Robust SAA

Dimitris Bertsimas · Vishal Gupta · Nathan Kallus

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Abstract Sample average approximation (SAA) is a widely popular approach to data-driven decision-making under uncertainty. Under mild assumptions, SAA is both tractable and enjoys strong asymptotic performance guarantees. Similar guarantees, however, do not typically hold in finite samples. In this paper, we propose a modification of SAA, which we term Robust SAA, which retains SAA’s tractability and asymptotic properties and, additionally, enjoys strong finite-sample performance guarantees. The key to our method is linking SAA, distributionally robust optimization, and hypothesis testing of goodness-of-fit. Beyond Robust SAA, this connection provides a unified perspective enabling us to characterize the finite sample and asymptotic guarantees of various other data-driven procedures that are based upon distributionally robust optimization. We present examples from inventory management and portfolio allocation, and demonstrate numerically that our approach outperforms other data-driven approaches in these applications.

Keywords Sample average approximation of stochastic optimization · Data-driven optimization · Goodness-of-fit testing · Distributionally robust optimization · Conic programming · Inventory management · Portfolio allocation

1 Introduction

In this paper, we treat the stochastic optimization problem

$$z_{\text{stoch}} = \min_{x \in X} \mathbb{E}_F[c(x; \xi)],$$

where $c(x, \xi)$ is a given cost function depending on a random vector $\xi$ following distribution $F$ and a decision variable $x \in X \subseteq \mathbb{R}^n$. This is a widely used modeling paradigm in operations research, encompassing a number of applications [8, 36].

In real-world applications, however, the distribution $F$ is unknown. Rather, we are given data $\xi^1, \ldots, \xi^N$, which are typically assumed to be drawn IID from $F$. The most common approach in these settings is the sample average approximation (SAA). SAA approximates the true, unknown distribution $F$ by the
empirical distribution $\hat{F}_N$, which places $1/N$ mass at each of the data points. In particular, the SAA approach approximates (1) by the problem

$$z_{\text{SAA}} = \min_{x \in X} \frac{1}{N} \sum_{j=1}^{N} c(x, \xi^j).$$

(2)

Variants of the SAA approach in this and other contexts are ubiquitous throughout operations research, often used tacitly without necessarily being referred to by this name.

Under mild conditions on the cost function $c(x; \xi)$ and the sampling process, SAA enjoys two important properties:

**Asymptotic Convergence:** As the number of data points $N \to \infty$, both the optimal value $z_{\text{SAA}}$ of (2) and an optimal solution $x_{\text{SAA}}$ converge to the optimal value $z_{\text{stoch}}$ of (1) and an optimal solution $x_{\text{stoch}}$ almost surely (e.g. [22, 24]).

**Tractability:** Finding the optimal value of and an optimal solution to (2) is computationally tractable (e.g. [9]).

In our opinion, these two features – asymptotic convergence and tractability – underly SAA’s practical success in data-driven settings. Similar performance guarantees, however, do not hold for SAA for finite $N$, except in certain special cases (e.g. [24, 26]).

In this paper, we propose a novel approach to (1) in data-driven settings which we term Robust SAA. Robust SAA inherits SAA’s favorable asymptotic convergence and tractability. Unlike SAA, however, Robust SAA enjoys a strong finite sample performance guarantee for a wide class of optimization problems. The key idea of Robust SAA is to approximate (1) by a particular data-driven, distributionally robust optimization approach.

More specifically, a distributionally robust optimization (DRO) problem is

$$\tau = \min_{x \in X} \bar{C}(x, F),$$

(3)

where

$$\bar{C}(x, F) = \sup_{F_0 \in F} \mathbb{E}_{F_0}[c(x; \xi)],$$

(4)

where $F$ is a set of potential distributions for $\xi$. We call such a set a distributional uncertainty set or DUS whose first few moments of a distribution or other structural features, but did not explicitly consider the data-driven setting. Recently, the authors of [11, 13] took an important step forward proposing data-driven DRO formulations in which the DUS $F$ is a function of the data, i.e., $F = F(\xi^1, \ldots, \xi^N)$, and showing that (3) remains tractable. Loosely speaking, their DUSs consist of distributions whose first few moments are close to the sample moments of the data. The authors show how to tailor these DUSs so that for any $0 \leq \alpha \leq 1$, the probability (with respect to data sample) that the true (unknown) distribution $F \in F(\xi^1, \ldots, \xi^N)$ is at least $1 - \alpha$. Consequently, solutions to (3) based on these DUSs enjoy a distinct, finite-sample guarantee:

**Finite-Sample Performance Guarantee:** With probability at least $1 - \alpha$ with respect to the data sampling process, for any optimal solution $\tau$ to (3), $\tau \geq \mathbb{E}_{F}[c(\tau, \xi)]$, where the expectation is taken with respect to the true, unknown distribution $F$.

In contrast to SAA, however, the methods of [11, 13] do not generally enjoy asymptotic convergence. (We make this claim precise Section 4.3.2).

Our approach, Robust SAA, is a particular type of data-driven DRO. Unlike existing approaches, however, our DUSs are not defined in terms of the sample moments of the data, but rather are specified as the confidence region of a goodness-of-fit (GoF) hypothesis test. Intuitively, our DUSs consist of all distributions which are “small” perturbations of the empirical distribution – hence motivating the name Robust SAA – where the precise notion of “small” is determined by the choice of GoF test. Different GoF tests yield different DUSs with different computational and statistical properties.

We prove that like other data-driven DRO proposals, Robust SAA also satisfies a finite-sample performance guarantee. Moreover, we prove that for a wide-range of cost functions $c(x; \xi)$, Robust SAA can be reformulated as a single-level convex optimization problem suitable for off-the-shelf solvers and is tractable theoretically and practically. Unlike other data-driven DRO proposals, however – and this is key – we prove that Robust SAA also satisfies an asymptotic convergence property similar to SAA. In other words, Robust SAA combines the strengths of both the classical SAA and data-driven DRO. Computational experiments in inventory management and portfolio allocation confirm that these properties translate into higher quality solutions for these applications in both small and large sample contexts.
In addition to proposing Robust SAA as an approach to addressing (1) in data-driven settings, we highlight a connection between GoF hypothesis testing and data-driven DRO more generally. Specifically, we show that any DUS that enjoys a finite-sample performance guarantee, including the methods of [11, 13], can be recast as the confidence region of some statistical a hypothesis test. Thus, hypothesis testing provides a unified viewpoint. Adopting this viewpoint, we characterize the finite-sample and asymptotic performance of DROs in terms of certain statistical properties of the underlying hypothesis test, namely significance and consistency. This characterization highlights an important, new connection between statistics and data-driven DRO. From a practical perspective, our results allow us to describe which DUSs are best suited to certain applications, providing important modeling guidance to practitioners. Moreover, this connection motivates the use of well-established statistical procedures like bootstrapping in the DRO context. Numerical experimentation confirms that these procedures can significantly improve upon existing algorithms and techniques.

To summarize our contributions:

1. We propose a new approach to optimization in data-driven settings, termed Robust SAA, which enjoys both finite sample and asymptotic performance guarantees for a wide-class of problems.
2. We develop new connections between SAA, DRO and statistical hypothesis testing. In particular, we characterize the finite-sample and asymptotic performance of data-driven DROs in terms of certain statistical properties of a corresponding hypothesis test, namely its significance and consistency.
3. Leveraging the above characterization, we shed new light on the finite sample and asymptotic performance of existing DRO methods and Robust SAA. In particular, we provide practical guidelines on designing appropriate DRO formulations for specific applications.
4. We prove that Robust SAA yields tractable optimization problems that are solvable in polynomial time for a wide class of cost functions. Moreover, for many cases of interest, including two-stage convex optimization with linear recourse, Robust SAA leads a single-level convex optimization formulations that can be solved using off-the-shelf software for linear or second-order optimization.
5. Through numerical experiments in inventory management and portfolio allocation, we illustrate that Robust SAA leads to better performance guarantees than existing data-driven DRO approaches and has performance similar to classical SAA in the large-sample regime.
6. Finally, we show how Robust SAA can be used to obtain approximations to the “price of data” – the price one would be willing to pay in a data-driven setting for additional data.

The remainder of this paper is structured as follows. We next provide a brief literature review and describe the model setup. In Section 2, we illustrate the fundamental connection between DRO and the confidence regions of GoF tests and explicitly describe Robust SAA. Section 3 connects the significance of the hypothesis test to the finite-sample performance of a DRO. Section 4 connects the consistency of the hypothesis test to the asymptotic performance of the DRO. Section 5 proves that for the tests we consider, Robust SAA leads to a tractable optimization problem for many choices of cost function. Finally, Section 7 presents an empirical study and Section 8 concludes. All proofs except that for Theorem 2 are in the appendix.

1.1 Literature review

DRO was first proposed by the author in [32], where $\mathcal{F}$ is taken to be the set of distributions with a given mean and covariance in a specific inventory context. DRO has since received much attention in the literature, with many authors focusing on DUSs $\mathcal{F}$ defined by fixing the first few moments of the distribution [6, 10, 28, 29], although some also consider other structural information such as unimodality [16]. In [43], the authors characterized the computational tractability of (3) for a wide range of DUSs $\mathcal{F}$ by connecting tractability to the geometry of $\mathcal{F}$.

As mentioned, in [11, 13], the authors extended DRO to the data-driven setting. In [11], the authors studied chance constraints, but their results can easily be cast in the DRO setting. Both papers focus on tractability and the finite-sample guarantee of the resulting formulation. Neither considers asymptotic performance. In [21], the authors also propose a data-driven approach to chance constraints, but do not discuss either finite sample guarantees or asymptotic convergence. Using our hypothesis testing viewpoint, we are able to complement these existing works and establish a unified set of conditions under which the above methods will enjoy a finite-sample guarantee and/or be asymptotically convergent.

Recently, several other authors have considered hypothesis testing in certain, specific optimization contexts. In [5], the authors show how hypothesis tests can be used to construct uncertainty sets for robust, linear optimization problems, and establish a finite-sample guarantee that is similar in spirit to our own. They do not, however, consider asymptotic performance. In [1], the authors consider robust optimization
problems described by phi-divergences over uncertain, discrete probability distributions with finite support and provides tractable reformulations of these constraints. The authors mention that these divergences are related to GoF tests for discrete distributions, but do not explicitly explore asymptotic convergence of their approach to the full-information optimum or the case of continuous distributions. Similarly, in [23], the authors study a stochastic lot-sizing problem under discrete distributional uncertainty described by Pearson’s χ² GoF test and develop a dynamic programming approach to this particular problem. The authors establish conditions for asymptotic convergence for this problem but do not discuss finite sample guarantees.

By contrast, we provide a systematic study of GoF testing and data-driven DRO. By connecting these problems with the existing statistics literature, we provide a unified treatment of both discrete and continuous distributions, finite-sample guarantees, and asymptotic convergence. Moreover, our results apply in a general optimization context for a large variety of cost functions. We consider this viewpoint to both unify and extend these previous results.

1.2 Setup

In the remainder, we denote the support of ξ by Ξ. We assume Ξ ⊆ ℝᵈ is closed, and denote by ℙ(Ξ) the set of (Borel) probability distributions over Ξ. For any probability distribution F₀ ∈ ℙ(Ξ), F₀(A) denotes the probability of the event ξ ∈ A. To streamline the notation when d = 1, we let F₀(t) = F₀((−∞, t]). When d > 1 we also denote by F₀,i the univariate marginal distribution of the i-th component, i.e., F₀,i(A) = F₀(⟨ξ : ξ_i ∈ A⟩). We assume that X ⊆ ℝᵈ is closed and that for any x ∈ X, E_F₀[Σ(x; ξ)] < ∞ with respect to the true distribution, i.e., the objective function of the full-information stochastic problem (1) is well-defined.

When Ξ is unbounded, (3) may not admit an optimal solution. (We will see non-pathological examples of this behavior in Section 3.2.) To be completely formal in what follows, we first establish sufficient conditions for the existence of an optimal solution. First, recall the definition of equicontinuity:

**Definition 1** A set of functions ℋ = {h : ℝᵐ₁ → ℝᵐ₂} is equicontinuous if for any given x ∈ ℝᵐ₁ and ε > 0 there exists δ > 0 such that for all h ∈ ℋ, ||h(x) − h(x′)|| < ε for any x′ with ||x − x′|| < δ.

In words, equicontinuity generalizes the usual definition of continuity of a function to continuity of a set of functions.

Our sufficient conditions constitute an analogue of the classical Weierstrass Theorem for deterministic optimization (see, e.g., [3], pg. 669):

**Theorem 1** Suppose there exists x₀ ∈ X such that C(x₀; ℋ) < ∞ and that c(x; ξ) is equicontinuous in x over all ξ ∈ Ξ. If either X is compact or \( \lim_{||x|| \to \infty} c(x; \xi) = \infty \) for any ξ, then the optimal value π of (3) is finite and is achieved at some π ∈ X.

2 Goodness-of-Fit testing and Robust SAA

In this section, we provide a brief review of GoF testing as it relates to Robust SAA. For a more complete treatment, including the wider range of testing cases possible, we refer the reader to [12, 39].

Given IID data ξ¹, ..., ξᴺ and a distribution F₀, a GoF test considers the hypothesis

\[
H₀ : \text{The data } ξ¹, ..., ξᴺ \text{ were drawn from } F₀
\]

and rejects it if there is sufficient evidence against it, otherwise making no particular conclusion. A test is said to be of significance level α if the probability of incorrectly rejecting H₀ is at most α.

A typical test specifies a statistic

\[
S_N = S_N(F₀, ξ¹, ..., ξᴺ)
\]

that depends on the data ξ¹, ..., ξᴺ and hypothesis F₀ and also specifies a threshold Q_{Sₐ}(α) that depends only on α. The test rejects H₀ if S_N > Q_{Sₐ}(α).

The threshold Q_{Sₐ}(α) is usually the (1 − α)th quantile of the distribution of Sₐ under the assumption that the data have the distribution F₀. For some tests, Q_{Sₐ}(α) can be computed (or bounded) in closed-form. More generally, Q_{Sₐ}(α) can be approximated numerically using techniques like the bootstrap, in particular when it may depend on F₀ (see [18]). Implementations of bootstrap procedures for computing thresholds Q_{Sₐ}(α) are available in many popular software packages, e.g., the function `one.boot` in the [R] package `simpleboot`. 

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**Scientific Name**:&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&n
Fig. 1: A visualization of an example confidence region of the Kolmogorov-Smirnov test at significance 20%. The dashed curve is the true cumulative distribution function, that of a standard normal. The solid curve is the empirical cumulative distribution function having observed 100 draws from the true distribution. The confidence region contains all distributions with cumulative distribution functions that take values inside the grey region.

One example is the Kolmogorov-Smirnov (KS) test for univariate distributions. The KS test uses the statistic
\[ D_N = \max_{i=1,\ldots,N} \left\{ \max \left\{ \frac{i}{N} - F_0(\xi^{(i)}), F_0(\xi^{(i)}) - \frac{i-1}{N} \right\} \right\}. \]

Tables for \( Q_{DN}(\alpha) \) are widely available (see e.g. [12, 38]).

The set of all distributions \( F_0 \) that pass a test is called the confidence region of the test and is denoted by
\[ F_{SN}(\xi^1, \ldots, \xi^N) = \left\{ F_0 \in \mathcal{P}(\Xi) : S_N(F_0, \xi^1, \ldots, \xi^N) \leq Q_{SN}(\alpha) \right\}. \quad (6) \]

As an example, Figure 1 illustrates the confidence region of the KS test. Observe that by construction, the confidence region of a test with significance level \( \alpha \) is a DUS which contains the true, unknown distribution \( F \) with probability at least 1 - \( \alpha \).

2.1 The Robust SAA approach

Given data \( \xi^1, \ldots, \xi^N \), the Robust SAA approach involves the following steps:

1. Choose a significance level \( 0 < \alpha < 1 \) and goodness-of-fit test at level \( \alpha \) independently of the data.
2. Let \( \mathcal{F} = \mathcal{F}_N(\xi^1, \ldots, \xi^N) \) be the confidence region of the test.
3. Solve
\[ \pi = \arg \min_{\pi \in \mathcal{F}} \sup_{F_0 \in \mathcal{F}_N(\xi^1, \ldots, \xi^N)} \mathbb{E}_{F_0}[c(x; \xi)] \]
and let \( \pi \) be an optimal solution.

Section 5 illustrates how to solve the optimization problem in the last step for various choices of goodness-of-fit test and classes of cost functions.

2.2 Connections to existing methods

As observed in Section 2, we can use a GoF test at significance level \( \alpha \) to construct a DUS that contains the true distribution \( F \) with probability at least 1 - \( \alpha \) via its confidence region. It is possible to do the reverse as well. Given a data-driven DUS \( \mathcal{F}_N(\xi^1, \ldots, \xi^N) \) that contains the true distribution with probability at least 1 - \( \alpha \) with respect to the sampling distribution, we can construct a GoF test with significance level \( \alpha \) that rejects the hypothesis (5) whenever \( F_0 \notin \mathcal{F}_N(\xi^1, \ldots, \xi^N) \). This is often termed the duality between hypothesis tests and confidence regions (see for example §9.3 of [30]).
This reverse construction can be applied to existing data-driven DUSs in the literature such as [11, 13] to construct their corresponding hypothesis tests. In this way, hypothesis testing provides a common ground on which to understand and compare the methods.

In particular, the hypothesis tests corresponding to the DUSs of [11, 13] test only the first moments of the true distribution (cf. Section 4.3.2). By contrast, we will for the most part focus on tests (and corresponding confidence regions) that test the entire distribution, not just the first two moments. This feature is key to achieving both finite-sample and asymptotic guarantees.

