Asset Markets with Heterogeneous Information

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Abstract

I define a notion of competitive equilibrium for asset markets where assets are heterogeneous and traders have heterogeneous information about them. Markets are defined by a price and a procedure for clearing trades. Any asset can in principle be traded in any market but traders can use their information to impose acceptance rules which specify which goods they are willing to trade in each market. I then apply this notion to a model of distressed sales under asymmetric information and examine whether it can account for fire sales: sharp drops in prices when distressed agents need to sell assets. Standard models of asymmetric information with informed sellers, heterogeneous assets and identical uninformed buyers predict the opposite phenomenon, as more distressed sellers on average sell less-adversely-selected pools of assets. With heterogeneity among buyers in their ability to distinguish assets of different qualities, the possibility of fire sales depends on the joint distribution of wealth and ability.

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1 Introduction

I study competitive asset markets where traders have different information about the goods being traded. Sellers own a portfolio of assets of heterogeneous quality and there are potential gains from trade in selling them to a group of buyers. Since Akerlof (1970), one special case has been studied in great detail: where sellers are informed and buyers are uninformed. Instead, I allow for different buyers to have different information about each of the assets.

Wilson (1980) and Hellwig (1987) first showed that in simple trading environments with asymmetric information the predictions are sensitive to the exact way that competition is modeled: the order of decisions, who proposes prices, etc. Faced with this difficulty, one approach has been to study these problems as games where all of these features are spelled out completely (Rothschild and Stiglitz 1976, Wilson 1977, Miyazaki 1977, Stiglitz and Weiss 1981, Arnold and Riley 2009). An alternative approach has been to attempt to abstract from the details of how trading takes place and adapt the notion of Walrasian competitive equilibrium (Gale 1992, 1996, Dubey and Geanakoplos 2002, Bisin and Gottardi 2006) or competitive search equilibrium (Guerrieri et al. 2010, Guerrieri and Shimer 2012, Chang 2011) to settings with asymmetric information.

Existing definitions of competitive equilibrium for asymmetric information environments typically define a set of markets and allow any good to be traded in any market; traders’ decision problem is then to choose supply or demand in each market. This construct is not enough to handle environments with many differently-informed traders. The reason is that when different goods trade in the same market, some traders (on the same side of the market) may have enough information to tell them apart while other traders on the same side of the market do not. Analyzing this possibility requires developing a new notion of competitive equilibrium where traders can act on this differential information in a way that’s not reducible to just choosing quantities. In the equilibrium definition below, traders can act on their information by imposing acceptance rules. These specify which goods the trader is willing to trade in each market. Each trader’s acceptance rules must be consistent with his own information; in particular, if a trader’s information is not sufficient to tell two goods apart, his acceptance rule cannot discriminate between them.

Allowing different traders to impose their own acceptance rules in a given market can give rise to situations where there is more than one possible way to clear the market. This indeterminacy can be resolved by defining the set of all possible market-clearing algorithms and allowing traders to direct their trades to markets that use the algorithm they prefer. Thus, the set of markets is defined as the set of all price-algorithm pairs and equilibrium is
defined in terms of quantities and acceptance rule for each market.

Two special cases are analyzed in detail. In both of them there are two asset qualities in known proportions, a fraction of sellers are impatient and buyers observe an imperfect binary signal from each asset. In one case (“false positives”), buyers may observe good signals from bad assets while in the other (“false negatives”) they may observe bad signals from good assets. In both cases, buyers can be ranked by their expertise, i.e. their probability of making mistakes.

For the false positives case, the equilibrium can be characterized quite simply. All assets trade at the same price; sellers of high-quality assets can sell as many units as they choose at that price but sellers of low quality assets face rationing. Low quality assets that are more likely to be mistaken for high quality assets face less rationing than easily detectable ones, and some assets cannot be traded at all. Only buyers who observe sufficiently informative signals choose to trade, while the rest stay out of the market. For the false negatives case, different high-quality assets trade at different prices, which depend on how many buyers are able to realize that the asset is of high quality. Thus more transparent assets command a premium.

One question of applied interest for this example has to do with what happens if the number of impatient sellers increases. Will there be fire-sale effects, with prices falling with the number of impatient sellers? Uhlig (2010) has shown that in a pure asymmetric information case with equally uninformed buyers prices should go up with the number of distressed sellers because these are the only ones that sell high-quality assets. In the case of differentially informed buyers, there are countervailing effects. Under a false positives information structure, when more assets are sold the marginal buyer has lower quality information; as a result, the net effect depends on the joint distribution of buyers’ wealth and information quality in a way that is easily characterized. Under a false negatives information structure a form of cash-in-the-market pricing arises, which may lead to both fire sale effects and increasing transparency premia.

2 The Economy

Dates

There are two periods, \( t = 1 \) and \( t = 2 \). Consumption at time \( t \) is denoted \( c_t \).

There is actually nothing about the model that requires the temporal interpretation. It could be “apples” and “oranges” rather than “today” and “tomorrow”. The key will be that
oranges come in boxes called “assets” not everyone knows how many oranges are contained in each box.

**Agents and preferences**

Agents are divided into buyers and sellers. Buyers are divided into a finite number of types \( b \in B \). Preferences for buyers are

\[
u(c_1, c_2) = c_1 + c_2
\]

and their consumption is constrained to be nonnegative.

Sellers are divided into a finite number of types \( v \in V \). Preferences for sellers are

\[
u(c_1, c_2, v) = c_1 + \beta(v) c_2
\]

with \( \beta(v) \leq 1 \). Whenever \( \beta(v) < 1 \) there are potential gains from intertemporal trade between buyers and sellers.

Linearity in the preferences of sellers makes things simple because it means that the decision of what to do with one asset does not depend on what the seller does with any other asset. For buyers, linearity is not so essential and is there mostly for simplicity.

**Endowments and Assets**

Buyer \( b \) has an endowment of \( w(b) \) goods at \( t = 1 \).

There is a finite set of assets indexed by \( i \in I \). Asset \( i \) will produce \( q(i) \) goods at \( t = 2 \). I refer to \( q(i) \) as the quality of asset \( i \). Assume w.l.o.g. that \( q(i) \) is weakly increasing. Seller \( v \) is endowed with \( e(i, v) \) assets of type \( i \).

**Information**

Each seller knows the index \( i \) (and therefore the quality \( q(i) \)) of each asset he owns. Buyers do not observe \( i \) but instead buyer \( b \) observes a signal \( x(i, b) \) whenever he analyzes asset \( i \). If \( x(i, b) \neq x(i', b) \) whenever \( q(i) \neq q(i') \) then buyer \( b \) is perfectly informed about asset qualities. Otherwise, he cannot tell apart some asset qualities from others. The interesting case is when at least some buyers are not perfectly informed about asset qualities.
3 Equilibrium

Markets

There is no market for trading $t = 1$ goods against $t = 2$ goods. If there was such a market, which can be interpreted as a market for uncollateralized borrowing, then the gains from trade would be exhausted and the resulting allocation would be first-best efficient. Instead, the only way to achieve some sort of intertemporal trade is to trade $t = 1$ goods for assets. These assets will in turn produce $t = 2$ goods.

There are many markets, operating simultaneously, where agents can exchange goods for assets. Each market $m$ is defined by a price $p(m)$ of assets in terms of goods and a clearing algorithm, described in more detail below. In principle, any asset can be traded in any market. Let $M$ be the set of markets.

Gale (1996) uses a similar construct: rather than letting the price clear markets, all possible prices coexist and at each price there is pro-rata rationing of excess supply or excess demand. There are two main differences with Gale’s setup. First, the current setup allows more elaborate clearing algorithms than simply rationing the long end of the market. These algorithms make it possible to describe which trades take place when different buyers place different types of orders in the same market (more on clearing algorithms below). Second, I allow agents to trade in as many markets as they want rather than limiting them to a single market. This is meant to capture the idea of anonymous markets and is clearly more appropriate in some applications than in others.

Seller’s problem

Sellers must choose how much to supply of each asset in each market. Formally, each seller chooses, for each asset type $i$, a measure $s(\cdot; i)$ on the set of markets $M$; $s(M_0; i)$ represents the total amount of assets of type $i$ that the seller supplies in the subset of markets $M_0 \subseteq M$.

From the point of view of the sellers, each market is characterized by a price $p(m) \in \mathbb{R}^+$ and a probability of actually being able to sell asset $i$, denoted by $\eta(i, m)$, where $\eta(\cdot, m) : I \rightarrow [0, 1]$. $\eta(\cdot, m)$ is an endogenous object, which results from the clearing algorithm and from the equilibrium supply and demand in market $m$. Each seller simply takes it as given.
Seller $v$ solves the following problem:

$$\max_{c_1, c_2, s} u(c_1, c_2, v)$$

$$\text{s.t.}$$

$$c_1 = \sum_{i \in I} \int M p(m) \eta(i, m) ds(m; i)$$

$$c_2 = \sum_{i \in I} q(i) \left[ e(i, v) - \int M \eta(i, m) ds(m; i) \right]$$

$$s(m; i) \leq e(i, v) \quad \forall i, m$$

$$\int M \eta(i, m) ds(m; i) \leq e(i, v) \quad \forall i$$

Constraint (2) computes how many goods the seller gets at $t = 1$ as a result of his sales. For each asset $i$, he supplies $s(m; i)$ in market $m$, and succeeds in selling with probability $\eta(i, m)$, in which case he gets $p(m)$. Adding up over all markets and qualities results in (2). Constraint (3) computes how many goods the seller gets at $t = 2$ as a result of the assets which he does not sell. For each quality $i$ his unsold assets are equal to his endowment $e(i, v)$ minus what he sold in all markets, and each yields $q(i)$ goods. Constraint (4) says that he can at most attempt to sell his entire endowment of each asset in any given market. This is important when $\eta(i, m) < 1$. It rules out a strategy of offering, say, 100 units for sale when he only owns 30 because he knows that due to rationing, only 30% of the units are actually sold. Constraint (5) just says that the total sales of any given quality (added across all markets) are constrained by the seller’s endowment. Note that this embodies the assumption that sellers can attempt to sell the same asset in many markets, i.e. I do not impose

$$s(M; i) \leq e(i, v) \quad \forall i$$

If I imposed (6) instead of (5), then a unit that is offered in one market could no longer be offered in other markets, and this commitment could be used as a signal of quality. Gale (1992, 1996), Guerrieri et al. (2010), Guerrieri and Shimer (2012) and Chang (2011) all make assumptions similar to (6).

