share \( \phi^i(Y) \).

Notice that requiring that the current government passes a state to the future government such that it does not have a strict incentive to default is without loss of generality. In fact, suppose to the contrary that the current government chooses \( B' \) such that \( v(B') < V \). Then it follows that the price of external debt equals zero and so no resources can be raised from foreign creditors. Moreover, since there is a default next period, private agents optimality implies that \( a^{i'} = 0 \) for all \( i \). Hence, the current government is better off defaulting today, strictly if external debt is strictly positive or there is domestic asset inequality, contradicting optimality for the current government.

A \textit{Markov equilibrium} are policy rules \( \bar{P}, C_0, \bar{Y}, \bar{B}' \) and value functions \( \bar{V} \) and \( \bar{v} \) such they satisfy

\[
P(B|\bar{P}, \bar{v}) = \bar{P}(B), \quad C_0(B|\bar{P}, \bar{v}) = \bar{C}_0(B), \quad Y(B|\bar{P}, \bar{v}) = \bar{Y}(B), \quad \bar{B}'(B|\bar{P}, \bar{v}) = \bar{B}'(B), \quad \text{and } v(B|\bar{P}, \bar{v}) = \bar{v}(B)
\]
for all $\mathcal{B}$ such that $\bar{v}(\mathcal{B}) \geq V$ where $P, C_0, Y, B', V$, and $v$ are the policy rule and the value associated with the problem in (60) and (61) given $\bar{P}$ and $\bar{v}$.

**Characterization.** We now turn to show that outcomes obtained from the policy rules associated with (22) and (??) are a Markov equilibrium outcome.

First we prove some properties of $w$ in (22) that we will use in the proofs below. We cannot prove that the operator implicitly defined in (22) is a contraction. We will then focus on the largest fixed point of (22).

**Claim 1** $w(\hat{B})$ is strictly decreasing, concave and differentiable.

**Proof.** We are going to use techniques similar to APS. The operator implicitly defined by the right hand side of (22) is monotone. Then, we can show along the lines of Theorem 5 in APS that if we start iterating on it from $w_0 \geq w$ then the sequence $\{w_n\}$ converges to $w$.

Note that of $w_n$ is concave then $w_{n+1}$ is concave as well because the constraint set in (22) is convex and the objective function is convex. Then the limit must be convex as well. A similar argument can be applied to show that $w$ is decreasing. Differentiability follows from Benveniste-Scheinkman theorem. ■

**Claim 2** The political sustainability constraint is always binding.

**Proof.** Suppose by way of contradiction that the current government chooses a policy $(P, C_0, Y, C_1', B)$ such that (64) is slack. Then the government can increase the value of current generation by increasing foreign debt by a small amount (still respecting incentive of next period government not to default) and with debt issuance increase consumption of young agents. This variation clearly increases the value. Hence the sustainability is always binding. ■

Letting $\lambda$ and $\eta$ be the multiplier on the consolidated budget constraint and the political sustainability constraint, the first order necessary condition for an optimum in (22) are

\[
\frac{1}{C_0} - \lambda = 0 \tag{66}
\]

\[
\Omega'(Y) - \eta \omega H'(Y) - \lambda Z = 0 \tag{67}
\]

\[
\beta \frac{1}{C_1} + \eta \left[ \frac{\omega}{C_1} + w' (B') \right] = 0 \tag{68}
\]

\[
\frac{\lambda}{1 + \tau} + \eta w' (B') = 0 \tag{69}
\]

and the envelope condition is $w' (B) = -\lambda$. Notice for later reference that the multiplier
on the political sustainability constraint \( \eta \) can be expressed in terms of allocations as
\[
\eta = \frac{\Omega' (Y) - \frac{1}{\zeta_0} Z}{\omega H' (Y)} = \left[ \frac{\psi}{1 - \psi} - (1 + \psi + \beta) H' (Y) \right] - \frac{1}{\zeta_0} Z > 0
\]
(70)

Note that \( \eta > 0 \) since the political sustainability constraint is always binding.