3 Finite-sample performance guarantees

We first study the implication of a test’s significance on the finite-sample performance of Robust SAA. Let us define the following random variables expressible as functions of the data $\xi^1, \ldots, \xi^N$:

The DRO solution: $\pi \in \arg\min_{x \in X} \sup_{F_0 \in F_N(\xi^1, \ldots, \xi^N)} \mathbb{E}_{F_0} [c(x; \xi)].$

The DRO value: $\tau = \min_{x \in X} \sup_{F_0 \in F_N(\xi^1, \ldots, \xi^N)} \mathbb{E}_{F_0} [c(x; \xi)].$

The true cost of the DRO solution: $z = \mathbb{E}_F [c(\pi(\xi); \xi^1, \ldots, \xi^N)].$

The following is an immediate consequence of significance.

**Theorem 2** If $F_N(\xi^1, \ldots, \xi^N)$ is the confidence region of a valid GoF test at significance $\alpha$, then, with respect to the data sampling process,

$$P(\tau \geq z) \geq 1 - \alpha.$$  

**Proof** Suppose $F \in F_N$. Then $\sup_{F_0 \in F_N} \mathbb{E}_{F_0} [c(x; \xi)] \geq \mathbb{E}_F [c(x; \xi)]$ for any $x \in X$. Therefore, we have $\tau \geq z$. In terms of probabilities, this implication yields,

$$P(\tau \geq z) \geq P(F \in F_N) \geq 1 - \alpha. \qed$$

This makes explicit the connection between the statistical property of significance of a test with the objective performance of the corresponding Robust SAA decision in the full-information stochastic optimization problem.

Next we review the particular GoF tests we will employ.

3.1 Tests for distributions with known discrete support

When $\xi$ has known finite support $\Xi = \{\hat{\xi}^1, \ldots, \hat{\xi}^n\}$ there are two popular tests of GoF: Pearson’s $\chi^2$ test and the G-test (see [12]). Let $p(j) = F(\{\hat{\xi}^j\})$, $p_0(j) = F_0(\{\hat{\xi}^j\})$, and $\hat{p}_N(j) = \frac{1}{N} \sum_{i=1}^N 1[\xi^i = \hat{\xi}^j]$ be the true, hypothetical, and empirical probabilities of observing $\hat{\xi}^j$, respectively.

Pearson’s $\chi^2$ test uses the statistic

$$X_N = \left( \sum_{j=1}^n \frac{(p(j) - \hat{p}_N(j))^2}{p_0(j)} \right)^{1/2},$$

whereas the G-test uses the statistic

$$G_N = \left( 2 \sum_{j=1}^N \hat{p}_N(j) \log \left( \frac{\hat{p}_N(j)}{p_0(j)} \right) \right)^{1/2}.$$ 

The confidence regions of these take the form of (6) for $S_N$ being either $X_N$ or $G_N$. An illustration of these $(n = 3, N = 50)$ is given in Figure 2. Intuitively, these can be seen as being generalized balls around the empirical distribution $\hat{p}_N$. The metric is given by the statistic $S_N$ and the radius diminishes as $Q_{S_N}(\alpha) = O(N^{-1/2})$ (see [12]).
Fig. 2: Distributional uncertainty sets (projected onto the first two components) for the discrete case with \(n = 3, \alpha = 0.8, N = 50, p = (0.5, 0.3, 0.2)\). The dot denotes the true frequencies \(p\) and the triangle the observed fractions \(\hat{p}\).

### 3.2 Tests for univariate distributions

Suppose \(\xi\) is a univariate continuous random variable that is known to have lower support greater than \(\xi\) and upper support less than \(\xi\). These bounds could possibly be infinite. The most commonly used GoF tests in this setting are the Kolmogorov-Smirnov (KS) test, the Kuiper test, the Cramér-von Mises (CvM) test, the Watson test, and the Anderson-Darling (AD) test. The KS (\(D_N\)), the Kuiper (\(V_N\)), the CvM (\(W_N\)), the Watson (\(U_N\)), and the AD (\(A_N\)) tests use the statistics (see \([12, 38]\))

\[
D_N = \max_{i=1, \ldots, N} \left\{ \max \left\{ \frac{i}{N} - F_0(\xi^{(i)}), F_0(\xi^{(i)}) - \frac{i-1}{N} \right\} \right\},
\]

\[
V_N = \max_{1 \leq i \leq N} \left( F_0(\xi^{(i)}) - \frac{i-1}{N} \right) + \max_{1 \leq i \leq N} \left( \frac{i}{N} - F_0(\xi^{(i)}) \right),
\]

\[
W_N = \left( \frac{1}{12N^2} + \frac{1}{N} \sum_{i=1}^{N} \left( \frac{2i-1}{2N} - F_0(\xi^{(i)}) \right)^2 \right)^{1/2},
\]

\[
U_N = \left( W_N^2 - \frac{1}{N} \sum_{i=1}^{N} F_0(\xi^{(i)}) - \frac{1}{2} \right)^{1/2},
\]

\[
A_N = \left( -1 - \sum_{i=1}^{N} \frac{2i-1}{N^2} \left( \log F_0(\xi^{(i)}) + \log(1 - F_0(\xi^{(N+1-i)})) \right) \right)^{1/2}.
\]

We let \(S_N \in \{D_N, W_N, A_N, V_N, U_N\}\) be any one of the above statistics and \(Q_{S_N}(\alpha)\) the corresponding threshold. Tables for \(Q_{S_N}(\alpha)\) are widely available (see \([12, 38]\)). Moreover, \(Q_{S_N}(\alpha)\) can be computed by simulation as the \((1 - \alpha)^{th}\) percentile of the distribution of \(S_N\) when \(F_0(\xi)\) in (7) are replaced by IID uniform random variables on \([0, 1]\).

The confidence regions of these tests take the form of (6). Recall Figure 1 illustrated \(F_{D_N}\). As in the discrete case, \(F_{S_N}\) can also be seen as a generalized ball around the empirical distribution \(\hat{F}_N\). Again, the radius diminish as \(Q_{S_N}(\alpha) = O(N^{-1/2})\) (see \([12]\)).

When \(\xi\) and \(\xi\) are finite, we take \(F_{S_N}\) to be our DUS corresponding to these tests. When either \(\xi\) or \(\xi\) is infinite, however, \(\tau\) in (3) may also be infinite as seen in the following proposition.

**Proposition 1** Fix \(x, \alpha, \) and \(S_N \in \{D_N, W_N, A_N, V_N, U_N\}\). If \(c(x; \xi)\) is continuous but unbounded on \(\Xi\) then \(C(x; F_{S_N}^\alpha) = \infty\) almost surely.
The conditions of Proposition 1 are typical in many applications. For example, in [42], the authors briefly propose a data-driven DRO formulation of the newsvendor problem that is equivalent to our Robust SAA formulation using the KS test. Using Proposition 1, however, one can show that if the uncertain demand is supported on the positive real-line, the optimal value of this formulation is infinite. We will return to the data-driven newsvendor in Example 2 below.

Consequently, when either $\xi$ or $\overline{\xi}$ is infinite, we will employ an alternative, non-standard, GoF test in Robust SAA. The confidence region of our proposed test will satisfy the conditions of Theorem 1, and, therefore, (3) will attain a finite, optimal solution.

Our proposed test combines one of the above GoF tests with a second test for a generalized moment of the distribution. Specifically, fix any function $\phi : \Xi \to \mathbb{R}_+$ such that $E_F[\phi(\xi)] < \infty$ and $|c(x_0, \xi)| = O(\phi(\xi))$ for some $x_0 \in X$. For a fixed $\mu_0$, consider the null hypothesis

$$H'_0 : E_F[\phi(\xi)] = \mu_0.$$  \hspace{1cm} (8)

There are many possible hypothesis tests for (8). Any of these tests can be used as the second test in our proposal. For concreteness, we focus on a test with rejects (8) if

$$M_N = \left| \mu_0 - \frac{1}{N} \sum_{i=1}^{N} \phi(\xi^i) \right| > Q_{M_N}(\alpha).$$  \hspace{1cm} (9)

As mentioned in Section 2, the threshold $Q_{M_N}(\alpha)$ can be computed via the bootstrap. In our numerical experiments in Section 7.1 we approximate $Q_{M_N}(\alpha)$ as $\bar{\sigma} N T_{N-1}(\alpha/2)/\sqrt{N}$ where $T_{N-1}(\alpha/2)$ is the $(1 - \alpha/2)^{th}$ quantile of the Student-T distribution with $N-1$ degrees of freedom and $\bar{\sigma}^2$ is the sample variance of $\phi(\xi)$. This is a widely used approximation in statistics which is known to perform well in similar applications [30].

Given $0 < \alpha_1, \alpha_2 < 1$, combining $S_N$ and (9), we propose the following GoF test:

Reject $F_0$ if either $S_N > Q_{S_N}(\alpha_1)$ or $E_{F_0}[\phi(\xi)] - \frac{1}{N} \sum_{i=1}^{N} \phi(\xi^i) > Q_{M_N}(\alpha_2)$.

By the union bound, the probability of incorrectly rejecting $F_0$ is at most

$$\mathbb{P}(S_N > Q_{S_N}(\alpha_1)) + \mathbb{P} \left( \left| E_{F_0}[\phi(\xi)] - \frac{1}{N} \sum_{i=1}^{N} \phi(\xi^i) \right| > Q_{M_N}(\alpha_2) \right) \leq \alpha_1 + \alpha_2.$$

Thus, our proposed test has significance level $\alpha_1 + \alpha_2$.

The confidence region of the above test is given by the intersection of the confidence region of our original goodness-of-fit test and the confidence region of our test for (8):

$$F_{S_N,M_N}^{a_1,a_2} = F_{S_N}^{a_1} \cap F_{M_N}^{a_2} = \left\{ F_0 \in \mathcal{P}(\Xi) : S_N \leq Q_{S_N}(\alpha_1), \left| E_{F_0}[\phi(\xi)] - \frac{1}{N} \sum_{i=1}^{N} \phi(\xi^i) \right| \leq Q_{M_N}(\alpha_2) \right\}. $$  \hspace{1cm} (10)

Observe that since $|c(x_0; \xi)| = O(\phi(\xi))$, i.e., $\exists \nu, \eta$ such that $|c(x_0; \xi)| \leq \nu + \eta \phi(\xi)$, we have

$$C(x_0; F_{S_N,M_N}^{a_1,a_2}) = \sup_{F_0 \in F_{S_N,M_N}^{a_1,a_2}} \mathbb{E}_{F_0}[c(x_0; \xi)] \leq \nu + \eta \sup_{F_0 \in F_{S_N,M_N}^{a_1,a_2}} \mathbb{E}_{F_0}[\phi(\xi)] \leq \nu + \frac{\eta N}{N} \sum_{i=1}^{N} \phi(\xi^i) + \eta Q_{M_N}(\alpha_2) < \infty,$$

so that unlike $F_{S_N}^{a_1}$, our new confidence region $F_{S_N,M_N}^{a_1,a_2}$ does indeed satisfy the conditions of Theorem 1, even if $\xi$ or $\overline{\xi}$ are infinite.

3.3 Tests for multivariate distributions

In this section, we propose two different tests for the case $d \geq 2$. The first is a standard test based on testing marginal distributions. The second is a new test we propose that tests the full joint distribution.
3.3.1 Testing marginal distributions

Let $\alpha_1, \ldots, \alpha_d > 0$ be given such that $\alpha = \alpha_1 + \cdots + \alpha_d < 1$. Consider the test for the hypothesis $F = F_0$ that proceeds by testing the hypotheses $F_i = F_{0,i}$ for each $i = 1, \ldots, d$ by applying a test from the previous two sections at significance level $\alpha_i$ to the sample $\xi_i^1, \ldots, \xi_i^N$ and rejecting $F_0$ if any of these fail. The corresponding confidence region is

$$F_{\text{marginals}}^\alpha = \left\{ F_0 \in \mathcal{P}(\Xi) : F_{0,i} \in F_i^{\alpha_i}(\xi_i^1, \ldots, \xi_i^N) \forall i = 1, \ldots, d \right\},$$

where $F_i^{\alpha_i}$ denotes the confidence region corresponding to the test applied to the $i^{th}$ component. By the union bound we have

$$P\left( F \notin F_{\text{marginals}}^\alpha \right) \leq \sum_{i=1}^{d} P\left( F_i \notin F_i^{\alpha_i} \right) \leq \sum_{i=1}^{d} \alpha_i = \alpha,$$

so the test has significance $\alpha$.

3.3.2 Testing linear-convex ordering

In this section, we first provide some background on the linear-convex ordering (LCX) of random vectors first proposed in [33], and then use LCX to motivate a new GoF test for multivariate distributions. To the best of our knowledge, we are the first to propose GoF tests based on LCX.

Given two multivariate distributions $G$ and $G'$, we write

$$G \preceq_{\text{LCX}} G' \iff E_G[\phi(a^T \xi)] \leq E_{G'}[\phi(a^T \xi)] \quad \forall a \in \mathbb{R}^d \text{ and convex functions } \phi : \mathbb{R} \to \mathbb{R},$$

(11)

$$\iff E_G[\max\{a^T \xi - b, 0\}] \leq E_{G'}[\max\{a^T \xi - b, 0\}] \quad \forall |a_1| + \ldots + |a_d| + |b| \leq 1,$$

(12)

where the second equivalence follows from Theorem 3.A.1 of [34].

Our interest in LCX stems from the following result from [33]. Assuming $E_G[\|\xi\|_2^2] < \infty$,

$$E_G[\|\xi\|_2^2] \geq E_{G'}[\|\xi\|_2^2] \quad \text{and} \quad G \preceq_{\text{LCX}} G' \implies G = G'.$$

(13)

Equation (13) motivates our GoF test. Intuitively, the key idea of our test is that if $F \neq F_0$, i.e., we should reject $F_0$, then by (13) either $E_{F_0}[\|\xi\|_2^2] < E_F[\|\xi\|_2^2]$ or $F_0 \not\preceq_{\text{LCX}} F$. Thus, we can create a GoF test by testing for each of these cases separately.

More precisely, for a fixed $\mu_0$, first consider the hypothesis

$$H_0^\mu : E_F[\|\xi\|_2^2] = \mu_0.$$  

(14)

As in Section 3.2, there are many possible tests for (14). For concreteness, we focus on a one-tailed test which rejects (14) if $R_N = \frac{1}{N} \sum_{i=1}^{N} \|\xi_i\|_2^2 - \mu_0 > Q_{R_N}(\alpha)$, where $Q_{R_N}(\alpha)$ is a threshold which can be computed by bootstrapping.

Next, define the statistic

$$C_N(F_0) = \sup_{|a_1| + \cdots + |a_d| + |b| \leq 1} \left( E_{F_0}[\max\{a^T \xi - b, 0\}] - \frac{1}{N} \sum_{i=1}^{N} \max\{a^T \xi_i^1 - b, 0\} \right).$$

From (12), $C_N(F_0) \leq 0 \iff F_0 \preceq_{\text{LCX}} \hat{F}_N$. (Recall that $\hat{F}_N$ denotes the empirical distribution.)

Finally, combining these pieces and given $0 < \alpha_1, \alpha_2 < 1$, our LCX-based GoF test is

$$\text{Reject } F_0 \text{ if either } C_N(F_0) > Q_{C_N}(\alpha_1) \text{ or } E_{F_0}[\|\xi\|_2^2] < \frac{1}{N} \sum_{i=1}^{N} \|\xi_i\|_2^2 - Q_{R_N}(\alpha_2).$$

(15)

The threshold $Q_{C_N}(\alpha_1)$ can be computed by bootstrapping or exactly bounded explicitly. See Section 9.3 in the appendix for further discussion. In our numerical experiments in Section 7.3, we use bootstrapped thresholds.

From a union bound we have that the LCX-based GoF test (15) has significance level $\alpha_1 + \alpha_2$. The confidence region of the LCX-based GoF test is

$$F_{C_N,R_N}^{\alpha_1,\alpha_2} = \left\{ F_0 \in \mathcal{P}(\Xi) : C_N(F_0) \leq Q_{C_N}(\alpha_1), E_{F_0}[\|\xi\|_2^2] \geq \frac{1}{N} \sum_{i=1}^{N} \|\xi_i^2\|_2 - Q_{R_N}(\alpha_2) \right\}.$$
4 Convergence

Had we known the true distribution $F$ we would solve problem (1). As we gather more data, we know more and more about $F$. Therefore, it is clearly desirable that our decisions converge to the optimal solutions of (1).

In this section, we study the relationship between the GoF test underlying an application of Robust SAA and convergence properties of the Robust SAA optimal values $\Sigma$ and solutions $\mathcal{F}$. Recall from Section 2.2 that since many existing DRO formulations can be recast as confidence regions of hypothesis tests, our analysis will simultaneously also allow us to study the convergence properties of these methods as well.

The convergence conditions we seek are

\begin{align*}
\text{Convergence of objective function:} & \quad \mathcal{C}(x; \mathcal{F}_N) \to E_F[c(x; \xi)] \quad (17) \\
& \quad \text{uniformly over any compact subset of } X, \\
\text{Convergence of optimal values:} & \quad \min_{x \in X} \mathcal{C}(x; \mathcal{F}_N) \to \min_{x \in X} E_F[c(x; \xi)], \quad (18) \\
\text{Convergence of optimal solutions:} & \quad \text{Every sequence } x_N \in \arg \min_{x \in X} \mathcal{C}(x; \mathcal{F}_N) \text{ has at least one limit point, and all of its limit points are in } \arg \min_{x \in X} E_F[c(x; \xi)], \quad (19)
\end{align*}

all holding almost surely (a.s.). The key to these will be a restricted form of statistical consistency that we term uniform consistency.

