Asymptotically Optimal Inventory Control for Assemble-to-Order Systems with Identical Lead Times

Martin I. Reiman
Alcatel-Lucent Bell Labs, Murray Hill, NJ 07974, marty@research.bell-labs.com

Qiong Wang
Industrial and Enterprise Systems Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, qwang04@illinois.edu

May 22, 2013

Optimizing multi-product assemble-to-order (ATO) inventory systems is a long-standing difficult problem. We consider ATO systems with identical component lead times and a general ‘bill of materials’. We use a related two-stage stochastic program (SP) to set a lower bound on the average inventory cost and develop inventory control policies for the dynamic ATO system using this SP. We apply the first-stage SP optimal solution to specify a base-stock replenishment policy, and the second-stage SP recourse linear program to make allocation decisions. We prove that our policies are asymptotically optimal on the diffusion scale, so the percentage gap between the average cost from its lower bound diminishes to zero as the lead time grows.

Key words: Assemble-to-Order, Inventory Management, Stochastic Linear Program, Stochastic Control, Asymptotic Optimality, Diffusions Scale.

1. Introduction

The assemble-to-order (ATO) inventory system, where multiple components are used to produce multiple products, is a classical and much studied model in inventory theory. Demand for the products is random, while components are obtained from an uncapacitated supplier after a deterministic (component dependent) lead time. The components that are used in each product are specified in the ‘bill of materials’ (BOM) matrix. A key assumption is that assembly is instantaneous, so that inventory is kept at the component rather than the product level. Any unfulfilled demand is backlogged and each backlogged product incurs a product-dependent constant backlog cost per unit of time. Each component in inventory incurs a component-specific inventory holding cost per unit of time. The objective is to find a control policy, which can conveniently be described via replenishment (ordering from supplier) and allocation (assembling to meet demand) policies, that minimizes the long-run average expected total cost.
As described briefly below, it has long been recognized that for systems with multiple products, exact optimization is difficult, so most prior work has focused on optimizing within particular policy classes and has yielded sub-optimal policies in general.

In this paper we continue the development and analysis, begun in Doğru et al. [4], of an approach that uses the solution to a particular stochastic program (SP) as the basis for constructing control policies for ATO inventory systems. This paper, similarly to [4], restricts attention to the situation where all components have the same lead time, \( L \). The SP that we use is a two-stage stochastic linear program with complete recourse that is a particular relaxation of the SP that arises in the context of a one-period ATO model, cf. Song and Zipkin [26]. The first stage decision is how many components to order. The second stage decision, taken after demand has been realized, is how much of each product to assemble. The relaxation, introduced in [4], was shown there to provide a lower bound on the achievable cost in the dynamic ATO system under any feasible policy. It was suggested in [4] that a base-stock policy be used for replenishment, with the optimal first stage solution of the SP used as base-stock levels. In addition, it was shown in [4] that, for a simple ATO structure known as the W model (where 3 components are used to assemble 2 products - see Figure 1), the solution of the second stage (recourse) LP motivates, in a natural manner, a simple priority policy for allocation. While it was noted in [4] that there should be a way, in principal, to translate the solution of the recourse LP into an allocation policy, the development of this translation was left for future work. One contribution of this paper is to fill that gap. In particular, we display an LP, which is equivalent to the recourse LP, that provides a ‘backlog target’, and introduce an associated ‘allocation principle’ that allows the identification of allocation policies that properly track the backlog target. It was noted in [4] that the numerical experiments carried out on the W system there were consistent with asymptotic optimality of our policy as the lead time grows, and this asymptotic optimality was conjectured there. The main contribution of this paper is to show that this asymptotic optimality holds, not only for the W model, but for any ATO system where all components have identical lead times.

The rest of this paper is organized as follows. In the remainder of this section we describe related work and describe our ‘SP-based approach’ as a four step process. We define the inventory control problem in Section 2. We discuss the use of the SP to set a lower bound on the inventory cost and to develop inventory policies in Section 3. We carry out the asymptotic analysis in Section 4. Section 5 contains most of the proofs. A clear future goal is to extend the results of this paper to systems with non-identical lead times. To this end, we discuss related challenges in Section 6.

1.1. Related Work

There is a tremendous gap between optimizing inventory control in one-product ATO systems and doing so in multi-product ATO systems. The former is a completely solved problem. For
systems with one component, Karlin and Scarf [14] showed that the optimal control is a base-stock policy with the base-stock level determined by a newsvendor model. For systems with multiple components with different lead times, Rosling [23] found the optimal control policy by drawing an analogy with the optimal control of multi-echelon systems.

For systems with multiple products, the search for an optimal policy is hindered by the allocation decision that distributes components to demands for different products. To be optimal, the allocation policy may in principle depend on the entire status of the pipeline, which includes not only the total numbers of ordered components that are yet to arrive, but also the exact times when such replenishment orders were placed. Tracking this information would require a tremendous state space that grows exponentially with the lead time. Except in a few rare cases in which special parameter values along with simple structure render the allocation decision inconsequential (Doğru et al. [4], Lu et al. [18], Lu et al. [15], and Reiman and Wang [21]), currently there appears to be no computationally feasible way to obtain an optimal policy.

As is discussed in [4], where a more detailed discussion of prior work is given, to various degrees most existing approaches rely on a First-In-First-Out (FIFO) scheme to simplify allocation decisions (see e.g., Lu and Song [16], Lu et al. [17], for the continuous review cases, and Agrawal and Cohen [1], Akçay and Xu [2], Hausman et al. [12], and Zhang [27] for discrete review cases.) Among these papers it is worth singling out [16], which finds the optimal base-stock levels for FIFO allocation with an arbitrary BOM. There are a few departures from FIFO in recent development. For instance, ‘no hold-back’ policies proposed in Lu et al. [18] require that all demands should be served when components are available, and hence violate the definition of FIFO, which holds components for demands that arrive earlier regardless of whether they can be served. However, optimization over this class of policies, including the replenishment decision, is not considered in [18].

Another stream of related work involves asymptotic analysis. Plambeck and Ward [20] consider an ATO production/inventory system. Their model differs from ours in that component production is subject to capacity constraints. They consider a broader optimization problem where product pricing, production capacity and inventory management are all subject to control. Of specific interest to us here is their inventory management, which involves allocation of components to products. They provide an asymptotically optimal (on diffusion scale) allocation policy for minimizing total discounted cost in the so-called ‘high-volume’ asymptotic regime. Their policy uses a ‘Big-Step’ (cf. Harrison [8]) inspired discrete review policy. Three points about this paper relative to [20] are worth mentioning: (1) We consider long run average rather than discounted costs; (2) We use a continuous review policy; and (3) Our results also hold in the high volume limit and can be translated to that regime by holding $L$ fixed and letting the demand arrival rate grow.
The discrete review approach used in [20] was applied by Lu et al. [15] to a particular ATO system. They considered the so-called N model (2 components and 2 products) where the components are allowed to have different (deterministic) lead times. They obtained an asymptotically optimal discrete review policy for minimizing total discounted cost in the high volume asymptotic regime.

Finally, Goldberg et al. [5] showed that a very simple control policy (order a fixed amount in each period) is asymptotically optimal for a lost sales model as the lead time grows with all other parameters held fixed.

1.2. The SP-based Approach

The approach taken in this paper can be summarized in the following four-steps:

1. introduce a two-stage SP whose optimal solution is a lower bound on the average inventory cost that can be achieved by any feasible inventory policy;
2. solve the SP;
3. use the SP optimal solution to design a dynamic control policy for the associated ATO system; and
4. prove that the policy developed in step 3 is asymptotically optimal on the diffusion scale.

As we mentioned above, the SP that we need to solve is a two-stage stochastic linear program with complete recourse. Hence while an SP can in general be difficult to solve, ours is one of the easiest types, and can be handled by standard methods such as Sample Average Approximation (SAA), cf. Shapiro et al. [25]. In this sense, our approach replaces an impossibly complicated dynamic control problem by a solvable optimization problem.

Our approach is similar to the ‘four-step’ method introduced in Harrison [7] and used repeatedly since then, e.g. in Harrison [8], Harrison and López [9], Harrison and Wein [11], Plambeck and Ward [20], and many others. In that approach the original problem involves control of a stochastic processing network and the (asymptotic) lower bound is given by a Brownian control problem.

It is worthwhile to describe the contributions of this paper in terms of the four steps above. We resort to the use of the lower bound SP provided in [4] for Step 1. The SP is stated in terms of an infimum, and this infimum may not be attained. A transformed SP, with a minimum that is attained, was provided in [4] for the W system. We provide, in Theorem 2, a transformed SP that works for all BOMs. This result makes solution of the SP possible, and hence fits in Step 2. As mentioned above, we provide both a replenishment and allocation policy based on the SP solution. This is Step 3. Finally, our proof of asymptotic optimality is Step 4.

Although from a superficial point of view it appears that all four steps have been completed for the case with identical lead times, it is worth pointing out that the application of this approach to real ‘industrial scale’ problems has not yet been undertaken. In addition, the case with non-identical
lead times remains. A lower bound SP was provided for this case in Reiman and Wang [21], where a replenishment policy was also provided. Some of the challenges involved in proving asymptotic optimality in this case are described in Section 6.

2. The Inventory Control Problem

We consider an ATO system that has $m$ products and $n$ components. The bill of materials is given by the matrix $A$, of which element $a_{ji}$ represents the amount of component $j$ ($1 \leq j \leq n$) needed to assemble one unit of product $i$ ($1 \leq i \leq m$), and the $j$th column, $A_j$ ($1 \leq j \leq n$), gives the use of component $j$ by all products. We assume that $a_{ji}$ ($1 \leq i \leq m, 1 \leq j \leq n$) are non-negative integers. (We can thus handle any rational quantities by appropriate definition of a component.) We denote by $\bar{a}$ the largest element of $A$ and $a$ the smallest non-zero element. As alluded to in the Introduction, we assume that all components have the same replenishment lead time $L$. Although this paper focuses on continuous-review models, our analysis can be extended to periodic-review models with little effort.

There are three semi-infinite time intervals that arise in our model and analysis. Demand for products arrives over the interval $[0, \infty)$. Orders for components can be placed over the interval $[-L, \infty)$. Finally, we do our cost accounting over the interval $[L, \infty)$, which we call our optimization horizon. Demand is modeled by the vector process $D = \{D(t), t \geq 0\}$, where

$$D(t) = (D_1(t), \ldots, D_m(t)), \quad t \geq 0,$$

$D_i(t)$ is the amount of demand for product $i$ ($1 \leq i \leq n$) that arrives within the interval $[0, t]$, and $D(0^-) = 0$. (All sample paths are taken to be right continuous.) We assume that $D$ is compound Poisson: the number of orders is a Poisson process $\Lambda = \{\Lambda(t), t \geq 0\}$ with $E[\Lambda(1)] = \lambda$, and there is an associated i.i.d. sequence of random vectors that give order sizes. A generic element of this sequence denoted by $S = (S_1, \ldots, S_m)$, where $S_i$ is the order size for product $i$ ($1 \leq i \leq m$). Although the order size vectors are independent, the components $S_i$ ($1 \leq i \leq m$) can be dependent. We assume that $S$ has a finite $(2 + \delta)$ moment, i.e.,

$$\eta_i \equiv E[S_i^{2+\delta}] < \infty, \quad 1 \leq i \leq m,$$

where $\delta > 0$ can be arbitrary small. Thus demands arriving per unit time have finite means

$$\mu = (\mu_1, \ldots, \mu_m) \quad \text{where} \quad \mu_i = \lambda E[S_i], \quad 1 \leq i \leq m,$$

and a finite covariance matrix $\Sigma$. We denote the variances by $\sigma_{ii}$, $1 \leq i \leq m$. 