**Lemma 9** Given an initial condition \((B, 0)\) such that it is not optimal to default in period zero, the outcome obtained from \(C_{1,0} = \sum_i \mu_i \log (\alpha^i + \hat{P} (B))\) and for all \(t \geq 0\)
\[
(C_{0t}, Y_t, C_{1t+1}, B) = (C_{0w}, Y^w, C_{1w}, B) (C_{1t} + B)
\]
is a Markov equilibrium outcome of the political economy game.

**Proof.** Consider \(w\) that solves the functional equation defined by the right hand side of (22). For all \(B\) such that \(v (B) \geq V_{AUT}\), let \(\hat{P} (B)\) be the decision rule associated with problem (??). We can construct the other equilibrium objects for a Markov equilibrium as follows:
\[
\bar{P} (B) = \hat{P} (B),\quad \bar{C}_0 (B) = C_{0w} (x (B)),\quad \bar{Y} (B) = Y^w (x (B)),\quad \bar{B}' (B) = B_{1w} (x (B)),
\]
\[
\bar{P}^e (B) = \varphi^1 (Y^w (x (B))) C_{1w} (x (B)),\quad \bar{a}^i (B) = \varphi^i (Y^w (x (B))) C_{1w} (x (B)) - \bar{P}^e (B),
\]
and \(\bar{v} (B) = \hat{v} (B)\) where
\[
x (B) \equiv B + \sum_i \mu_i \alpha^i + \hat{P} (B).
\]
To show that our constructed policy rules and values constitute a Markov equilibrium, we have to show that for all \(B\) such that \(v (B) \geq V\) we have that \(P (B|\bar{P} (\cdot), \bar{v} (\cdot)) = \hat{P} (B)\) and \(v (B|\bar{P} (\cdot), \bar{v} (\cdot)) = \hat{v} (B)\).

First, notice that we can combine problems (22) and (??) to obtain
\[
\hat{v} (B) = \max_{(\bar{P}, C_{0w}, Y^w, C_{1w}, B')} \omega \sum_i \mu_i \alpha^i \log (b^i + \bar{P}) + \log C_0 - \Omega (Y) + \beta \left[ H (Y) + \log (C_{1w}) \right]
\]
subject to
\[
\sum_i \mu_i (b^i + \bar{P}) + C_0 + B + G \leq ZY + \frac{B'}{1 + r}
\]
(72)
\[
\omega [H(Y) + \log C_{1w}] + w (B' + C_1) \geq V
\]
(73)
Second, since the sustainability is always binding, it is without loss of generality to replace
the constraint (73) with
\[ \hat{\nu}(B') \geq V \]
where \( B' = (B', \{ \phi^i(Y) C_1 - P^e \}, P^e) \). In fact, we know that in general
\[ \hat{\nu}(B') \geq \omega [H(Y) + \log C_1] + w(B' + C_1) \]
but at the optimum in (22) the above is an equality. To see this, suppose by way of contradiction that
\[ \hat{\nu}(B') > \omega [H(Y) + \log C_1] + w(B' + C_1) = V \]
Note that for this to be the case we must have that \( \hat{P}' > P^e \) otherwise we have an equality. So the next period government wants to redistribute resources toward the old. So we can equivalently write
\[ \omega [H(Y) + \log C_1] + w(B' + C_1) = \omega \sum_i \mu_i \alpha_i \log (\hat{\phi}^i(Y) C_1) + w(B' + C_1) < \]
\[ < \max_{\Delta \geq 0} \omega \sum_i \mu_i \alpha_i \log (\hat{\phi}^i(Y) C_1 + \Delta) + w(B' + C_1 - \Delta) \]
where \( \Delta \geq 0 \) are the additional pension payments provided by the future government. Given concavity of \( w \), the optimal \( \Delta^* \) is greater than zero if and only if
\[ \frac{\Theta \omega}{C_1} > -w'(\hat{B}') \]  \hspace{1cm} (74)
where \( \Theta \equiv \sum_i \mu_i \alpha_i / \hat{\phi}^i(Y) > 1 \). Consider now the following variation that increases pensions by \( \epsilon \), decreases output by \( \epsilon_y \) and increases external debt by \( \epsilon^* \). The variation is feasible in (22) if
\[ -\omega H' \epsilon_y + \frac{\omega}{C_1} \epsilon = \Theta \frac{\omega}{C_1} \epsilon \Rightarrow \epsilon_y = \frac{(\Theta - 1) \frac{\omega}{C_1} \epsilon}{-\omega H'} > 0 \]
\[ \left[ \Theta \frac{\omega}{C_1} + w'(\hat{B}') \right] \epsilon + w'(\hat{B}') \epsilon^* = 0 \Rightarrow \epsilon^* = \frac{\left[ \Theta \frac{\omega}{C_1} + w'(\hat{B}') \right]}{-w'(\hat{B}')} \epsilon \]
where \( \epsilon^* > 0 \) given (74). If the original allocation is optimal it must be that this variation does not increase the objective value so
\[ \Delta \text{obj} = \frac{1}{C_0} \left( \frac{\epsilon^*}{1 + r} - Z \epsilon_y \right) + \beta \frac{\epsilon}{C_1} + \left[ \frac{\psi}{1 - Y} - (1 + \psi + \beta) H'(Y) \right] \epsilon_y \leq 0 \]
\[ \Rightarrow \frac{\Delta \text{obj}}{\epsilon} = \frac{1}{C_0} \left[ \Theta \frac{\omega}{C_1} + w'(\hat{B}') \right] + \beta \frac{\epsilon}{C_1} + \left[ \frac{\psi}{1 - Y} - (1 + \psi + \beta) H'(Y) - Z \frac{1}{C_0} \right] \frac{(\Theta - 1) \frac{\omega}{C_1}}{-\omega H'} \leq 0 \]
Note that under (74), the first term is positive, the second is positive. For the right hand side to be negative it is necessary that the last term is negative but from (70) we know that also the last term is positive. Hence at the optimum (74) cannot hold. Then it must be that future government does not want to redistribute toward the old by increasing pension payments beyond what promised by the previous government. Hence we can rewrite (71) as