4.1 Uniform consistency and convergence of optimal solutions

In statistics, consistency of a test (see Def. 2 below) is a well-studied property that a GoF test may exhibit. In this section, we define a new property of GoF tests that we call uniform consistency. Uniform consistency is a strictly stronger property than consistency, in the sense that every uniformly consistent test is consistent, but some consistent tests are not uniformly consistent. More importantly, we will prove that uniform consistency of the underlying GoF test tightly characterizes when conditions (17)-(19) hold. In particular, we show that when $X$ and $\Sigma$ are bounded, uniform consistency of the underlying test implies conditions (17)-(19) for any cost function $c(x, \xi)$ which is equicontinuous in $x$, and if the test is not uniformly consistent, then there exist cost functions (equicontinuous in $x$) for which conditions (17)-(19) do not hold. When $X$ or $\Sigma$ are unbounded, the same conclusions hold for all cost functions which are equicontinuous in $x$ and satisfy an additional, mild, regularity condition. (See Theorem 3 for a precise statement.) In other words, we can characterize the convergence of Robust SAA and other data-driven, DRO formulations by studying if their underlying GoF test is uniformly consistent. In our opinion, these results highlight a new, fundamental connection between statistics and data-driven optimization. We will use this result to assess the strength of various DRO formulations for certain applications in what follows.

First, we recall the definition of consistency of a GoF test (cf. entry for consistent test in [14]):

**Definition 2** A GoF test is consistent if, for every $F_0 \neq F$, the probability of rejecting $F_0$ approaches 1 as $N \to \infty$.

Observe

**Proposition 2** If a test is consistent, then any $F_0 \neq F$ is a.s. rejected infinitely often (i.o.) as $N \to \infty$.

**Proof**

\[ P(F_0 \text{ rejected i.o.}) = P \left( \limsup_{N \to \infty} \{ F_0 \notin \mathcal{F}_N \} \right) \geq \limsup_{N \to \infty} P(F_0 \notin \mathcal{F}_N) = 1, \]

where the first inequality follows from Fatou’s Lemma, and the second since the test is consistent. $\square$

Consistency describes the test’s behavior with respect to a single, fixed distribution $F_0$. In particular, the conclusion of Proposition 2 holds only when we consider the same, fixed distribution $F_0$ for each $N$. We would like to extend consistency to describe the test’s behavior with respect to many alternatives $F_0$ simultaneously. Motivated by an alternate definition of local uniform convergence,\(^3\) we define uniform consistency by requiring that a condition similar to the conclusion of Proposition 2 hold for almost every sequence of distributions:

\[ \text{Recall: A sequence of functions } g_n : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2} \text{ converges locally uniformly to a continuous function } g : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2} \text{ if and only if for any convergent sequence } x_n \to x \text{ we have that } g_n(x_n) \to g(x). \]
Definition 3 A GoF test is uniformly consistent if, a.s., every sequence $F_N$ that does not converge weakly to $F$ is rejected i.o.

The requirement that $F_N$ does not converge weakly to $F$ parallels the requirement that $F_0 \neq F$.

Uniform consistency is a strictly stronger requirement than consistency.

Proposition 3 If a test is uniformly consistent, then it is consistent. Moreover, there exist tests which are consistent, but not uniformly consistent.

Uniform consistency is the key property for the convergence of Robust SAA. Besides uniform consistency, convergence will be contingent on three assumptions.

Assumption 1 $c(x; \xi)$ is equicontinuous in $x$ over all $\xi \in \Xi$.

Assumption 2 $X$ is closed and either
a. $X$ is bounded or
b. $\lim_{\|x\| \to \infty} c(x; \xi) = \infty$ uniformly over $\xi$ in some $D \subseteq \Xi$ with $F(D) > 0$ and $\liminf \inf_{\|x\| \to \infty} c(x; \xi) > -\infty$.

Assumption 3 Either
a. $\Xi$ is bounded or
b. $\exists \phi : \Xi \to \mathbb{R}_+$ such that $\sup_{F_0 \in \mathcal{F}_N} \left| \frac{1}{N} \sum_{i=1}^{N} \phi(\xi^i) \right| \to 0$ almost surely and $c(x; \xi) = O(\phi(\xi))$ for each $x \in X$.

Assumptions 1 and 2 are only slightly stronger than those required for the existence of an optimal solution in Theorem 1. The second portion of Assumption 2b is trivially satisfied by cost functions which are bounded from below. Finally, observe that in the case that $\Xi$ is unbounded, our proposed DUS in (10) satisfies Assumption 3b by construction.

Under these assumptions, the following theorem provides a tight characterization of convergence.

Theorem 3 Assumptions 1, 2, and 3 imply conditions (17)-(19) hold a.s. if and only if $F_N$ is the confidence region of a uniformly consistent test.

Thus, in one direction, we can guarantee convergence (i.e., conditions (17)-(19) hold a.s.) if Assumptions 1, 2, and 3 are satisfied and we use a uniformly consistent test in applying Robust SAA. In the other direction, if the test is not uniformly consistent, there will exist instances satisfying Assumptions 1, 2, and 3 for which convergence fails.

Some of the GoF tests in Section 3 are not consistent, and therefore, cannot be uniformly consistent. By Theorem 3, DROs built from these tests cannot exhibit asymptotic convergence for all cost functions. One might argue, then, that these DRO formulations should be avoided in modeling and applications in favor of DROs based on uniformly consistent tests.

In most applications, however, we are not concerned with asymptotic convergence for all cost functions, but rather only for the given cost function $c(x; \xi)$. It may happen a DRO may exhibit asymptotic convergence

Table 1: Summary of convergence results.

<table>
<thead>
<tr>
<th>GoF test</th>
<th>Support</th>
<th>Uniformly consistent</th>
<th>(17)-(19) hold a.s. for any $c(x; \xi)$ that is equicontinuous in $x$</th>
<th>separable (20)</th>
<th>as in (25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^2$ and G-test</td>
<td>Finite</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>KS, Kuiper, CvM, Watson, and AD tests</td>
<td>Univariate</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Test of marginals using the above tests</td>
<td>Multivariate</td>
<td>No</td>
<td>No*</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>LCX-based test</td>
<td>Multivariate</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Tests implied by DUSs of [11, 13]</td>
<td>Multivariate</td>
<td>No</td>
<td>No*</td>
<td>No*</td>
<td>Yes</td>
</tr>
</tbody>
</table>

* denotes the result is tight in the sense that there are examples in this class that do not converge.
for this particular cost function, even when its DUS is given by the confidence region of an inconsistent test. (We will see an example of this behavior with the multi-item newsvendor problem in Section 7.2.)

To better understand when this convergence may occur despite the fact that the test is not consistent, we introduce a more relaxed form of uniform consistency.

**Definition 4** Given \( c(x; \xi) \), we say that \( F_N \) \( c \)-converges to \( F \) if \( \mathbb{E}_{F_N}[c(x; \xi)] \to \mathbb{E}_F[c(x; \xi)] \) for all \( x \in X \).

**Definition 5** A test is \( c \)-consistent if, a.s., every sequence \( F_N \) that does not \( c \)-converge to \( F \) is rejected i.o.

This notion may potentially be weaker than consistency, but is sufficient for convergence for a given instance as shown below.

**Theorem 4** Suppose Assumptions 1 and 3 hold and that \( F_N \) always contains the empirical distribution. If \( F_N \) is the confidence region of a \( c \)-consistent test, then conditions (17)-(19) hold a.s.

In the next sections we will explore the consistency of the various tests introduced in Section 3. We summarize our results in Table 1.

### 4.2 Tests for distributions with discrete or univariate support

All of the classical tests we considered in Section 3 are uniformly consistent.

**Theorem 5** The \( \chi^2 \) and G-tests are uniformly consistent.

**Theorem 6** The KS, Kuiper, CvM, Watson, and AD tests are uniformly consistent.

### 4.3 Tests for multivariate distributions

#### 4.3.1 Testing marginal distributions

We first claim that the test of marginals is not consistent. Indeed, consider a multivariate distribution \( F_0 \neq F \) which has the same marginal distributions, but a different joint distribution. By construction, the probability of rejecting \( F_0 \) is at most \( \alpha \) for all \( N \), and hence does not converge to 1. Since the test of marginals is not consistent, it cannot be uniformly consistent.

We next show that the test is, however, \( c \)-consistent whenever the cost is separable over the components of \( \xi \).

**Proposition 4** Suppose \( c(x; \xi) \) is separable over the components of \( \xi \), that is, can be written as

\[
 c(x; \xi) = \sum_{i=1}^{d} c_i(x; \xi_i), \tag{20}
\]

and Assumptions 1, 2, and 3 hold for each \( c_i(x; \xi_i) \). Then, the test of marginals is \( c \)-consistent if each univariate test is uniformly consistent.

That is to say, if the cost can be separated as in (20), applying the tests from Section 3.2 to the marginals is sufficient to guarantee convergence.

It is important to note that some cost functions may only be separable after a transformation of the data, potentially into a space of different dimension. If that is the case, we may transform \( \xi \) and apply the tests to the transformed components in order to achieve convergence.

#### 4.3.2 Tests implied by DUSs of [11, 13]

The DUS of [13] has the form

\[
 F_{BY,N} = \left\{ F_0 \in \mathcal{P}(\Xi) : \begin{array}{c}
 (\mathbb{E}_{F_0}[\xi] - \hat{\mu}_N)^T \hat{\Sigma}_N^{-1} (\mathbb{E}_{F_0}[\xi] - \hat{\mu}_N) \leq \gamma_1,N(\alpha), \\
 \gamma_3,N(\alpha) \hat{\Sigma}_N \preceq \mathbb{E}_{F_0}[\xi^T (\xi - \hat{\mu}_N)(\xi - \hat{\mu}_N)^T] \preceq \gamma_2,N(\alpha) \hat{\Sigma}_N
\end{array} \right\} \tag{21}
\]

where \( \hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} \xi_i, \hat{\Sigma}_N = \frac{1}{N} \sum_{i=1}^{N} (\xi_i - \hat{\mu}_N)(\xi_i - \hat{\mu}_N)^T. \tag{22} \)
The thresholds $\gamma_{1,N}(\alpha), \gamma_{2,N}(\alpha), \gamma_{3,N}(\alpha)$ are developed therein (for $\Xi$ bounded) so as to guarantee a significance of $\alpha$ (in our GoF interpretation) and, in particular, have the property that

$$0 \leq \gamma_{1,N}(\alpha) \rightarrow 0, \ 1 \leq \gamma_{2,N}(\alpha) \rightarrow 1, \ 1 \geq \gamma_{3,N}(\alpha) \rightarrow 1.$$  \hspace{1cm} (23)

The DUS of [11] has the form

$$F_{\text{CEG},N} = \left\{ F_0 \in \mathcal{P}(\Xi) : \frac{\|E_{F_0}[\xi] - \bar{\mu}_N\|_F}{\Sigma N} \leq \gamma_{1,N}(\alpha), \|E_{F_0}[(\xi - E_{F_0}[\xi])(\xi - E_{F_0}[\xi])^T] - \Sigma N\|_{\text{Frobenius}} \leq \gamma_{2,N}(\alpha) \right\}.$$  \hspace{1cm} (24)

The thresholds $\gamma_{1,N}(\alpha), \gamma_{2,N}(\alpha)$ are developed in [37] (for $\Xi$ bounded) so as to guarantee a significance of $\alpha$ and with the property that

$$0 \leq \gamma_{1,N}(\alpha) \rightarrow 0, \ 0 \leq \gamma_{2,N}(\alpha) \rightarrow 0.$$  \hspace{1cm} (25)

The GoF tests implied by these DUSs consider only the first two moments of a distribution (mean and covariance). Therefore, the probability of rejecting a multivariate distribution different from the true one but with the same mean and covariance is by construction never more than $\alpha$, instead of converging to 1. That is, these tests are not consistent and therefore they are not uniformly consistent. We next provide conditions on the cost function that guarantee that the tests are nonetheless $c$-consistent.

**Proposition 5** Suppose $c(x; \xi)$ can be written as

$$c(x; \xi) = c_0(x) + \sum_{i=1}^{d} c_i(x)\xi_i + \sum_{i=1}^{d} \sum_{j=1}^{d} c_{ij}(x)\xi_i\xi_j$$  \hspace{1cm} (26)

and that $E_F[\xi_i\xi_j]$ exists. Then, the tests with confidence regions given by $F_{\text{DY},N}$ or $F_{\text{CEG},N}$ are $c$-consistent.

Note that because we may transform the data to include components for each pairwise multiplication, the conditions on the cost function in Proposition 5 are stronger than those in Proposition 4. In particular, in one dimension, separability is trivially always true whereas the decomposition (25) is clearly not.

### 4.3.3 Testing linear-convex ordering

The previous two multivariate GoF tests were neither consistent, nor uniformly consistent. By contrast,

**Proposition 6** The LCX-based test is consistent.

**Proposition 7** Suppose $\Xi$ is bounded. Then the LCX-based test is uniformly consistent.

It is an open question whether the LCX-based test is uniformly consistent – in addition to being consistent – for unbounded $\Xi$. We conjecture that it is. Moreover, in our numerical experiments involving the LCX test, we have observed convergence of the Robust SAA solutions to the full-information optimum even when $\Xi$ is unbounded. (See Section 7.3 for an example.)

### 5 Tractability

In this section, we characterize conditions under which problem (3) is theoretically tractable, i.e., can be solved with a polynomial-time algorithm. Additionally, we are interested in cases where (3) is practically tractable, i.e., can be solved using off-the-shelf linear or second-order cone optimization solvers. In the case of one problem – the newsvendor problem – we show that Robust SAA using the KS test admits a closed-form solution.
5.1 Tests for distributions with known discrete support

We begin this section with a reformulation of (3) as a single-level optimization problem for $F_{X_N}^*$ and $F_{G_N}^*$, from which tractability results will follow. The confidence regions of the discrete GoF tests we consider are a special case of those considered in [1]. As direct corollaries of the results therein we have the following:

**Theorem 7** Under the assumptions of Theorem 1, we have

$$C(x; F_{X_N}^*) = \min_{r,s,t,c} \left( \frac{1}{2} (Q_{X_N}(\alpha))^2 - 1 \right) s - \sum_{j=1}^{n} \hat{p}_N(j)t_j$$

s.t. \quad r \in \mathbb{R}, \quad s \in \mathbb{R}_+, \quad t \in \mathbb{R}^n, \quad c \in \mathbb{R}^n

\begin{align*}
& s + r \geq c_j, \quad \forall j = 1, \ldots, n \\
& (2s - c_j - r, 2t_j, c_j - r) \in C_{SOC}^3 \\
& c_j \geq c \left( x; \hat{\xi}^j \right), \quad \forall j = 1, \ldots, n
\end{align*}

where $C_{SOC}^3 = \left\{ (x,y,z) \in \mathbb{R}^3 : x \geq \sqrt{y^2 + z^2} \right\}$ is the three-dimensional second-order cone and $C_{XC} = \left\{ (x,y,z) : ye^{x/y} \leq z, \quad y > 0 \right\}$ is the exponential cone.

The DRO problem (3) is $\min_{x \in X} C(x; F)$. Therefore, for $F_{X_N}^*$ and $F_{G_N}^*$, (3) can formulated as a single-level optimization problem by augmenting the corresponding minimization problem above with the control variable $x \in X$. Note that apart from the constraints $x \in X$ and

$$c_j \geq c \left( x; \hat{\xi}^j \right), \quad \forall j = 1, \ldots, n$$

the rest of the constraints, as seen in the problems in Theorem 7, are convex. The following result characterizes in general when solving these problems is tractable in a theoretical sense.

**Theorem 8** Suppose that $X \subseteq \mathbb{R}^{d_x}$ is a closed convex set for which a weak separation oracle is given and that

$$c \left( x; \hat{\xi}^j \right) = \max_{k=1, \ldots, K_j} c_{jk}(x)$$

where each $c_{jk}(x)$ is a convex function in $x$ for which evaluation and subgradient oracles are given. Then, under the assumptions of Theorem 1, we can find an $\epsilon$-optimal solution to (3) in the discrete case for $S_N = X_N, G_N$ in time and oracle calls polynomial in $n, d_x, K_1, \ldots, K_n, \log(1/\epsilon)$.

For some problems the constraints $x \in X$ and (27) can also be conically formulated as the Example 1 below shows. In such a case, the DRO can be solved directly as a conic optimization problem. Optimization over the exponential cone — a non-symmetric cone — although theoretically tractable, is numerically challenging. Fortunately, the particular exponential cone constraints (26) can be recast as second-order cone constraints, albeit with constraint complexity growing in both $n$ and $N$ (see [27]).

**Example 1** Two-stage problem with linear recourse and a non-increasing, piece-wise-linear convex disutility.

Consider the following problem

$$c(x; \hat{\xi}) = \max_{k=1, \ldots, K} \left( \gamma_k R_j(x) + \beta_k \right), \quad \gamma_k \leq 0$$

where $R_j(x) = \min_{y \in \mathbb{R}^{d_y}} f_j^T y$

s.t. $A_jx + B_jy = b_j$

$X = \{ x \geq 0 : Hx = h \}.$
This problem was studied in a non-data-driven DRO settings in [4, 17, 44]. To formulate (3), we may introduce variables $y \in \mathbb{R}_+^{n \times d_y}$ and replace (27) with

$$
c_j \geq \gamma_k (c^T x + f_j^T y_j) + \beta_k \quad \forall j = 1, \ldots, n, \forall k = 1, \ldots, K,
$$

$$
A_j x + B_j y_j = b_j \quad \forall j = 1, \ldots, n.
$$

The resulting problem is then a second-order cone optimization problem for $F^2_{\mathcal{X}}$ and $F^2_{\mathcal{O}}$.