The choice of $s(m; i)$ for values of $i, m$ such that $\eta(i, m) = 0$ has no effect on the utility obtained by the seller. The interpretation of this is that if he is not going to be able to sell, it doesn’t matter whether or not he tries. Formally, this means that program (1) has multiple
solutions. I am going to assume that when this is the case, the solution has to be robust to small positive $\eta(i,m)$, meaning that the seller would attempt to sell in all the markets where if he could he would want to.

**Definition 1.** A solution to program (1) is robust if there exists a sequence of functions $\eta^n(i,m) > 0$ and a sequence of consumption and selling decisions $c^n_1, c^n_2, s^n$ such that

1. $c^n_1, c^n_2, s^n$ solve program

$$\max_{c_1, c_2, s} u(c_1, c_2, v)$$

s.t.

$$c_1 = \sum_{i \in I} \left[ \int_M p(m) \eta^n(i,m) ds^n(m;i) \right]$$

$$c_2 = \sum_{i \in I} q(i) \left[ e(i,v) - \int_M \eta^n(i,m) ds^n(m;i) \right]$$

$$s^n(m;i) \leq e(i,v) \quad \forall i, m$$

$$\int \eta^n(i,m) ds^n(m;i) \leq e(i,v) \quad \forall i$$

2. $\eta^n(i,m) \to \eta(i,m)$

3. $c^n_1 \to c_1$, $c^n_2 \to c_2$ and $s^n(M_0;i) \to s(M_0;i)$ for any measurable $M_0 \subseteq M$.

**Buyer’s orders**

When buyers place orders in a market, they can specify both the quantity of assets that they demand and what subset of assets they are willing to accept. An example of an order will be “I offer to buy 5 assets as long as the indices $i$ of those assets satisfy $i \in \{1,2,4\}$”. I formalize the idea that buyers can be selective by defining acceptance rules:

**Definition 2.** An acceptance rule is a function $\chi : I \to \{0, 1\}$.

$\chi(i) = 1$ means that a buyer is willing to accept asset $i$ and $\chi(i) = 0$ means he is not. Buyers cannot just impose any selection rule that they want, such as accepting only the highest-quality assets. They are not necessarily able to tell different assets apart from each other since do not observe $i$ but just the imperfect signal $x(i,b)$. Feasible acceptance rules are those that only discriminate between assets that buyers can actually tell apart.
Definition 3. An acceptance rule $\chi$ is feasible for buyer $b$ if it is measurable with respect to buyer $b$'s information set, i.e. if

$$\chi(i) = \chi(i') \quad \text{whenever} \quad x(i, b) = x(i', b)$$

In general, since different buyers observe different signals, the set of feasible acceptance rules will be different for each of them and in equilibrium they will end up imposing different acceptance rules.

I denote the set of possible acceptance rules by $\Xi$ and the set of acceptance rules that are feasible for buyer $b$ by $\Xi_b$.

Allocation functions

Buyers understand that the mix of assets they will get if they buy from market $m$ depends on the acceptance rule they impose. In order to describe this, I introduce the concept of an allocation function.

Definition 4. An allocation function $A$ assigns a measure $A(\cdot, \chi)$ on $I$ to each acceptance rule $\chi \in \Xi$.

Allocation functions are endogenous objects, which result from the clearing algorithm and from the equilibrium supply and demand in each market. Each buyer simply takes the allocation function in each market as given. I denote the allocation function in market $m$ by $A(\cdot, \cdot, m)$

A buyer that demands one asset with acceptance rule $\chi$ in a market $m$ where the allocation function is $A(\cdot, \cdot, m)$ will obtain a measure $A(i, \chi, m)$ of assets of type $i$. Whenever a market $m$ is such that $A(I, \chi, m) < 1$, this means that buyers who impose acceptance rule $\chi$ in market $m$ are rationed, in the sense that they obtain less than one asset for each asset they demand.

Buyer’s problem

Buyers must choose their entire demand pattern, which involves how much to demand in each market and what acceptance rules to impose. Formally, each buyer chooses a measure $d$ over markets and acceptance rules.
Table 1: Example of supplies and demands in a market

<table>
<thead>
<tr>
<th>i</th>
<th>q(i)</th>
<th>χ(i) of buyer b₁</th>
<th>χ(i) of buyer b₂</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1.5</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.5</td>
</tr>
</tbody>
</table>

db₁ = 1  db₂ = 1

Buyer b solves the following problem:

\[
\max_{c_1, c_2} u(c_1, c_2)
\]

s.t.

\[
c_1 = w(b) - \int_{\Xi \times M} p(m) A(I, \chi, m) dd(\chi, m)
\]

\[
c_2 = \int_{\Xi \times M} \left( \sum_{i \in I} q(i) A(i, \chi, m) \right) dd(\chi, m)
\]

\[
d(\Xi_b, M) = d(\Xi, M)
\]

\[
c_1 \geq 0 \quad c_2 \geq 0
\]

Constraint (8) says that t = 1 consumption is equal to the buyer’s endowment minus what he spends on buying assets. In market m, upon demanding one asset and imposing acceptance rule χ he obtains A(I, χ, m) assets and pays p(m) for each of them. His total spending is given by integrating these expenditures using the demand measure he chooses. Constraint (9) computes the total amount of t = 2 goods that the buyer will obtain. Constraint (10) restricts the buyer to place positive demand measure only on feasible acceptance rules.

Clearing algorithms

Each market is defined by a price p(m) and a clearing algorithm. A clearing algorithm is a rule that determines what trades take place as a function of what trades are proposed by buyers and sellers.

To see why different clearing algorithms would lead to different results, consider the example in Table 1. There are three types of assets. Assets 1 and 2 will pay zero at t = 2 while asset 3 will pay one. There are two buyers in market m, with types b₁ and b₂. Type b₁ cannot tell apart assets 2 and 3 so he must either accept both of them or reject both of
them; assume he is willing to accept both of them but rejects asset 1. Type $b_2$ can distinguish the worthless assets 1 and 2 from the good asset 3 so he can impose that he will only accept asset 3. Each of the buyers demands a single unit. The total supply from all sellers is 1.5 units of each asset.

One possible clearing algorithm would say: “let $b_1$ choose first and take a representative sample of the assets he is willing to accept; then $b_2$ can do the same”. This would result in the following allocation function and selling probability:

$$A(i, \chi) = \begin{cases} 
0 & \text{if } \chi = \{0, 1, 1\} \\
0 & \text{if } i = 1 \\
0.5 & \text{if } i = 2 \\
0.5 & \text{if } i = 3 \\
0 & \text{if } \chi = \{0, 0, 1\} \\
0 & \text{if } i = 1 \\
0.75 & \text{if } i = 2 \\
0.25 & \text{if } i = 3 \
\end{cases}$$

$$\eta(i) = \begin{cases} 
0 & \text{if } i = 1 \\
0.5 & \text{if } i = 2 \\
1 & \text{if } i = 3 
\end{cases}$$

Type $b_1$ picks randomly from the sample excluding the rejected asset 1. Since there are equal amounts of assets 2 and 3 and the total exceeds his demand, he gets a measure 0.5 of each. After that, type $b_2$ gets to pick. He only accepts asset 3 and there is one unit left, which is exactly what he wants. From the sellers’ point of view, all assets 3 are sold but only $\frac{1}{3}$ of assets 2 and no asset 1 are sold.

Another possible clearing algorithm would say “let $b_2$ choose first and take a representative sample of the assets he is willing to accept; then $b_1$ can do the same”. This results in:

$$A(i, \chi) = \begin{cases} 
0 & \text{if } \chi = \{0, 1, 1\} \\
0.75 & \text{if } i = 1 \\
0.25 & \text{if } i = 2 \\
0 & \text{if } \chi = \{0, 0, 1\} \\
0 & \text{if } i = 1 \\
0.5 & \text{if } i = 2 \\
0.5 & \text{if } i = 3 \
\end{cases}$$

$$\eta(i) = \begin{cases} 
0 & \text{if } i = 1 \\
0.5 & \text{if } i = 2 \\
0.6 & \text{if } i = 3 
\end{cases}$$

After $b_2$ picks one unit of asset 3, there are only 0.5 units left, and there are still 1.5 units of asset 2. A representative sample from this remainder will give type $b_1$ a total of 0.75 units of asset 2 and 0.25 units of asset 3.

Clearly different algorithms result in different allocations and it is necessary to determine which algorithm will be used. The equilibrium definition below assumes that there exist separate markets for each possible clearing algorithm and traders can choose which of these markets they wish to trade in. To make this statement precise, I need to describe the set of possible clearing algorithms.

Which trades will eventually take place depends on the order in which buyers’ orders are executed; clearing algorithms are rules for determining this order. The algorithm defines
several rounds of trading. Each buyer will execute his trades in one of those rounds, or perhaps split among several rounds. When a buyer’s trade is executed, the buyer picks a representative sample of the acceptable assets, if any, that remain on sale in the market. The algorithm specifies, for each acceptance rule, in which round(s) a buyer who imposes that rule will execute his trades. Therefore a clearing algorithm can be represented by a mapping that assigns to each clearing rule a measure over the rounds of trading; the measure indicates what fraction of the requested trades will be executed in each round. The set of clearing algorithms is the set of all such mappings.

**Definition 5.** A clearing algorithm consists of:

1. a finite number of rounds $K$

2. a probability measure $\omega(\cdot, \chi)$ on $\{1, \ldots, K\}$ for each possible acceptance rule $\chi \in \Xi$.