The demand that arrives at a particular time \( t \) (if any) is denoted by
\[
d(t) \equiv D(t) - D(t^-), \quad t \geq 0,
\]
while demand that arrives between two distinct time points is denoted by
\[
D(t_1, t_2) \equiv D(t_2) - D(t_1), \quad t_2 > t_1 \geq 0.
\]
With a slight abuse of notation, let
\[
D(t) \equiv D(t - L, t), \quad t \geq L,
\]
denote demand that arrives within the lead time immediately before time \( t \). Since the arrival process is compound Poisson, \( D(t) \) have the same distribution for all \( t \geq L \). Let \( D = (D_1, ..., D_m) \) denote a random vector that has this distribution. We refer to \( D \) as the lead time demand, and note that
\[
E[D] = L\mu \quad \text{and} \quad E[(D - E[D])(D - E[D])'] = L\Sigma.
\] (1)

As previously mentioned, the control policy can conveniently be described via a replenishment policy and an allocation policy. We denote the replenishment policy by \( \gamma \) and the allocation policy by \( p \), so the control policy is given by \( (\gamma, p) \). A replenishment policy \( \gamma \) gives rise to the process \( \{R(t), t \geq -L\} \), where
\[
R(t) = (R_1(t), ..., R_n(t)),
\]
and \( R_j(t) \) represents the amount of component \( j \) (\( 1 \leq j \leq n \)) ordered during \([-L, t]\). Let \( R(-L^-) = 0 \) and define
\[
r(t) \equiv R(t) - R(t^-), \quad t \geq -L,
\]
\[
R(t_1, t_2) \equiv R(t_2) - R(t_1), \quad t_2 > t_1 \geq -L,
\]
and \( R(t) \equiv R(t - L, t), \quad t \geq 0, \)
to be orders placed at time \( t \), during the period \((t_1, t_2)\), and within the lead time immediately before time \( t \) respectively. Since one cannot order a negative quantity, each element of \( R(t) \) is non-decreasing over \( t \) (\( t \geq -L \)). Recall that we allow replenishment orders starting at \( t = -L \).

An allocation policy \( p \) gives rise to the process \( \{Z(t), t \geq 0\} \), where
\[
Z(t) = (Z_1(t), ..., Z_m(t)),
\]
and \( Z_i(t) \) is the amount of product \( i \) (\( 1 \leq i \leq n \)) served during \([0, t]\). Let \( Z(0^-) = 0 \) and define
\[
z(t) \equiv Z(t) - Z(t^-), \quad t \geq 0,
\]
\[
Z(t_1, t_2) \equiv Z(t_2) - Z(t_1), \quad t_2 > t_1 \geq 0,
\]
and \( Z(t) \equiv Z(t - L, t), \quad t \geq L, \)
to be demand served at time $t$, during the period $(t_1, t_2]$, and within the lead time immediately before time $t$ respectively. Since one can only serve a positive amount of demand, $Z(t)$ is element-wise non-decreasing over time.

The event sequence at any time is as follows: arrival of new demands, receipt of previously-ordered components, allocation of available components to serve demands, and placement of new orders. Not all events happen at each time. Unsatisfied demands are backlogged and unused components stay in inventory. Let

$$B(t) = (B_1(t), \ldots, B_m(t)) \quad \text{and} \quad I(t) = (I_1(t), \ldots, I_n(t))$$

be the backlog and inventory levels at $t$ ($t \geq 0$) after these events. Then

$$B(t) = B(t^-) + d(t) - z(t) \quad (2)$$
and
$$I(t) = I(t^-) + r(t - L) - A z(t), \quad (3)$$

where we define $B(0^-) = 0$ and $I(0^-) = 0$. Between two distinct time points $0 \leq t_1 < t_2$,

$$B(t_2) = B(t_1) + D(t_1, t_2) - Z(t_1, t_2) \quad (4)$$
and
$$I(t_2) = I(t_1) + R(t_1 - L, t_2 - L) - A Z(t_1, t_2). \quad (5)$$

Specializing these conditions to $t_1 = t - L$ and $t_2 = t$,

$$B(t) = B(t - L) + D(t) - Z(t) \quad (6)$$
and
$$I(t) = I(t - L) + R(t - L) - A Z(t). \quad (7)$$

For discussions below, we define $B^-(t)$ and $I^-(t)$ as backlog and inventory levels at time $t$ ($t \geq 0$) after demand arrival and components receipt but before the allocation of components. Hence,

$$B(t) = B^-(t) - z(t), \quad t \geq 0,$$
$$I(t) = I^-(t) - A z(t), \quad t \geq 0.$$

We define

$$Q(t) \equiv AB^-(t) - I^-(t) = AB(t) - I(t), \quad t \geq 0,$$

as the component shortage at time $t$: there are more components $j$ on-hand than the amount needed to clear the existing backlog if $Q_j(t) \leq 0$ and less than enough if $Q_j(t) > 0$ ($1 \leq j \leq n$).

Let $h_j$ be the cost of holding a unit of component $j$ ($1 \leq j \leq n$) in inventory per unit of time and $b_i$ be the cost of keeping a unit of demand for product $i$ ($1 \leq i \leq m$) in backlog per unit of time. Without loss of generality, we assume that $h_j > 0$ ($1 \leq j \leq n$) and $b_i > 0$ ($1 \leq i \leq m$). (We
can remove all components with zero holding costs and all products with zero backlog costs from
the model with no effect on the total cost.) Satisfying a unit of demand for product \( i \) \((1 \leq i \leq m)\) removes from the system a cost of
\[ c_i = b_i + \sum_{j=1}^{n} a_{ji} h_j \]
per-unit of time. We refer to \( c_i \) as the unit inventory cost of product \( i \) \((1 \leq i \leq m)\). Let
\[ b = (b_1, ..., b_m), \ h = (h_1, ..., h_n) \text{ and } c = (c_1, ..., c_m). \]
Hence the total expected inventory plus backlog cost at time \( t \) is
\[ h \cdot I(t) + b \cdot B(t). \]
The goal of inventory management is to develop a policy \((\gamma, p)\) to minimize the following long-run average expected total inventory cost:
\[ C_{\gamma,p} = \limsup_{T \to \infty} \frac{1}{T} \int_{L}^{T+L} E[b \cdot B(t) + h \cdot I(t)] \, dt. \] (9)
Note that the integral in (9) is over the interval \([L, T+L]\), which is consistent with our optimization horizon, rather than \([0, T]\) as in equation (8) of [4]. The difference between these intervals becomes immaterial when the limit \( T \to \infty \) is taken. To be feasible, the policy cannot serve more demand than the amount arrived or the amount allowed by the available supply of required components, i.e., for any \( 0 \leq t_1 < t_2 \),
\[ Z(t_1, t_2) \leq B(t_1) + D(t_1, t_2), \]
(10)
\[ A Z(t_1, t_2) \leq I(t_1) + R(t_1 - L, t_2 - L). \]
(11)
The policy needs to be non-anticipating, i.e., \( r(t) \) and \( z(t) \) can depend only on information available at \( t \), given by \( B(0^-), I(0^-), \{D(s), 0 \leq s \leq t\}, \{Z(s), 0 \leq s < t\}, \) and \( \{R(s), -L \leq s < t\} \).

We conclude this section by showing two ATO systems (see Figure 1) that have been commonly-studied in the literature. In the W system (e.g., [4]), two products are assembled from three components, and both products use the same amount of the common component 0. In the M system (e.g., [16], [19]), two components are used to build three products where the middle product 0 uses the same amount of each component as the side product that uses that component exclusively.

3. Stochastic Program: Lower Bound and Policy Development
We present the lower bound SP of [4] along with an alternate lower bound SP in Section 3.1 and use the alternate lower bound SP to design the inventory policy in Section 3.2.
3.1. The SP Lower Bound

The one-period ATO model considered in Song and Zipkin [26] corresponds to the following SP:

$$\min_{y \geq 0} \{ h \cdot y + b \cdot E[D] - E[\phi(y; D)] \},$$

$$\phi(y; D) = \max_{z \geq 0} \{ c \cdot z | z \leq D, Az \leq y \}. \quad (12)$$

This is also a simple version of ‘Newsvendor network’ of Harrison and van Mieghem [10]. This SP involves minimizing the inventory cost for a particular point in time in the absence of any past history. Hence the replenishment policy reduces to a one-time decision, $y = (y_1, \ldots, y_n)$ where $y_j$ is the order quantity of component $j$ ($1 \leq j \leq n$). The allocation decision also simplifies to the choice of a vector $z = (z_1, \ldots, z_m)$ where $z_i$ is the amount of product $i$ demand ($1 \leq i \leq m$) to serve. Demand is given by a random vector $D$. Cost parameters $b$, $h$, and $c$ are analogous to those in Section 2. Components are allocated after observing all demands, an arrangement that is impossible for a dynamic ATO system where demand arrivals continue into the indefinite future.

Although intuitively it may seem that the SP (12) provides, at each point in time, a relaxation of the dynamic inventory control problem, it was shown in Doğru et al. [4] that this is not the case. In particular, in the dynamic system, past decisions may lead to backlogs of lower value products, allowing the manager to divert components ordered to clear these backlogs to serve more valuable new arrivals if needed. The SP in (12) has no such flexibility. Doğru et al. [4] introduced the following relaxed version of (12) where the initial backlog $(\alpha)$ can be chosen optimally:

$$\underline{C} = \inf_{\alpha \geq 0} \Phi(\alpha) \quad (13)$$
\[ \Phi(\alpha) \equiv \inf_{y \geq 0} \{ h \cdot y + b \cdot E[(\alpha + D)] - E[\phi(y; \alpha + D)] \}, \]
\[ \phi(y; \alpha + D) = \max_{z \geq 0} \{ c \cdot z \mid z \leq \alpha + D, Az \leq y \}. \]  

(14)

By Theorem 2.1 in [4], if \( D \) has the same distribution as the lead time demand, then \( \underline{C} \) in (14) is a lower bound on the cost objective in (9). Below is a re-statement of their result.