5.2 Tests for univariate distributions

We now consider the case where $\xi$ is a general univariate random variable. We proceed by reformulating (3) as a single-level optimization problem by leveraging semi-infinite conic duality. This leads to corresponding tractability results. In the following we will use the notation $\xi^{(0)} = \xi$ and $\xi^{(N+1)} = \xi$.

The first observation is that the constraint $S_N (\zeta_1, \ldots, \zeta_N) \leq Q_{S_N} (\alpha)$ is convex in $\zeta = F_0 (\xi^{(0)})$ and representable using canonical cones. By a canonical cone, we mean any cartesian product of the cones $\mathbb{R}, \{0\}, \mathbb{R}_+^k$ (positive orthant), $C_{SO_2}$ (second-order cone), and semidefinite cone. Optimization over canonical cones is tractable both theoretically and practically using state-of-the-art interior point algorithms [2].

**Theorem 9** For each of $S_N \in \{D_N, V_N, W_N, U_N, A_N\}$

$$
S_N (\zeta_1, \ldots, \zeta_N) \leq Q_{S_N} (\alpha) \iff A_S \zeta - b_{S_N, \alpha} \in K_{S_N}
$$

for convex cones $K_{S_N}$, matrices $A_{S_N}$, and vectors $b_{S_N, \alpha}$ as follows:

$$
K_{D_N} = \mathbb{R}_+^{2N}, \quad b_{D_N, \alpha} = \begin{pmatrix} \frac{1}{N} - Q_{D_N} (\alpha) \\ \vdots \\ - \frac{N-1}{N} - Q_{D_N} (\alpha) \end{pmatrix}, \quad A_{D_N} = \begin{pmatrix} [N] \\ [-N] \end{pmatrix},
$$

$$
K_{V_N} = \{ (x, y) \in \mathbb{R}^{2N} : \min_i x_i + \min_i y_i \geq 0 \}, \quad b_{V_N, \alpha} = \begin{pmatrix} \frac{1}{N} - Q_{V_N} (\alpha)/2 \\ \vdots \\ - \frac{N-1}{N} - Q_{V_N} (\alpha)/2 \end{pmatrix}, \quad A_{V_N} = \begin{pmatrix} [N] \\ [-N] \end{pmatrix},
$$

$$
K_{W_N} = C_{SO_2}^{N+1}, \quad b_{W_N, \alpha} = \begin{pmatrix} \sqrt{N (Q_{W_N} (\alpha))^2 - \frac{1}{2N}} \\ \frac{1}{2N} \\ \vdots \\ \frac{2N-1}{2N} \end{pmatrix}, \quad A_{W_N} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots \end{pmatrix},
$$

$$
K_{U_N} = C_{SO_2}^{N+2}, \quad b_{U_N, \alpha} = \begin{pmatrix} \frac{1}{2} + \left( \frac{N}{2} - \frac{N}{2} (Q_{U_N} (\alpha))^2 \right) \\ \frac{1}{2} - \left( \frac{1}{2} - \frac{N}{2} (Q_{U_N} (\alpha))^2 \right) \\ 0 \\ \vdots \end{pmatrix}, \quad A_{U_N} = \begin{pmatrix} \frac{1}{2} & \frac{3-N}{2N} & \cdots & \frac{N-1}{2N} \\ \frac{1}{2} & \frac{3-N}{2N} & \cdots & \frac{N-1}{2N} \\ 0 & \cdots & 0 \vdots \end{pmatrix},
$$

$$
K_{A_N} = \left\{ (z, x, y) \in \mathbb{R} \times \mathbb{R}_+^{2N} : |z| \leq \prod_{i=1}^{N} (x_i y_i)^{2^{2-N}/(2N)} \right\}, \quad b_{A_N, \alpha} = \begin{pmatrix} e^{-(Q_{A_N} (\alpha))^2 - 1} \\ 0 \\ \vdots \end{pmatrix}, \quad A_{A_N} = \begin{pmatrix} [N] \\ [-I_N] \end{pmatrix},
$$

$$
K_{F_N} = \left\{ (z, x, y) \in \mathbb{R} \times \mathbb{R}_+^{2N} : |z| \leq \prod_{i=1}^{N} (x_i y_i)^{2^{2-N}/(2N)} \right\}, \quad b_{F_N, \alpha} = \begin{pmatrix} e^{-(Q_{F_N} (\alpha))^2 - 1} \\ 0 \\ \vdots \end{pmatrix}, \quad A_{F_N} = \begin{pmatrix} [N] \\ [-I_N] \end{pmatrix}.
$$
where $I_N$ is the $N \times N$ identity matrix, $\tilde{I}_N$ is the skew identity matrix ($[\tilde{I}_N]_{ij} = I[i = N - j]$), and $E_N$ is the $N \times N$ matrix of all ones.

Note that the cones $K_{D_N}, K_{W_N}, K_{U_N}$ are canonical cones. The other cones can be expressed using canonical cones. The cone $K_{V_N}$ is an orthogonal projection of an affine slice of $\mathbb{R}^{2n+2} \times \mathbb{R}^3$. The cone $K_{A_N}$ is an orthogonal projection of an affine slice of the product of $2^{[\log_2(2N^2)]+1} - 2 = O(N^2)$ three-dimensional second-order cones (see [27]). Therefore, the constraint $A_N \zeta - b_{N, \alpha} \in K_{S_N}$ can be expressed using canonical cones in each case.

Problem (3) is a two-level optimization problem. To formulate it as a single-level problem, we dualize the inner problem, $C(x; F)$. For a cone $K \subseteq \mathbb{R}^k$, we use the notation $K^*$ to denote the dual cone $K^* = \{y \in \mathbb{R}^k : y^T z \geq 0 \forall z \in K\}$. The following is a direct consequence of Proposition 3.4 of [35].

**Theorem 10** Let $S_N \in \{D_N, W_N, V_N, U_N\}$. Under the assumptions of Theorem 1,  
\[
C \left( x; \mathcal{F}_{SN}^x \right) = \min_{r,t,c} b_{SN, \alpha}^T r + c_{N+1} + \sup_{x, \xi} \left( A_{SN}^T r \right) = c_i - c_{i+1} \quad \forall i = 1, \ldots, N
\]

\[
C \left( x; \mathcal{F}_{SN, M_N}^{\alpha_1, \alpha_2} \right) = \min_{r,t,c} b_{SN, \alpha_1}^T r + c_{N+1} + \sup_{x, \xi} \left( A_{SN}^T r \right) = c_i - c_{i+1} \quad \forall i = 1, \ldots, N
\]

Note that the cones $K_{D_N}, K_{W_N}, K_{U_N}$ are self-dual ($K^* = K$) and therefore the dual cones remain canonical cones. For $K_{V_N}$ and $K_{A_N}$, the dual cones are

\[
K_{V_N}^* = \left\{ (x, y) \in \mathbb{R}^{2N} : \sum_{i=1}^N x_i = \sum_{i=1}^N y_i \right\}
\]

\[
K_{A_N}^* = \left\{ (z, x, y) : (z, x, y) \in K_{A_N} \right\}
\]

and therefore they remain expressible using canonical cones.

Note that in the case of $\mathcal{F}_{SN}^x$, the worst-case distribution has discrete support on no more than $N + 1$ points. This is because shifting probability mass inside the interval $[\xi^{(i-1)}, \xi^{(i)}]$ does not change any $F_0(\xi^{(i)})$. In the worst-case, all mass in the interval (if any) will be placed on the point in the interval with the largest cost (including the left endpoint in the limit).

The DRO problem (3) is min$_{x \in X} C(x; F)$. Therefore, for $\mathcal{F}_{SN}^x$ and $\mathcal{F}_{SN, M_N}^{\alpha_1, \alpha_2}$, (3) can be formulated as a single-level optimization problem by augmenting the corresponding minimization problem above with the control variable $x \in X$. We next give general conditions that ensure the theoretical tractability of the problem.

**Theorem 11** Suppose that $X \subseteq \mathbb{R}^{d_x}$ is a closed convex set for which a weak separation oracle is given and that

\[
c(x; \xi) = \max_{k=1, \ldots, K} c_k(x; \xi)
\]

where each $c_k(x; \xi)$ is convex in $x$ for each $\xi$ and continuous in $\xi$ for each $x$ and for which an oracle is given for the subgradient in $x$. If $F = \mathcal{F}_{SN}^x$, suppose also that an oracle is given for maximizing $c_k(x; \xi)$ over $x$ in any closed (possibly infinite) interval for fixed $x$. If $F = \mathcal{F}_{SN, M_N}^{\alpha_1, \alpha_2}$, suppose also that an oracle is given for maximizing $c_k(x; \xi) + \nabla \phi(\xi)$ over $\xi$ in a closed interval for fixed $x$ and $\eta \in \mathbb{R}$. Then, under the assumptions of Theorem 1, we can find an $\epsilon$-optimal solution to (3) in time and oracle calls polynomial in $N, d_x, K, \log(1/\epsilon)$ for $F = \mathcal{F}_{SN}^x$ or $F = \mathcal{F}_{SN, M_N}^{\alpha_1, \alpha_2}$.
As in the discrete case, when the constraints \( x \in X \) and (29) (or, (30)) can be conically formulated, Theorem 10 provides an explicit single-level conic optimization formulation of the problem (3). In Examples 2, 3, and 4 below, we consider specific problems for which this is the case and study this formulation.

**Example 2 The newsvendor problem.** In the newsvendor problem, one orders in advance \( x \geq 0 \) units of a product to satisfy an unknown future demand for \( \xi \geq 0 \) units. Unmet demand is penalized by \( b > 0 \), representing either backlogging costs or lost profit. Left over units are penalized by \( h > 0 \), representing either holding costs or recycling costs. The cost function is therefore \( c(x; \xi) = \max \{ b(\xi - x), h(x - \xi) \} \), the lower support of \( \xi \) is \( \xi \geq 0 \), and the space of controls is \( X = \mathbb{R}_+ \). In this case the constraints (29) for bounded-support case become

\[
  c_i \geq b(\xi^{(i)} - x), \quad c_i \geq h(x - \xi^{(i-1)}) \quad \forall i = 1, \ldots, N + 1
\]

and \( x \in X \) becomes \( x \in \mathbb{R}_+ \). In the unbounded case, we may use \( \phi(\xi) = |\xi| \) in the construction of (10). Because \( \xi \geq 0 \), we have \( \phi(\xi) = \xi \). The constraints (30) then become

\[
  c_i \geq b(\xi^{(i)} - x) - (t - s)\xi^{(i)}, \quad c_i \geq h(x - \xi^{(i-1)}) - (t - s)\xi^{(i-1)} \quad \forall i = 1, \ldots, N + 1
\]

where the \((N + 1)^{th}\) left constraint is equivalent to \( b \leq t - s \) because \( \xi^{(N+1)} = 0 \). Substituting these constraints in this way the DRO (3) becomes a conic optimization problem.

In the specific case of bounded support and \( F = F_{\mathcal{D}_N} \), this reformulation yields a linear optimization problem, which admits a closed-form solution given next.

**Proposition 8** Suppose that \( \Xi = [\xi_1, \xi] \) is compact, and \( N \) is large enough so that \( Q_{\mathcal{D}_N}(\alpha) < \min \{ b, h \} \). Then, the DRO (3) for the newsvendor problem with \( F = F_{\mathcal{D}_N} \) admits the closed-form solution:

\[
  \pi = (1 - \theta)\xi^{(i_0)} + \theta\xi^{(i_1)}
\]

\[
  \pi = \frac{1}{N} \sum_{1 \leq i \leq i_0, i_1 \leq i \leq N} c(\pi; \xi^{(i)}) + Q_{\mathcal{D}_N}(\alpha) c(\pi; \xi) + Q_{\mathcal{D}_N}(\alpha) c(\pi; \xi)
\]

\[
  - \left( \frac{N(\theta - Q_{\mathcal{D}_N}(\alpha))}{N} \right) c(\pi; \xi^{(i_0)}) - \left( \theta + Q_{\mathcal{D}_N}(\alpha) \right) c(\pi; \xi^{(i_1)})
\]

where \( \theta = b/(b + h) \), \( i_0 = \lfloor N(\theta - Q_{\mathcal{D}_N}(\alpha)) \rfloor \), and \( i_1 = \lfloor N(\theta + Q_{\mathcal{D}_N}(\alpha)) \rfloor \).

Important, this means that solving the Robust SAA newsvendor problem is no more difficult than solving the SAA newsvendor problem.

**Example 3 Max of bilinear functions.** More generally, we may consider cost functions of the form (31) with bilinear parts \( c_k(x; \xi) = p_{k0} + p_{k1}^T x + p_{k2}^T \xi + p_{k3}^T x \). In this case, (29) is equivalent to

\[
  c_i \geq p_{k0} + p_{k1}^T x + p_{k2}^T \xi + p_{k3}^T x, \quad \forall i = 1, \ldots, N, \forall k = 1, \ldots, K
\]

\[
  c_i \geq p_{k0} + p_{k1}^T x + p_{k2}^T \xi + p_{k3}^T x, \quad \forall i = 1, \ldots, N, \forall k = 1, \ldots, K
\]

If the cost is fully linear, \( p_{ik} = 0 \) (as in the case of the newsvendor example), then (29) can be written in one linear inequality:

\[
  c_i \geq p_{k0} + \max \left( p_{k2}^T \xi + p_{k3}^T x \right) \quad \forall i = 1, \ldots, N, \forall k = 1, \ldots, K
\]

For \( F = F^{\gamma, \alpha}_{\mathcal{S}_N, M_N} \) we may use \( \phi(\xi) = |\xi| \) and simply add \( |\xi^{(i-1)}| \) and \( |\xi^{(i)}| \) to the left-hand sides of (32) and (33), respectively, or to the corresponding branches of the max in (34).

**Example 4 Two-stage problem.** Consider a two-stage problem similar to the one studied in Example 1:

\[
  c(x; \xi) = \max_{k = 1, \ldots, K} \left( \gamma_k R(x; \xi) + \beta_k \right), \quad \gamma_k \leq 0
\]

where \( R(x; \xi) = \min_{y \in \mathbb{R}_+^p} (f + g\xi)^T y \)

\[
  \text{s.t. } Ax + By = b + p\xi
\]

\[
  X = \{ x \geq 0 : Hx = h \}.
\]
When only the right-hand-side vector is uncertain \((g = 0)\), the recourse \(R(x; \xi)\) is convex in \(\xi\) so that the supremum in (29) is taken at one of the endpoints and we may use a similar construction as in Example 3.

When only the cost vector is uncertain \((p = 0)\), the recourse \(R(x; \xi)\) is concave in \(\xi\). By linear optimization duality we may reformulate (29) by introducing variables \(R \in \mathbb{R}^{N+1}\), \(y \in \mathbb{R}^{d_y \times (N+1)}\), \(\eta \in \mathbb{R}^{N+1}\), \(\theta \in \mathbb{R}^{N+1}\) and constraints

\[
\begin{align*}
    c_i \geq \gamma_k c^T x + \gamma_k R_i + \beta_k & \quad \forall i = 1, \ldots, N + 1, \quad \forall i = k, \ldots, K \\
    \eta_i - \theta_i = f^T y_i, \quad Ax + By_i \leq b & \quad \forall i = 1, \ldots, N + 1 \\
    R_i \geq g^T y_i + \xi^{(i)} \eta_i - \xi^{(i-1)} \theta_i & \quad \forall i = 1, \ldots, N + 1.
\end{align*}
\]

5.3 Tests for multivariate distributions

5.3.1 Testing marginal distributions

Recall that when \(c(x; \xi)\) is separable over the components of \(\xi\), i.e.,

\[c(x; \xi) = \sum_{i=1}^{d} c_i(x; \xi_i),\]

Robust SAA converges for the test of marginals (cf. Section 4.3.1). We next show that Robust SAA is also tractable in this case. When \(\mathcal{F} = \mathcal{F}_{\text{marginals}}\) and costs are separable, (3) can be written as

\[
\min_{x \in X} \sup_{F_0 \in \mathcal{F}} \mathbb{E}_{F_0}[c(x; \xi)] = \min_{x \in X} \sum_{i=1}^{d} \sup_{F_{0,i} \in \mathcal{F}_{\xi_i}} \mathbb{E}_{F_{0,i}}[c_i(x; \xi_i)].
\]

Applying Theorems 7 and 10 separately to these \(d\) subproblems yields a single-level optimization problem. This problem is theoretically tractable when each subproblem satisfies the corresponding conditions in Theorems 8 and 11. Similarly, when each subproblem is of one of the forms treated in Examples 1, 2, 3, and 4, (3) can be formulated as a linear or second-order cone optimization problem.

5.3.2 Testing linear-convex ordering

Next, we consider the case of the test based on LCX. For this section we restrict our attention to cost functions of the form

\[
c(x; \xi) = \max_{k=1, \ldots, K} \left\{ p_{k0} + p_{k1}^T x + p_{k2}^T \xi + x^T p_k \right\}.
\]

The following result provides a semi-infinite linear optimization reformulation of (3) and a polynomial-time separation algorithm.