In the example from Table 1, the first clearing algorithm consists of $K = 3$ and:

$$\omega(k, \chi) = \begin{cases} 
1 & \text{if } \chi = \{0, 1, 1\} \\
0 & \text{if } \chi = \{0, 0, 1\} \\
0 & \text{any other } \chi \\
0 & \text{if } k = 1 \\
1 & \text{if } k = 2 \\
0 & \text{if } k = 3 \\
\end{cases}$$

(14)

This says that the acceptance rule $\chi_1 = \{0, 1, 1\}$ will execute its trades in the first round while the acceptance rule $\chi_2 = \{0, 0, 1\}$ will execute its trades in the second round (and any other rule, which nobody in the example imposes, would come later).

The second clearing algorithm instead consists of $K = 3$ and:

$$\omega(k, \chi) = \begin{cases} 
0 & \text{if } \chi = \{0, 1, 1\} \\
1 & \text{if } \chi = \{0, 0, 1\} \\
0 & \text{any other } \chi \\
1 & \text{if } k = 1 \\
0 & \text{if } k = 2 \\
0 & \text{if } k = 3 \\
\end{cases}$$

(15)

Allocation functions and selling probabilities in a market result from applying the market’s clearing algorithm to the demand and supply in that market. Demand consists of a measure $D$ on $\Xi$. For each subset $\Xi_0$ of acceptance rules $D(\Xi_0)$ is the total amount of assets demanded by buyers who impose acceptance rules $\chi \in \Xi_0$. Supply consists of a function $S : I \to \mathbb{R}^+$. $S(i)$ is the total amount of $i$-type assets offered in the market.

Given demand $D$, supply $S$ and a clearing algorithm $\{K, \omega\}$, allocation functions $A$ and selling probabilities $\eta$ are computed as follows:
1. Denote the residual supply when the algorithm reaches round $k$ by $S^k(i)$. In the $k$th-round, the algorithm allocates to acceptance rule $\chi$ the following amounts of each asset per unit demanded:

$$A^k(i, \chi) = \begin{cases} \frac{\chi(i)S^k(i)}{\sum_{\chi \in \Xi} \chi(i)S^k(i)d_i} & \text{if } \sum_{i \in I} \chi(i)S^k(i)d_i > 0 \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

This states that, as long as there is a positive measure of assets that are acceptable, the acceptance rule is assigned a representative sample of those assets, while if there are no acceptable assets left the demand associated with that acceptance rule is left unsatisfied.\footnote{Allocating this amount might not be feasible: For instance, suppose that in the example from Table 1 buyer $b_2$ demanded 2 units instead of 1 and the clearing algorithm prescribed the measures $\omega$ given by (14). In round 2, the algorithm would attempt to assign 2 units of asset 2 to buyer $b_2$ when only 1 remains.

To complete the description of the algorithm, it is necessary to describe what happens if this arises. Let

$$r(k, i) \equiv \frac{S^k(i)}{\sum_{\chi \in \Xi} A^k(i, \chi)\omega(k, \chi)D(\chi)}$$

$$r(k) \equiv \left( \min_{i} \min \{r(k, i), 1\} \right)$$

$r(k, i)$ is the ratio of the residual supply of asset $i$ as of round $k$ to the total demand that the algorithm attempts to allocate during round $k$. The amount the algorithm actually allocates per unit of demand is given by:

$$A^k(i, \chi) = r(k)$$

This means that if in some round there is insufficient supply to meet demand for some asset, the allocation received by all acceptance rules in that round is reduced in the same proportion. This could leave some acceptable assets un-allocated. Therefore if $r(k) < 1$ and there are any assets remaining such that $\chi(i) = 1$ for an acceptance rule $\chi$ such that round $k$ is in the support of $\omega(k, \chi)$, then round $k$ is repeated until this is no longer the case.

For example, suppose the assets and acceptance rules are as in Table 1, buyer $b_1$ demands 2 units and buyer $b_2$ demands 8 units. Suppose further that the allocation algorithm is $K = 3$ and the following measures:

$$\omega(k, \chi) = \begin{cases} \text{if } \chi = \{0, 1, 1\} & \text{if } \chi = \{0, 0, 1\} & \text{any other } \chi \\ 0 & 0.25 & 0 & \text{if } k = 1 \\ 1 & 0.75 & 0 & \text{if } k = 2 \\ 0 & 0 & 1 & \text{if } k = 3 \end{cases}$$

In the first round, the algorithm attempts to allocate the following amounts per unit demanded:

$$A^k(i, \chi) = \begin{cases} \text{if } \chi = \{0, 1, 1\} & \text{if } \chi = \{0, 0, 1\} \\ 0 & 0 & \text{if } i = 1 \\ 0.5 & 0 & \text{if } i = 2 \\ 0.5 & 1 & \text{if } i = 3 \end{cases}$$
2. The allocation received by acceptance rule $\chi$ up to round $k$ is:

$$A(i, \chi, k) = \sum_{j=1}^{k} A^j(i, \chi) \omega(j, \chi)$$  \hspace{1cm} (17)

and the overall allocation received by acceptance rule $\chi$ is:

$$A(i, \chi) = \sum_{j=1}^{K} A^j(i, \chi) \omega(j, \chi)$$  \hspace{1cm} (18)

3. It remains to compute the evolution of residual supply, which is simply given by:

$$S^1(i) = S(i)$$  \hspace{1cm} (19)

$$S^k(i) = S(i) - \sum_{\chi \in \Xi} A(i, \chi, k-1) D(\chi) \quad \text{for } k \geq 2$$

However, this results in

$$r(k = 1, i = 3) = \frac{1.5}{0.5 \times 1 \times 2 + 1 \times 0.25 \times 8} = 0.5$$

and therefore pro-rated demand is

$$A^k(i, \chi) r(k) = \begin{cases} 
0 & \text{if } \chi = \{0, 1, 1\} \\
0 & \text{if } \chi = \{0, 0, 1\} \\
0.25 & \text{if } i = 1 \\
0 & \text{if } i = 2 \\
0.25 & \text{if } i = 3 
\end{cases}$$

This allocates a total of 0.5 units of each of assets 2 and 3 to buyer $b_1$ and 1 unit of asset 3 to buyer 2. This exhausts the supply of asset 3 but not of asset 2, which is acceptable to buyer $b_1$. Therefore the first round is repeated on the remaining supply of $\{1.5, 1, 0\}$ and demands of 1 unit for buyer $b_1$ and 1 unit $(1 = 8 \times 0.25 - 1)$ for buyer $b_2$, resulting in

$$A^k(i, \chi) = \begin{cases} 
0 & \text{if } \chi = \{0, 1, 1\} \\
0 & \text{if } \chi = \{0, 0, 1\} \\
1 & \text{if } i = 1 \\
0 & \text{if } i = 2 \\
0 & \text{if } i = 3 
\end{cases}$$

which requires no further pro-rating. Since the supply of acceptable assets has been exhausted, buyer $b_2$ receives nothing further in the second round. Overall, the allocation functions that result from this combination of supply, demand and clearing algorithm are:

$$A(i, \chi) = \begin{cases} 
0 & \text{if } \chi = \{0, 1, 1\} \\
0 & \text{if } \chi = \{0, 0, 1\} \\
0.75 & \text{if } i = 1 \\
0 & \text{if } i = 2 \\
0.25 & \text{if } i = 3 
\end{cases}$$

$$0.125$$
4. The selling probabilities are found as follows. First, for any asset \( i \), let \( \Xi_i \equiv \{ \chi \in \Xi : \chi(i) = 1 \} \) be the set of acceptance rules that accept asset \( i \). Then

\[
\eta(i) = \begin{cases} 
\frac{\sum_{\chi \in \Xi} A(i,\chi) D(\chi)}{S(i)} & \text{if } S(i) > 0 \\
1 & \text{if } S(i) = 0 \text{ and } D(\Xi_i) > 0 \\
0 & \text{if } S(i) = 0 \text{ and } D(\Xi_i) = 0 
\end{cases}
\] (20)

Equation (20) says that, if an asset is in positive supply, then the probability of selling it can simply be computed as the ratio of the total measure that gets allocated (which could be zero) and the measure of supply. If an asset is in zero supply, then there are two possibilities. If there is a positive measure of demand with acceptance rules that would accept it, then any seller who supplied asset \( i \) would be able to sell it, so \( \eta(i) = 1 \). Instead, if there is no demand that finds asset \( i \) acceptable, then it would remain unsold.

The set of markets \( M \) is the set of all possible pairs of a nonnegative price and a feasible clearing algorithm.

**Definition of equilibrium**

An equilibrium consists of:

1. Consumption and supply decisions \( c_{1,v}, c_{2,v}, s_v \) by sellers
2. Supply of each asset in each market \( S(\cdot,m) \)
3. Consumption and demand decisions \( c_{1,b}, c_{2,b}, d_b \) by buyers
4. Demand in each market \( D(\cdot,m) \)
5. An allocation function \( A(\cdot,\cdot,m) \) for each market
6. A probability of selling function \( \eta(\cdot,m) \) for each market

such that

1. \( c_{1,v}, c_{2,v}, s_v \) are a robust solution to program (1) for each seller \( v \), taking \( \eta(\cdot,m) \) as given
2. $c_{1,b}, c_{2,b}, d_b$ solve program (7) for each buyer $b$, taking $A(\cdot,\cdot,m)$ as given

3. For any market $m$:

   (a) supply is
   \[ S(i,m) = \sum_{s} s(m,i) \quad \forall i \]

   (b) demand is
   \[ D(\Xi_0,m) = \sum_{b} d_b(\Xi_0,m) \quad \forall \Xi_0 \subseteq \Xi \]

   and

   (c) $A(\cdot,\cdot,m)$, $\eta(\cdot,m)$, $S(\cdot,m)$ and $D(\cdot,m)$ satisfy equations (18) and (20)

4 Two examples

The examples below have a continuum of both assets and agents, but the definitions above extend straightforwardly to this setting. The set of assets is $I = [0,1]$, sellers’ types are $v \in [0,1]$ and buyers’ types are $b \in [0,1]$. In both examples, the distribution of discount factors for sellers is

$$
\beta(v) = \begin{cases} 
1 & \text{if } v \geq \mu \\
0 & \text{if } v < \mu 
\end{cases}
$$

so there is a fraction $\mu$ of sellers who are really desperate to consume at $t = 1$ (perhaps banks facing runs or hedge funds facing margin calls) while the rest are just as patient as the buyers. Asset qualities are $q(i) = I(i > \lambda)$ for some $\lambda \in (0,1)$. This means that a fraction $1 - \lambda$ of assets (those with indices $i > \lambda$) are high quality assets and will pay a dividend of 1 at $t = 2$ and a fraction $\lambda$ (those with indices $i \leq \lambda$) are “lemons” and will pay nothing. Each seller is endowed with one unit of each asset, i.e. $e(i,v) = 1$ and buyers’ endowments $w(b)$ are unspecified for now.