**Theorem 1.** Let \( \gamma, p \) be any feasible replenishment and allocation policies. Let \( C^{(\gamma,p)} \) be the resulting long-run average total expected inventory cost as defined in (9). Let \( \underline{C} \) be given by (14). Then

\[ C \leq C^{(\gamma,p)}. \]  

(15)

It is easy to verify that \( \Phi(\alpha) \) is non-increasing in \( \alpha \). Due to the unbounded support of \( D \), the infimum in (13) may not be attained at finite values of \( \alpha \) and \( y \) even though \( \underline{C} \) is finite. To deal with this issue we consider an alternative relaxation of (12). Instead of optimizing the initial backlog level, we allow \( y \) and \( z \) to be negative, giving rise to the following SP:

\[ C^* = \inf_{y \in \mathbb{R}^n} C(y) \]
\[ C(y) = h \cdot y + b \cdot E[D] - E[\phi(y; D)], \]
\[ \phi(y; D) = \max_{z \in \mathbb{R}^m} \{ c \cdot z \mid z \leq D, Az \leq y \}. \]  

(16)

The absence of non-negativity constraints in the above formulation allows the manager to carry out an infeasible feat of undoing past ordering and allocation decisions that she may regret after observing new demands. Thus it is not surprising that \( C^* \) is also a lower bound on \( C^{(\gamma,p)} \). Moreover, the following result, whose proof is in Section 5, shows that \( C^* \) attains the same value as \( \underline{C} \), but unlike the latter SP, the solution is attained at a finite optimal value, \( y^* \) (possibly not unique).

**Theorem 2.** There exists \( y^* \in \mathbb{R}^n \) such that

\[ C(y^*) = C^* = \underline{C}. \]

Moreover, for any \( y^* \) such that \( C(y^*) = C^* \),

\[ |y^*_j| \leq M, \quad 1 \leq j \leq n, \]  

(17)

where \( M \) is a constant that depends only on \( A, b, h, \) and \( E[D] \).

Following the theorem, we denote \( \underline{C} \) as the cost lower bound and formalize the SP as

\[ \underline{C} = \min_{y \in \mathbb{R}^n} C(y) \]
\[ C(y) = h \cdot y + b \cdot E[D] - E[\phi(y; D)] \]
\[ \phi(y; D) = \max_{z \in \mathbb{R}^m} \{ c \cdot z \mid z \leq D, Az \leq y \}. \]  

(18)

This SP has complete recourse. We will use the recourse problem \( \phi(y; D) \) to design an allocation policy and the optimal solution \( y^* \) to specify a replenishment policy.
3.2. Policy Development

We first show how the SP above naturally leads us to propose (as in [4]) a base-stock replenishment policy, and then focus our efforts on the more complex development of an allocation policy, which was left open in [4].

In (18), the optimal first-stage solution, $y^*$, specifies the amounts of components to order for serving demand $D$. Since $D$ is the lead time demand, the solution can be naturally imitated by a base-stock policy in an ATO system by letting $y^*$ be the base-stock levels. If $y^*$ is not unique, it does not matter which one we select, as long as it is used consistently to set base-stock levels.

A base-stock policy keeps the inventory position of each component, i.e., the total inventory (on-hand and in pipeline) in excess of the amount needed to clear existing backlogs, at a constant level. Without loss of generality, we assume that the optimal inventory positions $y^*$ are reached at $t = 0$. To keep inventory positions at that level at all subsequent times,

$$R(t) = AD(t), \quad t \geq L,$$

i.e., amounts ordered during the past lead time are the same as the amounts needed to satisfy demands that arrived during that period. The quantity $R(t)$ is also the total inventory in the pipeline at time $t$. Thus

$$y^* = I(t) + R(t) - AB(t), \quad t \geq L,$$

which gives rise to the following expression of the on-hand inventory:

$$I(t) = y^* + AB(t) - AD(t), \quad t \geq L.$$  

(20)

Combining (20) with (8) yields the following simple expression for the component shortage process under a base-stock policy:

$$Q(t) = AD(t) - y^*, \quad t \geq L.$$  

(21)

The allocation outcome in (18) typically cannot be exactly replicated in inventory control of ATO systems. Components in the latter case cannot wait to be allocated until after all demands have been observed. Nor can previously-allocated components be ‘clawed back’. Nevertheless, the recourse LP of (18) does suggest the following approach: given $Q(t)$ as component shortage at time $t$ ($t \geq 0$), we set backlog target $B^*(t)$ by solving

$$B^*(t) = \arg \min \{ c \cdot B \mid B \geq 0, AB \geq Q(t) \}, \quad t \geq 0.$$  

(22)

This LP is equivalent to the recourse LP in (18): denote $B$ by $D - z$ and $Q(t)$ by $AD - y$,

$$\min_{B \in \mathbb{R}^m} \{ c \cdot B \mid B \geq 0, AB \geq Q(t) \} = \min_{z \in \mathbb{R}^m} \{ c \cdot (D - z) \mid D - z \geq 0, AZ \geq AD - y \} = c \cdot D - \max_{z \in \mathbb{R}^m} \{ c \cdot z \mid z \leq D, Az \leq y \}. $$  

(23)
The LP (23) optimizes the allocation of components while the LP (22) optimizes the allocation of component shortage. The value of this transformation will become obvious when we give the intuition for proving asymptotic optimality in Section 4.3. To use this result to induce an asymptotically optimal policy and to avoid wide fluctuations of the backlog target in response to small perturbations of component shortage, we need $B^*(t)$ to be uniformly Lipschitz continuous with respect to $Q(t)$. If the optimal solution of (22) is a singleton for all possible values of $Q(t)$, then the condition follows from Hoffman’s Lemma [13] (also see Theorem 10.5 of Schrijver [24]). If the optimal solution is not unique, these results only imply that the solution sets are Lipschitz continuous, and we solve the following quadratic programming problem to select the solution with minimum Euclidean norm:

$$\min_{B \geq 0} \{\|B\| | AB \geq Q(t), c \cdot B \leq \psi^*\}$$

(24)

where $\psi^*$ is the optimal objective value of (22) and $\|B\|$ denotes Euclidean norm of $B$. In general Lipschitz continuity of sets does not imply Lipschitz continuity of the minimum norm selection, cf. Aubin and Frankowska [3]. However, for the special case of (24), Theorem 4.1.d in Han et al. [6] shows that the optimal solution is unique and Lipschitz continuous.

Using the backlog targets, we establish the following Verification Lemma, which presents a sufficient condition for achieving exact optimality under the aforementioned base-stock policy.

**Lemma 1. (Verification Lemma)** Any inventory policy that uses a base-stock replenishment policy with base-stock levels $y^*$ is optimal if the resulting backlog levels satisfy

$$c \cdot E[B(t)] = c \cdot E[B^*(t)] \quad \text{for all } t \geq L.$$  

(25)

Although this lemma is not quite a special case of the Lemma 2 in [21] because we are using a base-stock policy here, the proof is immediate from (8), (20), (23), and (25):

$$b \cdot E[B(t)] + h \cdot E[I(t)] = c \cdot E[B(t)] + h \cdot (y^* - AE[D(t)])$$

$$= c \cdot E[B^*(t)] + h \cdot y^* - h \cdot (AE[D])$$

$$= b \cdot E[D] + h \cdot y^* - E[\max_z \{c \cdot z | z \leq D, Az \leq y^*\}],$$

i.e., the integrand in (9) reaches the lower bound given by (18) at all $t \geq L$.

The lemma suggests that in an ideal situation, product backlogs should always stay at their targets. While the condition cannot always be satisfied in general, it motivates us to define the following principle for designing implementable policies that myopically aim to keep backlog levels close to their targets.
**Allocation Principle:** Backlog levels exceeding their targets, $B_i(t) > B^*_i(t)$ $(1 \leq i \leq m, \ t \geq 0)$, are not allowed to persist if the required components are all available, i.e., the allocation policy must yield

$$ (B_i(t) - B^*_i(t))^+ \cdot [\min_{j:a_{ji} > 0} \{(I_j(t) - a_{ji} + 1)^+\}] = 0, \ 1 \leq i \leq m. \quad (26) $$

In addition, any product whose backlog level is below or at the target is not served, i.e.,

$$ z_i(t) \leq (B^-_i(t) - B^*_i(t))^+, \ 1 \leq i \leq m. \quad (27) $$

Specializing this principle to the W system, when $c_1 > c_2$, the optimal target in (22) becomes

$$ B^*(t) = \arg\min\{c_1B_1 + c_2B_2 | B_i \geq 0, B_i \geq Q_i(t), B_1 + B_2 \geq Q_0(t), \ i = 1, 2\} \quad (28) $$

$$ = (Q^+_1(t), Q^+_2(t) \lor (Q_0(t) - Q^+_1(t))^+). $$

Hence to meet this target, product 1 should have no backlog if there is no shortage of the side component 1, which implies exactly the same priority rule as in [4]: give the common component to product 2 only if the same component cannot be used to serve product 1.

The allocation principle also dictates that when a product has a strictly positive backlog that does not exceed its target, its demand is not served even if all required components are available. Hence reservation ('hold back') in some situations is a feature of our principle. Such a situation can happen in the M system, e.g., when $c_0 > c_1 + c_2$.

The allocation principle is not a specific policy, but rather a condition that may be satisfied by many different policies. We show below that, coupled with a base-stock policy that uses the 'correct' base-stock levels, any allocation policy that satisfies the allocation principle is asymptotically optimal. In systems with more than two products sharing one component, the definition in (8) and the LP in (22) indicate that when the backlog of one product is strictly below its target, backlogs of other products may all exceed their targets. In this case, the allocation principle may allow a family of policies that differ in their selections of the latter products to serve first.

We can also relax the allocation principle to admit more policies. For instance, it is shown in [4] that in the W system with a short lead time, if $c_1$ vastly exceeds $c_2$, then withholding some common component 0 from product 2 when there is no component shortage can result in a lower inventory cost than the myopic priority policy that serves as much demand as possible. To allow the former situations, we can generalize (26) to

$$ (B_i(t) - B^*_i(t))^+ \cdot [\min_{j:a_{ji} > 0} \{(I_j(t) - a_{ji} + 1 - w_{ji})^+\}] = 0, \ 1 \leq i \leq m, \quad (29) $$

where $w_{ji}$ is the amount of component $j$ $(1 \leq j \leq n)$ withheld from product $i$ $(1 \leq i \leq m)$. As we show in Section 4.3, if $w_{ji}$ $(1 \leq i \leq m; 1 \leq j \leq n)$ remains a constant or grows slowly enough in $L$, policies that satisfy (29) will also be asymptotically optimal.
4. Asymptotic Analysis

For the purposes of our asymptotic analysis, we introduce a family of ATO systems indexed by the lead time $L$. All parameters other than $L$ are held fixed, while $L \to \infty$. (As was noted in the Introduction, our results also apply to the high-volume limit considered in [20], where the lead time stays constant while the order arrival rate $\lambda$ increases.)

Let $C_L$ denote the long run average cost for our policy, and let $C_L$ denote the lower bound, both for lead time $L$. Then our main result (Theorem 4) states that

$$\lim_{L \to \infty} \frac{C_L}{C_L} = 1.$$  \hspace{1cm} (30)

In Theorem 3 we show that $L^{-\frac{1}{2}}C_L$ converges to a finite positive constant. Thus (30) is equivalent to

$$\lim_{L \to \infty} \frac{C_L - C_L}{\sqrt{L}} = 0,$$

and (31) is actually what we show. The scaling of the costs and various stochastic processes in this system is basically that of a (functional) central limit theorem. This reflects a simple fact: As the lead time $L$ grows, the total demand over a lead time (when properly centered and scaled) converges to a normally distributed random variable.