**Theorem 12** Suppose that we can express \(c(x; \xi)\) as in (36), that \(X = \{x \in \mathbb{R}^{d_x} : x \geq 0, Hx = h\}\) with \(h \in \mathbb{R}^{d_h}\), and that \(\Xi = \mathbb{R}^{d}\). Under the assumptions of Theorem 1, the optimal value of (3) for \(\mathcal{F} = \mathcal{F}_{C_N, R_N}\) is given by the semi-infinite linear optimization problem

\[
\max_{r, s, t} \sum_{k=1}^{K} (p_{k0} r_k + p_{k2} s_k) + h^T t
\]

s.t.

\[
\begin{align*}
    r & \in \mathbb{R}^{k_+}, \quad s \in \mathbb{R}^{k \times d}, \quad t \in \mathbb{R}^{d} \\
    \sum_{k=1}^{K} \max \{a^T s_k - br_k, 0\} & \leq Q_{C_N}(\alpha_1) + \frac{1}{N} \sum_{i=1}^{N} \max \{a^T \xi^i - b, 0\} \quad \forall \|a\|_1 + \|b\| \leq 1 \\
    \sum_{k=1}^{K} r_k & = 1 \\
    H^T t - \sum_{k=1}^{K} (r_k p_{k1} - p_k z_k) & \leq 0,
\end{align*}
\]

and the optimal solution \(\pi\) is given by the dual variable associated with constraint (38).
6 Estimating the price of data

Our framework allows one to compute the price one would be willing to pay for further data gathering. Given the present dataset, we define the price of data (PoD) as follows:

$$\text{PoD} = \mathbb{E} \left[ \xi^1, \ldots, \xi^N \right] - \mathbb{E} \left[ \xi^1, \ldots, \xi^N, \xi^{N+1} \right] | \xi^1, \ldots, \xi^N. \right.$$  

(40)

PoD is equal to the expected marginal benefit of one additional data point in reducing our bound on costs.

One way to estimate the above quantity is via resampling:

$$\text{PoD} \approx \mathbb{E} \left[ \xi^1, \ldots, \xi^N \right] - \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \xi^i, \xi^N, \xi^i \right].$$  

(41)

The resampled average can also be, in turn, estimated by an average over a smaller random subsample from the data. This approach is illustrated numerically in Section 7.3.
In the case of the newsvendor problem using the KS test, the closed form solution yields a simpler approximation. Observe that in Proposition 8, small changes to the data change $x$ very little and the costs for $\varepsilon$ near $x$ (in particular, between $i_{lo}$ and $i_{hi}$) are small compared to costs far away from $x$. Thus, we suggest the approximation

$$\text{PoD} \approx (Q_{D_n}(\alpha) - Q_{D_{n+1}}(\alpha)) \left( c(x; \overline{\varepsilon}) + c(x; \underline{\varepsilon}) \right).$$

(42)

This approximation is illustrated numerically in section 7.1.

We can write a more explicit approximation using the asymptotic approximation of $Q_{D_n}(\alpha)$ (see [39]) and $1/\sqrt{N} - 1/\sqrt{N} + 1 \approx 1/(2N^{3/2})$ for large $N$:

$$\text{PoD} \approx \frac{q_\alpha}{2N^{3/2}} \left( c(x; \overline{\varepsilon}) + c(x; \underline{\varepsilon}) \right) \quad \text{where} \quad q_\alpha = \begin{cases} 1.36, & \alpha = 0.05, \\ 1.22, & \alpha = 0.1, \\ 1.07, & \alpha = 0.2. \end{cases}$$

7 Empirical study

We now turn to an empirical study of Robust SAA as applied to specific problems in inventory and portfolio management. The cost functions are all specific cases of the examples studied in Section 5. Recall that in these examples the resulting formulations were all linear and second-order cone optimization problems.

7.1 Single-item newsvendor

We begin with an application to the classic newsvendor problem with continuous demand distribution, as studied in Example 2. We will consider both bounded and unbounded distributions. In the latter case we employ the Student’s T-test to ensure a finite solution. We implement (3) in closed form for the KS test with bounded support using Proposition 8, using IPOPT 3.11 [41] for the AD test, and using GUROBI 5.5 [20] otherwise.

We consider a 95% service-level requirement ($b = 19$, $h = 1$) and each of the following distributions, truncated above at 250 units in the bounded case:

1. Normal distribution with mean 100 and standard deviation 50, truncated to be nonnegative.
2. Right-skewed Gumbel distribution with location 70 and scale $30/\gamma$ (the Euler constant), truncated to be nonnegative.
3. Mixture model of 40% normal with mean 40 and standard deviation 25 and 60% right-skewed Gumbel with location 125 and scale $15/\gamma$, truncated to be nonnegative.

We plot their PDFs in Figure 3. In the bounded case we use a significance level of 20% (i.e., 80% confidence). In the unbounded case we use a significance level of 15% for the GoF test and 5% for the Student’s T-test (yielding total significance of 20%).

In Figure 4 we consider the bounded normal distribution and compare the values of the full-information problem (1), SAA estimates (2), Robust SAA bounds $\overline{\varepsilon}$ using the KS test, the data-driven DRO bound of [13], and the non-data-driven DRO bound of [32]. We note that the SAA estimates converge to the true
Fig. 4: Convergence of SAA estimates and Robust SAA guarantees compared with the data-driven DRO of [13] and non-data-driven DRO of [32]. The error bars denote the 20th and 80th percentiles. All data-driven DRO guarantee bounds are solved at a significance of 20%. Note the horizontal log scale.

optimum, but very often underestimate the true costs of SAA’s recommended control (e.g. 65% of time for \( N = 100 \)), which is necessarily above the full-information optimum. These estimates are biased (the mean is below the full-information optimum) and have the peculiar property that the estimated costs grow with \( N \), contradicting the value of data collection.

The data-driven guarantees of [13] do not converge to the full information optimum. Instead, they converge upon the bound of [32], in which one restricts mean and variance to their exact true values and letting all else vary. These data-driven bounds, however, decrease with \( N \), consistent with true costs improving as more data is gathered. Interpreting the DUS of [13] as a hypothesis test, we also attempt to apply the bootstrap to estimate valid thresholds \( \gamma_{1,N}(\alpha), \gamma_{2,N}(\alpha), \gamma_{3,N}(\alpha) \) (see (21)). The result is plotted in the same figure. We note that the bound is much smaller, but still non-convergent.

Our proposed method provides an order quantity, cost guarantee, and true costs that converges to the full information optimum control and value. In this particular case, computation of the bound and order quantity is done in closed form. The value of the bounds decrease with \( N \), in agreement with the value of data collection, and their convergence is consistent with the notion that we discover \( F \) as we get more data. The magnitude of the effect of data collection on the bounds is what we termed the Price of Data, or PoD, in Section 6. In Figure 5, we compare the true PoD (40) and the approximation we developed (42). Notice the fit is quite tight.

In Figure 6, we consider the behavior of Robust SAA for a wider range of distributions and tests. First and foremost, the numerical results confirm the guarantees and convergence as \( N \to \infty \) irrespective of what

Fig. 5: The price of data in the newsvendor problem: average of true PoD in solid black and of its distribution-agnostic approximation (42) in dashed grey. Note the log scales.
is the true, unknown distribution $F$. Different tests also seem to yield mostly comparable results, with the AD test providing slightly better results when $N$ is at least 100. With small $N$, the Kuiper and Watson tests seem to perform the best. These observations should not, however, be taken as general conclusions about the relative performance of these tests for general problems. The conservatism of the guarantees depends both on the structure of the cost function as well as the true, unknown distribution and how we test it. For practical purposes, if the convergence rates are comparable as they are here, we recommend to choose the test that yields the simplest optimization problem, which in this case is the KS test.

7.2 Multi-item newsvendor

We now consider the multi-item newsvendor problem, which is a special case of a separable cost function as considered in Section 5.3.1. Recall that in the multi-item newsvendor we have $X = \{x \in \mathbb{R}_+^d : \sum_{i=1}^d x_i \leq \overline{x}\}$.

Fig. 7: The probabilistic guarantees of Robust SAA for the multi-item newsvendor problem. The vertical lines denote the span from the 20th to the 80th percentile. The dashed line denotes the full-information optimum.
Fig. 8: The PDFs of the distributions of security returns considered for the portfolio allocation problem.

(a) \( i = 1 \)

(b) \( i = 5 \)

(c) \( i = 10 \)

for some capacity \( \sum x_i \leq \pi \) and

\[
c(x; \xi) = \sum_{i=1}^{d} c_i(x_i; \xi_i),
\]

where each \( c_i \) takes the form of a newsvendor cost function with its own parameters \( b_i, h_i \).

We consider the case of three items, each having demand distributed as one of the three bounded distributions considered in the single-item case, with the parameters \( \pi = 250, r_1 = 15, r_2 = 10, r_3 = 5, c_1 = 6, c_2 = 4, c_3 = 2, b_1 = 3, b_2 = 2, b_3 = 1 \). In our application of Robust SAA we employ the test based on marginals where, for different choices of univariate test, we use the same GoF test for each marginal, each at significance of 6.67% (total significance 20%).

We present the results in Figure 7. Again, we plot both the mean and 20th and 80th percentiles of probabilistic guarantees as the size of the sample grows and compare these to the full-information optimum. As predicted by the theory, we observe convergence of guarantees even though testing marginals is not generally a uniformly consistent test.

7.3 Portfolio Allocation

We now consider the minimum-CVaR portfolio allocation problem as studied in Example 5. We minimize the 10%-level CVaR of negative returns of a portfolio of \( d = 10 \) securities. The random returns are supported on the unbounded domain \( \mathbb{R}^{10} \) and given by the factor model

\[
\xi_i = \frac{i}{11} t + \frac{11 - i}{11} \xi_i, \quad i = 1, \ldots, 10
\]

Fig. 9: Comparison of performance of various data-driven approaches to the portfolio allocation problem (39). The vertical bars denote the span from the 20th to the 80th percentile. The full-information optimum is shown as a dashed line.
Fig. 10: The price of data in portfolio optimization: average of true PoD in solid black and of its resampling-based approximation (41) in dashed grey. Note the vertical log scale.

where $\tau$ is a common market factor following a normal distribution with mean 2.5% and standard deviation 3%, and $\zeta_i$’s are independent idiosyncratic contributions all following a negative Pareto distribution with upper support 3.7%, mean 2.5%, and standard deviation 3.8% (i.e. $\zeta_i \sim 0.05 - \text{Pareto}(0.013, 2.05)$). All securities have the same average return. Lower indexed securities are more volatile but are also more diversified. We plot the PDFs of the returns of a few of the securities in Figure 8.

For samples drawn from this distribution, we consider data-driven solutions by the SAA, the DRO of [13], and our method using the test for LCX. We use the bootstrap to compute $Q_{\mathcal{C}_n}(\alpha)$ (see Section 9.3). Since the constants $\gamma_{1,N}(\alpha)$, $\gamma_{2,N}(\alpha)$, and $\gamma_{3,N}(\alpha)$ (see (21)) for the DRO of [13] are only developed therein for the case of known bounded support and in order to offer a fair comparison, we also bootstrap these thresholds. We implement the DRO (3) using GUROBI 5.5 [20].

We present the results in Figure 9. As can be seen the SAA underestimates the risk of its recommended portfolios. The method of [13] provides valid bounds but they do not appear to converge and are also highly variable. In comparison, our method using the LCX test provides apparently convergent guarantees and its guarantees are tightly concentrated.

In Figure 10 we compare the true price of data (40) for the LCX-based DRO bound and the resampling based approximation of it (41).

8 Conclusion

In this paper, we proposed a novel, tractable approach to data-driven optimization called robust sample average approximation (Robust SAA). Robust SAA enjoys the tractability and finite-sample performance guarantees of many existing data-driven methods, but, unlike those methods, additionally exhibits asymptotic behavior similar to traditional sample average approximation (SAA). The key to the approach is a novel connection between SAA, DRO, and statistical hypothesis testing.

In particular, we were able to link properties of a data-driven optimization problem, i.e., its finite sample and asymptotic performance, to statistical properties of an associated goodness-of-fit hypothesis test, i.e., its significance and consistency. As a theoretical consequence, this connection allow us to describe the finite sample and asymptotic performance of both Robust SAA and other data-driven DRO formulations. As a practical consequence, our hypothesis testing perspective first, sheds light on which data-driven DRO formulations are likely to perform well in particular applications and second, enables us to use powerful, numerical tools like bootstrapping to improve their performance. Numerical experiments in inventory management and portfolio allocation confirm that our new method Robust SAA is tractable and can outperform existing data-driven methods in these applications.

References

9 Appendix

9.1 Proof of Theorem 1

Proof Fix any \( x \in X \). Let \( \epsilon > 0 \) be given. By equicontinuity of the cost at \( x \) there is a \( \delta > 0 \) such that any \( y \in X \) with \( ||x - y|| < \delta \) has \( |c(x; \xi) - c(y; \xi)| \leq \epsilon \) for all \( \xi \in \Xi \). Fix any such \( y \). Then

\[
C(y; F) = \sup_{F_0 \in F} E_{F_0}[c(y; \xi)] \leq \sup_{F_0 \in F} E_{F_0}[c(x; \xi)] + \epsilon = C(x; F) + \epsilon, \tag{43}
\]

\[
C(x; F) = \sup_{F_0 \in F} E_{F_0}[c(x; \xi)] \leq \sup_{F_0 \in F} E_{F_0}[c(y; \xi)] + \epsilon = C(y; F) + \epsilon. \tag{44}
\]

Let \( S = \{ x \in X : C(x; F) < \infty \} \). By assumption \( x_0 \in S \), so \( S \neq \emptyset \). (44) implies that \( S \) is closed relative to \( X \), which is closed, and therefore closed relative to \( \mathbb{R}^d \). Since the objective is only finite on \( S \) we restrict our attention to \( S \). (43) and (44) imply that \( C(x; F) \) is continuous in \( x \) on \( S \).

If \( X \) is compact then \( S \) is compact. Suppose \( S \) is not compact and let \( x_i \in S \) be any sequence such that \( \lim_{i \to \infty} ||x_0 - x_i|| = \infty \). Then by coerciveness, \( c_i(\xi) = c(x_i; \xi) \) diverges pointwise to infinity. Fix any \( F_0 \in F \). Let \( \xi_i = \inf_{j \geq i} \xi_j \), which is then pointwise monotone nondecreasing and pointwise divergent to infinity. Then, by Lebesgue’s monotone convergence theorem, \( \lim_{i \to \infty} E_{F_i}[c_i(\xi)] = \infty \). Since \( \xi_i < \xi_j \) pointwise for any \( j \geq i \), we have \( E_{F_0}[c_i(\xi)] \leq \lim_{i \to \infty} E_{F_i}[c_i(\xi)] \) and therefore

\[
\infty = \lim_{i \to \infty} E_{F_0}[c_i(\xi)] \leq \lim_{i \to \infty} \inf_{i \geq j} E_{F_i}[c_i(\xi)] = \lim_{i \to \infty} \inf E_{F_0}[c_i(\xi)].
\]

Thus \( C(x; F) \geq E_{F_0}[c(x; \xi)] \) is also coercive in \( x \) over \( S \).

By the usual extreme value theorem, with either compactness or coerciveness, the continuous \( C(x; F) \) attains its minimal (finite) value at an \( x \in S \subseteq X \).

9.2 Proof of Proposition 1

Proof Suppose that \( c(x; \xi) \to \infty \) as \( \xi \to \infty \). The case of unboundedness in the negative direction is similar.

Let \( M \) be given. Choose \( \rho > 0 \) small so that \( \xi^{(i)} - \xi^{(i-1)} \geq 2\rho \) for all \( i \). For \( \delta > 0 \) and \( \xi' \geq \xi(N) + \rho \), let \( F_{\delta, \xi'} \) be the measure with density function

\[
f(\xi; \delta, \xi') = \begin{cases}
1/(2N\rho) & \xi^{(i)} - \rho \leq \xi \leq \xi^{(i)} + \rho \text{ for } 1 \leq i \leq N - 1, \\
1/(2N\rho) & \xi^{(N)} - \rho + \delta \rho \leq \xi \leq \xi^{(N)} + \rho - \delta \rho, \\
1/(2N\rho) & \xi' \leq \xi \leq \xi' + 2\delta \rho, \\
0 & \text{otherwise}.
\end{cases}
\]

Notice that for any \( \xi' \), \( F_{\delta, \xi'} \) (i.e., take \( \delta = 0 \)) minimizes \( S_N(F_0) \) over distributions \( F_0 \). Since \( \alpha > 0 \), \( Q_{S_N}(\alpha) \) is strictly greater than this minimum. Since \( S_N(F_{\delta, \xi'}) \) increases continuously with \( \delta \) independently of \( \xi' \), there must exist \( \delta > 0 \) small enough so that \( F_{\delta, \xi'} \in \mathcal{F}_{S_N} \) for any \( \xi' > \xi(N) + \rho \). By infinite limit of the cost function, there exists \( \xi' > \xi(N) + \rho \) sufficiently large such that \( c(x; \xi) \geq MN/\delta \) for all \( \xi \geq \xi' \). Then, we have \( C(x; \mathcal{F}_{S_N}) \geq E_{F_0}[c(x; \xi)] \geq \mathbb{P}(\xi \geq \xi') MN/\delta = M \).

Since we have shown this for every \( M > 0 \), we have \( C(x; \mathcal{F}_{S_N}) = \infty \).

9.3 Computing a threshold \( Q_{C_N}(\alpha) \)

We provide two ways to compute \( Q_{C_N}(\alpha) \) for use with the LCX-based GoF test. One is an exact, closed form formula, but which may be loose. Another uses the bootstrap to compute a tighter, but approximate threshold.