The two examples differ in the information that buyers have, illustrated in Figure (1). In the “false positives” case, buyer $b$ observes $x(i,b) = I(i > b\lambda)$. When an asset is of high quality, every buyer observes $x(i,b) = 1$. When an asset is a lemon, those buyers of types $b \leq \frac{i}{\lambda}$ will observe $x(i,b) = 1$, so they cannot distinguish it from a high quality asset; instead, buyers with $b > \frac{i}{\lambda}$ will observe $x(i,b) = 0$ and conclude that the asset is a lemon. $b$ can therefore be thought of as an index of expertise: higher values of $b$ means that there is a smaller subset of lemons that the buyer might misidentify as high quality assets.
Furthermore, expertise is nested: if type $b$ can identify asset $i$ as a lemon, then so can all types $b' > b$.

Conversely, in the “false negatives” case, buyer $b$ observes $x(i, b) = \mathbb{I}(i > 1 - b(1 - \lambda))$. When an asset is a lemon all buyers observe $x(i, b) = 0$ but when it is of high quality only those buyers with $b \geq \frac{1 - i}{1 - \lambda}$ observe $x(i, b) = 1$ and realize it is of high quality. Again, $b$ can be thought of as an index of expertise.

In these examples, all trades take place in markets that use one of the following clearing algorithms:

**Definition 6.**

1. The *less-restrictive-first* algorithm uses a $[0, 1]$ continuum of trading rounds and assigns measure

   $$\omega(K_0) = \begin{cases} 
   1 & \text{if } g \in K_0 \\
   0 & \text{otherwise}
   \end{cases}$$

   to acceptance rules of the form $\chi(i) = \mathbb{I}(i \geq g)$ and measure

   $$\omega(K_0) = \begin{cases} 
   1 & \text{if } 1 \in K_0 \\
   0 & \text{otherwise}
   \end{cases}$$

   for any other acceptance rule.

2. The *more-restrictive-first* algorithm uses a $[0, 1]$ continuum of trading rounds and assigns measure

   $$\omega(K_0) = \begin{cases} 
   1 & \text{if } 1 - g \in K_0 \\
   0 & \text{otherwise}
   \end{cases}$$
to acceptance rules of the form \( \chi(i) = \mathbb{1}(i \geq g) \) and measure

\[
\omega(K_0) = \begin{cases} 
1 & \text{if } 1 \in K_0 \\
0 & \text{otherwise}
\end{cases}
\]

for any other acceptance rule.

3. The non-restrictive-first-then-more-restrictive-first algorithm uses a \([0, 1]\) continuum of trading rounds and assigns measure

\[
\omega(K_0) = \begin{cases} 
1 & \text{if } 0 \in K_0 \\
0 & \text{otherwise}
\end{cases}
\]

to acceptance rule \( \chi(i) = 1 \forall i \); measure

\[
\omega(K_0) = \begin{cases} 
1 & \text{if } 1 - g \in K_0 \\
0 & \text{otherwise}
\end{cases}
\]

to acceptance rules of the form \( \chi(i) = \mathbb{1}(i \geq g) \) if \( g > 0 \), and measure

\[
\omega(K_0) = \begin{cases} 
1 & \text{if } 1 \in K_0 \\
0 & \text{otherwise}
\end{cases}
\]

for any other acceptance rule.

The first two of these algorithms distinguish between acceptance rule that take the form of a simple cutoff value and any other rules. Cutoff rules are ordered from least restrictive (lower cutoff for acceptance) to most restrictive (higher cutoff for acceptance) or vice versa; any acceptance rules that takes a different form is left to the last round. The last algorithm is the same as the more-restrictive-first except that the rule that accepts all assets is given priority.

**False Positives Case**

There is an (essentially) unique equilibrium. All trades take place in the same market \( m^* \) where the price is \( p^* \) and the clearing algorithm is less-restrictive-first. Distressed sellers (those with \( v < \mu \)) supply all their assets to this market while non-distressed sellers only supply lemons, so total supply of asset \( i \) is \( \mu \) for \( i > \lambda \) and 1 for \( i \geq \lambda \). There is a cutoff \( b^* \)
such that buyers with $b < b^*$ do not trade at all while buyers with $b > b^*$ spend their entire endowment buying assets. Buyers only accept assets for which they observe $x(i, b) = 1$, which means buyer $b$ only accepts assets in the range $[b \lambda, 1]$, so lower-$b$ buyers accept more lemons mixed in with their high-quality assets. Buyers of types $b < b^*$ below some cutoff prefer to stay out of the market and do not trade at all. $b^*$ and $p^*$ are given by the solution to:

\[
\int_{b^*}^{1} \frac{1}{\lambda(1-b) + \mu(1-\lambda)} \frac{w(b)}{p^*} db = 1
\]

(21)

\[
p^* = \frac{\mu(1-\lambda)}{\lambda(1-b^*) + \mu(1-\lambda)}
\]

(22)

Formally, the equilibrium is:

1. Supply decisions

\[
s_v(m, i) = \begin{cases} 
1 & \text{if } \begin{cases} 
i \leq \lambda \\
p(m) \geq p^* \text{ and } v < \mu \\
or \\
p(m) \geq 1 \\
otherwise 
\end{cases} 
0 & \text{otherwise} 
\end{cases}
\]

(23)

leading to consumption decisions

\[
c_{1,v} = \begin{cases} 
p^* \int_{0}^{1} \eta(i, m^*) di & \text{if } v < \mu \\
p^* \int_{0}^{\lambda} \eta(i, m^*) di & \text{if } v \geq \mu 
\end{cases}
\]

\[
c_{2,v} = \begin{cases} 
0 & \text{if } v < \mu \\
1-\lambda & \text{if } v \geq \mu 
\end{cases}
\]

(24)

2. Supply

\[
S(i, m) = \begin{cases} 
1 & \text{if } i \leq \lambda \\
1 & \text{if } i > \lambda, p(m) \geq 1 \\
\mu & \text{if } i > \lambda, p(m) \in [p^*, 1) \\
0 & \text{if } i > \lambda, p(m) < p^*
\end{cases}
\]

(25)
3. Demand decisions

\[ d_b(\chi, m) = \begin{cases} 
\frac{w(b)}{p^*} & \text{if } m = m^*, \, \chi(i) = x(i, b) \text{ and } b > b^* \\
0 & \text{otherwise}
\end{cases} \] (26)

leading to consumption

\[ c_1(b) = \begin{cases} 
w(b) & \text{if } b < b^* \\
0 & \text{if } b \geq b^*
\end{cases} \]
\[ c_2(b) = \begin{cases} 
0 & \text{if } b < b^* \\
\frac{w(b)}{p^*} \frac{\mu(1-\lambda)}{\lambda(1-b^*+\mu(1-\lambda))} & \text{if } b \geq b^*
\end{cases} \] (27)

4. Demand

\[ D(\Xi_0, m) = \begin{cases} 
\frac{1}{p^*} \int_{\{b: \chi(i) = x(i, b) \in \Xi_0\}} w(b) \, db & \text{if } m = m^* \\
0 & \text{otherwise}
\end{cases} \] (28)

5. Allocation functions

\[ A(I_0, \chi, m) = \begin{cases} 
\frac{\int_{i \in I_0} \chi(i) \mathbb{1}(i \leq \lambda) \, di}{\int_{i \in I} \chi(i) \mathbb{1}(i \leq \lambda) \, di} & \text{if } p(m) < p^* \text{ and } \int_{i \in I} \chi(i) \mathbb{1}(i \leq \lambda) \, di > 0 \\
0 & \text{if } p(m) < p^* \text{ and } \int_{i \in I} \chi(i) \mathbb{1}(i \leq \lambda) \, di = 0
\end{cases} \]
\[ \int_{i \in I} \chi(i) \mathbb{1}(i > \lambda) \, di & \text{if } p(m) \in [p^*, 1] \text{ and } \int_{i \in I} \chi(i) \mathbb{1}(i \leq \lambda) + \mu \mathbb{1}(i > \lambda) \, di > 0 \\
\frac{\int_{i \in I_0} \chi(i) \, di}{\int_{i \in I} \chi(i) \, di} & \text{if } p(m) \geq 1 \text{ and } \int_{i \in I} \chi(i) \, di > 0 \\
0 & \text{if } p(m) \geq p^* \text{ and } \int_{i \in I} \chi(i) \, di = 0
\] (29)

6. Probability of selling

\[ \eta(i, m) = \begin{cases} 
1 & \text{if } m = m^* \text{ and } i > \lambda \\
\frac{1}{p^*} \frac{w(b)}{\lambda(1-b^*+\mu(1-\lambda))} \, db & \text{if } m = m^* \text{ and } i \in [\lambda b^*, \lambda] \\
0 & \text{if } m = m^* \text{ and } i < \lambda b^*
\end{cases} \] (30)

\[ 0 & \text{if } m \neq m^*
\]
Proposition 1. Equations (21)-(30) describe an equilibrium.