Observe that (31) is more stringent than the ‘fluid-scale’ asymptotic optimality criterion. The definition of the latter is the same as in (31) except that $\sqrt{L}$ is replaced by $L$. A policy that is optimal only on the fluid scale may have an average cost that differs from the exact optimum by a quantity on the order of $\sqrt{L}$ (or possibly larger). In this case, the ratio in (30) may not converge to unity. Asymptotic optimality on the fluid scale is an easy criterion that is satisfied by every existing approach that we are aware of. However, for each policy that we can find in the literature, there is always a counterexample, constructed rather easily, showing that it is not asymptotically optimal on the diffusion scale. For example, the numerical experiments conducted in [4] indicated that (30) does not hold for the policy developed in [16].

To indicate their dependence on $L$, the various random variables and processes will assume the same definitions as before except that they may have a superscript ($L$) or be assigned an argument $L$ to specify the system under discussion. Some variables are centered (i.e., taking the difference from its mean). As a rule of our notation, we use $\tilde{X}$ (attaching a tilde) to denote a scaled but not centered value of variable $X$, and $\hat{X}$ (attaching a hat) to denote a centered and scaled value of $X$. For instance, analogous to the definition of $D$ in Section 2, $D^{(L)}$ denotes the random vector that has the same distribution as demands arriving over a lead time in system $L$. Its centered and scaled version is given by

$$D^{(L)} = \frac{D^{(L)} - L\mu}{\sqrt{L}}.$$  \hspace{1cm} (32)
Following (1),

$$E[\hat{D}^{(L)}] = 0 \text{ and } E[\hat{D}^{(L)}(\hat{D}^{(L)\prime})] = \Sigma.$$ 

Furthermore, for the interest of our discussion, we scale without centering demand arrivals, orders placed, and demand served at time $t$ ($t \geq 0$), between times $t_1$ and $t_2$ ($t_2 > t_1 \geq 0$), and within a lead time immediately preceding time $t$ ($t \geq 0$):

$$\tilde{d}^{(L)}(t) \equiv \frac{d^{(L)}(Lt)}{\sqrt{L}}, \quad \hat{D}^{(L)}(t_1, t_2) \equiv \frac{D^{(L)}(Lt_1, Lt_2)}{\sqrt{L}}, \quad \tilde{D}^{(L)}(t) \equiv \frac{D^{(L)}(Lt)}{\sqrt{L}}$$

$$\tilde{r}^{(L)}(t) \equiv \frac{r^{(L)}(Lt)}{\sqrt{L}}, \quad \tilde{R}^{(L)}(t_1, t_2) \equiv \frac{R^{(L)}(Lt_1, Lt_2)}{\sqrt{L}}, \quad \tilde{R}^{(L)}(t) \equiv \frac{R^{(L)}(Lt)}{\sqrt{L}},$$

and

$$\tilde{z}^{(L)}(t) \equiv \frac{z^{(L)}(Lt)}{\sqrt{L}}, \quad \tilde{Z}^{(L)}(t_1, t_2) \equiv \frac{Z^{(L)}(Lt_1, Lt_2)}{\sqrt{L}}, \quad \tilde{Z}^{(L)}(t) \equiv \frac{Z^{(L)}(Lt)}{\sqrt{L}},$$

respectively. We also scale the backlog and inventory levels at each time by

$$\tilde{B}^{(L)}(t) \equiv \frac{B^{(L)}(Lt)}{\sqrt{L}} \quad \text{and} \quad \tilde{I}^{(L)}(t) \equiv \frac{I^{(L)}(Lt)}{\sqrt{L}}, \quad t \geq 0,$$

respectively. Combining these two quantities gives rise to the scaled version of the shortage process, i.e.,

$$A\tilde{B}^{(L)}(t) - \tilde{I}^{(L)}(t) = \tilde{Q}^{(L)}(t) \equiv \frac{Q^{(L)}(Lt)}{\sqrt{L}}, \quad t \geq 0.$$

The next two subsections present some preliminary lemmas. In Section 4.1, we discuss properties of scaled (and sometimes also centered) versions of demand processes. In Section 4.2, we address properties of our inventory policy. The discussion culminates in Section 4.3 where we prove that our policy is asymptotically optimal on the diffusion scale.

### 4.1. The Demand Process

We first consider demands arriving at a given point in time, that is, order sizes. As $L$ increases, even though the distribution of the order size $S$ does not change, the maximum order size over all arrivals that occur within a lead time will increase. However, the following lemma shows that its expected value is negligible on the leading order of average cost ($\sqrt{L}$) that is of interest to us.

**Lemma 2.** Under the assumption that $S$ has finite $(2 + \delta)$ moment ($\delta > 0$),

$$E \left[ \sup_{t-1 \leq \tau \leq t} d^{(L)}_i(\tau) \right] \leq 3\lambda \int_0^t (1 + \eta) \alpha \frac{t^2}{\sqrt{L}}, \quad 1 \leq i \leq m. \quad (33)$$

Besides the size of single arrivals, we are also interested in the total demand arriving between two distinct time points, $D^{(L)}(Lt_1, Lt_2)$, $0 \leq t_1 < t_2$. Their centered and scaled values are

$$\tilde{D}^{(L)}(t_1, t_2) \equiv \frac{D^{(L)}(Lt_1, Lt_2) - L(t_2 - t_1)\mu}{\sqrt{L}}, \quad 0 \leq t_1 < t_2,$$
where
\[ E[\hat{D}^{(L)}(t_1, t_2)] = 0 \quad \text{and} \quad E[\hat{D}^{(L)}(t_1, t_2)\hat{D}^{(L)}(t_1, t_2)'] = (t_2 - t_1)\Sigma. \]

Although we don’t explicitly use this fact, it should be noted that a functional central limit theorem indicates that when \( L \to \infty \), \( \hat{D}(0, t) \) converges to a Brownian motion. In the following lemma, we establish upper bounds on the centered-and-scaled version of the expected maximum demand that occurs within certain subintervals of a lead time. Note that although the result is stated with respect to the time interval \([0, 1]\), due to stationarity this result holds for any interval of the form \([t, t+1]\) for \( t \geq 0 \).

**Lemma 3.** For all \( i = 1, \ldots, m \),
\[
E \left[ \sup_{0 \leq \tau \leq L^{-1/4}} |\hat{D}_i^{(L)}(0, \tau)| \right] \leq (1 + \sigma_i^2) L^{-1/8},
\]
\[
E \left[ \sup_{L^{-1/4} \leq \tau \leq 1} \left( |\hat{D}_i^{(L)}(0, \tau)| - \sqrt{L} \tau \kappa \right)^2 \right] \leq \frac{\sigma_i^2}{\kappa} L^{-1/4},
\]
where \( \kappa \) can be any positive constant.

We also center and scale demand over a lead time, \( \hat{D}^{(L)}(t) \equiv \hat{D}^{(L)}(t-1, t), t \geq 1 \). The distribution of \( \hat{D}^{(L)}(t) \) is the same as \( \hat{D}^{(L)} \) in (32).

**4.2. Inventory Policy**

Following our replenishment policy in each system \( L \), we solve the following SP:
\[
\min_{y \in \mathbb{R}^m} C^{(L)}(y)
\]
where \( C^{(L)}(y) = h \cdot y + b \cdot (L\mu) - E[\varphi(y; \hat{D}^{(L)})] \)
\[
\varphi(y; \hat{D}^{(L)}) = \max_{z \in \mathbb{R}} \{ c \cdot z | z \leq \hat{D}^{(L)}, A z \leq y \}
\]
and use the optimal solution \( y^{(L)*} \) to set base-stock levels. Alternatively, we can center and scale decision variables by letting
\[
\hat{y} = \frac{y - L A \mu}{\sqrt{L}} \quad \text{and} \quad \hat{z} = \frac{z - L \mu}{\sqrt{L}}
\]
to transform (36) into
\[
\min_{\hat{y} \in \mathbb{R}^m} \hat{C}^{(L)}(\hat{y})
\]
where \( \hat{C}^{(L)}(\hat{y}) = h \cdot \hat{y} - E[\varphi(\hat{y}; \hat{D}^{(L)})] \)
\[
\varphi(\hat{y}; \hat{D}^{(L)}) = \max_{\hat{z} \in \mathbb{R}} \{ c \cdot \hat{z} | \hat{z} \leq \hat{D}^{(L)}, A \hat{z} \leq \hat{y} \}.
\]

One may easily verify that
\[
\hat{C}^{(L)}(\hat{y}) = \frac{C^{(L)}(L^{1/2} \hat{y} + L A \mu)}{\sqrt{L}}.
\]
Therefore the optimal solution(s) of (37), \( \hat{y}^{(L)*} \), is related to that of (36) as follows
\[
\hat{y}^{(L)*} = \frac{y^{(L)*} - LA\mu}{\sqrt{L}}.
\]
(39)
Here \( y^{(L)*} \) sets base-stock levels and \( \hat{y}^{(L)*} \) gives their differences from mean demands, measured on the \( \sqrt{L} \) scale. The former grow without bound as the lead time increases while by the lemma below, whose proof is in Section 5, the latter are bounded regardless how long the lead time is.

**Lemma 4.** There exists a constant \( M \) such that for all \( L > 0 \),
\[
|\hat{y}^{(L)*}_j| \leq M, \quad 1 \leq j \leq n.
\]
(40)
We can similarly specify the allocation principle for the \( L \)th system. Operating at the lead order, for each \( L \), we solve the following (scaled) version of (22)
\[
\tilde{B}^{(L)*}(t) = \arg \min \{ c \cdot B \mid B \geq 0, AB \geq \tilde{Q}^{(L)}(t) \}, \quad t \geq 0,
\]
(41)
to set the backlog target. Recall that in (22), we apply a minimum norm selection to keep the optimal solution \( B^*(t) \) uniformly Lipschitz continuous with respect to \( Q(t) \). Here we continue the same approach to make \( \tilde{B}^{(L)*}(t) \) uniformly Lipschitz continuous with respect to \( \tilde{Q}^{(L)}(t) \). As a consequence, the following lemma (whose proof is in Section 5) shows that we may use demand fluctuations to bound the changes of backlog target over time.

**Lemma 5.** There exists a constant \( g \) that depends only on \( A \) and \( c \), such that for any \( t_2 > t_1 \geq 1 \),
\[
|\tilde{B}^{(L)*}_i(t_2) - \tilde{B}^{(L)*}_i(t_1)| \leq g \sum_{l=1}^m |\tilde{D}^{(L)}(t_1,t_2) - \tilde{D}^{(L)}(t_1-1,t_2-1)|, \quad 1 \leq i \leq m,
\]
(42)
and for \( t \geq 1 \),
\[
|\tilde{B}^{(L)*}_i(t) - \tilde{B}^{(L)*}_i(t^-)| \leq g \sum_{l=1}^m |\tilde{d}^{(L)}(t) - \tilde{d}^{(L)}(t-1)|, \quad 1 \leq i \leq m.
\]
(43)

### 4.3. Asymptotic Optimality

To prove the asymptotic optimality of our policy, we first apply the central limit theorem to show that \( L^{-\frac{1}{2}} C^{(L)} \) converges to a finite positive constant, so that (30) is satisfied if our policy is asymptotically optimal on the diffusion scale, i.e., it satisfies (31). We then introduce an asymptotic version of the Verification Lemma to present a sufficient condition for the latter optimality, followed by a proof that our policy satisfies this condition.