The theorem below employs a bound on \( E_{\mathcal{F}} \left( ||\xi||_2^2 \right) \) to provide a valid threshold. This bound could either stem from known support bounds or from changing (14) to a two-sided hypothesis with two-sided confidence interval, using the lower bound in (16) and the upper bound in (45) given below.

Theorem 13 Let \( N \geq 2 \). Suppose that with probability at least \( 1 - \alpha_2 \), \( E_{\mathcal{F}} \left( ||\xi||_2^2 \right) \leq Q_{\mathcal{F}_{\alpha_2}} \). Let \( \alpha_1 \in (0, 1) \) be given and suppose \( F_0 \succeq_{LCX} F \). Then, with probability at least \( 1 - \alpha_1 - \alpha_2 \),

\[
E_{\mathcal{F}} \left( ||\xi||_2^2 \right) \leq Q_{\mathcal{F}_{\alpha_2}}(\alpha_2) \quad \text{and}
\]

\[
C_N(F_0) \leq \left( 1 + Q_{\mathcal{F}_{\alpha_2}}(\alpha_2) \right) \left( 1 + \frac{p}{2 - p} \right) \frac{2^{1/2 + 1/2}}{N^{1 - 1/2}} \sqrt{d + 1} \left( d + 1 \right) \log \left( \frac{N}{d + 1} \right) \log \left( \frac{1}{\alpha_1} \right), \tag{45}
\]

\[
\text{for } d \geq 0.
\]
where
\[
p = \frac{1}{2} \left( \sqrt{\log(256) + 8 \log(N) + (\log(2N))^2} - \log(2N) \right) \in (1, 2).
\] (46)

Hence, defining \( Q_{C_N}(\alpha) \) equal to the right-hand side of (45), we get a valid threshold for \( C_N \) in testing \( F_0 \preceq_{LCX} F \) at level \( \alpha \).

Proof Fix any \( p \in (1, 2) \). Since \( \mathcal{S} = \{ \xi \in \Xi : \max \{ a^T \xi - b, 0 \} \leq t \} : ||a||_1 + |b| \leq 1, t \in \mathbb{R} \} \) is contained in the class of the empty set and all halfspaces, it has Vapnik-Chervonenkis dimension at most \( d + 1 \).

Notice that for any \( ||a||_1 + |b| \leq 1 \), \( 0 \leq \max \{ a^T \xi - b, 0 \} \leq \max \{ 1, ||\xi||_\infty \} \leq \max \{ 1, ||\xi||_2 \} \). Therefore
\[
E_F[\max \{ a^T \xi - b, 0 \}]^2 \leq 1 + E_F[||\xi||_2^2] \quad \text{and}
\]
\[
\int_0^\infty (\mathbb{P}_F(\max \{ a^T \xi - b, 0 \} > t))^{1/p} dt \leq 1 + \int_1^\infty \left( E_F[\max \{ a^T \xi - b, 0 \}]^2 \right)^{1/p} dt
\]
\[
\leq \left( 1 + E_F[||\xi||_2^2] \right)^{1/p} \left( 1 + \frac{p}{2-p} \right) \leq \left( 1 + E_F[||\xi||_2^2] \right) \left( 1 + \frac{p}{2-p} \right)
\]
by Markov’s inequality and \( 1 < p < 2 \). Observe
\[
C_N(F_0) \leq \sup_{||a||_1 + |b| \leq 1} \left( E_{F_0}[\max \{ a^T \xi - b, 0 \}] - E_F[\max \{ a^T \xi - b, 0 \}] \right)
\]
\[
+ \sup_{||a||_1 + |b| \leq 1} \left( E_F[\max \{ a^T \xi - b, 0 \}] - \frac{1}{N} \sum_{i=1}^N \max \{ a^T \xi^i - b, 0 \} \right)
\]
\[
\leq \sup_{||a||_1 + |b| \leq 1} \left( E_F[\max \{ a^T \xi - b, 0 \}] - \frac{1}{N} \sum_{i=1}^N \max \{ a^T \xi^i - b, 0 \} \right),
\]
where the second inequality follows because \( F_0 \preceq_{LCX} F \). By applying Theorem 5.2 of [40] to the bottom-rightmost end of (47), we conclude that (45) holds for any \( p \in (1, 2) \). The \( p \) given in (46) optimizes the bound for \( N \geq 2 \). \( \square \)

Next we show how to bootstrap an approximate threshold \( Q_{C_N}(\alpha) \). Recall that we seek a threshold \( Q_{C_N}(\alpha) \) such that \( \mathbb{P}(C_N(F_0) > Q_{C_N}(\alpha)) \leq \alpha \) whenever \( F_0 \preceq_{LCX} F \). Employing (47), we see that a sufficient threshold is the \( (1 - \alpha)^{th} \) quantile of
\[
\sup_{||a||_1 + |b| \leq 1} \left( E_F[\max \{ a^T \xi - b, 0 \}] - \frac{1}{N} \sum_{i=1}^N \max \{ a^T \xi^i - b, 0 \} \right),
\]
where \( \xi^i \) are drawn IID from \( F \). The bootstrap [18] approximates this by replacing \( F \) with the empirical distribution \( \hat{F}_N \). In particular, given an iteration count \( B \), for \( t = 1, \ldots, B \) it sets
\[
Q^t = \sup_{||a||_1 + |b| \leq 1} \left( \frac{1}{N} \sum_{i=1}^N \max \{ a^T \xi^i - b, 0 \} - \frac{1}{N} \sum_{i=1}^N \max \{ a^T \hat{\xi}^i - b, 0 \} \right)
\] (48)
where \( \hat{\xi}^i \) are drawn IID from \( \hat{F}_N \), i.e., IID random choices from \( \{ \xi^1, \ldots, \xi^N \} \). Then the bootstrap approximates \( Q_{C_N}(\alpha) \) by the \( (1 - \alpha)^{th} \) quantile of \( \{ Q^1, \ldots, Q^B \} \). However, it may be difficult to compute (48) as the problem is non-convex. Fortunately (48) can be solved with a standard MILP formulation or by discretizing the space and enumerating (the objective is Lipschitz).

In particular, our bootstrap algorithm for computing \( Q_{C_N}(\alpha) \) is as follows:

\textbf{Input:} \( \xi^1, \ldots, \xi^N \) drawn from \( F \), significance 0 < \( \alpha < 1 \), precision \( \delta > 0 \), iteration count \( B \)
\textbf{Output:} Threshold \( Q_{C_N}(\alpha) \) such that \( \mathbb{P}(C_N(F_0) > Q_{C_N}(\alpha)) \leq \alpha \) whenever \( F_0 \preceq_{LCX} F \).

For \( t = 1, \ldots, B \):
1. Draw \( \hat{\xi}^{t,1}, \ldots, \hat{\xi}^{t,N} \) IID from \( \hat{F}_N \).
2. Solve \( Q^t = \sup_{||a||_1 + |b| \leq 1} \left( \frac{1}{N} \sum_{i=1}^N \max \{ a^T \xi^i - b, 0 \} - \frac{1}{N} \sum_{i=1}^N \max \{ a^T \hat{\xi}^i - b, 0 \} \right) \) to precision \( \delta \).
Sort \( Q^{(1)} \leq \cdots \leq Q^{(B)} \) and return \( Q^{\lfloor (1-\alpha)B \rfloor} + \delta \).
9.4 Proof of Proposition 3

Proof We first prove that a uniformly consistent test is consistent. Let \( G_0 \neq F \) be given. Denote by \( d \) be the Lévy-Prokhorov metric, which metrizes weak convergence \([7]\), and observe that \( d(G_0, F) > 0 \).

Next, define \( R_N = \sup_{i \in \mathbb{N}} d(F_0, F) \). We claim that if the test is uniformly consistent, then \( \mathbb{P}(R_N \rightarrow 0) = 1 \). Indeed, suppose for some sample path, \( R_N \not\rightarrow 0 \). By the definition of the supremum, there must exist \( \delta > 0 \) and a sequence \( F_N \in \mathcal{F}_N \) such that \( d(F_N, F) \geq \delta \) i.o. Since \( d \) metrizies weak convergence, \( F_N \) does not converge to \( F \). However, \( F_N \in \mathcal{F}_N \) for all \( N \), i.e. it is never rejected, which contradicts what must hold a.s. under uniform consistency.

Finally, since \( \mathbb{P}(R_N \rightarrow 0) = 1 \) and a.s. convergence implies convergence in probability, we have that \( \mathbb{P}(R_N < \epsilon) \rightarrow 1 \) for every \( \epsilon > 0 \), and, in particular, for \( \epsilon = d(G_0, F) \). Then, \( \mathbb{P}(G_0 \text{ rejected}) = \mathbb{P}(G_0 \notin \mathcal{F}_N) \geq \mathbb{P}(R_N < d(G_0, F)) \rightarrow 1 \). This proves the first part of the proposition.

For the second part, we describe a test which is consistent but not uniformly consistent. Consider testing a continuous distribution \( F \) with the following univariate GoF test:

Given data \( \xi^1, \ldots, \xi^N \) drawn from \( F \) and a hypothetical continuous distribution \( F_0 \):

Let \( j = \lfloor \log_2 N \rfloor, \ i = N - 2^j \).

If \( \frac{i}{2^j} \leq F_0(\xi^i) \leq \frac{i + 1}{2^j} \) then \( F_0 \) is not rejected.

Otherwise, reject \( F_0 \) if it is rejected by the KS test at level \( \frac{\alpha}{1 - 2^{-j}} \) applied to the data \( \xi^2, \ldots, \xi^N \).

Notice that under the null-hypothesis, the probability of rejection is

\[
\mathbb{P}(\text{F rejected}) = \mathbb{P}(F_0(\xi^i) \notin \left[ \frac{i}{2^j}, \frac{i + 1}{2^j} \right]) \mathbb{P}(F_0 \text{ is rejected by the KS test}) = (1 - 2^{-j}) \frac{\alpha}{1 - 2^{-j}} = \alpha,
\]

where we’ve used that \( \xi^i \) is independent of the rest of the sample, and \( F_0(\xi^i) \) is uniformly distributed for \( F_0 \) continuous. Consequently, the test is a valid GoF test and it has significance \( \alpha \).

We claim this test is also consistent. Specifically, consider any \( F_0 \neq F \). By continuity of \( F_0 \) and consistency of the KS test,

\[
\mathbb{P}(\text{F rejected}) = \mathbb{P}(F_0(\xi^i) \notin \left[ \frac{i}{2^j}, \frac{i + 1}{2^j} \right]) \mathbb{P}(F_0 \text{ is rejected by the KS test}) \rightarrow 1.
\]

However, the test is not uniformly consistent. Fix any continuous \( F_0 \neq F \) and let

\[
F_N = \begin{cases} 
F_0 & \text{if } \frac{i}{2^j} \leq F_0(\xi^i) \leq \frac{i + 1}{2^j}, \\
F_N & \text{otherwise.}
\end{cases}
\]

Observe that \( 0 \leq F_0(\xi^i) \leq 1 \) and \( [0, 1] = \bigcup_{j=0}^{j-1} \left[ \frac{i}{2^j}, \frac{i + 1}{2^j} \right] \]. That is, for every \( j \), \( F_N = F_0 \) at least once for \( N \in \{2^j, \ldots, 2^{j+1} - 1\} \). Therefore \( F_N = F_0 \) i.o., so it does not converge weakly to \( F \). However, as constructed, \( F_N \) is never rejected by the above test. This is done for every sample path so the test cannot be uniformly consistent. \( \square \)

9.5 Proofs of Theorems 3 and 4

We first establish two useful results.

Proposition 9 Suppose \( F_N \) is the confidence region of a uniformly consistent test and that Assumptions (1) and (2) hold. Then, almost surely, \( \mathbb{E}_{F_N}[c(x; \xi)] \rightarrow \mathbb{E}_F[c(x; \xi)] \) for any \( x \in X, F_N \in \mathcal{F}_N \).

Proof Restrict to the a.s. event that \( (F_N \not\rightarrow F \implies F_N \notin \mathcal{F}_N \) i.o.). Fix \( F_N \in \mathcal{F}_N \). Then the contrapositive gives \( F_N \not\rightarrow F \). Fix \( x \in \Xi \). If \( \Xi \) is bounded (Assumption 2a) then the result follows from the portmanteau lemma (see for example Theorem 2.1 of \([7] \)). Suppose otherwise (Assumption (2)b). Then \( \mathbb{E}_{F_N}[\phi(\xi)] \rightarrow \mathbb{E}_F[\phi(\xi)] \). By Theorem 3.6 of \([7] \), \( \phi(\xi) \) is uniformly integrable over \( \{F_1, F_2, \ldots\} \). Since \( c(x; \xi) = O(\phi(\xi)) \), it is also uniformly integrable over these. Then the result follows by Theorem 3.5 of \([7] \). \( \square \)

Proposition 10 Suppose Assumption 1 holds and \( C(x_N; \mathcal{F}_N) \rightarrow \mathbb{E}_F[c(x; \xi)] \) for any convergent sequence \( x_N \rightarrow x \). Then \((17)\) holds.
Proof Let $E \subseteq X$ compact be given and suppose to the contrary that $\sup_{x \in E} |C(x; F_N) - \mathbb{E}_F[c(x; \xi)]| \not\rightarrow 0$. Then $\exists \varepsilon > 0$ and $x_N \in E$ such that $|C(x_N; F_N) - \mathbb{E}_F[c(x_N; \xi)]| \geq \varepsilon$ i.o. This, combined with compactness, means that there exists a subsequence $N_1 < N_2 < \ldots < N_k \rightarrow \infty$ such that $x_{N_k} \rightarrow x \in E$ and $|C(x_{N_k}; F_N) - \mathbb{E}_F[c(x_{N_k}; \xi)]| \geq \varepsilon \forall k$. Then,

$$0 < \varepsilon \leq |C(x_{N_k}; F_N) - \mathbb{E}_F[c(x_{N_k}; \xi)]| \leq |C(x_{N_k}; F_N) - \mathbb{E}_F[c(x; \xi)]| + |\mathbb{E}_F[c(x; \xi)] - \mathbb{E}_F[c(x_{N_k}; \xi)]|.$$ 

By assumption, $\exists k_1$ such that $|C(x_{N_k}; F_N) - \mathbb{E}_F[c(x; \xi)]| \leq \varepsilon/4 \forall k \geq k_1$. By equicontinuity and $x_{N_k} \rightarrow x$, $\exists k_2$ such that $|c(x; \xi) - c(x_{N_k}; \xi)| \leq \varepsilon/4 \forall \xi, k \geq k_2$. Then,

$$|\mathbb{E}_F[c(x; \xi)] - \mathbb{E}_F[c(x_{N_k}; \xi)]| \leq \mathbb{E}_F[|c(x; \xi) - c(x_{N_k}; \xi)|] \leq \varepsilon/4 \forall \xi, k \geq k_2.$$

Combining and considering $k = \max \{k_1, k_2\}$, we get the contradiction $\varepsilon \leq \varepsilon/2$ for strictly positive $\varepsilon$.

We prove the “if” and “only if” sides of Theorem 3 separately.

Proof (Proofs of Theorem 4 and the “if” side of Theorem 3) For either theorem restrict to the a.s. event that

$$\mathbb{E}_{F_N}[c(x; \xi)] \rightarrow \mathbb{E}_F[c(x; \xi)] \quad \text{for every } x \in X, F_N \in F_N$$

(using Proposition 9 for Theorem 3 or by assumption of $c$-consistency for Theorem 4).

Let any convergent sequence $x_N \rightarrow x$ and $\varepsilon > 0$ be given. By equicontinuity and $x_N \rightarrow x$, $\exists N_1$ such that $|c(x_N; \xi) - c(x; \xi)| \leq \varepsilon/2 \forall \xi, N \geq N_1$. Then, $|C(x_N; F_N) - C(x; F_N)| \leq \sup_{F \in F_N} |c(x_N; \xi) - c(x; \xi)| \leq \varepsilon/2 \forall N \geq N_1$. By definition of supremum, $\exists F_N \in F_N$ such that $C(x; F_N) \leq \mathbb{E}_{F_N}[c(x; \xi)] + \varepsilon/4$. By (49), $\mathbb{E}_{F_N}[c(x; \xi)] \rightarrow \mathbb{E}_F[c(x; \xi)]$. Hence, $\exists N_2$ such that $|\mathbb{E}_{F_N}[c(x; \xi)] - \mathbb{E}_F[c(x; \xi)]| \leq \varepsilon/4 \forall N \geq N_2$. Combining these with

$$|C(x; F_N) - \mathbb{E}_F[c(x; \xi)]| \leq |C(x; F_N) - C(x; F_N)| + |C(x; F_N) - \mathbb{E}_F[c(x; \xi)]|,$$

we get

$$|C(x; F_N) - \mathbb{E}_F[c(x; \xi)]| \leq \varepsilon \quad \forall N \geq \max \{N_1, N_2\}.$$

Thus, by Proposition 10, we get (17) holds.