In equilibrium, all trades take place in market \( m^* \), where the clearing algorithm is less-restrictive-first. As in example (12), being preceded by less-restrictive trades is not a problem for buyers because these trades don’t change the relative proportions of acceptable assets in the residual supply faced by a more-restrictive buyer. Therefore all buyers receive a representative sample of the overall supply of assets they are willing to accept. This makes the less-restrictive-first market attractive for buyers; otherwise, as in example (13) illustrates and Lemma 1 below proves, any buyer faces more adverse selection if higher-\( b \) buyers have been cleared before him. All markets besides \( m^* \) have zero demand, so no matter what the clearing algorithm, a buyer in those markets would receive a representative sample of the assets he accepts, just as in market \( m^* \). This means that buyers are indifferent between buying in market \( m^* \) or in other markets where the price is also \( p^* \), but sticking to \( m^* \) is one of the optimal choices.

Sellers, for their part, are indifferent regarding what algorithm is used to clear trades: they just care about the price and the probability of selling. Therefore they supply the same assets in all markets that have the same price.

Buying at prices other than \( p^* \) is never optimal for buyers. At prices lower than \( p^* \), the supply includes only lemons, so buyers prefer to stay away, whereas at prices above \( p^* \), the supply of assets is exactly the same as at \( p^* \) but the price is higher.

This does not settle the question of whether a buyer chooses to buy at all. Consider a buyer of type \( b \). The sample of assets he accepts includes all the high quality assets that are supplied, which are \( \mu (1 - \lambda) \), as well as all lemons with indices \( i \in (b \lambda, \lambda] \), which total \( \lambda (1 - b) \). Therefore the terms of trade (in terms of \( t = 2 \) goods per \( t = 1 \) good spent) for buyer \( b \) if he buys at price \( p^* \) are:

\[
\tau (b) = \frac{1}{p^* \lambda (1-b) + \mu (1-\lambda)}
\]

Condition (22) implies that the terms of trade for type \( b^* \) are \( \tau (b^*) = 1 \), which leave him indifferent. Buyers with \( b > b^* \) get \( \tau (b) > 1 \), so they spend all their endowment buying assets and buyers with \( b < b^* \) would get \( \tau (b) < 1 \), so they prefer not to buy at all.

Because \( \beta (v) = 0 \) for distressed sellers, they will supply all their assets in market \( m \) unless they can sell them for sure in markets where the price is higher than \( p(m) \). The probability of selling asset \( i \) in market \( m^* \) is given by the ratio of the total allocation of that asset across of buyers to the supply of that asset. For high quality assets, the supply
is $\mu (1 - \lambda)$ and buyer $b$ (with $b \geq b^*$) obtains $\frac{u(b)}{p} \frac{\mu(1-\lambda)}{\lambda(1-b)+\mu(1-\lambda)}$ units, so the probability of selling the asset is given by the left-hand-side of equation (21). This equation implies that sellers can sell high quality assets for sure in market $m^*$ and therefore do not supply them in markets with prices below $p^*$. For assets $i \in (\lambda b^*, \lambda]$, the supply is 1 and buyer $b$ obtains $\frac{u(b)}{p} \frac{1}{\lambda(1-b)+\mu(1-\lambda)}$ as long as $b \in (b^*, \frac{1}{2})$; lower types demand nothing and higher types reject asset $i$. This implies the selling probability (30). Notice that the selling probability is continuous. Lemons with indices just below $\lambda$ fool almost all buyers into thinking they are likely to be high quality and therefore sellers are able to sell them with high probability; assets with indices just above $\lambda b^*$ fool very few buyers and are sold with low probability.

**Proposition 2.** In any equilibrium, the price and allocations are those of the equilibrium described by equations (21)-(30).

**False Negatives Case**

In this case, trade takes place in many different markets. Seller decisions are

$$s_v (m, i) = \begin{cases} 
1 & \text{if } \begin{cases} 
i \leq \lambda \\
p (m) \geq p^* (i) \text{ and } v < \mu \\
or \\
p (m) \geq 1 \\
\text{otherwise}
\end{cases} 
\end{cases} \quad (31)$$

This is just like (23), except that the threshold price at which distressed sellers manage to sell their high-quality assets (an therefore below which they do not supply them) varies by asset. This price function $p^* (i)$ can be found as follows.

As a first pass, the price of an asset depends on the endowment of the buyers who are able to observe that it is of high quality. For any function $p$, let

$$E (i, p) \equiv \int_{\lambda}^{i} \mu p (j) \, dj \quad (32)$$

$E (i, p)$ is the amount of $t = 1$ goods it takes to buy all the assets in the interval $(\lambda, i)$ that
are held by distressed sellers if prices are \( p(i) \). Also define

\[
W(i) \equiv \int_{\hat{b}(i)}^{1} w(b) \, db \tag{33}
\]

\[
\hat{b}(i) \equiv \frac{1 - i}{1 - \lambda}
\]

\( \hat{b}(i) \) is the lowest buyer type that observes \( x(i, b) = 1 \), i.e. the least expert buyer who realizes that asset \( i \) is of high quality. \( W(i) \) is the total endowment of \( t = 1 \) goods of buyers who are able to realize that asset \( i \) is of high quality. If all high-quality assets were bought by buyers who can identify them as high quality, the maximum price they would be able to pay would satisfy

\[
E(i, p) = W(i) \tag{34}
\]

Using (32) and (33), the price function that satisfies (34) is

\[
p^E(i) = \frac{1}{\mu(1 - \lambda)} w \left( \frac{1 - i}{1 - \lambda} \right) \tag{35}
\]

In order to find the prices at which prices actually trade, \( p^E \) must be adjusted in three ways: (i) imposing monotonicity, (ii) imposing 1 as an upper bound and (iii) checking whether non-selective buyers may want to demand positive amounts.

**Monotonicity**

Let \( \lambda < i < i' \). Both assets \( i \) and \( i' \) are of high quality but \( i' \) is easier to identify as high quality (i.e. everyone who observes \( x(i, b) = 1 \) observes \( x(i', b) = 1 \) but not vice versa). It cannot be the case that \( i \) trades at a higher price than \( i' \) because any no buyer who can indentify asset \( i \) as a high-quality asset would prefer to buy from the lower-price markets where \( i' \) is traded. This implies that the function \( p^*(i) \) at which assets trade must be monotonic.

Define \( p^M \) as the result of applying the transformation \( M \) defined in the Appendix on the function \( p^E \). Transformation \( M \), illustrated in Figure 2, transforms the function \( p^E \) into a monotonic function that satisfies the following properties:

1. \( E(i, p^M) \leq W(i) \). This implies that at prices \( p^M(i) \), buyers who recognize assets as good have enough wealth to buy them.

2. \( p^M \) is constant in any interval \( (i, i') \) where \( E(i, p^M) < W(i) \). This implies that easier-
to-recognize high-quality assets do not command a premium over harder-to-recognize assets over the range where the monotonicity constraint binds.

Upper bound

The function $p^M(i)$ could take values above 1 if buyers’ endowments are sufficiently large. Define

$$p^M(i) \equiv \min\{p^M(i), 1\}$$

to take into account that buyers will never demand assets in markets where $p(m) > 1$.

Non-selective buyers

Suppose distressed sellers are supplying asset $i$ in a market $m$ where the clearing algorithm is less-restrictive-first. A buyer who demands assets in this market and imposes $\chi(i) = 1$ (i.e. accept everything) will obtain a representative sample from a pool of at least $\mu(i - \lambda)$ good assets (because if sellers supply asset $i$ they also supply all assets is $(i, \lambda)$) mixed with $\lambda$ bad assets. Therefore such a buyer would obtain a fraction of at least

$$p^N(i) \equiv \frac{\mu(i - \lambda)}{\mu(i - \lambda) + \lambda}$$

\hspace{1cm} \text{(36)}

good assets and would prefer that over not trading as long as $p(m) < p^N(i)$. Hence as long as there is enough total endowment of $t = 1$ goods that some buyers do not trade (a maintained assumption), $p^N(i)$ is a lower bound on the price function. Define $i^N$ as the
highest $i$ for which this bound is binding, i.e.

\[
i^N \equiv \begin{cases} \max \{ i : p^N(i) \geq \bar{p}^M(i) \} & \text{if this set is nonempty} \\ \lambda & \text{otherwise} \end{cases}
\]

Overall, the price function $p^*(i)$ is given by

\[
p^*(i) = \begin{cases} p^N(i^N) & \text{if } i \leq i^N \\ \bar{p}^M(i) & \text{otherwise} \end{cases}
\] (37)

as illustrated in Figure 3.

Buyers can be ranked into three groups of decreasing expertise.

1. The highest-expertise group is defined by $b \geq \frac{1-i^N}{1-\lambda}$. These buyers are able to detect high-quality assets in the markets where the price is sufficiently low that non-selective buyers are also willing to buy, i.e. in the range $(\lambda, i^N)$ in Figure 3. They demand $d = \frac{w(b)}{p^N(i^N)}$ from the market where the price is $p^N(i^N)$ and the clearing algorithm is non-restrictive-first-then-more-restrictive-first, imposing the rule $\chi(i) = \ldots$
\[ \mathbb{I}(i > 1 - b (1 - \lambda)). \]

2. The next group is defined by \( b \in \left( \frac{1 - i^1}{1 - \lambda}, \frac{1 - i^N}{1 - \lambda} \right) \), where \( i^1 \) is defined by

\[
i^1 \equiv \begin{cases} 
\min \{ i : p^* (i) = 1 \} & \text{if this set is nonempty} \\
1 & \text{otherwise}
\end{cases}
\]

These buyers are able to detect high-quality assets in markets where the price is below 1 but too high to attract nonselective buyers, i.e. in the range \((i^N, i^1)\) in Figure 3. They demand \( d = \frac{w(b)}{p(m)} \) from the market where the price is \( p^* (1 - b (1 - \lambda)) \) and the clearing algorithm is more-restrictive-first, imposing the rule \( \chi (i) = x (i, b) \).