We introduce the ‘limit’ stochastic program as follows. Let
\[
\xi = (\xi_1, \ldots, \xi_m)
\]
be a normally-distributed random vector with mean \(0\) and covariance \(\Sigma\). For \(y \in \mathbb{R}^n\), let
\[
\hat{C}(y) \equiv h \cdot y - E[\phi(y; \xi)]
\]
where
\[
\phi(y; \xi) \equiv \max_{z \in \mathbb{R}^m} \{ c \cdot z | z \leq \xi, Az \leq y \}.
\]
Note that for given \((y, \xi)\), \(\phi(y; \xi)\) is always feasible, so
\[
\hat{C}^* \equiv \inf_{y \in \mathbb{R}^m} \hat{C}(y)
\]
is also a two-stage SP with complete recourse. Let
\[
\hat{C}_L \equiv \frac{C_L}{\sqrt{L}}.
\]

**Theorem 3.** (See Section 5 for proof) The optimal value of the scaled SP converges to that of the limit SP:
\[
\lim_{L \to \infty} \hat{C}_L = \hat{C}^*.
\] (44)

From the above it is clear that our policy satisfies (30) if it is asymptotically optimal on the diffusion scale, i.e. satisfies (31).

Below we present a sufficient condition for our approach to satisfy (diffusion-scale) asymptotic optimality condition (31). We refer to our conclusion as an Asymptotic Verification Lemma to highlight that it is a parallel to the Verification Lemma in Section 3.2. In both cases, the condition is given by the excesses of backlog levels at each time over their targets. While the exact condition in Lemma 1 cannot hold in general, its asymptotic version in the lemma below (whose proof is in Section 5) is much easier to satisfy.

**Lemma 6. Asymptotic Verification Lemma** Any family of inventory policies that use base-stock replenishment with base-stock levels \(y^{(L)*}\) in the system with lead time \(L\) is asymptotically optimal if
\[
\lim_{L \to \infty} \sup_{t \geq 1} \{ |E[\tilde{B}^{(L)}(t)] - E[\tilde{B}^{(L)*}(t)]| \} = 0.
\] (45)

Using the above lemma, we can prove asymptotic optimality of a policy by showing that the difference between the expected backlog levels and their targets is negligible for all products and at all times. Such condition can be satisfied by policies that satisfy the aforementioned allocation principle under the more-broadly defined condition (29). The specialization of the latter condition to the \(L^{th}\) system is
\[
(B_i^{(L)}(t) - B_i^{(L)*}(t))^+ \cdot \min_{j, a_{ji} > 0} \{ (I_j^{(L)}(t) - a_{ji} + 1 - w_{ji}^{(L)*})^+ \} = 0, \quad 1 \leq i \leq m.
\] (46)

For the policy to be asymptotically optimal, we show that it is sufficient that
\[
\lim_{L \to \infty} \frac{w_{ji}^{(L)}}{\sqrt{L}} = 0, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.
\] (47)
Theorem 4. Let \( \{\gamma^*(L), p^*(L)\}, L > 0 \) denote a family of policies that use base-stock replenishment with base-stock levels \( y^{(L)*} \) and use an allocation policy that satisfies the allocation principle along with (47). Then
\[
\lim_{L \to \infty} \frac{C_{L}^{\gamma^*(L), p^*(L)}}{C_{L}} = 1.
\] (48)

Before presenting the proof, an intuitive explanation is in order. We refer to the positive difference of a backlog level from its target as the excess, and the negative difference as the deficit. The opening part of the proof makes a simple but useful observation: the definition of backlog target (24) and the requirement of the allocation principle (46) (with condition (47)) imply that no product will have a non-trivial amount of excess unless some other product has a non-trivial amount of deficit. Hence our conclusion holds if the expected deficit of any product never passes a negligible level, i.e., when condition (51) in the proof below applies. To prove the condition, we first note that the backlog targets, \( B^{(L)} \), are on the same order (\( \sqrt{L} \)) as the component shortage, \( Q^{(L)} \). Demand arrival rates (with time scaled, but not space) are on the order of \( L \). Since products with a deficit are not served, as \( L \) increases, the probability that a product has a deficit persisting for more than a lead time becomes asymptotically negligible. So do the expected deficits, a situation shown by the discussion of Case 1 in (53)-(56). For sample paths where the deficit persists for less than a lead time, taken care of in (57)-(63), Lemma 3 is used to show that, again, the expected deficit is asymptotically negligible.

The proof of the above arguments is facilitated by stationary demand processes and the base-stock replenishment policy. The latter renders the past states irrelevant after a lead time, so we can develop a bound on the expected deficit over the finite time interval \([0, 2]\) and apply the bound uniformly to the infinite time horizon.

**Proof** Following Lemma 6, we prove the result by showing that (45) holds.

For given \( L \) and \( t \geq 1 \), let
\[
S^+_L(t) = \left\{ i : \tilde{B}^{(L)}_i(t) > \tilde{B}^{(L)*}_i(t), \ 1 \leq i \leq m \right\},
\]
\[
S^-_L(t) = \left\{ i : \tilde{B}^{(L)}_i(t) < \tilde{B}^{(L)*}_i(t), \ 1 \leq i \leq m \right\}.
\]

Since \( \tilde{B}^{(L)*}_i(t) \) is feasible for (24),
\[
A_j \cdot \tilde{B}^{(L)*}_i(t) \geq \tilde{Q}^{(L)}(t) = A_j \cdot \tilde{B}^{(L)}(t) - \tilde{I}^{(L)}(t).
\]

Therefore for all \( j = 1, ..., n \),
\[
\sum_{i \in S^+_L(t)} a_{ji} (\tilde{B}^{(L)}_i(t) - \tilde{B}^{(L)*}_i(t)) \leq \tilde{I}^{(L)}_j(t) + \sum_{i \in S^-_L(t)} a_{ji} (\tilde{B}^{(L)*}_i(t) - \tilde{B}^{(L)}_i(t)).
\] (49)
Observe that for every \( i' \in S^*_L(t) \), there exists some \( j' \) such that \( a_{j'i'} > 0 \) and

\[
\sum_{i \in S^*_L(t)} a_{j'i'}(\tilde{B}_{i'}^{(L)}(t) - \tilde{B}_{i'}^{(L)*}(t)) \leq \sum_{i \in S^*_L(t)} a_{j'i'}(\tilde{B}_{i'}^{(L)*}(t) - \tilde{B}_{i'}^{(L)}(t)) - \frac{1 - a_{j'i'} - w_{j'i'}^{(L)}}{\sqrt{L}}. \tag{50}
\]

If not, then (49) implies that \( B_{j'}^{(L)}(t) > B_{j'}^{(L)*}(t) \) and \( I_{j'}^{(L)}(t) - a_{j'i'} + 1 - w_{j'i'}^{(L)} > 0 \) for all \( j \) such that \( a_{j'i'} > 0 \), which violates the allocation principle (46). Since all terms on the left-hand side of (50) are strictly positive, the inequality implies that

\[
\frac{1 - a_{j'i'} - w_{j'i'}^{(L)}}{\sqrt{L}} < a_{j'i'}(\tilde{B}_{i'}^{(L)}(t) - \tilde{B}_{i'}^{(L)*}(t)) + \frac{1 - a_{j'i'} - w_{j'i'}^{(L)}}{\sqrt{L}} \leq \sum_{i \in S^*_L(t)} a_{j'i'}(\tilde{B}_{i'}^{(L)*}(t) - \tilde{B}_{i'}^{(L)}(t)).
\]

Using (47),

\[
\lim_{L \to \infty} \frac{1 - a_{j'i'} - w_{j'i'}^{(L)}}{\sqrt{L}} = 0,
\]

and we prove the theorem by showing that for any \( \epsilon > 0 \), if \( L \) is large enough, then for any \( i \in S^*_L(t) \),

\[
E[\tilde{B}_{i'}^{(L)*}(t) - \tilde{B}_i^{(L)}(t)] < \epsilon \text{ for all } t \geq 1. \tag{51}
\]

For a given \( i \in S^-_L(t) \), let \( t_i^{(L)} = \sup\{ \tau : 0 \leq \tau \leq t \text{ and } \tilde{B}_i^{(L)*}(\tau) < \tilde{B}_i^{(L)}(\tau) \} \)

We can write

\[
E[\tilde{B}_i^{(L)*}(t) - \tilde{B}_i^{(L)}(t)] = E[(\tilde{B}_i^{(L)*}(t) - \tilde{B}_i^{(L)}(t))1(t_i^{(L)} < t - 1)] + E[(\tilde{B}_i^{(L)*}(t) - \tilde{B}_i^{(L)}(t))1(t_i^{(L)} \geq t - 1)] \tag{52}
\]

and prove (51) by considering the two situations on the right-hand side separately.

In cases where \( t_i^{(L)} < t - 1 \), \( \tilde{B}_i^{(L)}(\tau) \leq \tilde{B}_i^{(L)*}(\tau) \) for all \( \tau \in [L(t - 1), Lt] \). Under our policy, no demand for product \( i \) is served during this lead time. Therefore

\[
\tilde{B}_i^{(L)}(t) \geq \tilde{D}_i^{(L)}(t) = \tilde{D}_i^{(L)}(t) + \sqrt{L}\mu_i. \tag{53}
\]

Let \( j(i) \) be any \( j \) such that

\[
a_{j(i)i}^{-1}[\tilde{Q}_{j(i)}^{(L)}(t)]^+ = \max_{j, a_{j'i} > 0} [a_{j'i}^{-1}[\tilde{Q}_{j}^{(L)}(t)]^+] \geq \tilde{B}_i^{(L)*}(t) \geq 0, \tag{54}
\]

where the first inequality holds because otherwise \( \tilde{B}_i^{(L)*}(t) \) is not optimal for (41). From (20),

\[
\tilde{Q}_{j(i)}^{(L)}(t) = A_{j(i)} \cdot \tilde{B}_i^{(L)}(t) - \tilde{D}_i^{(L)}(t) = A_{j(i)} \cdot \tilde{D}_i^{(L)}(t) - y_{j(i)}^{(L)*} / \sqrt{L} = A_{j(i)} \cdot \tilde{D}_i^{(L)}(t) - \hat{q}_{j(i)}^{(L)*}. \tag{55}
\]
Define
\[ \tilde{\beta} = \frac{\overline{a}}{a} \text{ and } \tilde{y}_{\text{min}}^{(L)} = \min_{i,j} \left\{ -\frac{|y_{ji}^{(L)}|}{a_{ji}} \right\}. \]