Let $A_N = \arg \min_{x \in X} C(x; F_N)$. We now show that $\bigcap N A_N$ is bounded. If $X$ is compact (Assumption 3a) then this is trivial. Suppose $X$ is not compact (Assumption 3a). Using the same arguments as in the proof of Theorem 1, we have in particular that $\lim_{||x|| \rightarrow \infty} \mathbb{E}_F[c(x; \xi)] = \infty$, $z_{\text{stoch}} = \min_{x \in X} \mathbb{E}_F[c(x; \xi)] < \infty$, that $A = \arg \min_{x \in X} \mathbb{E}_F[c(x; \xi)]$ is compact, and each $A_N$ is compact. Let $x^* \in A$. Fix $\varepsilon > 0$. By definition of supremum $\exists F_N \in F_N$ such that $C(x^*; F_N) \leq \mathbb{E}_{F_N}[c(x^*; \xi)] + \varepsilon$. By (49), $\mathbb{E}_{F_N}[c(x^*; \xi)] \rightarrow \mathbb{E}_F[c(x^*; \xi)] = z_{\text{stoch}}$. Since true for any $\varepsilon$ and since $\min_{x \in X} C(x; F_N) \leq C(x^*; F_N)$, we have $\limsup_{N \rightarrow \infty} \min_{x \in X} C(x; F_N) \leq z_{\text{stoch}}$. Now, suppose for contradiction that $\bigcap N A_N$ is unbounded, i.e. there is a subsequence $N_1 < N_2 < \ldots < N_k \rightarrow \infty$ and $x_{N_k} \in A_N$ such that $|x_{N_k}| \rightarrow \infty$. Let $\delta = \limsup_{N \rightarrow \infty} \inf_{x \in X} c(x_N; \xi) \geq \inf_{x \in X} \inf_{D} c(x_N; \xi) > -\infty$ and $\delta = \min \{0, \delta^* \}$. By $D$-uniform coercivity, $\exists k_0$ such that $c(x_N; \xi) \geq (z_{\text{stoch}} + 1 - \delta)/F(D) \forall \xi \in D, k \geq k_0$. In the case of Theorem 3, let $F_N$ be any $F_N \in F_N$. In the case of Theorem 4, let $F_N$ be the empirical distribution $F_N = F_N \in F_N$. In either case, we get $F_N \rightarrow F$ weakly. In particular, $F_N(D) \rightarrow F(D)$. Then $\mathbb{E}_{F_N}[c(x_N; \xi)] \geq F_N(D) \times (z_{\text{stoch}} + 1 - \delta)/F(D) + \min \{0, \inf_{D} c(x_N; \xi)\} \forall k \geq k_0$. Thus $\limsup_{N \rightarrow \infty} \min_{x \in X} C(x; F_N) \geq z_{\text{stoch}} + 1 - \delta + \delta = z_{\text{stoch}} + 1$, yielding the contradiction $z_{\text{stoch}} + 1 \leq z_{\text{stoch}}$.

Thus $\exists \delta_0$ compact such that $A \subseteq A_\infty$, $A_N \subseteq A_\infty$. Then, by (17),

$$\delta_0 = \lim_{x \in X} C(x; F_N) - \min_{x \in X} \mathbb{E}_F[c(x; \xi)] = \min_{x \in A_\infty} C(x; F_N) - \min_{x \in A_\infty} \mathbb{E}_F[c(x; \xi)] 
\leq \sup_{x \in A_\infty} |C(x; F_N) - \mathbb{E}_F[c(x; \xi)]| \rightarrow 0,$$

yielding (18). Let $x_N \in A_N$. Since $A_\infty$ is compact, $x_N$ has at least one convergent subsequence. Let $x_{N_k} \rightarrow x$ be any convergent subsequence. Suppose for contradiction $x \not\in A$, i.e. $\varepsilon = \mathbb{E}_F[c(x; \xi)] - z_{\text{stoch}} > 0$. Since $x_{N_k} \rightarrow x$ and by equicontinuity, $\exists k_1$ such that $|c(x_{N_k}; \xi) - c(x; \xi)| \leq \varepsilon/4 \forall \xi, k \geq k_1$. Then,

$$|\mathbb{E}_F[c(x_{N_k}; \xi)] - \mathbb{E}_F[c(x; \xi)]| \leq \mathbb{E}_F[|c(x_{N_k}; \xi) - c(x; \xi)|] \leq \varepsilon/4 \forall k \geq k_1.$$

Also $\exists k_2$ such that $\delta_N \leq \varepsilon/4 \forall k \geq k_2$. Then, for $k \geq \max \{k_1, k_2\}$,

$$\min_{x \in X} C(x; F_N) = C(x_{N_k}; F_N) \geq \mathbb{E}_F[c(x_{N_k}; \xi)] - \delta_N \geq \mathbb{E}_F[c(x; \xi)] - \varepsilon/2 \geq z_{\text{stoch}} + \varepsilon/2.$$

Taking limits, we contradict (18).

□
Proof (Proof of the “only if” side of Theorem 3) Consider any $\Xi$ bounded ($R = \sup_{\xi \in \Xi} |\xi| < \infty$). Let $X = \mathbb{R}^d$, and

$$
c_1(x; \xi) = ||x|| \left( 2 + \text{Re} \left( e^{ix^T \xi} \right) \right), \quad c_2(x; \xi) = ||x|| \left( 2 - \text{Re} \left( e^{ix^T \xi} \right) + 2 \right), \quad c_3(x; \xi) = ||x|| \left( 2 + \text{Im} \left( e^{ix^T \xi} \right) \right), \quad c_4(x; \xi) = ||x|| \left( 2 - \text{Im} \left( e^{ix^T \xi} \right) + 2 \right).
$$

Since $|c_i((x, y), \xi)| \leq 3 ||x||$, expectations exist. The gradient of each $c_i$ at $x$ has magnitude bounded by $R ||x|| + 3$ uniformly over $\xi$, so equicontinuity is satisfied. Also, $\lim_{||x|| \to \infty} c_i(x, y; \xi) \geq \lim_{||x|| \to \infty} ||x|| = \infty$ uniformly over all $\xi \in \Xi$ and $c_i(x; \xi) \geq 0$, so Assumption 3 is satisfied. Restrict to the a.s. event that (17) applies simultaneously for $c_1, c_2, c_3, c_4$. Then we have that, for every $x \in \mathbb{R}^d$,

$$
2 ||x|| + ||x|| \sup_{F \in \mathcal{F}_N} \text{Re} \left( \mathbb{E}_{F} \left[ e^{ix^T \xi} \right] \right) \to 2 ||x|| + ||x|| \lim_{F \to 0} \text{Re} \left( \mathbb{E}_{F} \left[ e^{ix^T \xi} \right] \right)
$$

This implies that $\sup_{F \in \mathcal{F}_N} \left| \mathbb{E}_{F} \left[ e^{ix^T \xi} \right] - \mathbb{E}_F \left[ e^{ix^T \xi} \right] \right| \to 0$ for every $x$. Fix $F_N$ such that $F_N \in \mathcal{F}_N$ eventually. Then $\mathbb{E}_{F_N} \left[ e^{ix^T \xi} \right] \to \mathbb{E}_F \left[ e^{ix^T \xi} \right]$ for every $x$. By the Lévy continuity theorem, $F_N$ converge weakly to $F$. This is the contrapositive of the uniform consistency condition. \hfill \Box

9.6 Proof of Theorem 5

Proof In the case of finite support $\Xi = \{\hat{\xi}^1, \ldots, \hat{\xi}^n\}$, total variation metrizes weak convergence:

$$
d_{TV}(q, q') = \frac{1}{2} \sum_{j=1}^{n} |q(j) - q'(j)|.
$$

Restrict to the almost sure event $d_{TV}(\hat{\rho}_N, p) \to 0$ (see Theorem 11.4.1 of [15]). We need only show that now $\sup_{F \in \mathcal{F}_N} d_{TV}(\hat{\rho}_N, p_0) \to 0$, yielding the contrapositive of the uniform consistency condition.

By an application of the Cauchy-Schwartz inequality,

$$
d_{TV}(\hat{\rho}_N, p_0) = \frac{1}{2} \sum_{j=1}^{n} |\hat{\rho}_N(j) - p_0(j)| \leq \frac{1}{2} \sum_{j=1}^{n} \left| \hat{\rho}_N(j) - p_0(j) \right| \leq \frac{1}{2} \sqrt{\left( \sum_{j=1}^{n} \left( \frac{\hat{\rho}_N(j) - p_0(j)}{p_0(j)} \right)^2 \right)^{1/2}} = X_N(p_0).
$$

By [25],

$$
d_{TV}(\hat{\rho}_N, p_0) \leq \frac{1}{\sqrt{2}} \sqrt{\sum_{j=1}^{n} \sum_{j=1}^{n} \log \left( \frac{\hat{\rho}_N(j)}{p_0(j)} \right)^{1/2}} = G_N(p_0).
$$

Since both the $\chi^2$ and G-tests use a rejection threshold equal to $\sqrt{Q/N}$ where $Q$ is the $(1 - \alpha)^{th}$ quantile of a $\chi^2$ distribution with $n - 1$ degrees of freedom ($Q$ is independent of $N$), we have that $d_{TV}(\hat{\rho}_N, p_0)$ is uniformly bounded over $p_0 \in \mathcal{F}_N$ by a quantity diminishing with $N$. \hfill \Box

9.7 Proof of Theorem 6

Proof In the case of univariate support, the Lévy metric metrizes weak convergence:

$$
d_{Lévy}(G, G') = \inf \{ \epsilon > 0 : G(\xi - \epsilon) - \epsilon \leq G'(\xi) \leq F_0(\xi + \epsilon) + \epsilon \forall \xi \in \mathbb{R} \}.
$$

Restrict to the almost sure event $d_{Lévy}(\hat{F}_N, F) \to 0$ (see Theorem 11.4.1 of [15]). We need only show that now $\sup_{F \in \mathcal{F}_N} d_{Lévy}(\hat{F}_N, F_0) \to 0$, yielding the contrapositive of the uniform consistency condition.
Fix $F_0$ and let $0 \leq \varepsilon < d_{\text{KvM}}(\hat{F}_N, F_0)$. Then $\exists \delta_0$ such that either (1) $\hat{F}_N(\delta_0 - \varepsilon) - \varepsilon > F_0(\delta_0)$ or (2) $\hat{F}_N(\delta_0 + \varepsilon) + \varepsilon < F_0(\delta_0)$. Since $F_0$ is monotonically non-decreasing, (1) implies $D_N(F_0) \geq \hat{F}_N(\delta_0) - F_0(\delta_0 - \varepsilon) > \varepsilon$ and (2) implies $D_N(F_0) > F_0(\delta_0 + \varepsilon) - \hat{F}_N(\delta_0 + \varepsilon) > \varepsilon$. Hence $d_{\text{KvM}}(\hat{F}_N, F_0) \leq D_N(F_0)$. Moreover, $D_N \leq V_N$ by definition. Since $\sup_{F_0 \in \mathcal{F}_N} S_N(F_0) = Q_{\mathcal{S}_N}(\alpha) = O(N^{-1/2})$ for either statistic, both the KS and Kuiper tests are uniformly consistent.

Consider $D_N'(F_0) = \max_{i=1, \ldots, N} \{ F_0(\xi^{(i)}) - 2\frac{i-1}{N} \} = \sigma (F_0(\xi^{(i)}) - 2\frac{j-1}{N})$, where $j$ and $\sigma$ are the maximizing index and sign, respectively. Suppose $D_N'(F_0) \geq 1/\sqrt{N} + 1/N$. If $\sigma = +1$, this necessarily means that $1 - 2\frac{j-1}{N} \geq 1/\sqrt{N} + 1/N$ and therefore $N - j \geq \lceil \sqrt{N} \rceil + 1$. By monotonicity of $F_0$ we have for $0 \leq k \leq \lceil \sqrt{N} \rceil$ that $j + k \leq N$ and 
\[ F_0(\xi^{(j+k)}) - \frac{2(j+k)}{2N} \geq F_0(\xi^{(j)}) - \frac{2j}{2N} = D_N'(F_0) - \frac{k}{N} \geq 0. \]
If instead $\sigma = -1$, this necessarily means that $2\frac{j-1}{N} \geq 1/\sqrt{N} + 1/N$ and therefore $j \geq \lceil \sqrt{N} \rceil + 1$. By monotonicity of $F_0$ we have for $0 \leq k \leq \lceil \sqrt{N} \rceil$ that $j - k \geq 1$ and 
\[ F_0(\xi^{(j-k)}) - \frac{2(j-k)}{2N} \leq F_0(\xi^{(j)}) - \frac{k}{N} = D_N'(F_0) - \frac{k}{N} \geq 0. \]
In either case we have that 
\[ W_N^2 = \frac{1}{12N^2} + \frac{1}{N} \sum_{i=1}^{N} \left( F_0(\xi^{(i)}) - 2\frac{i-1}{2N} \right)^2 \geq \frac{1}{12N^2} + \frac{1}{N} \sum_{k=0}^{\lceil \sqrt{N} \rceil} \left( D_N' - \frac{k}{N} \right)^2 \geq D_N^2 \]
using $D_N'(F_0) \geq 1/\sqrt{N} + 1/N$ and $|D_N'(F_0) - D_N(F_0)| \leq 1/(2N)$ in the last inequality. Therefore, 
\[ D_N^2(F_0) \leq \max \left\{ \frac{1}{\sqrt{N}} + \frac{3}{2N}, \sqrt{N} W_N^2 + \frac{2}{N} \right\}. \]
Since $F_0(\xi)(1 - F_0(\xi)) \leq 1$ and by using the integral formulation of CvM and AD (see [39]) the same is true replacing $W_N^2$ by $A_N^2$. Since $Q_{\mathcal{S}_N}(\alpha) = O(N^{-1/2})$ so that $\sup_{F_0 \in \mathcal{F}_N} S_N^2(F_0) = O(N^{-1})$ for either statistic, both the CvM and AD tests are uniformly consistent.

\[ W_N^2 - U_N^2 = \frac{1}{N} \sum_{i=1}^{N} F_0(\xi^{(i)}) - \frac{1}{2} \leq \max \left\{ \left( \frac{1}{N} \sum_{i=1}^{N} \min \left\{ 1, 2\frac{i-1}{2N} + D_N'(F_0) \right\} - \frac{1}{2} \right)^2, \right. \]
\[ \left. \left( \frac{1}{N} \sum_{i=1}^{N} \max \left\{ 0, 2\frac{i-1}{2N} + D_N'(F_0) \right\} - \frac{1}{2} \right)^2 \right\} \]
Letting $M = \lceil \frac{1}{4} + N(1 - D_N'(F_0)) \rceil$ we have $\sum_{i=1}^{N} \min \left\{ 1, 2\frac{i-1}{2N} + D_N'(F_0) \right\} = \frac{M^2}{2N} + MD_N'(F_0) + N - M$ so that in the case of $D_N'(F_0) \geq 1/\sqrt{N} + 1/N$, \( \left( \frac{1}{N} \sum_{i=1}^{N} \min \left\{ 1, 2\frac{i-1}{2N} + D_N'(F_0) \right\} - \frac{1}{2} \right)^2 = O(1/N) \). Thus, the Watson test is also uniformly consistent. $\square$

9.8 Proof of Proposition 4

Proof. Apply Theorem 3 to each $i$ and restrict to the almost sure event that (17) holds for all $i$. Fix $F_N$ such that $F_N \in \mathcal{F}$ eventually. Then, (17) yields $\mathbb{E}_{F_N}[\mathbb{E}_F[c_i(x; \xi_i)] \rightarrow \mathbb{E}_F[c_i(x; \xi_i)]$ for every $x \in \mathcal{X}$. Summing over $i$ yields the contrapositive of the $c$-consistency condition. $\square$

9.9 Proof of Proposition 5

Proof. Restrict to a sample path in the almost sure event $\mathbb{E}_{F_N}[\xi_i] \rightarrow \mathbb{E}_F[\xi_i], \mathbb{E}_{F_N}[\xi_i, \xi_j] \rightarrow \mathbb{E}_F[\xi_i, \xi_j]$ for all $i, j$. Consider any $F_N$ such that $F_N \in \mathcal{F}_{\text{CEG}, N}$ eventually. Then clearly $\mathbb{E}_{F_N}[\epsilon_i] \rightarrow \mathbb{E}_F[\epsilon_i], \mathbb{E}_{F_N}[\epsilon_i, \epsilon_j] \rightarrow \mathbb{E}_F[\epsilon_i, \epsilon_j]$.