3. Finally, buyers with \( b < \frac{1 - i^1}{1 - \lambda} \) can only detect high-quality assets in markets where the price is 1. They are indifferent between:

(a) Buying from markets where \( p (m) = 1 \) and imposing \( \chi (i) = \mathbb{I}(i > 1 - b (1 - \lambda)) \)

(b) Buying from markets where \( p (m) = p^N (i^N) \) and the clearing algorithm is non-restrictive-first, and imposing \( \chi (i) = 1 \)

(c) Not Buying at all

Among the possible patterns of demand they are indifferent between, one that is consistent with equilibrium is:

(a) Buyers in the range \((b^1, \frac{1 - i^1}{1 - \lambda})\), where \( b^1 \) is defined by

\[
\int_{b^1}^{\frac{1 - i^1}{1 - \lambda}} w (b) \, db = \mu (1 - i^1)
\]

demand \( d = w (b) \) from the market where \( p (m) = 1 \) and the algorithm is more-restrictive-first, imposing \( \chi (i) = x (i, b) \). In this way they buy all the assets in \((i^1, 1)\) that distressed sellers want to sell.

(b) Buyers with \( b < b^1 \) demand a total of

\[
\frac{\left[ \mu (i^N - \lambda) - \frac{W_0}{p(i^N)} \right] \left[ \mu (i^N - \lambda) + \lambda \right]}{\mu (i^N - \lambda)}
\]
where
\[
W_0 \equiv \int_{\{b p^* (1-b(1-\lambda))=p^N(i^N)\}} w(b) \, db
\]
from the market where \( p(m) = p^* (i^N) \) and the clearing algorithm is non-restrictive-first-then-more-restrictive-first, imposing \( \chi(i) = 1 \). In this way they demand enough total assets that they end up buying all the assets in the range \((\lambda, i^N)\) that buyers with \( b \geq \frac{1-i^N}{1-\lambda} \) cannot afford.

What makes buyers direct their trades to markets with these specific clearing algorithms? For markets where selective buyers with different degrees of expertise trade, such as the markets where assets in the ranges \([\lambda, i^N]\), \([i^A, i^B]\) and \([i^1, 1]\) in Figure 3 trade, the less expert traders will be more selective, since they are only able to identify a smaller subset of assets as high-quality. Therefore they want to make sure they clear first lest the assets that they want run out; they more-expert, less-selective buyers are indifferent. Hence the appeal of more-restrictive-first. For non-selective buyers, allowing selective buyers to clear before them would worsen the adverse selection, as in the false-positive example. Therefore they want to make sure that they trade in a market where non-selective trades clear first.

The resulting allocation functions and selling probabilities that complete the description of equilibrium are:

Allocation functions:

1. For markets where \( p(m) < p^* (i^N) \): 
   \[
   A(I_0, \chi, m) = \begin{cases} 
   \int_{i \in I_0} \chi(i) \mathbb{1}(i \leq \lambda) \, di 
   & \text{if } \int_{i \in I} \chi(i) \mathbb{1}(i \leq \lambda) \, di > 0 \\
   0 & \text{otherwise}
   \end{cases}
   \]

2. For markets where \( p(m) \in (p^* (i^N), 1) \), the clearing algorithm is more-restrictive first and there exists more than one \( i \) such that \( p^* (i) = p(m) \), such as the markets where assets in the range \([i^A, i^B]\) in Figure 3 trade, then
   
   (a) For acceptance rules such that \( \chi(\lambda) = \mathbb{1}(i \geq g) \) with \( g > \lambda \), the allocation function is
   \[
   A(I_0, \chi, m) = \begin{cases} 
   1 & \text{if } g \in I_0 \\
   0 & \text{otherwise}
   \end{cases}
   \]

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(b) For any other acceptance rule

\[ A(I_0, \chi, m) = \begin{cases} \int_{i \in I_0} \chi(i) \mathbb{I}(i \leq \lambda) di & \text{if } \int_{i \in I} \chi(i) \mathbb{I}(i \leq \lambda) di > 0 \\ 0 & \text{otherwise} \end{cases} \]

3. For all other markets:

Define \( i(m) \) by

\[ i(m) \equiv \max \{ i : p^*(i) \leq p(m) \} \]

then

\[ A(I_0, \chi, m) = \begin{cases} \int_{i \in I_0} \chi(i) \mathbb{I}(i \leq \lambda) + \mu \mathbb{I}(i > \lambda) di & \text{if } \int_{i \in I} \chi(i) \mathbb{I}(i \leq \lambda) + \mu \mathbb{I}(i > \lambda) di > 0 \\ 0 & \text{otherwise} \end{cases} \]

Probability of selling:

1. In markets where \( p(m) = p^*(i) \) for a single \( i > i^N \) and the clearing algorithm is less-restrictive-first:

\[ \eta(j) = \begin{cases} 1 & \text{if } j \geq i \\ 0 & \text{otherwise} \end{cases} \]

2. In markets where there exist more than one \( i > i^N \) such that \( p^*(i) = p(m) \) and the clearing algorithm is more-restrictive-first:

\[ \eta(j) = \begin{cases} 1 & \text{if } j \geq \min \{ i : p^*(i) = p(m) \} \\ 0 & \text{otherwise} \end{cases} \]

3. In the market where \( p(m) = p^*(i^N) \) and the clearing algorithm is non-restrictive-first-then-more-restrictive-first:

\[ \eta(j) = \begin{cases} 1 - \frac{1}{p(i^N) \mu(i^N - \lambda)} & \text{if } j \geq \lambda \\ 0 & \text{otherwise} \end{cases} \]

Two features of the equilibrium are worth noting. First, as in the false positives example, all high-quality assets held by distressed Sellers end up being sold while lemons face rationing. Second, among the high-quality assets, those that are easily
seen to be of high quality trade at higher prices. There is therefore a premium for transparency in asset prices.

5 Fire Sales

The term “fire sales” is sometimes used to refer to situations where a trader’s urgency for funds leads him to sell assets at prices that are far below their usual price. In the false positives example, something like a fire-sale would arise if the price \( p^* \) at which all trades take place were decreasing in \( \mu \), so that when more sellers are distressed, asset prices fall. From equations (21) and (22), one can compute:

\[
\frac{dp^*}{d\mu} = \frac{\lambda (p^*)^2}{\mu^2 (1 - \lambda)} \left[ 1 - b^* - \mu \frac{\lambda (1 - b^*)}{\mu^2 (1 - \lambda)} p^* + \frac{1}{p^*} \int_{b^*}^{1} \frac{(1 - \lambda)w(b)}{\lambda (1 - b) + \mu (1 - \lambda)} \, db \right]
\]

(39)

Inspection of equation (39) leads to the following result, which relates the possibility of fire sales to the joint distribution of wealth and expertise.

**Proposition 3.**

1. \( \lim_{w(b^*) \to \infty} \frac{dp^*}{d\mu} = \frac{\lambda (p^*)^2}{\mu^2 (1 - \lambda)} (1 - b^*) > 0 \)

2. If \( w(b^*) = 0 \), then \( \frac{dp^*}{d\mu} = -\frac{1}{\mu} \int_{b^*}^{1} \frac{(1 - \lambda)w(b)}{\lambda (1 - b) + \mu (1 - \lambda)} \, db < 0 \)

3. If \( w(b) = w \) for all \( b \), then \( \frac{dp^*}{d\mu} = 0 \)

In general, there are two opposing effects when more sellers become distressed. On the one hand, distressed sellers are the only ones who are willing to sell high-quality assets. Other things being equal, this should improve the pool of assets being sold and thus lead to higher, not lower, prices. This is the effect emphasized by Uhlig (2010). On the other hand, more distressed sellers mean that more assets are being offered for sale. Given that the more expert buyers have exhausted their wealth, it is necessary to resort to less expert buyers. These less expert buyers are aware that they are less clever at filtering out the lemons so, other things being equal, they will make up for this by only entering the market if prices are lower.

Proposition 3 shows that which effect dominates (locally) depends on the density of wealth at the equilibrium cutoff level of expertise. If \( w(b^*) \) is high, this means that a large amount of wealth would enter the market if the cutoff level of expertise was lowered slightly. In this case, the direct selection effect dominates and prices rise, meaning there are no fire
sales. Instead when \( w(b) \) is low, cutoff level of expertise needs to fall a lot in order to attract sufficient wealth to buy the extra units supplied. In this case, the changing-threshold effect dominates and prices fall. Interestingly, for the special case where wealth is evenly distributed across all levels of expertise, the price is the same for any \( \mu \), so both effects cancel out.

The model is useful for exploring the relationship among other theories of fire-sales in the existing literature. One class of theories (Shleifer and Vishny 1992, 1997, Kiyotaki and Moore 1997) emphasizes that the marginal buyer of an asset can be a second-best user with diminishing marginal product. If first-best users need to sell more units, asset prices will fall along the marginal-product curve of second-best users. Evidence consistent with this pattern has been documented by Pulvino (1998) in the market for used aircraft. But financial assets are not aircraft. The holder of a financial asset does not need to use his expertise and/or complementary assets in order to extract value from it, so the idea of a second-best user does not naturally fit fire-sales in financial markets. However, the current model illustrates that expertise may be relevant in the trade itself, and moving along a gradient of expertise can induce to fire-sale effects.

A second class of theories (Allen and Gale 1994, 1998, Acharya and Yorulmazer 2008) relies on the notion of cash-in-the-market pricing. There is a given amount of purchasing-power available, so if more units are to be sold, the price must fall. But these class of models typically leave unanswered the question of why buyers with deep pockets (for instance, rich individuals) stay out of the market. The current model provides an explanation for buyers staying out of the market: even though there are good deals available for those who have expertise, those who do not have expertise are rationally worried that they are not able to select the deals among all the assets on offer. In other words, given their expertise, buying from this market does not provide excess returns, even though it does for experts.