Then \( \tilde{\beta} \geq 1 \) and \( \tilde{y}_{\text{min}}^{(L)} \) is finite by Lemma 4. From (54) and (55),
\[ \tilde{B}_i^{(L)*} (t) \leq \max_{j : a_{ji} > 0} \frac{\tilde{Q}_j^{(L)} (t)}{a_{ji}} \leq \max_{j : a_{ji} > 0} \frac{A_j \cdot \tilde{D}_i^{(L)} (t) - \tilde{y}_{ji}^{(L)*}}{a_{ji}} \leq \beta \sum_{l=1}^{m} |\tilde{D}_i^{(L)} (t)| - \tilde{y}_{\text{min}}^{(L)}. \]

Applying the above inequality and using (53),
\[
E \left[ (B_i^{(L)*} (t) - \tilde{B}_i^{(L)} (t)) 1 (t_i^{(L)} < t - 1) \right] \leq E \left[ \left( \tilde{\beta} \sum_{l=1}^{m} |\tilde{D}_i^{(L)} (t)| - \tilde{y}_{\text{min}}^{(L)} + |\tilde{D}_i^{(L)} (t)| - \sqrt{L \mu_i} \right)^+ \right]
\leq E \left[ \left( (\tilde{\beta} + 1) \sum_{l=1}^{m} |\tilde{D}_i^{(L)} (t)| - (\tilde{y}_{\text{min}}^{(L)} + \sqrt{L \mu_i}) \right)^+ \right]
\leq (\tilde{\beta} + 1) \sum_{l=1}^{m} E \left[ \left( |\tilde{D}_i^{(L)} (t)| - \frac{\tilde{y}_{\text{min}}^{(L)} + \sqrt{L \mu_i}}{m(\beta + 1)} \right)^+ \right]
\leq \frac{(\tilde{\beta} + 1)^2 m}{\tilde{y}_{\text{min}}^{(L)} + \sqrt{L \mu_i}} \sum_{l=1}^{m} \sigma_{ll}^2,
\]
where the last inequality comes from Chebyshev’s inequality.

If \( t_i^{(L)} \geq t - 1 \), we first establish that
\[
\tilde{B}_i^{(L)*} (t) - \tilde{B}_i^{(L)} (t) \leq (\tilde{B}_i^{(L)*} (t) - \tilde{B}_i^{(L)*} (t_i^{(L)})) - (\tilde{B}_i^{(L)} (t) - \tilde{B}_i^{(L)} (t_i^{(L)}))
+ |\tilde{B}_i^{(L)*} (t_i^{(L)}) - \tilde{B}_i^{(L)*} (t_i^{(L)} -)|
\]
by considering the following two scenarios. In scenario one, demand for product \( i \) is served at time \( t_i^{(L)} \). This implies that the product’s preallocation backlog exceeds its target. Following (27) and the definition of \( t_i^{(L)} \),
\[ \tilde{B}_i^{(L)} (t_i^{(L)}) = \tilde{B}_i^{(L)*} (t_i^{(L)}), \]
so (57) holds. In scenario two, demand for product \( i \) is not served at time \( t_i^{(L)} \), so
\[ \tilde{B}_i^{(L)} (t_i^{(L)}) = \tilde{B}_i^{(L)} (t_i^{(L)} -) + \tilde{a}_i^{(L)} (t_i^{(L)}). \]
By the definition of \( t_i^{(L)} \) and because \( \tilde{a}_i^{(L)} (t_i^{(L)}) \geq 0 \),
\[ \tilde{B}_i^{(L)*} (t_i^{(L)} -) < \tilde{B}_i^{(L)} (t_i^{(L)} -) \leq \tilde{B}_i^{(L)} (t_i^{(L)}), \]
which also leads to (57).
We now consider the right-hand side of (57), starting from the last term. From Lemma 2 and (43), there exists some constant \( \theta_1 \) such that

\[
E \left[ \sup_{t-1 \leq \tau \leq t} \{|\tilde{B}_i^{(L)}(\tau) - \tilde{B}_i^{(L)}(\tau^-)|\} \right] \leq \theta_1 L^{-\frac{1}{5m+1}}. \tag{58}
\]

For the first two terms, applying (42) with \( t_1 = t_i^{(L)} \) and \( t_2 = t \),

\[
\tilde{B}_i^{(L)}(t) - \tilde{B}_i^{(L)}(t_i^{(L)}) \leq g \sum_{l=1}^{m} |\hat{B}_i^{(L)}(t_i^{(L)}, t) - \hat{B}_i^{(L)}(t_i^{(L)} - 1, t - 1)|. \tag{59}
\]

Under our inventory policy, no product \( i \) demand is served during \( (t_i^{(L)}, t] \), so

\[
\tilde{B}_i^{(L)}(t) - \tilde{B}_i^{(L)}(t_i^{(L)}) \geq \hat{B}_i^{(L)}(t_i^{(L)}, t) = \hat{D}_i^{(L)}(t_i^{(L)}, t) + \sqrt{L} \mu_i (t-t_i^{(L)}).
\]

Let \( \tau = t - t_i^{(L)} \). Then \( 0 \leq \tau \leq 1 \), and

\[
\sqrt{L} \mu_i (t-t_i^{(L)}) = \sqrt{L} \mu_i \tau, \quad \hat{D}_i^{(L)}(t_i^{(L)}, t) \overset{d}{=} \hat{D}_i^{(L)}(1, 1+\tau),
\]

and for all \( l = 1, \ldots, m \),

\[
\hat{D}_i^{(L)}(t_i^{(L)}, t) - \hat{D}_i^{(L)}(t_i^{(L)} - 1, t - 1) \overset{d}{=} \hat{D}_i^{(L)}(1, 1+\tau) - \hat{D}_i^{(L)}(0, \tau).
\]

Thus applying (59), (60) and above equivalences in distribution,

\[
E \left[ (\tilde{B}_i^{(L)}(t) - \tilde{B}_i^{(L)}(t_i^{(L)})) - (\tilde{B}_i^{(L)}(t) - \tilde{B}_i^{(L)}(t_i^{(L)})) \right] \leq E \left[ \sup_{0 \leq \tau \leq 1} \mathcal{V}^{(L)+}(\tau) \right] \tag{61}
\]

where

\[
\mathcal{V}^{(L)}(\tau) = g \sum_{l=1}^{m} |\hat{D}_i^{(L)}(1, 1+\tau) - \hat{D}_i^{(L)}(0, \tau)| + |\hat{D}_i^{(L)}(1, 1+\tau) - \sqrt{L} \mu_i \tau |.
\]

Let \( \kappa = \mu_i/(2mg+1) \) and define

\[
u_i^{(L)}(\tau) = |\hat{D}_i^{(L)}(0, \tau)| - \sqrt{L} \tau \kappa
\]

and

\[
u_i^{(L)}(\tau) = |\hat{D}_i^{(L)}(1, 1+\tau)| - \sqrt{L} \tau \kappa, \quad 1 \leq l \leq m.
\]

Then

\[
\mathcal{V}^{(L)}(\tau) \leq g \sum_{l=1}^{m} \nu_i^{(L)}(\tau) + g \sum_{l=1}^{m} \nu_i^{(L)}(\tau) + \nu_i^{(L)}(\tau),
\]

and thus

\[
E \left[ \sup_{0 \leq \tau \leq 1} \mathcal{V}^{(L)+}(\tau) \right] \leq g \sum_{l=1}^{m} E \left[ \sup_{0 \leq \tau \leq 1} \nu_i^{(L)+}(\tau) \right] + (g+1) \sum_{l=1}^{m} E \left[ \sup_{0 \leq \tau \leq 1} \nu_i^{(L)+}(\tau) \right]. \tag{62}
\]
Since $0 \leq \tau \leq 1,$

$$E \left[ \sup_{0 \leq \tau \leq 1} u_i^{(L)}(\tau) \right] = E \left[ \sup_{0 \leq \tau \leq 1} \left( |\hat{D}_i^{(L)}(0, \tau)| - \sqrt{L\tau} \right)^+ \right]$$

$$\leq E \left[ \sup_{0 \leq \tau \leq L^{-1/4}} |\hat{D}_i^{(L)}(0, \tau)| \right] + E \left[ \sup_{L^{-1/4} \leq \tau \leq 1} \left( |\hat{D}_i^{(L)}(0, \tau)| - \sqrt{L\tau} \right)^+ \right]$$

$$\leq (1 + \sigma_\mu^2) L^{-1/8} + \sigma_\mu^2 L^{-1/4}, \quad 1 \leq l \leq m,$$

where the last inequality is a direct application of Lemma 3. Since $u_i^{(L)}(\tau) = v_i^{(L)}(\tau)$ $(1 \leq l \leq m, 0 \leq \tau \leq 1),$ we conclude from (57), (58), (61), (62), and (63) that

$$E \left[ \left( \hat{D}_i(t^{(L)}(t)) - \hat{D}_i^{(L)}(t) \right) I(t^{(L)}(t) \geq t - 1) \right] \leq \theta_1 L^{-d \delta / (2d+1)} + \theta_2 L^{-1/8} + \theta_3 L^{-1/4},$$

where $\theta_1,$ $\theta_2$ and $\theta_3$ are constants. This inequality and (56) prove (51), and thus the theorem.

5. Proofs

Proof of Theorem 2 We first prove that if $y^*$ optimizes (16), then

$$|y_j^*| \leq M, \quad 1 \leq j \leq n,$$

where $M \equiv \max_{1 \leq j \leq n} \{ A_j \cdot E[D] + (h_j^{-1} \land \zeta_j^{-1}) b \cdot E[D] \}$ and $\zeta_j \equiv \min_{i:a_{ji}>0} \{ b_i/a_{ji} \}.$

Suppose that in (16), for given $y$ and $D,$ $z^*(y; D)$ optimizes $\varphi(y; D).$ Hence the objective value satisfies

$$C(y) = h \cdot E[y - Az^*(y; D)] + b \cdot E[D - z^*(y; D)].$$

(65)

For simplicity, denote $z^*(y^*; D)$ by $z^*$ $(1 \leq j \leq n).$ Since $Az^* \leq y$ and $z^* \leq D,$ for all $j = 1, \ldots, n,$

$$C(y) \geq b \cdot E[D - z^*] \geq \sum_{i:a_{ji}>0} a_{ji} b_{ji} E[D_i - z^*_i] \geq \zeta_j A_j \cdot E[D - z^*] \geq \zeta_j (A_j \cdot E[D] - y_j),$$

$$C(y) \geq h \cdot E[y - Az^*] \geq h_j (y_j - A_j \cdot E[z^*]) \geq h_j (y_j - A_j \cdot E[D]).$$

(66)

Observe that the objective value of (16) under a feasible solution $y = 0$ and $z(y; D) = 0$ is $b \cdot E[D].$ The above inequalities indicate that a necessary condition for $C(y) \leq b \cdot E[D]$ is that

$$y_j \geq A_j \cdot E[D] - \frac{b \cdot E[D]}{\zeta_j} \quad \text{and} \quad y_j \leq A_j \cdot E[D] + \frac{b \cdot E[D]}{h_j},$$

which cannot hold if $|y_j| > M$ for some $j = 1, \ldots, n.$ Thus to be optimal, $y^*$ must satisfy (64)

We now prove $C^* = C.$ By replacing $z$ with $z^* + \alpha$ and $y$ with $y^* + A\alpha,$ we transform (14) into

$$C = \inf_{\alpha \geq 0} \left\{ \inf_{y' \geq -A\alpha} \{ h \cdot y' + b \cdot E[D] - E[\phi(y', \alpha; D)] \} \right\}$$

(67)

where $\phi(y', \alpha; D) \equiv \max_{z' \geq -\alpha} \{ c \cdot z' |z' \leq D, A\alpha \leq y' \}.$
Since any feasible solution for (67) is also feasible for (16), \( C^* \leq C \).