$$\hat{v}(B) = \max_{(P, C_0, Y, C_1, B', P^e)} \omega \sum_i \mu_i \alpha_i \log \left( b_i + P \right) + \log C_0 - \Omega(Y) + \beta [H(Y) + \log (C'_1) ]$$

subject to

$$\sum_i \mu_i \left( b_i + P \right) + C_0 + B + G \leq ZY + \frac{B'}{1+r}$$

(75)

$$\hat{v}(B') \geq V$$

$$B' = \left( B', \left\{ \phi^i(Y) C_1 - P^e \right\}, P^e \right)$$

(76)

It is then evident that (75) is equivalent to (61) and so

$$P(B|\hat{P}(\cdot), \hat{v}(\cdot)) = \hat{P}(B)$$

and

$$v(B|\hat{P}(\cdot), \hat{v}(\cdot)) = \hat{v}(B).$$

Q.E.D. ■

C.3.1 Proof of Proposition 4

Part i): The fact that $C_0$ is decreasing in $\hat{B}$ follows from combining (66) with the envelope condition and using concavity of $w$.

Part ii): Suppose by way of contradiction that $\hat{B}'$ is increasing in $\hat{B}$. From (68) and concavity of $v$ it follows immediately that $C_1$ is then decreasing in $\hat{B}$. Hence from the political sustainability constraint it follows that $Y$ must be decreasing in $\hat{B}'$. But totally differentiating the first order condition for output (67) we obtain

$$\left[ \Omega'' + \frac{\lambda H''}{(1+r)v'} \right] \frac{dY}{dB} = \frac{\lambda H'v''}{(1+r)(v')^2} \frac{dB'}{dB} + \left[ \frac{Z + H'(Y)}{-(1+r)v'} \right] \frac{d\lambda}{dB}$$

(79)

implying that $Y$ is increasing in $\hat{B}$ since $\lambda$ is also increasing in $\hat{B}$ and all coefficients are positive. This is a contradiction. Hence it must be that $\hat{B}'$ is decreasing.