Consider any $F_N$ such that $F_N \in \mathcal{F}_{\text{DY}, N}$ eventually. Because covariances exist, we may restrict to $N$ large enough so that \[ \| \Sigma_N \|_2 \leq M \text{ (operator norm)} \] and $F_N \in \mathcal{F}_{\text{DY}, N}$. Then we get 
\[ \| \mathbb{E}_{F_N}[\epsilon] - \mu_N \|_2 \leq M \gamma_1, N(\alpha) \rightarrow 0 \]
and
\[(\gamma_{3,N}(\alpha) - 1) \hat{\Sigma}_N \preceq \mathbb{E}_{F_0}[(\xi - \hat{\mu}_N) (\xi - \hat{\mu}_N)^T] - \Sigma_N \preceq (\gamma_{2,N}(\alpha) - 1) \hat{\Sigma}_N,\]
which gives \[\left\| \mathbb{E}_{F_0}[(\xi - \hat{\mu}_N) (\xi - \hat{\mu}_N)^T] - \Sigma_N \right\|_2 \leq M \max \{\gamma_{2,N}(\alpha) - 1, 1 - \gamma_{3,N}(\alpha)\} \to 0.\] Then again, we have \[\mathbb{E}_{F_0}[\xi_i] \to \mathbb{E}_F[\xi_i], \mathbb{E}_{F_0}[\xi_i \xi_j] \to \mathbb{E}_F[\xi_i \xi_j].\]

In either case we get \[\mathbb{E}_{F_0}[\mathcal{C}(x; \xi)] \to \mathbb{E}_{F_0}[\mathcal{C}(x; \xi)]\] for any \(x\) due to factorability as in (25). This yields the contrapositive of the \(c\)-consistency condition. \(\square\)

9.10 Proof of Proposition 6

Proof If \(F_0 \neq F\) then Theorem 1 of [33] yields that either \(F_0 \not\perp_{\text{LCX}} F\) or there is some \(j = 1, \ldots, d\) such that \(\mathbb{E}_{F_0}[\xi_j^2] \neq \mathbb{E}_F[\xi_j^2]\). If \(F_0 \not\perp_{\text{LCX}} F\) then power approaches one since \(C_N > 0\) but \(Q_{C_N}(a_1) \to 0\). Otherwise, \(F_0 \perp_{\text{LCX}} F\) yields \(\mathbb{E}_{F_0}[\xi_i^2] \leq \mathbb{E}_F[\xi_i^2]\) for all \(i\) via (11) using \(a = e_i\) and \(\phi(\xi) = \xi^2\). Then \(\mathbb{E}_{F_0}[\xi_i^2] \neq \mathbb{E}_F[\xi_i^2]\)

\[\] must mean that \(\mathbb{E}_{F_0} [\|\xi\|_2^2] < \mathbb{E}_F [\|\xi\|_2^2]\) and power still goes to one. \(\square\)

9.11 Proof of Proposition 7

Proof Let \(R = \sup_{\xi \in \mathbb{R}} \|\xi\|_2 < \infty\). Restrict to the almost sure event that \(\hat{F}_N \to F\). Consider \(F_N\) such that \(F_N \in \mathcal{F}_N\) eventually. Let \(N\) be large enough so that it is so. Fix \(\|a\|_2 = 1\). Let \(a_1 = a\) and complete an orthonormal basis for \(\mathbb{R}^d\): \(a_1, a_2, \ldots, a_d\). On the one hand we have \(Q_{R_N}(a_2) \geq \mathbb{E}_{F_N} \left[\sum_{i=1}^d (a_i^T \xi)^2\right] - \mathbb{E}_{F_N} \left[\sum_{i=1}^d (a_i^T \xi)^2\right]\). On the other hand, for each \(i\),

\[\mathbb{E}_{F_N} [(a_i^T \xi)^2] - \mathbb{E}_{F_N} [(a_i^T \xi)^2] = 2 \int_{b=-R}^0 \left( \mathbb{E}_{F_N} [\max \{b - a_i^T \xi, 0\}] - \mathbb{E}_{F_N} [\max \{b - a_i^T \xi, 0\}] \right) db + 2 \int_0^R \left( \mathbb{E}_{F_N} [\max \{a_i^T \xi - b, 0\}] - \mathbb{E}_{F_N} [\max \{a_i^T \xi - b, 0\}] \right) db \geq 4 \int_{b=0}^R (||a||_1 + |b|)Q_{C_N}(a_1) db \geq 4 \left( \sqrt{d + R^2}/2 \right) Q_{C_N}(a_1) = p_N.\]

Therefore, \(q_N = Q_{R_N}(a_2) + (d - 1)p_N \geq \mathbb{E}_{F_N} [(a^T \xi)^2] - \mathbb{E}_{F_N} [(a^T \xi)^2]\) and \(Q_{R_N}(a_2), Q_{C_N}(a_1), p_N, q_N \to 0\). Let \(G_N(t) = F_N(\{\xi : a_i^T \xi \leq t\}) \in [0, 1]\) and \(\hat{G}_N(t) = \hat{F}_N(\{\xi : a_i^T \xi \leq t\}) \in [0, 1]\) be the CDFs of \(a_i^T \xi\) under \(F_N\) and \(\hat{F}_N\), respectively. Then,

\[q_N \geq \mathbb{E}_{F_N} [(a^T \xi)^2] - \mathbb{E}_{F_N} [(a^T \xi)^2] = 2 \int_{b=-R}^0 \left( \mathbb{E}_{F_N} [\max \{b - a_i^T \xi, 0\}] - \mathbb{E}_{F_N} [\max \{b - a_i^T \xi, 0\}] \right) db + 2 \int_0^R \left( \mathbb{E}_{F_N} [\max \{a_i^T \xi - b, 0\}] - \mathbb{E}_{F_N} [\max \{a_i^T \xi - b, 0\}] \right) db = 2 \int_{b=-R}^0 \int_{t=-R}^{b} (G_N(t) - \hat{G}_N(t)) dt db + 2 \int_0^R \int_{t=b}^R (G_N(t) - \hat{G}_N(t)) dt db \geq p_N,\]

\[\int_{t=-R}^b (\hat{G}_N(t) - G_N(t)) dt \geq -(\sqrt{d} + R)Q_{C_N}(a_1) \forall b \in [-R, 0],\]

\[\int_0^R (\hat{G}_N(t) - G_N(t)) dt \geq -(\sqrt{d} + R)Q_{C_N}(a_1) \forall b \in [0, R].\]

Because \(\hat{F}_N \to F\), we get \(\hat{G}_N(t) \to F(\{\xi : a_i^T \xi \leq t\})\) and therefore \(G_N(t) \to F(\{\xi : a_i^T \xi \leq t\})\) at every continuity point \(t\). Because true for every \(a\), the Cramer-Wold device yields \(F_N \to F\). This is the contrapositive of the uniform consistency condition. \(\square\)
9.12 Proof of Theorem 8

Proof Problem (3) is equal to the optimization problems of Theorem 7 augmented with the variable $x \in X$ and weak optimization is polynomially reducible to weak separation (see [19]). Tractable weak separation for all constraints except $x \in X$ and (27) is given by the tractable weak optimization over these standard conic-affine constraints. A weak separation oracle is assumed given for $X$. We polynomially reduce separation over $c_j \geq \max_k c_{jk}(x)$ for fixed $c_j, x'$ to the oracles. We first call the evaluation oracle for each $k$ to check violation and if there is a violation and $k^* \in \arg \max_k c_{jk}(x')$ then we call the subgradient oracle to get $s \in \partial c_{jk^*}(x')$ with $||s||_{\infty} \leq 1$ and produce the separating hyperplane $0 \geq c_{jk^*}(x') - c_j + s^T(x - x')$. □

9.13 Proof of Theorem 9

Proof Substituting the given formulas for $K_{SN}, A_{SN}, b_{SN, \alpha}$ for each $S_N \in \{D_N, V_N, W_N, U, N, A_N\}$ in $A_{SN, \zeta} = b_{SN, \alpha} \in K_{SN}$ we obtain exactly $S_N(\zeta_1, \ldots, \zeta_N) \leq Q_{SN}(\alpha)$ for $S_N$ as defined in (7). We omit the detailed arithmetic.

9.14 Proof of Theorem 11

Proof Under these assumptions (3) is equal to the optimization problems of Theorem 10 augmented with the variable $x$ and weak optimization is polynomially reducible to weak separation (see [19]). Tractable weak separation for all constraints except $x \in X$ and (29) is given by the tractable weak optimization over these standard conic-affine constraints. A weak separation oracle is assumed given for $x \in X$. By continuity and given structure of $c(x; \xi)$, we may rewrite (29) as

$$c_j \geq \max_{\xi \in [\xi^{(i-1)}, \xi^{(i)}]} c_k(x; \xi) \ \forall k = 1, \ldots, K. \quad (50)$$

We polynomially reduce weak $\delta$-separation at the $k^{th}$ constraint at fixed $c_j', x'$ to the oracles. We call the $\delta$-optimization oracle to find $\xi' \in [\xi^{(i-1)}, \xi^{(i)}]$ such that $c_j(x'; \xi') \geq \max_{\xi \in [\xi^{(i-1)}, \xi^{(i)}]} c_k(x; \xi) - \delta$. If $c_j' \geq c_j(x'; \xi')$ then $(c_j' + \delta, x')$ satisfy the constraint and is within $\delta$ of $(c_j', x')$. If $c_j' < c_j(x'; \xi')$ then we call the subgradient oracle to get $s \in \partial c_k(x', \xi')$ with $||s||_{\infty} \leq 1$ and produce the hyperplane $c_j \geq c(x'; \xi') + s^T(x - x')$ that is violated by $(c_j', x')$ and for any $(c_j, x)$ satisfying (50) (in particular if it is in the $\delta$-interior) we have $c_j \geq \max_{\xi \in [\xi^{(i-1)}, \xi^{(i)}]} c_k(x; \xi) \geq c_k(x; \xi') \geq c(x'; \xi') + s^T(x - x')$ since $s$ is a subgradient. The case for constraints (30) is similar. □

9.15 Proof of Lemma 8

Proof According to Theorem 10, the observations in Example 2, and by renaming variables, the DRO (3) is given by

$$\min \ y + \sum_{i=1}^{N} \left(Q_{DN}(\alpha) + \frac{i-1}{N}\right) s_i + \sum_{i=1}^{N} \left(Q_{DN}(\alpha) - \frac{i}{N}\right) t_i$$

s.t. $x \in \mathbb{R}^+, \ y \in \mathbb{R}, \ s \in \mathbb{R}^N, \ t \in \mathbb{R}^N$

$$(r - c)x + y + \sum_{i=j}^{N} (s_i - t_i) \geq (r - c)\xi^{(j)} \quad \forall j = 1, \ldots, N + 1$$

$$-(c - b)x + y + \sum_{i=j}^{N} (s_i - t_i) \geq -(c - b)\xi^{(j-1)} \quad \forall j = 1, \ldots, N + 1.$$
Applying linear optimization duality we get that its dual is

\[
(D) \max (r - c) \sum_{i=1}^{N+1} \xi^{(i)} p_i - (c - b) \sum_{i=1}^{N+1} \xi^{(i-1)} q_i \\
\text{s.t. } p \in \mathbb{R}_{+}^{N+1}, q \in \mathbb{R}_{+}^{N+1} \\
(r - c) \sum_{i=1}^{N+1} p_i - (c - b) \sum_{i=1}^{N+1} q_i \leq 0 \\
\sum_{i=1}^{N+1} p_i + \sum_{i=1}^{N+1} q_i = 1 \\
\sum_{i=1}^{j} p_i + \sum_{i=1}^{j} q_i \leq Q_{D_N}(\alpha) + \frac{j-1}{N} \quad \forall j = 1, \ldots, N \\
- \sum_{i=1}^{j} p_i - \sum_{i=1}^{j} q_i \leq Q_{D_N}(\alpha) - \frac{j}{N} \quad \forall j = 1, \ldots, N.
\]

It can be directly verified that the following primal and dual solutions are respectively feasible

\[x = (1 - \theta)\xi^{(i_0)} + \theta\xi^{(i_0)} ,\]
\[y = (r - c)\xi^{(N+1)} - (r - c)x ,\]
\[s_i = \begin{cases} (c - b) (\xi^{(i)} - \xi^{(i-1)}) & i \leq i_0 \\
0 & \text{otherwise} \end{cases} \quad \forall i = 1, \ldots, N ,\]
\[t_i = \begin{cases} (r - c) (\xi^{(i+1)} - \xi^{(i)}) & i \geq i_h \\
0 & \text{otherwise} \end{cases} \quad \forall i = 1, \ldots, N ,\]
\[p_i = \begin{cases} 0 & i < i_h - 1 \\
\frac{i - \theta}{i - \frac{\alpha}{Q_{D_N}(\alpha)}} & i = i_h \\
\frac{i - \frac{\alpha}{Q_{D_N}(\alpha)}}{i - N + 1} & i \geq i_h + 1 \end{cases} \quad \forall i = 1, \ldots, N \]
\[q_i = \begin{cases} 0 & i < i_h - 2 \\
\frac{\alpha}{i - \frac{\alpha}{Q_{D_N}(\alpha)}} & i = i_h \\
\frac{i - \frac{\alpha}{Q_{D_N}(\alpha)}}{i - N + 1} & i \geq i_h + 2 \end{cases} \quad \forall i = 1, \ldots, N ,\]

and that both have objective cost in their respective programs of

\[z = - (c - b)Q_{D_N}(\alpha)\xi^{(0)} - \frac{c - b}{N} \sum_{i=1}^{i_0} \xi^{(i)} - (c - b) \left(\theta - Q_{D_N}(\alpha) - \frac{i_0 - 1}{N}\right) \xi^{(i_0)} + (r - c)Q_{D_N}(\alpha)\xi^{(N+1)} + \frac{r - c}{N} \sum_{i=i_h+1}^{N} \xi^{(i)} + (r - c) \left(\frac{i_h}{N} - Q_{D_N}(\alpha) - \theta\right) \xi^{(i_0)} .\]

This proves optimality of \(x\). Adding \(0 = (c - b)\theta x - (r - c)(1 - \theta)x\) to the above yields the form of the optimal objective given in the statement of the result. \(\square\)

9.16 Proof of Theorem 12

**Proof** Fix \(x\). Let \(S = \{(a, b) \in \mathbb{R}^{d+1} : \|a\|_1 + \|b\|_1 \leq 1\}\). Using the notation of \([35]\), letting \(C\) be the cone of nonnegative measures on \(\Xi\) and \(C'\) the cone of nonnegative measures on \(S\), we write the inner problem as

\[
\sup_{F} \left\langle F, c(x; \xi) \right\rangle \\
\text{s.t. } F \in C , \left(1, F \right) = 1 \\
\sup_{(a, b) \in S} \left( \left\langle F, \max\{a^T \xi - b, 0\} \right\rangle - \frac{1}{N} \sum_{i=i_h+1}^{N} \max\{a^T \xi^{(i)} - b, 0\} - Q_{C_N}(\alpha_1) \right) \leq 0 \\
\left\langle F, \|\xi\|_2^2 \right\rangle \geq Q_{R_N}^2.
\]
Invoking Proposition 2.8 of [35] (with the generalized Slater point equal to the empirical distribution), and using the representation (36) of the cost function, we have that the strongly dual minimization problem is

\[
\min_{G, r, \theta} \quad \theta + Q_{C, \phi}(\alpha)G\{S\} + \left\langle G, \frac{1}{N} \sum_{i=1}^{N} \max\{a^T \xi^i - b, 0\} \right\rangle - Q_{R^d, \tau}^{\max}
\]

\[
s.t. \quad G \in C', \quad r \in \mathbb{R}^+, \quad \theta \in \mathbb{R}
\]

\[
\inf_{\xi \in \mathbb{R}^d} \left( \langle G, \max\{a^T \xi - b, 0\} \rangle - (p_{k2} + P_k^T x)^T \xi - \tau \||\xi\||_2^2 \right) \geq p_{k0} + p_{k1} x - \theta \quad \forall k = 1, \ldots, K. \tag{52}
\]

We will now show that only \( \tau = 0 \) is feasible. Consider any feasible solution with \( \tau > 0 \). Notice that

\[
\langle G, \max\{a^T \xi - b, 0\} \rangle - (p_{k2} + P_k^T x)^T \xi \leq (G\{S\} + ||p_{k2} + P_k^T x||_\infty) (||\xi||_1 + 1),
\]

which grows linearly with growing \( \xi \). In contrast, \( ||\xi||_2^2 \) grows quadratically, i.e. strictly faster. Therefore, the left-hand side of (52) is negative infinity but the right hand side is finite. Therefore only \( \tau = 0 \) is feasible so fix it as such. Now rewrite the \( k \)th constraint in (52) as follows

\[
p_{k0} + p_{k1} x - \theta \leq \min_{\xi \in \mathbb{R}^d, g_k} \langle G, g_k \rangle - (p_{k2} + P_k^T x)^T \xi_k
\]

\[
s.t. \quad \inf_{(a, b) \in S} (g_k(a, b) - a^T \xi_k + b) \geq 0
\]

\[
\inf_{(a, b) \in S} g_k(a, b) \geq 0.
\]

Again invoking Proposition 2.8 of [35] (with the generalized Slater point \( \xi = e, g(a, b) = \max\{a^T e - b, 0\} + 1 \)) we see that the above is equivalent to

\[
\exists H_k s.t. \quad \langle H_k, -b \rangle \geq p_{k0} + p_{k1} x - \theta
\]

\[
H_k \in C', \quad (G - H_k) \in C'
\]

\[
\langle H_k, a \rangle = p_{k2} + P_k^T x.
\]

Thus, introducing these variables \( H_k \) into the problem (51) as well as the variable \( x \in X \) and invoking Proposition 2.8 of [35] again (with the generalized Slater point being all variables zero except \( \theta \) sufficiently large) we get the strongly dual maximization problem

\[
\max_{r, s, t, \psi} \sum_{k=1}^{K} (p_{k0} r_k + p_{k2} s_k) + h^T t
\]

\[
s.t. \quad r \in \mathbb{R}^k, \quad s \in \mathbb{R}^{k \times d}, \quad t \in \mathbb{R}^d
\]

\[
\inf_{(a, b) \in S} \psi_k(a, b) \geq 0 \quad \forall k = 1, \ldots, K
\]

\[
\inf_{(a, b) \in S} \left( \psi_k(a, b) - a^T s_k + br_k \right) \geq 0 \quad \forall k = 1, \ldots, K
\]

\[
\sup_{(a, b) \in S} \left( \sum_{k=1}^{K} \psi_k(a, b) - \frac{1}{N} \sum_{i=1}^{N} \max\{a^T \xi^i - b, 0\} - Q_{C, \phi}(\alpha_1) \right) \leq 0
\]

\[
\sum_{k=1}^{K} r_k = 1
\]

\[
H^T t - \sum_{k=1}^{K} (r_k p_{k1} - P_k z_k) \leq 0
\]

\[
(53)
\]

where \( x \) is the dual variable associated with (53). Recognizing that given any feasible solution we may set \( \psi_k(a, b) = \max\{a^T s_k - br_k, 0\} \) and remain feasible with the same objective, we arrive at the formulation in the statement of the theorem. \( \square \)