To prove \( C^* \geq C \), define
\[
G_1(y, \alpha) \equiv E[\phi'(y, \alpha; D)] = E\left[ \max_{z \geq -\alpha} \{c \cdot z | z \leq D, A z \leq y\} \right],
\]
\[
G_2(y) \equiv E[\varphi(y; D)] = E\left[ \max_{z \in \mathbb{R}^m} \{c \cdot z | z \leq D, A z \leq y\} \right].
\]

Let \( \alpha^{(k)} = (k, \ldots, k) \) and denote, as a feasible solution to (67),
\[
y_j^{(k)} = y_j^* \lor (-k \sum_{i=1}^m a_{ji}).
\]

Since \( y^* \) is bounded by (64), if \( k \) is sufficiently large, \( y^{(k)} = y^* \). Thus we only need to prove that
\[
\lim_{k \to \infty} G_1(y^*, \alpha^{(k)}) \geq G_2(y^*),
\]
(68)
because if it holds, then
\[
C \leq \lim_{k \to \infty} \{h \cdot y^* + b \cdot E[D] - G_1(y^*, \alpha^{(k)})\} \leq h \cdot y^* + b \cdot E[D] - G_2(y^*) = C^*.
\]

Given that \( z^* \) is the optimal solution that yields \( \varphi(y^*; D) \), \( z_i^* \geq -k \) \( (1 \leq i \leq m) \) if
\[
A_j \cdot D \leq y_j^* + ka, \quad 1 \leq j \leq n.
\]
(69)
Otherwise, if there exists some \( i \) \( (i = 1, \ldots, m) \) such that \( z_i^* < -k \), then under (69) and because \( z \leq D \), no capacity constraint that involves \( z_i^* \) is binding. Thus increasing \( z_i^* \) is feasible and strictly improves \( c \cdot z^* \), which contradicts the definition of \( z^* \).

Define the event
\[
\Omega_{k_0} \equiv \{ \omega : D_i(\omega) \leq k_0, i = 1, \ldots, m \}, \quad \text{where} \quad k_0 = \frac{ka - M}{am}.
\]

We let \( k \) be sufficiently large so that \( k_0 \geq 0 \) (which is feasible because \( M \) does not depend on \( k \)). Since \( y_j^* \geq -M \) \( (1 \leq j \leq n) \), for all \( \omega \in \Omega_{k_0} \),
\[
A_j \cdot D \leq \frac{ka - M}{am} \sum_{i=1}^m a_{ji} \leq ka - M \leq y_j^* + ka, \quad 1 \leq j \leq n.
\]

Therefore the probability that (69) does not hold satisfies
\[
P\{ \exists j \in \{1, \ldots, n\} : A_j \cdot D > y_j^* + ka \} \leq P\{ \omega \notin \Omega_{k_0} \}.
\]

Since \( z_i^* \geq -k \) \( (1 \leq i \leq m) \) is feasible for (67) on \( \Omega_{k_0} \) and \( z^* \leq D \),
\[
G_1(y^*, \alpha^{(k)}) - G_2(y^*) \geq -c \cdot E[z^* 1(\omega \notin \Omega_{k_0})] \geq -\sum_{i=1}^m c_i E[D_i 1(\omega \notin \Omega_{k_0})],
\]
(70)
where \( \mathbf{1}(\cdot) \) is the indicator function. Since for all \( i = 1, \ldots, m \),

\[
E[D_i \ast \mathbf{1}(\omega \notin \Omega_{k_0})] \leq k_0 \sum_{i=1}^{m} P\{D_i \geq k_0\} + E[D_i \ast \mathbf{1}(D_i \geq k_0)],
\]

we can prove the right-hand side of (70) converges to zero and thus (68) is true by showing that

\[
\lim_{k_0 \to \infty} E[D_i \ast \mathbf{1}(D_i \geq k_0)] = 0, \quad 1 \leq i \leq m,
\]

which holds for all \( i = 1, \ldots, m \) because

\[
E[D_i \ast \mathbf{1}(D_i \geq k_0)] = \int_{k_0}^{\infty} P\{D_i \geq x\} dx \leq E[D_i^2] \int_{k_0}^{\infty} x^{-2} dx = E[D_i^2]/k_0
\]

and \( E[D_i^2] (1 \leq i \leq m) \) is finite by assumption.

**Proof of Lemma 2** Let \( s_i^k \) be the \( k \text{th} \) \((k = 1, 2, \ldots)\) realization of \( S_i \) and

\[
s_i^{\max}(k) = \max\{s_i^1, s_i^2, \ldots, s_i^k\}.
\]

We first prove that for all \( k = 1, 2, \ldots \),

\[
\frac{E[s_i^{\max}(k)]}{\sqrt{k}} \leq (1 + \eta_i)k^{-\frac{\delta}{2(2+\delta)}}.
\]  

(71)

Let \( \tilde{F}_{i,d}(x) \) be the complimentary CDF of \( S_i \) \((1 \leq i \leq m)\). Then by Chebyshev’s Inequality,

\[
P\{s_i^{\max}(k) > \sqrt{k}x\} = 1 - \left[1 - \tilde{F}_{i,d}(\sqrt{k}x)\right]^k \leq 1 - \left[1 - \frac{\eta_i}{(\sqrt{k}x)^{2+\delta}}\right]^k
\]

\[
\leq 1 - \left(1 - \frac{k\eta_i}{(\sqrt{k}x)^{2+\delta}}\right) = \eta_i k^{-\delta/2}\exp[-(2+\delta)]
\]

Let \( \Delta = \delta/(2(2+\delta)) \) and use the above

\[
\frac{E[s_i^{\max}(k)]}{\sqrt{k}} = \int_{0}^{\infty} P\{s_i^{\max}(k) > x\} dx \leq \int_{0}^{\infty} P\{s_i^{\max}(k) > \sqrt{k}x\} dx
\]

\[
\leq k^{-\Delta} + \int_{k^{-\Delta}}^{\infty} \eta_i k^{-\delta/2}\exp[-(2+\delta)] dx
\]

\[
< k^{-\Delta} + \eta_i k^{-\delta/2+\Delta(1+\delta)} = (1 + \eta_i)k^{-\frac{\delta}{2(2+\delta)}}.
\]

To use (71) to prove the lemma, let \( \Lambda^{(L)} \) be the Possion random variable with mean \( L\lambda \) \((1 \leq i \leq m)\),

\[
\sup_{t-1 \leq \tau \leq t} \tilde{d}_i^{(L)}(\tau) \overset{d}{=} s_i^{\max}(\Lambda^{(L)}), \quad 1 \leq i \leq m.
\]

Let \( p_k^{(L)} = P\{\Lambda^{(L)} = k\} \). Then

\[
E\left[\sup_{t-1 \leq \tau \leq t} \tilde{d}_i^{(L)}(\tau)\right] = \sum_{k=0}^{\lceil L\lambda \rceil} p_k^{(L)} E[s_i^{\max}(k)] + \sum_{k=\lceil L\lambda \rceil + 1}^{\infty} p_k^{(L)} E[s_i^{\max}(k)] \quad \frac{1}{\sqrt{L}}.
\]  

(72)
For all \( k \leq \lfloor \lambda L \rfloor \), \( E[s_i^{\max}(k)] \leq E[s_i^{\max}(\lfloor \lambda L \rfloor)] \). Thus
\[
\sum_{k=0}^{\lfloor \lambda L \rfloor} p_k^{(L)} \frac{E[s_i^{\max}(k)]}{\sqrt{\lambda \lambda L}} \leq \sqrt{\lambda} \frac{E[s_i^{\max}(\lfloor \lambda L \rfloor)]}{\sqrt{\lambda \lambda L}}.
\]
By applying (71) and observing that \( \lambda L \leq 2 \lfloor \lambda L \rfloor \) (1 \( \leq i \leq m \))
\[
\lambda \frac{E[s_i^{\max}(\lfloor \lambda L \rfloor)]}{\sqrt{\lambda \lambda L}} \leq \sqrt{\lambda}(1 + \eta_i)(\lfloor \lambda L \rfloor)^{-\frac{k}{2(2+\eta)}}
\]
\[
\leq (1 + \eta_i)(\lambda L/2)^{-\frac{k}{2(2+\eta)}}
\]
\[
\leq 2(1 + \eta_i)\lambda^{\frac{k}{2(2+\eta)}} L^{-\frac{k}{2(2+\eta)}}.
\]
(72)

For all \( k > \lfloor \lambda L \rfloor \),
\[
\frac{p_k^{(L)}}{\sqrt{\lambda \lambda L}} = \frac{(\lambda L)^k e^{-\lambda L}}{k! \lambda \lambda L} \leq \sqrt{\lambda} \frac{(\lambda L)^{k-1}}{k (k-1)!} e^{-\lambda L} = \sqrt{\lambda} \frac{p_k^{(L)}}{k p_{k-1}^{(L)}}.
\]
Using (71) and observing that \( k^{-\frac{k}{2(2+\eta)}} \leq (\lambda L)^{-\frac{k}{2(2+\eta)}} \) when \( k > \lfloor \lambda L \rfloor \),
\[
\sum_{k=\lfloor \lambda L \rfloor + 1}^{\infty} p_k^{(L)} \frac{E[s_i^{\max}(k)]}{\sqrt{\lambda \lambda L}} \leq \sqrt{\lambda} \sum_{k=\lfloor \lambda L \rfloor + 1}^{\infty} p_k^{(L-1)} \frac{E[s_i^{\max}(k)]}{\sqrt{\lambda}}
\]
\[
\leq (1 + \eta_i) \sqrt{\lambda} \sum_{k=\lfloor \lambda L \rfloor + 1}^{\infty} p_k^{(L-1)} [k^{-\frac{k}{2(2+\eta)}}]
\]
\[
\leq (1 + \eta_i) \lambda^{\frac{k}{2(2+\eta)}} L^{-\frac{k}{2(2+\eta)}}.
\]
(73)

The lemma follows from (72), (73), and (74).