The false-negatives example is close to this cash-in-the-market logic. Effectively (before adjusting for monotonicity, etc.), the price at which an asset trades is determined by the endowment of the buyers who are able to realize that this asset is of high quality. As equation (35) shows, an increase in \( \mu \) translates directly into a fall in \( p^E \). If the lower bound \( p^N \) is non-binding, then the fire-sale effect is unambiguous: prices of all assets must fall with \( \mu \), as illustrated by the first example in Figure 4. Interestingly, the price differential between the easiest to detect high-quality assets and the rest of the assets widens increasing the transparency premium, a form of flight to quality. However, by equation (36), \( p^N (i) \) is increasing in \( \mu \) due to the improved-overall-pool effect. Therefore for cases where the lower
bound $p^N$ binds, an increase in $\mu$ will be accompanied by a simultaneous fall in prices for assets bought only by experts and an increase in the prices for assets that are bought by non-selective buyers, as illustrated by the second example in Figure 4.

References


Appendix

[Note: the proofs use notation from a previous draft and need to be updated]

Proof of Proposition 1

1. Seller optimization.

Selling probabilities (30) imply that sellers will be able to sell all assets \( i \in [\lambda, 1] \) and a fraction \( \eta(i, m^*) < 1 \) of assets \( i \in (\lambda b^*, \lambda] \) in market \( m^* \), and nothing else. A necessary and sufficient condition for a solution to program (1) is that in market \( m^* \) distressed sellers supply the maximum possible amount of assets \( i \in [\lambda b^*, 1] \) and non-distressed sellers supply the maximum possible amount of assets \( i \in [\lambda b^*, \lambda] \) and no assets \( i \in (\lambda, 1] \). Imposing robustness then implies (23). The level of consumption (24) then follows from the budget constraint.

2. Supply.

(25) follows from aggregating (23) over all sellers.

3. Buyer optimization.

The optimality of acceptance rule (??) is immediate. The only other possible rule would be to accept all assets, which would result in a higher fraction of lemons.

Define

\[
\tau(m, b) \equiv \begin{cases} \frac{\int q(i)a(i,\chi(\cdot,m,b),m)di}{p(m)A(I,\chi(\cdot,m,b),m)} & \text{if } A(I,\chi(\cdot,m,b),m) > 0 \\ 0 & \text{otherwise} \end{cases}
\] (40)
and let

\[ \tau_{\text{max}} (b) \equiv \max_m \tau (m, b) \]

\[ M_{\text{max}} (b) \equiv \arg \max_m \tau (m, b) \]

Buyer optimization implies that

(a) if \( \tau_{\text{max}} (b) < 1 \), then

\[ d (m, b) = 0 \quad \forall m \]

(b) if \( \tau_{\text{max}} (b) > 1 \), then

\[ \int_{m \in M_{\text{max}} (b)} \frac{d (m, b)}{p (m)} dm = w (b) \]

and

\[ d (m, b) = 0 \quad \forall m \notin M_{\text{max}} (b) \]

Using equation (29),

\[
\tau (m, b) = \begin{cases} 
\frac{1}{p (m)} \frac{\mu (1 - \lambda)}{\lambda (1 - b) + \mu (1 - \lambda)} & \text{if } p (m) \geq p^* \\
0 & \text{otherwise}
\end{cases}
\]

so for all buyers

\[ \tau_{\text{max}} (b) = \frac{1}{p^*} \frac{\mu (1 - \lambda)}{\lambda (1 - b) + \mu (1 - \lambda)} \]

and

\[ M_{\text{max}} (b) = \{ m \in M \text{ s.t. } p (m) = p^* \} \]

Together with condition (22), this implies that types \( b < b^* \) have \( \tau^* (b) < 1 \) so \( d (m, b) = 0 \) is optimal for them and types \( b > b^* \) spend their entire endowment on markets where \( p (m) = p^* \); within these they are indifferent so buying just in market \( m^* \) is optimal, which implies (26). The level of consumption (27) then follows from the budget constraint.

4. Demand.

(28) follows from aggregating (26) over all buyers.

5. Allocation function.
In all markets except $m^*$ demand is zero, so for any clearing algorithm, equation (19) implies

$$S^n (i, m) = S^{n-1} (i, m) \quad \forall i, n, m \neq m^*$$

for all clearing algorithms. By induction, this implies that

$$a^n (i, \chi, m) = a^1 (i, \chi, m) \quad \forall i, n, m \neq m^*$$

and therefore

$$a(i, \chi, m) = a^1 (i, \chi, m) = \begin{cases} \chi (i) S (i) \omega (\chi, n) dD (\chi) \\ 0 \end{cases} \quad \forall i, m \neq m^* \quad (41)$$

In market $m^*$, condition (21) together with (25) and (28), implies that $\omega (\chi, 1) = 1 \quad \forall \chi$, so (41) holds as well. (41) and (25) together imply (29).


(30) follows from direct application of formula (20).

**Proof of Proposition 2**

I first establish a series of preliminary results.

**Lemma 1.** Consider an arbitrary market $m$ and suppose $i < j$. Then in any equilibrium the residual supplies after $n$ rounds of clearing in market $m$ satisfy

$$\frac{S^n (j)}{S^{n-1} (j)} \leq \frac{S^n (i)}{S^{n-1} (i)}$$

**Proof.** By equation (19) and (16)

$$\frac{S^n (i)}{S^{n-1} (i)} = 1 - \frac{\int \chi (i) S^{n-1} (i) \omega (\chi, n) dD (\chi)}{S^{n-1} (i)}$$

$$= 1 - \int \frac{\chi (i) \omega (\chi, n) dD (\chi)}{S^{n-1} (i) \chi (i) S^{n-1} (i) dD (\chi)}$$

Therefore

$$\frac{S^n (j)}{S^{n-1} (j)} - \frac{S^n (i)}{S^{n-1} (i)} = \int \frac{\chi (i) - \chi (j) S (k) S^{n-1} (k) dD (\chi)}{\chi (k) S^{n-1} (k) dD (\chi)} \quad (42)$$
Given their information, each buyer has three feasible acceptance rules:

\[
\chi(i) = x(i, b) \\
\chi(i) = 1 \\
\chi(i) = 1 - x(i, b)
\]

The last rule will never be used in equilibrium, because it implies accepting only assets that are known to be lemons. This means that all buyers will either accept all assets or accept only those for which they observe \(x = 1\). Under either of these rules, \(\chi(i) \leq \chi(j)\), so the right hand side of (42) must be nonpositive.

Lemma 1 states that as the rounds of clearing progress, high-quality assets leave the pool at a greater rate than lemons. This implies that, other things being equal, buyers prefer to trade in markets where their trades will clear sooner rather than later. In fact, the following result implies that, given an acceptance rule, the best terms of trade that a buyer can obtain in a market are if his trades clear in the first round so that he obtains a representative sample of the acceptable assets supplied to that market.

**Lemma 2.** Let

\[
\tau(\chi, m) \equiv \left\{ \begin{array}{ll}
\frac{\int q(i) a(i, \chi, m) di}{p(m) A(I, \chi, m)} & \text{if } A(I, \chi, m) > 0 \\
0 & \text{otherwise}
\end{array} \right.
\]

\[
\tau_{\text{max}}(\chi, m) \equiv \left\{ \begin{array}{ll}
\frac{1}{p(m)} \frac{\int q(i) \chi(i) S(i, m) di}{\int \chi(i) S(i, m) di} & \text{if } \int \chi(i) S(i, m) di > 0 \\
0 & \text{otherwise}
\end{array} \right.
\]

In any equilibrium,

\[
\tau(\chi, m) \leq \tau_{\text{max}}(\chi, m) \quad \forall \chi, m
\]

with equality if \(\omega(\chi, 1) = 1\).

**Proof.** If if \(A(I, \chi, m) = 0\), then the result is immediate. Otherwise, using equation (17) we can write:

\[
\tau(\chi, m) = \frac{\int q(i) [\sum_{n=1}^{\infty} \omega(\chi, n) a^n(i, \chi, m)] di}{p(m) \sum_{n=1}^{\infty} \omega(\chi, n) A^n(I, \chi, m)}
\]
For any \( n \),
\[
A^n(I, \chi, m) = \begin{cases} 
1 & \text{if } \int \chi(i) S^{n-1}(i, m) \, di > 0 \\
0 & \text{otherwise}
\end{cases}
\]

Letting \( \bar{n} \) be the highest value of \( n \) such that \( A^n(I, \chi, m) = 1 \), we can rewrite (43) as:
\[
\tau(\chi, m) = \frac{1}{p(m)} \int q(i) \left[ \sum_{n=1}^{\bar{n}} \frac{\omega(\chi, n)}{\sum_{n=1}^{\bar{n}} \omega(\chi, n)} a^n(i, \chi, m) \right] \, di
\]
\[
= \frac{1}{p(m)} \sum_{n=1}^{\bar{n}} \left[ \frac{\omega(\chi, n)}{\sum_{n=1}^{\bar{n}} \omega(\chi, n)} \int_{\lambda}^{1} a^n(i, \chi, m) \, di \right]
\]
\[
= \frac{1}{p(m)} \sum_{n=1}^{\bar{n}} \left[ \frac{\omega(\chi, n)}{\sum_{n=1}^{\bar{n}} \omega(\chi, n)} \int_{\lambda}^{1} \chi(i) S^{n-1}(i, m) \, di \right]
\]

Lemma 1 implies that the term
\[
\frac{\int_{\lambda}^{1} \chi(i) S^{n-1}(i, m) \, di}{\int \chi(i) S^{n-1}(i, m) \, di}
\]
is weakly decreasing in \( n \) and therefore the right hand side is maximized at \( \omega(\chi, 1) = 1 \), where \( \tau(\chi, m) = \tau^{\max}(\chi, m) \).

Lemma 2 places an upper bound on the terms of trade than can be obtained in a market given an acceptance rule. It is also possible to compute an upper bound for a given buyer who can choose among all his feasible acceptance rules.

**Lemma 3.** Let
\[
\tau(b, m) \equiv \max_{\chi \text{ feasible for } b} \tau(b, m)
\]

In any equilibrium
\[
\tau(b, m) \leq \tau^{\max}(\chi_b, m)
\]
where \( \chi_b \) is defined by equation (??)