**Proof of Lemma 3** Since \( \hat{D}_i^{(L)}(0, \cdot) \) is a martingale (with respect to the filtration generated by \( \{\mathcal{D}(t), t \geq 0\} \)) and \( |x| \) is a convex function, \( |\hat{D}_i^{(L)}(0, \cdot)| \) is a sub-martingale (1 \( \leq i \leq m \)). We apply Doob’s inequality to prove both results. For (34),
\[
E \left[ \sup_{0 \leq \tau \leq L^{-1/4}} |\hat{D}_i^{(L)}(0, \tau)| \right]
\]
\[
= \int_0^{L^{-1/4}} P \left\{ \sup_{0 \leq \tau \leq L^{-1/4}} |\hat{D}_i^{(L)}(0, \tau)| > x \right\} dx + \int_{L^{-1/4}}^{\infty} P \left\{ \sup_{0 \leq \tau \leq L^{-1/4}} |\hat{D}_i^{(L)}(0, \tau)| > x \right\} dx
\]
\[
\leq L^{-1/8} + \int_{L^{-1/4}}^{\infty} E \left[ (\hat{D}_i^{(L)}(0, L^{-1/4}))^2 \right] x^{-2} dx
\]
\[
= L^{-1/8}(1 + \sigma_{i,0}^2) \quad 1 \leq i \leq m.
\]

For (35),
\[
E \left[ \sup_{L^{-1/4} \leq \tau \leq 1} \left( |\hat{D}_i^{(L)}(0, \tau)| - \sqrt{\lambda \lambda L} \right)^+ \right] \leq E \left[ \sup_{L^{-1/4} \leq \tau \leq 1} \left( |\hat{D}_i^{(L)}(0, \tau)| - L^{1/4} \kappa \right)^+ \right]
\]
\[
= \int_{L^{1/4} \kappa}^{\infty} P \left\{ \sup_{L^{-1/4} \leq \tau \leq 1} |\hat{D}_i^{(L)}(0, \tau)| > x \right\} dx
\]
and thus $\sigma_i$ where (37), since $\hat{\mathcal{E}}$

The above three equations imply that $E[\hat{\mathcal{D}}(L)] = 0$,

$$\hat{C}(L)(\bar{y}) \geq -\zeta_j \bar{y}_j \text{ and } \hat{C}(L)(\bar{y}) \geq h_j \bar{y}_j, \quad 1 \leq j \leq n.$$ 

Since $\bar{y}^{(L)} = 0$ and $\bar{z}^{(L)} = 0 \wedge \hat{\mathcal{D}}(L)$ (where the minimum is taken component-wise) are feasible for (37),

$$\hat{C}(L)(\bar{y}^{(L)*}) \leq -c \cdot E[0 \wedge \hat{\mathcal{D}}(L)] \leq c \cdot E[||\hat{\mathcal{D}}(L)||],$$

and thus

$$|\bar{y}_j^{(L)*}| \leq \frac{c \cdot E[||\hat{\mathcal{D}}(L)||]}{h_j \wedge \zeta_j}, \quad 1 \leq j \leq n,$$

and the lemma holds because

$$E[||\hat{D}_t^{(L)}||] \leq 2 + E[(\hat{D}_t^{(L)})^2] \leq 2 + \max\{\sigma_{11}^2, \ldots, \sigma_{mm}^2\},$$

where $\sigma_i^2$ $(1 \leq i \leq m)$ are finite.

Proof of Lemma 5  For $t_2 > t_1 \geq 1$,

$$\begin{align*}
\hat{\mathcal{B}}^{(L)}(t_2) & = \hat{\mathcal{B}}^{(L)}(t_1) + \hat{\mathcal{D}}^{(L)}(t_1, t_2) - \bar{Z}^{(L)}(t_1, t_2), \\
\hat{\mathcal{I}}^{(L)}(t_2) & = \hat{\mathcal{I}}^{(L)}(t_1) + \hat{\mathcal{R}}^{(L)}(t_1 - 1, t_2 - 1) - A\bar{Z}^{(L)}(t_1, t_2).
\end{align*}$$

Under a base-stock policy,

$$\hat{\mathcal{R}}^{(L)}(t_1 - 1, t_2 - 1) = A\hat{\mathcal{D}}^{(L)}(t_1 - 1, t_2 - 1).$$

The above three equations imply that

$$\hat{Q}^{(L)}(t_2) - \hat{Q}^{(L)}(t_1) = A\hat{\mathcal{B}}^{(L)}(t_2) - \hat{\mathcal{I}}^{(L)}(t_2) - (A\hat{\mathcal{B}}^{(L)}(t_1) - \hat{\mathcal{I}}^{(L)}(t_1))$$

$$= A(\hat{\mathcal{D}}^{(L)}(t_1, t_2) - \bar{D}^{(L)}(t_1 - 1, t_2 - 1))$$

$$= A(\hat{\mathcal{D}}^{(L)}(t_1, t_2) - \bar{D}^{(L)}(t_1 - 1, t_2 - 1)).$$

Since $\hat{\mathcal{B}}^{(L)*}(t)$ is Lipschitz continuous in $\hat{\mathcal{Q}}^{(L)}(t)$, there exists $\vartheta$ such that for all $i = 1, \ldots, m$,

$$|\hat{B}_i^{(L)*}(t_2) - \hat{B}_i^{(L)*}(t_1)| \leq \vartheta \sum_{j=1}^n |\hat{Q}_j^{(L)}(t_2) - \hat{Q}_j^{(L)}(t_1)| \leq g \sum_{i=1}^m |\hat{D}_i^{(L)}(t_1, t_2) - \hat{D}_i^{(L)}(t_1 - 1, t_2 - 1)| \quad (75)$$

where $g$ is a constant, so (42) holds. The proof of (43) is similar to the above, using

$$\hat{Q}^{(L)}(t) - \hat{Q}^{(L)}(t^-) = A\hat{d}^{(L)}(t) - \hat{r}^{(L)}(t - 1) = A(\hat{d}^{(L)}(t) - \hat{d}^{(L)}(t - 1)).$$
Proof of Theorem 3  This result follows from Theorem 2.2 in Robinson and Wets [22]. Specifically, \( \hat{D}^{(L)} (L > 0) \) weakly converges to \( \xi \). The recourse problem \( \varphi(\hat{y}; \hat{D}^{(L)}) \) is always feasible and by Hoffman’s Lemma, continuous in both \( \hat{y} \) and \( \hat{D}^{(L)} \). Since
\[
\varphi(\hat{y}; \hat{D}^{(L)}) \leq c \cdot \hat{D}^{(L)},
\]
and \( \hat{D}^{(L)} (L > 0) \) have the same finite covariance matrix \( \Sigma \), \( \varphi(\hat{y}; \hat{D}^{(L)}) \) is uniformly integrable over \( \{\hat{D}^{(L)}, L > 0\} \). The function \( h \cdot \hat{y}^{(L)} \) is continuous. By Lemma 4, the optimal solution \( \hat{y}^{(L)} \) \( (L > 0) \) is contained in a finite set. Thus all conditions of Theorem 2.2 in [22] are satisfied. Their conclusion (a) immediately leads to (44).

Proof of Lemma 6  Let \( \{ (\gamma(L), p(L)), L > 0 \} \) denote a family of policies that use base-stock replenishment with base-stock levels \( y^{(L)} \). Applying (20) to the \( L \)th system,
\[
 I^{(L)}(Lt) = y^{(L)*} + AB^{(L)}(Lt) - AD^{(L)}(Lt), \quad t \geq 1.
\]

Scaling \( I^{(L)}(Lt) \) in the above and observing that \( E[D^{(L)}(Lt)] = L \mu \),
\[
\frac{b \cdot E[B^{(L)}(Lt)] + h \cdot E[I^{(L)}(Lt)]}{\sqrt{L}} = c \cdot E[\hat{B}^{(L)}(t)] + h \cdot \hat{y}^{(L)*}, \quad t \geq 1. \quad (76)
\]

Applying (20) to the shortage process in the \( L \)th system,
\[
 Q^{(L)}(Lt) = AB^{(L)}(Lt) - I^{(L)}(t) = AD^{(L)}(Lt) - y^{(L)*} \overset{d}{=} AD^{(L)} - y^{(L)*}.
\]

Because of this equivalence in distribution, in the \( L \)th system, (23) yields
\[
c \cdot E[B^{(L)*}(t)] = c \cdot E[D^{(L)} - z^{(L)*}], \quad (77)
\]
where \( z^{(L)*} \) is the optimal solution of the second (maximizing) LP in (23), which is also the recourse problem of the lower bound SP (18).

Applying (76) to scaled inventory cost in the \( L \)th system,
\[
\frac{C^{(L), p(L)}_L}{\sqrt{L}} = \lim_{T \to \infty} \left\{ \frac{1}{T} \int_1^{T+1} \frac{b \cdot E[B^{(L)}(Lt)] + h \cdot E[I^{(L)}(Lt)]}{\sqrt{L}} dt \right\} = h \cdot \hat{y}^{(L)*} + \lim_{T \to \infty} \left\{ \frac{1}{T} \int_1^{T+1} c \cdot E[\hat{B}^{(L)}(t)] dt \right\}.
\]

Following (77) and (39),
\[
\frac{C^{(L)}_L}{\sqrt{L}} = \frac{h \cdot E[y^{(L)*} - Az^{(L)*}]}{\sqrt{L}} + \frac{b \cdot E[D^{(L)} - z^{(L)*}]}{\sqrt{L}} = h \cdot \hat{y}^{(L)*} + c \cdot E[\hat{B}^{(L)*}(t)].
\]

Therefore
\[
\frac{C^{(L), p(L)}_L - C^{(L)}_L}{\sqrt{L}} = \lim_{T \to \infty} \left\{ \frac{1}{T} \int_1^{T+1} c \cdot \left( E[\hat{B}^{(L)}(t)] - E[\hat{B}^{(L)*}(t)] \right) dt \right\},
\]
and the lemma follows as a consequence.
6. Conclusions and Open Problems

We have developed an inventory policy for ATO systems with a general BOM and identical component lead times. We have proved that our policy is asymptotically optimal on the diffusion scale, and consequently, as the lead time grows, the percentage difference between the average inventory cost of our policy and its lower bound converges to zero. Since the inventory cost of an ATO system and the complexity of its optimization grows with the lead time, our result addresses situations where an optimal inventory policy is most needed and most difficult to develop.

While this paper addresses ATO models with identical lead times, we conjecture that a similar SP-based approach can also lead to asymptotically optimal policies for general systems with non-identical lead times. The first step has already been completed in [21], which shows that the optimal solution of a particular $K+1$ stage SP is a lower bound on the average inventory cost of ATO systems with $K \geq 1$ different replenishment lead times. Although a replenishment policy, based on the solution of the $K+1$ stage SP is suggested in [21], this policy is quite complicated. It seems clear, based on the optimality of this replenishment policy in certain contexts (including that of Rosling [23], where it reduces to his), that when lead times are not identical, a sensible replenishment policy should set inventory positions for components with shorter lead times based on the availability of components with longer lead times. Such a policy would in general not be of base-stock form. Our analysis in this paper relies on properties of base-stock policies. In particular, $Q(t)$ has a simple expression in terms of the demand process under a base-stock policy (cf. (21)) that would no longer hold if the replenishment policy were not base-stock. We leave the task of dealing with this significant issue to future work. (Note that we would also need to prove suitably generalized versions of Theorems 2 and 3, along with Lemmas 4 and 6.) On the other hand, in terms of an allocation policy it seems that our target-based allocation policy should be transferable to systems with non-identical lead times.

In addition to the issue of proving asymptotic optimality, there are issues that arise in making the SP-based approach computationally feasible for ‘industrial size’ problems, both with identical and non-identical lead times. Given the amount of efforts required, we leave these tasks to future research.

Acknowledgments

We thank Jong-Shi Pang for directing us to Theorem 4.1 in [6].

References