**Proof.** By definition, there is some acceptance rule \( \chi \) that is feasible for \( b \) that attains the
maximum. Therefore

\[
\tau (b, m) = \tau (\chi, m) \\
\leq \tau_{\text{max}} (\chi, m) \\
\leq \tau_{\text{max}} (\chi_b, m)
\]

where the first inequality follows from Lemma 2 and the second holds because \( \chi_b \) maximizes \( \frac{\int q(i) \chi(i) S(i, m) di}{\int \chi(i) S(i, m) di} \) among feasible acceptance rules. \( \square \)

Knowing the upper bound on the terms of trade a buyer can obtain in a given market \( m \) is useful because if one can find a market \( m' \) where a buyer can obtain better terms of trade than \( \tau (b, m) \), then implies that buyer \( b \) will not buy from market \( m \). Using this fact, the following result establishes that in equilibrium all trades take place at the same price.

**Lemma 4.** In equilibrium there is trade at only one price

**Proof.** Assume the contrary, suppose there is trade at \( p_H \) and \( p_L \). If buyers are willing to buy in a market \( m \) where \( p = p_L \), then it means that distressed sellers are willing to sell some assets \( i > \lambda \) at a price \( p_L \). In a robust solution to problem (1), this means that if \( v < \mu \) and \( i > \lambda \), then \( s(i, m, v) = 1 \), i.e. all distressed sellers supply the maximum amount of any asset \( i > \lambda \) in all markets where \( p > p_L \). This implies that in any market \( m \) where \( p(m) \in (p_L, p_H] \),

\[
S(i, m) = \begin{cases} 
\mu & \text{if } i > \lambda \\
1 & \text{if } i \leq \lambda 
\end{cases}
\]

Fix any \( b \) and take a market \( m \) such that acceptance rule \( \chi_b \) is cleared in the first round and \( p(m) \in (p_L, p_H) \). In such a market

\[
\tau (\chi_b, m) = \tau_{\text{max}} (\chi, m) \\
> \tau (b, m') \text{ for any } m' \text{ such that } p(m') = p_H
\]

The first equality follows from Lemma 2 and the inequality follows from Lemma 3, the fact that supply is the same in markets \( m \) and \( m' \) and the fact that \( p(m) < p_H \). Therefore buyer \( b \) will not buy from any market where the price is \( p_H \). Since this applies to all \( b \), there can be no trade at \( p_H \). \( \square \)

Next I show that in any equilibrium where there is trade at price \( p^* \), sellers are able to sell all their high-quality assets.
Lemma 5. Define

\[ \eta(i, p) \equiv \int_{m:p(m)=p} \eta(i, m) \, dm \]

In any equilibrium where there is trade at \( p^* \), \( \eta(i, p^*) = 1 \) for all \( i > \lambda \).

Proof. Assume the contrary. Since no feasible acceptance rule for any buyers distinguish between different high quality assets, \( \eta(i, p^*) < 1 \) for some \( i > \lambda \) implies \( \eta(i, p^*) < 1 \) for all \( i > \lambda \). By Lemma 4, there is no trade at any other price, which means that a fraction of high-quality assets held by distressed sellers remains unsold. Therefore in a robust solution to program (1), distressed sellers will supply \( s(i, m, v) = 1 \) for all \( m \) for \( i > \lambda \), which implies \( S(i, m) = \mu \) for \( i > \lambda \) and \( p(m) \leq 1 \). Now take any buyer \( b \) and any market \( m' \) with \( p(m') < p^* \) which clears acceptance rule \( \chi_b \) in the first round. The terms of trade for buying in market \( m' \) are

\[
\tau(b, m') = \frac{1}{p(m')} \int_{b \lambda}^{\lambda} S(i, m') \, di + \mu (1 - \lambda) \]
\[
\geq \frac{1}{p^*} \int_{b \lambda}^{\lambda} S(i, m) \, di + \mu (1 - \lambda) \]
\[
> \frac{1}{p^*} \int_{b \lambda}^{\lambda} S(i, m) \, di + \mu (1 - \lambda) \]
\[
= \tau^{\max}(\chi_b, m) \]
\[
\geq \tau(b, m) \quad \text{for any} \ m \ \text{s.t.} \ p(m) = p^* \]

where the first inequality follows from the fact that robust solutions to program (1) imply \( S(i, m) \leq S(i, m') \) if \( p(m) < p(m') \), for all \( i \); the second from \( p(m') < p^* \) and the last from Lemma 3. This implies that buyer \( b \) prefers to buy from market \( m' \) rather than from any market where \( p(m) = p^* \). Since this is true for all \( b \), it contradicts the assumption that there is trade at \( p^* \). \( \square \)

Using Lemmas (1)-(5) Proposition 2 follows by the following argument.

Let \( p^* \) and \( b^* \) be defined by equations (21) and (22).

Suppose there was an equilibrium where trade took place at price \( p_H > p^* \). For any market \( m \) where \( p = p_H \), supply satisfies \( S(i, m) = 1 \) for \( i \leq \lambda \) and \( S(i, m) \leq \mu \) for \( i > \lambda \). Therefore

\[
\tau(b, m) \leq \frac{1}{p_H} \frac{\mu (1 - \lambda)}{\lambda (1 - b) + \mu (1 - \lambda)} \]

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so the lowest \( b \) that may be willing to buy is \( b_H \), defined by

\[
\frac{1}{p_H} \frac{\mu (1 - \lambda)}{\lambda (1 - b_H) + \mu (1 - \lambda)} = 1
\]

Equation (22) implies \( b_H > b^* \).

The maximum measure of high-quality assets that buyer \( b \) can get is

\[
\frac{\mu (1 - \lambda)}{\lambda (1 - b) + \mu (1 - \lambda)} \frac{w(b)}{p_H}
\]

which means that the probability of selling a high-quality asset is at most

\[
\int_{b_H}^{1} \frac{1}{\lambda (1 - b) + \mu (1 - \lambda)} \frac{w(b)}{p_H} \, db
\]

but because \( p_H > p^* \) and \( b_H > b^* \), equation (21) implies this is less than 1, which contradicts Lemma (5).

Suppose now that there was an equilibrium where trade took place at \( p_L < p^* \). For any market \( m \) where \( p(m) \in (p_L, 1) \), supply satisfies \( S(i, m) = 1 \) for \( i \leq \lambda \) and \( S(i, m) = \mu \) for \( i > \lambda \). This is true in particular for markets where acceptance rule \( \chi_b \) is cleared in the first round, so for any \( p \in (p_L, 1) \), buyer \( b \) can find a market where the terms of trade are

\[
\tau = \frac{1}{p} \frac{\mu (1 - \lambda)}{\lambda (1 - b) + \mu (1 - \lambda)}
\]

Therefore, in order for trade to only take place at \( p_L \), it must be that all buyers with \( b > b_L \) obtain terms of trade of at least

\[
\frac{1}{p_L} \frac{\mu (1 - \lambda)}{\lambda (1 - b) + \mu (1 - \lambda)}
\]

in markets where \( p = p_L \), where \( b_L \) is defined by

\[
\frac{1}{p_L} \frac{\mu (1 - \lambda)}{\lambda (1 - b_L) + \mu (1 - \lambda)} = 1
\]

For this to happen, it must be that all traders with \( b > b_L \) have their trades cleared in the
first round. This would demand a total of

$$\int_{b_L}^{1} \frac{\mu (1 - \lambda)}{\lambda (1 - b) + \mu (1 - \lambda)} \frac{w(b)}{p_L} db$$

high-quality assets, but because $p_L < p^*$ and $b_L < b^*$, equation (21) implies this is more than $\mu (1 - \lambda)$, which is the total supply of high-quality assets, so not all trades can clear in the first round.

This means that in any equilibrium, all trades take place at $p = p^*$ and high-quality assets can be sold for sure. The rest of the equilibrium objects follow immediately.

**Monotonicity transformation**

For any continuous function $p(i) : [a, 1] \rightarrow \mathbb{R}^+$, define the transformation $M$, which converts the function $p$ into the monotonic function $p^M$, as the result of the following procedure:

1. For each $i$, define

   $$\Delta (i) \equiv \max_{i'} \int_i^{i'} [p(i) - p(j)] dj$$

   s.t. $i \geq i'$

   Notice that

   (a) $\Delta (i) \geq 0$ since $i' = i$ is always a possible choice

   i. if $p$ is monotonically increasing to the right of any given point $i$, then $\Delta (i) = 0$

   ii. $\Delta$ is continuous

2. Define the set:

   $$P = \{ i \in [a, 1] : \Delta (i) > 0 \}$$

3. Continuity of $\Delta$ implies that the set $P$ will consist of a series of disjoint intervals. For each interval $k$, define $i_k$ as the lower bound of the interval.

4. There are two possibilities:
(a) \( \Delta (i_k) = 0 \)

(b) \( \Delta (i_k) > 0 \). This is only possible if \( i_k = a \); otherwise by continuity of \( \Delta \), \( i_k \) could not be the lower bound of the interval.

5. For every \( i_k \) define

(a) In case \( \Delta (i_k) = 0 \):

\[
p_k = p(i_k)
\]

and \( i'_k \) as the highest solution \( i' \) to:

\[
\int_{i_k}^{i'} (p_k - p(i)) \, di = 0
\]

Note that \( i' = i_k \) is always a solution, which is why I need to specify that I am looking for the highest solution. Furthermore, I know that a solution greater than \( i_k \) must exist because \( \Delta (i) > 0 \) in a neighbourhood to the right of \( i_k \) and \( \Delta (i) \) is continuous.

(b) In case \( \Delta (i_k) > 0 \):

i. If there is a solution \( p_k, i'_k \) to the pair of equations:

\[
\int_{i_k}^{i'} (p_k - p(i)) \, di = 0
\]

\[
p_k = p(i')
\]

then \( p_k, i'_k \) is the lowest solution

ii. Otherwise, \( i'_k = 1 \) and \( p_k \) is the solution to

\[
\int_{i_k}^{1} (p_k - p(i)) \, di = 0
\]

6. \( p^M \) results from replacing \( p(i) \) with

\[
p^M(i) = p_k \quad \text{for all } i \in [i_k, i'_k], \text{ for all } k
\]